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MC

AFDELING ZUIVERE WISKUNDE ZW 21/74 JANUARY

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THE TRUNCATED-AVERAGE LIMIT AND THE CESÀRO LIMIT ARE INDEPENDENT

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS (MOS) subject classification scheme (1970): 40G99

The truncated-average limit and the Cesàro limit are independent

by

P. van Emde Boas

Abstract

In [1] J. van de Lune introduced the truncated-average limit of a sequence of real numbers. By its definition this concept seems a generalization of the Cesàro limit. In fact the two limits are independent; existence of one limit does not imply the existence of the other and from the existence of both limits their equality cannot be concluded.

Let $\underline{a} = (a_n)_{n=1}^{\infty}$ be a sequence of non-negative real numbers. We define the functions g, ϕ and Φ by

$$g(n,A) = \frac{1}{n} \sum_{k=1}^{n} \min (a_k, A)$$

$$\phi(A) = \liminf_{n\to\infty} g(n,A) \quad \text{and} \quad \Phi(A) = \limsup_{n\to\infty} g(n,A).$$

Clearly both ϕ and Φ are non-decreasing functions.
If $\lim_{A\to\infty} \phi(A) = \lim_{A\to\infty} \Phi(A) = p < \infty$ then p is called the *truncated-average limit* (tal) of the sequence $\underline{a}$. This concept was introduced by J. van de Lune in [1].

For a convergent non-negative sequence one clearly has $\lim_{n\to\infty} q_n = \text{tal}(a)$, hence the tal generalizes the usual limit concept.

It is not difficult to show that the sequences $\underline{a}$ for which the tal is defined form a vector lattice E_t^+ on which tal acts as a linear functional. Consequently the tal can be extended to the complete vector space $E_t = E_t^+ - E_t^+$. It can be shown that the positive elements in E_t are all contained in E_t^+ (cf. [1]).

The Cesàro-limit of a sequence $\underline{a} = (a_n)_{n=1}^{\infty}$ by definition equals

$$cs(\underline{a}) = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} a_k \right)$$

whenever the limit on the right-hand side exists.

Since for each A, $g(n,A) \le \frac{1}{n} (\sum_{k=1}^{n} a_k)$ and, moreover, $\lim_{A\to\infty} g(n,A) = \frac{1}{n} (\sum_{k=1}^{n} a_k)$ one gets the impression that the two limit concepts are related. Below it will be shown that actually the two concepts are logically independent.

Our result is based upon the following

Lemma 1. There exists a non-negative sequence $\underline{a}$ with $cs(\underline{a}) = 1$ and $\text{tal}(\underline{a}) = 0$.

Proof. Define the sequence $(a_n)_{n=1}^{\infty} = \underline{a}$ as follows:

$$\begin{cases} a_n = 2m + 1 & \text{if} \quad n = m^2, \\ a_n = 0 & \text{if} \quad n \text{ is not a square.} \end{cases}$$

Consequently the sequence $(a_n)_{n=1}^{\infty}$ starts like

$$3,0,0,5,0,0,0,0,7,0,0,0,0,0,0,9,0,0,\ldots.$$

Let $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. Clearly $b_n = 1$ whenever $n = k^2 - 1$ and $b_n = 1 + \frac{2}{k}$ for $n = k^2$. For $k^2 \le n \le (k+1)^2 - 1 = k^2 + 2k$ one has $1 \le b_n \le 1 + \frac{2}{k}$. Consequently $\lim_{n\to\infty} \frac{1}{n} b_n = 1$, so that the Cesãro limit of $\underline{a}$ exists and equals 1.

To evaluate tal($\underline{a}$) we estimate $\min(a_n, A)$ by c_n where

$$\begin{cases} c_n = A & \text{if} \quad n \text{ is a square,} \\ c_n = 0 & \text{if} \quad n \text{ is not a square.} \end{cases}$$

Consequently $g(n,A) \le \frac{1}{n} A \,\#\, \{m \mid m^2 < n\} = \frac{A}{n}[\sqrt{n}]$. Hence $\limsup_{n\to\infty} g(n,A) = \liminf_{n\to\infty} g(N,A) = 0$ regardless of the value of A. This shows tal($\underline{a}$) = 0. □

Let $(a_n)_{n=1}^{\infty} = \underline{a}$ and $(b_n)_{n=1}^{\infty} = \underline{b}$ be to non-negative sequences. The sequence $(c_n)_{n=1}^{\infty} = \underline{c}$ is called a *mixture* of $\underline{a}$ and $\underline{b}$ if each initial segment of $\underline{c}$ consists of all terms of two initial segments of the sequences $\underline{a}$ and $\underline{b}$. In such a case there exist two non-decreasing integral sequences $(i_k)_{k=1}^{\infty}$ and $(j_k)_{k=1}^{\infty}$ such that $i_k + j_k = k$, and such that $c_k = a_{i_k}$ iff $i_k > i_{k-1}$ and $c_k = b_{j_k}$ iff $j_k > j_{k-1}$. The number $\theta_k = i_k/k$ indicates the portion of terms in $\underline{c}$, taken from the sequence $\underline{a}$.

Lemma 2. Suppose that $\underline{a}$ and $\underline{b}$ are two sequences with tal($\underline{a}$) = tal($\underline{b}$) = p (cs($\underline{a}$) = cs($\underline{b}$) = p), and let $\underline{c}$ be a mixture of $\underline{a}$ and $\underline{b}$. Then tal($\underline{c}$) exists and equals p. (cs($\underline{c}$)=p).

Proof. We prove this for the tal (the case for the Cesãro limit being analogous). Since

$$g_{\underline{c}}(n,A) = \frac{1}{n}\sum_{k=1}^{n} \min(c_k,A) =$$

$$= \frac{1}{n}\left(\sum_{k=1}^{i_k} \min(a_k,A) + \sum_{k=1}^{j_k} \min(b_k,A)\right) =$$

$$= \theta_n \, g_{\underline{a}}(i_n,A) + (1-\theta_n)\, g_{\underline{b}}(j_n,A),$$

one has

$$\liminf_{n\to\infty} g_{\underline{c}}(n,A) \geq \min(\phi_{\underline{a}}(A),\phi_{\underline{b}}(A))$$

and

$$\limsup_{n\to\infty} g_{\underline{c}}(n,A) \leq \max(\Phi_{\underline{a}}(A),\Phi_{\underline{b}}(A))$$

and consequently

$$\lim_{A\to\infty} \phi_{\underline{c}}(A) = \lim_{A\to\infty} \Phi_{\underline{c}}(A) = \mathrm{tal}(\underline{a}) = \mathrm{tal}(\underline{b}). \quad \square$$

Theorem 3. There exist sequences $\underline{x} = (x_n)_{n=1}^{\infty}$ and $\underline{y} = (y_n)_{n=1}^{\infty}$ such that

(i) tal($\underline{x}$) exists and cs($\underline{x}$) does not exist,

(ii) tal($\underline{y}$) does not exist but cs($\underline{y}$) exists.

Proof. Let $\underline{a}$ be the sequence constructed to prove lemma 1. Let $\underline{p}$ be a sequence with constant terms zero and let $\underline{q}$ be a sequence with constant terms one. By the preceding lemma we have tal($\underline{x}$) = 0 for each mixture $\underline{x}$ of $\underline{a}$ and $\underline{p}$ whereas cs($\underline{y}$) = 1 whenever $\underline{y}$ is a mixture of $\underline{a}$ and $\underline{q}$.

Let $\underline{x}$ be a mixture of $\underline{a}$ and $\underline{p}$ which is defined as follows: on the interval $(2n)! \leq i < (2n+1)!$ the terms of $\underline{x}$ are selected from $\underline{p}$ (i.e. $j_i = j_{i-1}+1$). On the interval $(2n+1)! \leq i < (2n+2)!$ all the terms of $\underline{x}$ are taken from $\underline{a}$.

Now

$$\frac{1}{n}\sum_{k\leq n} x_k = \frac{1}{n}\left(\sum_{k\leq i_n} a_k + \sum_{k\leq j_n} p_k\right) = \frac{i_n}{n}\cdot\frac{1}{i_n}\sum_{k\leq i_n} a_k =$$

$$= \theta_n \cdot \frac{1}{i_n}\sum_{k\leq i_n} a_k.$$

Since $cs(\underline{a}) = 1$ one has

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k\leq n} x_k = (\liminf_{n\to\infty} \theta_n) \cdot 1 = 0$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k\leq n} x_k = (\limsup_{n\to\infty} \theta_n) \cdot 1 = 1.$$

Consequently $cs(\underline{x})$ is not defined. This proves (i).

To prove (ii) one constructs a similar mixture of $\underline{a}$ and $\underline{q}$. □

Reference

[1] J. van de Lune, *The truncated average limit and some of its applications in analytic number theory*, Mathematical Centre Report ZW 20/74, Amsterdam, 1974.