Confluence of the Disjoint Union of Conditional Term Rewriting Systems

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ABSTRACT

Toyama proved that confluence is a modular property of term rewriting systems. This means that the disjoint union of two confluent term rewriting systems is again confluent. In this paper we extend his result to the class of conditional term rewriting systems. In view of the important role of conditional rewriting in equational logic programming, this result may be of relevance in integrating functional programming and logic programming.

Introduction

Two directions can be distinguished in the use of conditional term rewriting systems. Bergstra and Klop [1], Kaplan [10] and Zhang and Rémy [25] studied conditional term rewriting as a means of enhancing the expressiveness in the algebraic specification of abstract data types. Recently, serious efforts have been initiated for integrating functional and logic programming. It has been recognized that conditional term rewriting systems provide a natural computational mechanism for this integration, see Dershowitz and Plaisted [5, 6], Fribourg [7] and Goguen and Meseguer [8].

For ordinary term rewriting systems a sizeable amount of theory has been developed. Only a small part has been extended to conditional term rewriting systems, notably sufficient conditions for confluence and termination ([1], [3], [4], [9], [11]). In this paper we extend a result of Toyama [22], which states that if a term rewriting system can be partitioned into two confluent systems with disjoint alphabets then the original system is confluent, to conditional term rewriting systems.

Conditional term rewriting is introduced in the next section. In Section 2 we consider disjoint unions of term rewriting systems. In Section 3 we prove that confluence is a modular property of join systems, a particular form of conditional term rewriting introduced in the next section. In Section 4 we observe that confluence is also a modular property of so-called semi-equational and normal systems and we conclude with suggestions for further research.

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1. Conditional Term Rewriting Systems: Preliminaries

Before introducing conditional term rewriting, we review the basic notions of unconditional term rewriting. Term rewriting is surveyed in Klop [12] and Dershowitz and Jouannaud [2].

Let $\mathcal{V}$ be a countably infinite set of *variables*. An *unconditional term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$. The set $\mathcal{F}$ consists of *function symbols*; associated with every $f \in \mathcal{F}$ is its arity $n \geq 0$. Function symbols of arity 0 are called *constants*. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$, notation $\mathcal{T}(\mathcal{F}, \mathcal{V})$, is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F}, \mathcal{V})$ and if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Terms not containing variables are *ground* terms. The set $\mathcal{R}$ consists of pairs (l, r) with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ subject to two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Pairs (l, r) are called *rewrite rules* or *reduction rules* and will henceforth be written as $l \to r$. We usually present a TRS as a set of rewrite rules, without making explicit the set of function symbols.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that the set $\{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$ is finite (the symbol $\equiv$ stands for syntactic equality). This set is called the *domain* of σ and will be denoted by $\mathcal{D}(\sigma)$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e. $\sigma(f(t_1, \ldots, t_n)) \equiv f(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol f and terms $t_1, \ldots, t_n$. We call $\sigma(t)$ an *instance* of t. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). Occasionally we present a substitution σ as $\sigma = \{x \to \sigma(x) \mid x \in \mathcal{D}(\sigma)\}$. The *empty* substitution will be denoted by ε (here $\mathcal{D}(\varepsilon) = \varnothing$).

A *context* $C[\,, \ldots, \,]$ is a 'term' which contains at least one occurrence of a special symbol $\Box$. If $C[\,, \ldots, \,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

The *rewrite relation* $\to_{\mathcal{R}} \subset \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. The transitive-reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$; if $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t. We write $s \leftarrow_{\mathcal{R}} t$ if $t \to_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The symmetric closure of $\to_{\mathcal{R}}$ is denoted by $\leftrightarrow_{\mathcal{R}}$. The transitive-reflexive closure of $\leftrightarrow_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible*. Two terms t_1, t_2 are *joinable*, notation $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathcal{R}} t_3 \twoheadleftarrow_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. The relation $\downarrow_{\mathcal{R}}$ is called *joinability*. We often omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there are no terms t with $s \to t$. A TRS is *terminating* or *strongly normalizing* if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \ldots$. In other words, every reduction sequence eventually ends in a normal form. A TRS is *confluent* or has the *Church-Rosser* property if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$).

The rewrite rules of a *conditional term rewriting system* (CTRS) have the form

$$l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$$

with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The equations $s_1 = t_1, \ldots, s_n = t_n$ are the *conditions* of the rewrite rule. Depending on the interpretation of the =-sign in the conditions, different rewrite relations can be associated with a given CTRS. In this paper we restrict ourselves to the three most common interpretations.

(1) In a *join* CTRS the =-sign in the conditions is interpreted as *joinability*. Formally: $s \to t$ if there exists a conditional rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$, a substitution σ and a context $C[\;]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow \sigma(t_i)$ for all $i \in \{1, \ldots, n\}$. Rewrite rules of a join CTRS will henceforth be written as

$$l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n.$$

(2) *Semi-equational* CTRS's are obtained by interpreting the =-sign in the conditions as *conversion*.

(3) In a *normal* CTRS $\mathcal{R}$ the rewrite rules are subject to the additional constraint that every t_i is a ground normal form with respect to the unconditional TRS obtained from $\mathcal{R}$ by omitting the conditions. The rewrite relation associated with a normal CTRS is obtained by interpreting the equality sign in the conditions as *reduction* ($\twoheadrightarrow$).

This classification originates essentially from Bergstra and Klop [1]. The nomenclature stems from Dershowitz, Okada and Sivakumar [4].

The restrictions we impose on CTRS's $\mathcal{R}$ in any of the three formulations are the same as for unconditional TRS's: if $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$ is a rewrite rule of $\mathcal{R}$ then l is not a single variable and variables occurring in r also occur in l. In particular, extra variables in the conditions are perfectly acceptable. In Section 4 we will discuss the technical problems associated with a possible relaxation of this requirement.

Sufficient conditions for the termination of CTRS's were given by Kaplan [11], Jouannaud and Waldmann [9] and Dershowitz, Okada and Sivakumar [4]. Sufficient conditions for confluence can be found in Bergstra and Klop [1] and Dershowitz, Okada and Sivakumar [3].

EXAMPLE 1.1. The semi-equational CTRS

$$\mathcal{R}_1 = \left\{ \begin{array}{lllll} a & \to & b & & \\ a & \to & c & & \\ b & \to & c & \Leftarrow & b = c \end{array} \right.$$

is easily shown to be confluent. So conversion in that system coincides with joinability. However, the corresponding join CTRS

$$\mathcal{R}_2 = \left\{ \begin{array}{lllll} a & \to & b & & \\ a & \to & c & & \\ b & \to & c & \Leftarrow & b \downarrow c \end{array} \right.$$

is not confluent: the reduction step from b to c is no longer allowed.

The following inductive definition of $\rightarrow_{\mathcal{R}}$ is fundamental ([1, 3, 4]) for analyzing the behaviour of CTRS's.

DEFINITION 1.2. Let $\mathcal{R}$ be a join, semi-equational or normal CTRS. We inductively define TRS's $\mathcal{R}_i$ for $i \geq 0$ as follows ($\square$ denotes $\downarrow$, $=$ or $\twoheadrightarrow$):

$$\mathcal{R}_0 = \{l \rightarrow r \mid l \rightarrow r \in \mathcal{R}\}$$
$$\mathcal{R}_{i+1} = \{\sigma(l) \rightarrow \sigma(r) \mid l \rightarrow r \Leftarrow s_1 \square t_1, \ldots, s_n \square t_n \in \mathcal{R} \text{ and } \sigma(s_j) \square_{\mathcal{R}_i} \sigma(t_j) \text{ for } j = 1, \ldots, n\}.$$

Observe that $\mathcal{R}_i \subseteq \mathcal{R}_{i+1}$ for all $i \geq 0$. We have $s \rightarrow_{\mathcal{R}} t$ if and only if $s \rightarrow_{\mathcal{R}_i} t$ for some $i \geq 0$. The *depth* of a rewrite step $s \rightarrow_{\mathcal{R}} t$ is defined as the minimum i such that $s \rightarrow_{\mathcal{R}_i} t$. Depths of conversions $s =_{\mathcal{R}} t$ and valleys $s \downarrow_{\mathcal{R}} t$ are similarly defined.

2. Modular Properties

In this section we review some of the results that have been obtained concerning the disjoint union of TRS's. We will also give the necessary technical definitions and notations for dealing with disjoint unions. These are consistent with [22, 24, 15].

DEFINITION 2.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRS's with disjoint alphabets (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$). The *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ is the CTRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$.

DEFINITION 2.2. A property $\mathcal{P}$ of CTRS's is called *modular* if for all CTRS's $\mathcal{R}_1$, $\mathcal{R}_2$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P} \iff \text{both } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ have the property } \mathcal{P}.$$

All previous research on modularity has been carried out in the world of unconditional TRS's. This research can be characterized by the phrase "simple statements, complicated proofs". Confluence was the first property for which the modularity has been established.

THEOREM 2.3 (Toyama [22]). *Confluence is a modular property of* TRS'*s*. $\square$

Toyama also gave the following simple example showing that termination is not modular.

EXAMPLE 2.4 (Toyama [23]). Let $\mathcal{R}_1 = \{F(0, 1, x) \rightarrow F(x, x, x)\}$ and

$$\mathcal{R}_2 = \begin{cases} or(x, y) & \rightarrow & x, \\ or(x, y) & \rightarrow & y. \end{cases}$$

Both systems are terminating, but in $\mathcal{R}_1 \oplus \mathcal{R}_2$ the term $F(or(0, 1), or(0, 1), or(0, 1))$ reduces to itself.

Other modularity results are presented in [13, 14, 15, 16, 21, 23, 24]. Middeldorp [20] contains a comprehensive survey.

Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint CTRS's. Every term $t \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ can be viewed as an alternation of $\mathcal{F}_1$-parts and $\mathcal{F}_2$-parts. This structure is formalized in Definition 2.5 and illustrated in Figure 1.

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ to $\mathcal{T}$ and we will use $\mathcal{T}_i$ as a shorthand for $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$ $(i = 1, 2)$.

DEFINITION 2.5.

(1) The *root symbol* of a term t, notation $root(t)$, is defined by

$$root(t) = \begin{cases} F & \text{if } t \equiv F(t_1, \ldots, t_n), \\ t & \text{otherwise.} \end{cases}$$

(2) Let $t \equiv C[t_1, \ldots, t_n] \in \mathcal{T}$ with $C[\,, \ldots, \,] \not\equiv \square$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[\,, \ldots, \,]$ is a $\mathcal{F}_a$-context and $root(t_1), \ldots, root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. The t_i's are the *principal* subterms of t.

(3) If $t \in \mathcal{T}$ then

$$rank(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1 + \max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!]. \end{cases}$$

(4) A subterm s of t is *special* if $s \equiv t$ or s is a special subterm of a principal subterm of t.

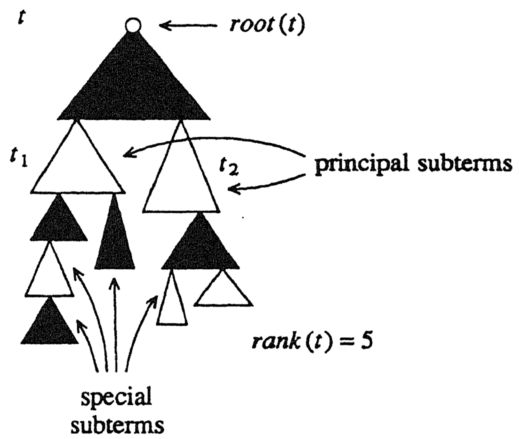

FIGURE 1.

To achieve better readability we will call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. Variables have no colour. A black (white) term does not contain white (black) function symbols, but may contain variables. In examples, black symbols will be printed as capitals and white symbols in lower case.

PROPOSITION 2.6. *If* $s \twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ *then* $rank(s) \geq rank(t)$.
PROOF. Straightforward. □

DEFINITION 2.7. Let $s \rightarrow t$ by application of a rewrite rule $l \rightarrow r$. We write $s \rightarrow^i t$ if $l \rightarrow r$ is being applied in one of the principal subterms of s and we write $s \rightarrow^o t$ otherwise. The relation $\rightarrow^i$ is called *inner* reduction and $\rightarrow^o$ is called *outer* reduction.

DEFINITION 2.8. Suppose σ and τ are substitutions. We write $\sigma \propto \tau$ if $\sigma(x) \equiv \sigma(y)$ implies $\tau(x) \equiv \tau(y)$ for all $x, y \in \mathcal{V}$. Notice that $\sigma \propto \varepsilon$ if and only if σ is injective. We write $\sigma \twoheadrightarrow \tau$ if $\sigma(x) \twoheadrightarrow \tau(x)$ for all $x \in \mathcal{V}$. Clearly $\sigma(t) \twoheadrightarrow \tau(t)$ whenever $\sigma \twoheadrightarrow \tau$.

DEFINITION 2.9. A substitution σ is called *black* (*white*) if $\sigma(x)$ is a black (white) term for every $x \in \mathcal{D}(\sigma)$. We call σ *top black* (*top white*) if the root symbol of $\sigma(x)$ is black (white) for every $x \in \mathcal{D}(\sigma)$.

Notice the subtle difference in handling variables: the substitution $\sigma = \{x \rightarrow F(y), y \rightarrow x\}$ is black but not top black. The following proposition is frequently used in the next section.

PROPOSITION 2.10. *Every substitution* σ *can be decomposed into* $\sigma_2 \circ \sigma_1$ *such that* σ_1 *is black* (*white*), σ_2 *is top white* (*top black*) *and* $\sigma_2 \propto \varepsilon$.
PROOF. Let $\{t_1, \ldots, t_n\}$ be the set of all maximal subterms of $\sigma(x)$ for $x \in \mathcal{D}(\sigma)$ with white root. Choose distinct fresh variables $z_1, \ldots, z_n$ and define the substitution σ_2 by $\sigma_2 = \{z_i \rightarrow t_i \mid 1 \leq i \leq n\}$. Let $x \in \mathcal{D}(\sigma)$. We define $\sigma_1(x)$ by case analysis.

(1) If the root symbol of $\sigma(x)$ is white then $\sigma(x) \equiv t_i$ for some $i \in \{1, \ldots, n\}$. In this case we define $\sigma_1(x) \equiv z_i$.
(2) If $\sigma(x)$ is a black term then we take $\sigma_1(x) \equiv \sigma(x)$.
(3) In the remaining case we can write $\sigma(x) \equiv C[\![t_{i_1}, \ldots, t_{i_k}]\!]$ for some $1 \leq i_1, \ldots, i_k \leq n$ and we define $\sigma_1(x) \equiv C[z_{i_1}, \ldots, z_{i_k}]$.

By construction we have $\sigma_2 \propto \varepsilon$, σ_1 is black and σ_2 is top white. □

3. Modularity of Confluence for Join Systems

In this section we show that confluence is a modular property of join CTRS's. To this end, we assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint confluent join CTRS's. We assume furthermore that all rewrite relations introduced in this section are defined on $\mathcal{T}$, unless stated otherwise. The same assumption is made for terms.

The fundamental property of the disjoint union of two unconditional TRS's $\mathcal{R}_1$ and $\mathcal{R}_2$, that is to say $s \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ implies either $s \rightarrow_{\mathcal{R}_1} t$ or $s \rightarrow_{\mathcal{R}_2} t$, does no longer hold for CTRS's, as can be seen from the next example.

EXAMPLE 3.1. Let $\mathcal{R}_1 = \{F(x, y) \rightarrow G(x) \Leftarrow x \downarrow y\}$ and $\mathcal{R}_2 = \{a \rightarrow b\}$. We have $F(a, b)$

$\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} G(a)$ because $a \downarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} b$, but neither $F(a, b) \rightarrow_{\mathcal{R}_1} G(a)$ nor $F(a, b) \rightarrow_{\mathcal{R}_2} G(a)$.

The problem is that when a rule of one of the CTRS's is being applied, rules of the other CTRS may be needed in order to satisfy the conditions. So the question arises how the rewrite relation $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ is related to $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$. In the example above we have $F(a, b) \rightarrow_{\mathcal{R}_2} F(b, b) \rightarrow_{\mathcal{R}_1} G(b) \leftarrow_{\mathcal{R}_2} G(a)$. This suggests that $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ corresponds to joinability with respect to the union of $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$. However, it turned out that $\rightarrow_{\mathcal{R}_1} \cup \rightarrow_{\mathcal{R}_2}$ is not an entirely satisfactory relation from a technical viewpoint. For instance, confluence of $\rightarrow_{\mathcal{R}_1} \cup \rightarrow_{\mathcal{R}_2}$ is not easily proved (cf. Lemma 3.6). We will define two more manageable rewrite relations $\rightarrow_1$ and $\rightarrow_2$ such that:

(1) their union is confluent (Lemma 3.6),

(2) reduction in $\mathcal{R}_1 \oplus \mathcal{R}_2$ corresponds to joinability with respect to $\rightarrow_1 \cup \rightarrow_2$ (Lemma 3.8).

From these two properties the modularity of confluence for join CTRS's is easily inferred (Theorem 3.9). The proof of the first property is a more or less straightforward reduction to Toyama's confluence result for the disjoint union of TRS's. The proof of the second property is rather technical but we believe that the underlying ideas are simple.

DEFINITION 3.2. The rewrite relation $\rightarrow_1$ is defined as follows: $s \rightarrow_1 t$ if there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a context $C[\,]$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i = 1, \ldots, n$, where the superscript o in $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ means that $\sigma(s_i)$ and $\sigma(t_i)$ are joinable using only *outer* $\rightarrow_1$-reduction steps. Notice that the restrictions of $\rightarrow_1$ and $\rightarrow_{\mathcal{R}_1}$ to $\mathcal{T}_1 \times \mathcal{T}_1$ coincide. The relation $\rightarrow_2$ is defined similarly.

EXAMPLE 3.3. Let

$$\mathcal{R}_1 = \begin{cases} F(x, y) & \rightarrow & G(x) & \Leftarrow & x \downarrow y \\ A & \rightarrow & B \end{cases}$$

and suppose $\mathcal{R}_2$ contains an unary function symbol g. We have $F(g(A), g(B)) \rightarrow_{\mathcal{R}_1} G(g(A))$ but not $F(g(A), g(B)) \rightarrow_1 G(g(A))$ because $g(A)$ and $g(B)$ are different normal forms with respect to $\rightarrow_1^o$. The terms $F(g(A), g(B))$ and $G(g(A))$ are joinable with respect to $\rightarrow_1$: $F(g(A), g(B)) \rightarrow_1 F(g(B), g(B)) \rightarrow_1 G(g(B)) \leftarrow_1 G(g(A))$.

NOTATION. The union of $\rightarrow_1$ and $\rightarrow_2$ is denoted by $\rightarrow_{1,2}$ and we abbreviate $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ to $\rightarrow$.

PROPOSITION 3.4. *If $s \rightarrow_{1,2} t$ then $s \rightarrow t$.*

PROOF. Trivial. □

The next proposition states a desirable property of $\rightarrow_1^o$-reduction. The proof however is more complicated than the analogical statement for TRS's (cf. Lemma 3.2 in [22]).

PROPOSITION 3.5. *Let s, t be black terms and suppose σ is a top white substitution such that $\sigma(s) \rightarrow_1^o \sigma(t)$. If τ is a substitution with $\sigma \propto \tau$ then $\tau(s) \rightarrow_1^o \tau(t)$.*

PROOF. We prove the statement by induction on the depth of $\sigma(s) \to_1^o \sigma(t)$. The case of zero depth is straightforward. If the depth of $\sigma(s) \to_1^o \sigma(t)$ equals $n+1$ ($n \geq 0$) then there exist a context $C[\,]$, a substitution ρ and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ in $\mathcal{R}_1$ such that $\sigma(s) \equiv C[\rho(l)]$, $\sigma(t) \equiv C[\rho(r)]$ and $\rho(s_i) \downarrow_1^o \rho(t_i)$ for $i = 1, \ldots, m$ with depth less than or equal to n. Proposition 2.10 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 is top white and $\rho_2 \propto \varepsilon$. The situation is illustrated in Figure 2. We define the substitution ρ^* by

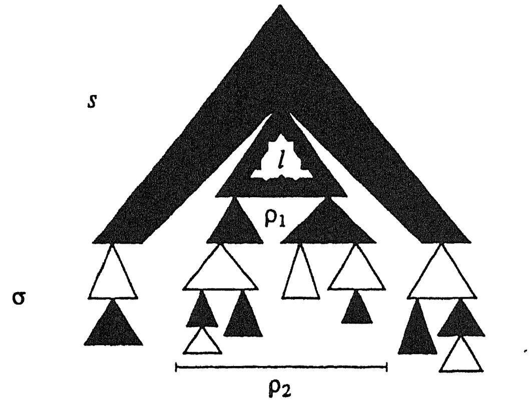

FIGURE 2.

$\rho^*(x) \equiv \tau(y)$ for every $x \in \mathcal{D}(\rho_2)$ and $y \in \mathcal{D}(\sigma)$ satisfying $\rho_2(x) \equiv \sigma(y)$. Notice that ρ^* is well-defined by the assumption $\sigma \propto \tau$. We have $\rho_2 \propto \rho^*$ since $\rho_2 \propto \varepsilon$ and $\varepsilon \propto \rho^*$. Combined with $\rho_2(\rho_1(s_i)) \downarrow_1^o \rho_2(\rho_1(t_i))$, the induction hypothesis and the observation that if $\rho_2(u_1) \to_1^o u_2$ and u_1 is a black term then $u_2 \equiv \rho_2(u_3)$ for some black term u_3, we obtain $\rho^*(\rho_1(s_i)) \downarrow_1^o \rho^*(\rho_1(t_i))$ by a straightforward induction on the length of the conversion $\rho_2(\rho_1(s_i)) \downarrow_1^o \rho_2(\rho_1(t_i))$ for $i = 1, \ldots, m$ (see Figure 3). Hence $\rho^*(\rho_1(l)) \to_1^o \rho^*(\rho_1(r))$. Let $C^*[\,]$ be the context obtained

$$
\begin{tikzcd}
\rho_2(\rho_1(s_i)) \arrow[r] \arrow[d, phantom, "\equiv"] & u_1 \arrow[r] \arrow[d, phantom, "\equiv"] & u_2 \arrow[d, phantom, "\equiv"] & \rho_2(\rho_1(t_i)) \arrow[l] \arrow[d, phantom, "\equiv"] & \text{observation} \\
\rho_2(\rho_1(s_i)) \arrow[r] \arrow[d, rightsquigarrow] & \rho_2(v_1) \arrow[r] \arrow[d, rightsquigarrow] & \rho_2(v_2) \arrow[d, rightsquigarrow] & \rho_2(\rho_1(t_i)) \arrow[l] \arrow[d, rightsquigarrow] & \text{induction hypothesis} \\
\rho^*(\rho_1(s_i)) \arrow[r] & \rho^*(v_1) \arrow[r] & \rho^*(v_2) & \rho^*(\rho_1(t_i)) \arrow[l] &
\end{tikzcd}
$$

$$\longrightarrow \;\equiv\; \longrightarrow_1^o$$

FIGURE 3.

from $C[\,]$ by replacing every principal subterm, which has the form $\sigma(x)$ for some variable $x \in \mathcal{D}(\sigma)$, by the corresponding $\tau(x)$. We leave it to the motivated reader to show that $\tau(s) \equiv C^*[\rho^*(\rho_1(l))]$ and $\tau(t) \equiv C^*[\rho^*(\rho_1(r))]$. We conclude that $\tau(s) \to_1^o \tau(t)$. □

LEMMA 3.6. *The rewrite relation $\rightarrow_{1,2}$ is confluent.*

PROOF. Define the unconditional TRS's $\mathcal{S}_1$ and $\mathcal{S}_2$ by $(i=1,2)$

$$\mathcal{S}_i = \{s \rightarrow t \mid s, t \in \mathcal{T}_i \text{ and } s \rightarrow^i t\}.$$

With some effort we can show that the restrictions of $\rightarrow_{\mathcal{S}_i}$, $\rightarrow_i$ and $\rightarrow_{\mathcal{R}_i}$ to $\mathcal{T}_i \times \mathcal{T}_i$ are the same†. Therefore $\mathcal{S}_1$ and $\mathcal{S}_2$ are confluent TRS's. Theorem 2.3 yields the confluence of $\mathcal{S}_1 \oplus \mathcal{S}_2$. We will show that $\rightarrow_{\mathcal{S}_i}$ and $\rightarrow^i$ coincide (on $\mathcal{T} \times \mathcal{T}$). Without loss of generality, we only consider the case $i=1$.

$\subseteq$ If $s \rightarrow_{\mathcal{S}_1} t$ then there exists a rewrite rule $l \rightarrow r$ in $\mathcal{S}_1$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. By definition $l \rightarrow_1 r$, from which we immediately obtain $s \rightarrow_1 t$.

$\supseteq$ If $s \rightarrow_1 t$ then there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i=1,\ldots,n$. According to Proposition 2.10 we can decompose σ into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white and $\sigma_2 \propto \varepsilon$. Induction on the number of rewrite steps in $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ together with Proposition 3.5 and the observation made in the proof of Proposition 3.5 yields $\sigma_1(s_i) \downarrow_1^o \sigma_1(t_i)$ $(i=1,\ldots,n)$. Hence $\sigma_1(l) \rightarrow_1 \sigma_1(r)$. Because $\sigma_1(l)$ and $\sigma_1(r)$ are black terms, $\sigma_1(l) \rightarrow \sigma_1(r)$ is a rewrite rule of $\mathcal{S}_1$. Therefore $s \equiv C[\sigma_2(\sigma_1(l))] \rightarrow_{S_1} C[\sigma_2(\sigma_1(r))] \equiv t$.

Now we have $\rightarrow_{\mathcal{S}_1 \oplus \mathcal{S}_2} = \rightarrow_{\mathcal{S}_1} \cup \rightarrow_{\mathcal{S}_2} = \rightarrow_1 \cup \rightarrow_2 = \rightarrow_{1,2}$. Therefore $\rightarrow_{1,2}$ is confluent. □

Due to space limitations, the reader is referred to the full version [17] of this paper for the complicated proof of the next proposition. The proof can also be found in [20].

PROPOSITION 3.7. *Let $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. For every substitution σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i=1,\ldots,n$ there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ for $i=1,\ldots,n$.* □

LEMMA 3.8. *If $s \rightarrow t$ then $s \downarrow_{1,2} t$.*

PROOF. We use induction on the depth of $s \rightarrow t$. The case of zero depth is trivial. Suppose the depth of $s \rightarrow t$ equals $n+1$ $(n \geq 0)$. By definition there exist a context $C[\,]$, a substitution σ and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ in $\mathcal{R}_1 \oplus \mathcal{R}_2$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow \sigma(t_i)$ $(i=1,\ldots,m)$ with depth less than or equal to n. Using the induction hypothesis and Lemma 3.6 we obtain $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i=1,\ldots,m)$, see Figure 4 where (1) is obtained from the induction hypothesis and (2) signals an application of Lemma 3.6. Without loss of generality we assume that the applied rewrite rule stems from $\mathcal{R}_1$. Proposition 3.7 yields a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i=1,\ldots,m)$. The next conversion shows that $s \downarrow_{1,2} t$:

$$s \equiv C[\sigma(l)] \twoheadrightarrow_{1,2} C[\tau(l)] \rightarrow_1 C[\tau(r)] \twoheadleftarrow_{1,2} C[\sigma(r)] \equiv t.$$

□

† A minor technical complication is caused by rewrite rules containing extra variables in the conditions.

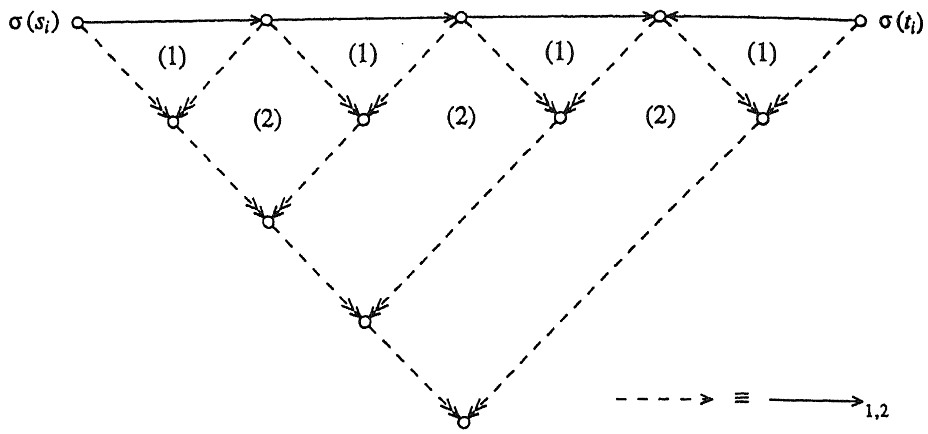

FIGURE 4.

THEOREM 3.9. *Confluence is a modular property of join* CTRS'*s*.

PROOF. Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint join CTRS's. We must prove the following equivalence: $\mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent $\Leftrightarrow$ both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent.

$\Rightarrow$ Trivial.

$\Leftarrow$ Easy consequence of Proposition 3.4, Lemma 3.6 and Lemma 3.8.

□

4. Concluding Remarks

In the previous section we have shown that confluence is modular property of join CTRS's. Since every normal CTRS can be viewed as a join CTRS, this result also holds for normal CTRS's. Confluence is also a modular property of semi-equational CTRS's. The proof has exactly the same structure, apart from the proof of Proposition 3.5, which is more complicated because the observation made in order to make the second induction hypothesis applicable is no longer sufficient. Details can be found in [17] or [20]. It is conceivable that we might prove a more general theorem from which we not only immediately obtain the above results, but also

(1) the modularity of confluence for other kinds of CTRS's like *normal-join* systems or *meta-conditional* systems (see [4]), and

(2) confluence results for the disjoint union of two different kinds of CTRS's.

This matter clearly has to be further pursued.

Another point which needs investigation is the syntactic restrictions imposed on the rewrite rules. From a programming point of view the assumption of a rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$ satisfying the requirement that r only contains variables occurring in l, is too restrictive. A semi-equational CTRS like ([3])

$$\mathcal{R} = \begin{cases} Fib(0) & \to \; <0, S(0)> \\ Fib(S(x)) & \to \; <z, A(y, z)> \; \Leftarrow \; Fib(x) = <y, z> \end{cases}$$

should be perfectly legitimate. The CTRS's $\mathcal{R}$ we are interested in, can be characterized by the phrase "if $s \rightarrow_{\mathcal{R}} t$ then $s \rightarrow t$ is a legal unconditional rewrite rule". However, the proofs in the preceding sections cannot easily be modified to cover these systems. For instance, Proposition 2.6 is no longer true and the proof of Proposition 3.5 seems insufficient.

In [18] and [19] we extended several other modularity results for TRS's to CTRS's.

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