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THE INFINITE HORIZON OPTIMAL CONTROL PROBLEM ON MANIFOLDS

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The infinite horizon optimal control problem on manifolds*)

by

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ABSTRACT

In this paper we present a differential geometric approach to the infinite horizon optimal control problem for nonlinear time-invariant control systems. It uses a recently proposed fibre bundle approach for the definition of nonlinear systems. The approach yields a coordinate free first order characterization of optimal curves without a priori regularity conditions. The usefulness of the approach is motivated and illustrated with the linear-quadratic optimal control problem.

KEY WORDS & PHRASES: *nonlinear system theory, optimal control problems on manifolds, infinite horizon problems, first order conditions, Lagrange multiplier rule*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Many problems in applied science can be formulated as optimal control problems. I.e. as a problem of steering a given system from a given initial point at time $t = 0$ to a prescibed target point (or -set) at time $t = T$, such that a certain cost functional is minimized. Examples can be found in engineering (control of satellites or distillation columns), see for instance ATHANS & FALB [1966], or economics (production planning, economic growth planning). In some of these examples, particularly in economics, it is natural to choose an infinite time interval ($T = \infty$). The following problem of optimal economic growth for a one sector closed economy with a single homogeneous good, described in INTRILIGATOR [1980], is a simple example. We seek to maximize with respect to the control c ($c(t)$ is the consumption per worker at time t), the social welfare functional:

$$W(c) = \int_{t_0}^{\infty} e^{-\delta(t-t_0)} U(k(t),c(t))dt,$$

with state variable k (capital per worker) satisfying the (nonlinear) system equations

$$\dot{k} = f(k) - \lambda k - c, \qquad k(t_0) = k_0$$

and restrictions: $0 \leq c \leq f(k)$, c piecewise continuous. We see that the welfare functional is defined by a utility function U with discounting at rate $\delta > 0$. The system equations are given by the observation that the rate of change of the capital per worker equals $f(k)$ (the level of output per worker, which is usually an increasing function of k) minus a constant λ (the depreciation rate of capital) times capital per worker, minus consumption per worker. One may choose the simplest form of the utility function depending on the consumption c only. However, other choices are imaginable too, depending on k but also depending on $\dot{c}$ of $\dot{k}$. For instance one might, for social stability reasons, want to avoid fast fluctuations in consumption and/or capital per worker and therefore add a penalizing term of the form $|\dot{c}|^2$ ($|\dot{k}|^2$). Note moreover that the restriction on the consumption might also be incorporated in the utility function by using an interior penalty function as known from mathematical programming techniques.

We see that generally an optimal control problem is given by:

- a *control system* (a differential equation in the state variables depending on controls),
- a *time interval* $[0,T]$ with $T \in \mathbb{R}_+$ or $T = \infty$,
- an *initial state* and a *target set* (a subset in state space, possibly consisting of one point),
- a *cost function* depending on the time, the state, the controls and, possibly, the derivatives of state and controls as functions of time,
- an *end cost function* defined on the target set (only appearing if $T < \infty$).

We shall use the following abbreviations for the various kinds of problems.

CEFHP: *clamped end point, finite horizon optimal control problem*; here the target set consists of one point and we have no end cost function; moreover $T < \infty$;

FEFHP: *free end point, finite horizon optimal control problem*; here the target set is some, at least one dimensional, subspace of the state space and $T < \infty$;

CEIHP (FEIHP): *clamped end point (free end point), infinite horizon optimal control problem*; $T = \infty$.

If the target set is not specified we talk of FHP (IHP) and if the time interval is not specified to be finite or not we may talk of CEP (FEP).
We shall use the variational approach for characterizing solutions of the optimal control problems. That means that we shall formulate optimal control problems as variational problems with restrictions and solve these.
A *variational problem* can be given by a Lagrangian function $L(x,\dot{x},t)$, an end cost function $h(x)$, a time interval $[0,T]$, possibly infinite, an initial point and a target set. Then one wants to minimize the action

$$J(x) = \int_0^T L(x(t),\dot{x}(t),t)dt + h(x(T))$$

over all possible curves $\{x(t),\ t \in [0,T]\}$. The problem is called *restricted* if the curves to be considered are restricted in some sense. We shall use similar abbreviations as above for variational problems, but with P replaced by VP. For instance IHVP means infinite horizon variational problem.

An optimal control problem can be formulated as a variational problem taking x to be both state and control variables and restricting the curves to be system trajectories with associated input.

In this paper we shall only consider first order conditions for optimality. Therefore we rather speak about *stationarity* and *stationary control* instead of optimality and optimal control.

In BUS [1982] we gave a differential geometric approach to the characterization of stationary solutions of the CEFHP (and CEFHVP). We proved there a generalization of the Lagrange multiplier rule on manifolds which appeared to be basic to this characterization. Moreover, we used a recent fibre bundle approach to nonlinear control systems on manifolds. In appendix to that paper we indicated the results for the FEFHP (FEFHVP). We shall summarize these results in section 2 and use these in section 3 to give a characterization of solutions for the IHP (IHVP). In section 4 we try to give some motivation for developping such a general formalism. We define the concepts of *equilibrium point* and *basin of extremal curves* and give a rough sketch along which lines existence and uniqueness results for stationary curves can be obtained. These ideas are illustrated with the example of linear-quadratic optimal control.

Our approach is coordinate free and uses no a priori regularity conditions. It is intended to contribute to the qualitative study of the IHP which might lead to general methods for solving the nonlinear optimal feedback control problem.

In this paper we use notations as given in BUS [1982], which are mostly consistent with the notations of SPIVAK [1979, vols I,II].

Thanks are due to H. Nijmeijer and Dr. J.H. van Schuppen for worthwile discussions about the subject.

2. SUMMARY OF EARLIER RESULTS

We shall first consider the finite horizon variational problem (FHVP) on a smooth (i.e. C^∞) manifold M. We assume that the *target set* S is either consisting of one point $x_T \in M$ (CEFHVP), or a smooth connected submanifold of M of dimension ≥ 1 (FEFHVP). We denote $I = [0,T]$ $(T \in \mathbb{R}_+)$. Instead of giving a Lagrangian $L(q,\dot{q},t)$ we give, more generally, an *action 1-form* α, which is a smooth differential 1-form on M. (The Lagrangian problem is obtained by choosing $M = TQ\times I$, $\alpha = L dt$, where Q is the so-called *configuration space*.) By $h : S \to \mathbb{R}$ we denote the *end cost function* and $x_0 \in M$ is the given initial point.

Now let us consider a smooth injective curve $\phi : I \to M$, with $\phi(0) = x_0$, $\phi(T) \in S$.

DEFINITION 2.1. A mapping $\tilde{\phi} : (-\delta,\delta) \times I \to M$ $(\delta \geq 0)$ is called a *variation* of ϕ for the FHVP iff

(i) $\tilde{\phi} = C^\infty$ in each variable, $\tilde{\phi}(\varepsilon,.)$ is injective for all $\varepsilon \in (-\delta,\delta)$;

(ii) $\tilde{\phi}(0,t) = \phi(t)$ for all $t \in I$;

(iii) $\tilde{\phi}(\varepsilon,0) = \phi(0)$, $\tilde{\phi}(\varepsilon,T) \in S$ for all $\varepsilon \in (-\delta,\delta)$.

We talk about CE(FE) variations, depending on S. We are interested in stationary solutions of the FHVP in the following sense.

DEFINITION 2.2. An injective curve $\phi : I \to M$ is *stationary* for the FHVP on M (defined by α, h, x_0, S) if $\phi(0) = x_0$, $\phi(T) \in S$ and either

(i)

(2.1) $$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(h(\phi_\varepsilon(T)) + \int_I \phi_\varepsilon^* \alpha\right) = 0,$$

for all variations $\phi_\varepsilon(\cdot) \underline{\underline{\nabla}} \tilde{\phi}(\varepsilon,.)$ of ϕ or, equivalently,

(ii)

(2.2) $$dh(V(T)) + \int_I \phi^* L_V \alpha = 0,$$

for all vector fields V along ϕ with $V(0) = 0$, $V(T) \in T_{\phi(T)}S$ (where L_V denote the Lie derivative with respect to V).

Recall that a vector field along a curve $\phi: I \to M$ is a smooth mapping $V: I \to TM$ such that for $t \in I$ $V(t) \in T_{\phi(t)}M$ and $\phi^* L_V\alpha$ is defined to be equal to $\phi^* L_X\alpha$ for any vector field X on M which is a smooth extension of V. For further details, as well as a proof of the equivalence of (i) and (ii) see BUS [1982].

We can give the following important characterization of stationary curves

PROPOSITION 2.3. An injective curve $\phi: I \to M$ is stationary for the FHVP iff $\phi(0) = x_0$, $\phi(T) \in S$ and

$$(2.3) \qquad \phi_*(\frac{\partial}{\partial t}|_t) \in \ker d\alpha, \quad \forall t \in I;$$

$$(2.4) \qquad (dh+\alpha)|_S(\phi(T)) = 0,$$

where $\ker d\alpha = \{v \in TM \mid d\alpha(v,w) = 0 \ \forall w \in T_{\pi(v)}M\}$ and $|_S$ denotes restriction to S.

The proof of this proposition is given in BUS [1982]. Condition (2.3) should be interpreted to be trivially satisfied for the CEVP. In this case S consists of one point and its tangent space reduces to the zero tangent vector. Condition (2.3) expresses the so-called *transversality condition* at the end point. Other references to this kind of approach to variational problems are GARDNER [1983] and GRIFFITHS [1983].

We may introduce restrictions on curves in M via smooth (C^∞) codistributions of fixed dimension on M. It is shown in BUS [1982] that for instance conditions that curves should be trajectories of a given (nonlinear) system can be easily expressed in such a way. We say that a curve $\phi: I \to M$ is *admissible* under restriction codistribution E if $\phi^*\beta = 0$ for all $\beta \in E$. Moreover, we call an injective curve $\phi: I \to M$ *stationary under restriction* E if it is stationary w.r.t. all admissible variations. However, the following more restrictive concept appears to be more useful.

DEFINITION 2.4. An injective curve $\phi: I \to M$ is *formally stationary* under restriction codistribution E(smooth and of fixed dimension) for the FHVP if $\phi(0) = x_0$, $\phi(T) \in S$, ϕ is admissible under E and either

$$(2.5) \qquad (\frac{d}{d\varepsilon}|_{\varepsilon=0} \phi_\varepsilon^*\beta=0 \ \forall\beta\in E) \Rightarrow \frac{d}{d\varepsilon}|_{\varepsilon=0} (h(\phi_\varepsilon(T))+\int_I \phi_\varepsilon^*\alpha) = 0$$

for all variations ϕ_ε of ϕ, or, equivalently

$$(2.6)\qquad (\phi^* L_V\beta=0\ \forall\beta\in E) \Rightarrow dh(V(T)) + \int_I \phi^* L_V\alpha = 0,$$

for all vector fields V along ϕ with $V(0) = 0$ and $V(T) \in T_{\phi(T)}S$.

The equivalence of the two conditions is shown in BUS [1982]. There it is also shown that at least for Lagrangian problems and for the free end point optimal control problem where the input appears linearly and the functions are analytic, the two notions stationarity and formal stationarity under restrictions are equivalent. For the clamped end point optimal control problem an additional technical condition is required. In the same reference a proof of the following generalized Lagrange multiplier rule has been given. First recall the definitions of two important 1-forms.
The *canonical 1-form* θ on T^*M is defined by

$$(2.7)\qquad \theta(\zeta) = \pi^*\zeta,$$

for all $\zeta \in T^*M$ ($\pi: T^*M \to M$ the natural projection). The *Cartan form* on E, a codistribution on M, associated with a 1-form α on M is defined by

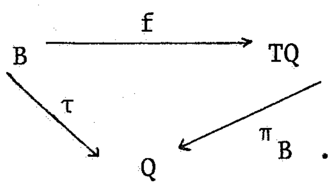
$$(2.8)\qquad \theta_\alpha = \pi_E^*\alpha + \theta_E,$$

with $\pi_E = \pi|_E$, $\theta_E = \theta|_E$ (restriction to E), θ the canonical 1-form.

<u>THEOREM 2.5</u>. *(Lagrange multiplier rule). An injective curve* ϕ: $I \to M$ *is formally stationary for the FHVP on* M *(with 1-form* α*, end cost* h*, target set* S*) under restriction codistribution* E*, if and only if there exists an injective curve* η: $I \to E$ *with* $\eta(t) \in \pi_E^{-1}(\phi(t))$ $(t\in I)$ *and* η *is stationary for the (unrestricted) FHVP on* E *with Cartan form* θ_α*, end cost* $h_E = h\circ\pi_E$ *and target set* $S_E = \chi(S)$ *for some section* χ: $M \to E$.

This theorem, which provides the basis for the following theory, shows that a restricted variational problem can be "reduced" to an unrestricted variational problem in a higher dimensional space. For the last problem we can use the characterization given by proposition 2.3. In order to use this theorem for the stationary control problem we first recall the notion of a general nonlinear control system introduced by BROCKETT [1977] and WILLEMS [1979] and worked out by NIJMEIJER & VAN DER SCHAFT [1982].

DEFINITION 2.6. A *nonlinear* (time-invariant) *control system* $\Sigma = \Sigma(Q,B,f)$ is defined by smooth manifolds Q and B, a fibre bundle τ: $B \to Q$ and a smooth map f: $B \to TQ$ such that the following diagram commutes



We call $\Sigma(Q,B,f)$ *affine* if B is a vector bundle and f, restricted to the fibres of B, is an affine map into the fibres of TQ. Σ is *analytic* if B and Q are analytic manifolds and f is an analytic map. We say that ψ: $I \to Q$ is a *trajectory* of Σ if ψ is absolutely continuous and

(2.9) $$\psi_*\left(\frac{\partial}{\partial t}\Big|_t\right) \in f(\tau^{-1}(\psi(t)), \text{ a.e. on } I.$$

With a trajectory we associate a *trajectory-input* ζ: $I \to B$ satisfying

(2.10) $$\tau(\zeta(t)) = \psi(t),\ \psi_*\left(\frac{\partial}{\partial t}\Big|_t\right) = f(\zeta(t)) \quad \text{a.e. on } I.$$

Note that the fibres of B represent the state dependent input spaces. If we choose local coordinates q for Q and u for the fibres $\tau^{-1}(q)$, then we obtain the familiar system equation $\dot{q} = f(q,u)$ (with abuse of notation for f: $(q,u) \to (q,f(q,u))$). Then a trajectory ψ: $I \to Q$ is a solution of this differential equation for some given initial point and some associated input. The pair: trajectory with associated input, is the trajectory-input and is in this coordinates often denoted by $\zeta(t) = (\psi(t),\nu(t))$. If Σ is affine then we can give a representation by

(2.11) $$f(q,u) = f_0(q) + \sum_{i=1}^{m} u_i f_i(q),$$

with $u_i \in \mathbb{R}$, f_0, $f_i \in X(Q)$ (smooth vector fields on Q), for i = 1,...,m. Such systems are sometimes called *input-linear*. In the sequel we shall always assume that f is an injective immersion.

We shall now translate a stationary control problem into a restricted variational problem. Note that we use the word "stationary" instead of "optimal" to indicate that we just consider first order conditions. We will use the same abbrevrations FHP etc. for the stationary as well as the optimal control problems. Referring to the introduction a FHP is defined by a control system $\Sigma(Q,B,f)$, a time interval $I = [0,T]$ ($T \in \mathbb{R}_+$), an initial state $q_0 \in Q$ and a target set $S \subset Q$, a cost function G: $TB \times I \to \mathbb{R}$ and an end cost function h: $S \to \mathbb{R}$. We restrict attention to two specific choices of S

1. S consists of one point q_T only (CEFHP);
2. $S = Q$ (FEFHP).

In the first case we assume $h \equiv 0$.
Then, the FHP is to find a trajectory-input ζ: $I \to B$ of Σ with $\tau(\zeta(0)) = q_0$, $\tau(\zeta(T)) \in S$ and such that ζ is stationary w.r.t. the cost

$$(2.12) \qquad J(\zeta) = h \circ \tau(\zeta(T)) + \int_I G(\zeta_*(\tfrac{\partial}{\partial t}|_t),t)dt.$$

REMARK 2.7. Allowing S to be a submanifold of Q of codimension > 0 would complicate the theory considerably. The main problem lies in the accessibility of points in a neighbourhood in S of the end point of a given trajectory, by trajectories of the system in fixed time T.

Now define a submanifold $M \subset TB \times I$ by $M = P \times I$ and

$$(2.13) \qquad P = \{w \in TB \mid f \circ \pi_B(w) = \tau_*(w)\},$$

where π_B: $TB \to B$ denotes natural projection. In canonical coordinates $(q,u,\dot{q},\dot{u})$ for TB P is the submanifold defined by $\dot{q} = f(q,u)$. Using the definition of trajectory-input we see that curves in $TB \times I$ which has the form

$$(2.14) \qquad t \to (\zeta_*(\tfrac{\partial}{\partial t}|_t),t),$$

for ζ a trajectory-input of Σ, lie in M. Define a codistribution E on M by

$$(2.15)\qquad E = \{\beta \mid \beta \text{ is 1-form on } M,\ \phi^*\beta = 0 \text{ for all } \phi \text{ of the form } \phi(t) = (\zeta_*(\tfrac{\partial}{\partial t}),t) \text{ with } \zeta \text{ a trajectory-input of } \Sigma\}.$$

In canonical coordinates $(q,u,\dot{u},t)$ for M, E is spanned by n+m 1-forms:

$$(2.16)\qquad \begin{aligned} \beta_i &= f_i(q,u)dt - dq_i \quad & i = 1,\dots,n,\\ \beta_{n+j} &= \dot{u}_j dt - du_j \quad & j = 1,\dots,m, \end{aligned}$$

with $f_i(q,u)$ the i-th coordinate of $f(q,u)$. For more details see BUS [1982]. We have from this reference the following results.

PROPOSITION 2.8. *A trajectory-input* ζ: I $\to$ B *is stationary for the FHP, with system* $\Sigma(Q,B,f)$, *cost* G, *end cost* h, *initial point* q_0 *and target set* $S(=\{q_T\}$ *or* Q), *if and only if* ϕ: I $\to$ M *given by*

$$(2.17)\qquad i\circ\phi = (\zeta_*(\tfrac{\partial}{\partial t}\big|_t),t),$$

with i *the embedding of* M *as submanifold in* TB $\times$ I, *is stationary for the restricted FHVP on* M *with restriction codistribution* E *(cf. (2.15)), 1-form* $(G\circ i)dt$, *end cost* $h_M = h\circ\tilde{\pi}$ ($\tilde{\pi}$: M $\to$ Q *natural projection*), *initial point* $\phi(0)$ *and target set* $\{\phi(T)\}$ *(CEFHP) or* P $\times$ {T} *(see (2.13)) (FEFHP). If* Σ *is affine analytic then formal stationary curves of the FEFHVP are stationary and vice versa. For the CEFHVP an additional condition is required (see BUS [1982]).*

Together with theorem 2.5 we obtain the following corollary.

COROLLARY 2.9. *Let be given the affine analytic FEFHP as in Proposition 2.8 and let an injective curve* ζ: I $\to$ B *be a trajectory-input of the system. Then the following statements are equivalent:*

(i) ζ *is stationary for this FEFHP;*

(ii) *there exists an injective curve* $\eta: I \to E$ *(cf. (2.15)) with* $\pi_E \circ \eta = \phi$ *(ϕ defined by (2.17) and* $\pi_E: E \to M$ *natural projection) such that* η *is stationary for the (unrestricted) FEFHVP with Cartan form (cf. (2.8)):*

$$(2.18) \qquad \theta_G = \pi_E^*((G\circ i)dt) + \theta_E,$$

end cost $h_M \circ \pi_E$ *and target set* $S_E = \chi(P\times\{T\})$ *for some section* $\chi: M \to E$.

(iii) *there exists an injective curve* $\eta: I \to E$ *with* $\pi_E \circ \eta = \phi$ *such that*

$$(2.19) \qquad \eta_*\left(\frac{\partial}{\partial t}\Big|_t\right) \in \ker d\theta_G$$

and

$$(2.20) \qquad (d(h_M \circ \pi_E) + \theta_G)\Big|_{S_E}(\eta(T)) \equiv 0.$$

Clearly a similar corollary can be derived for the CEFHP if an additional condition is satisfied to guarantee equivalence between stationarity and formal stationarity. In this case $S_E = \{\eta(T)\}$ and (2.20) disappears.

To illustrate this approach we shall work out these conditions in coordinates. Choose coordinates (q,u) in B and denote canonical coordinates on $TB \times I$ by $(q,u,\dot{q},\dot{u},t)$. Choose coordinates $(q,u,\dot{u},t)$ on M such that $i(q,u,\dot{u},t) = (q,u,f(q,u),\dot{u},t) \in TB \times I$. An element $\beta \in E$ given by

$$(2.21) \qquad \beta = \sum_{i=1}^{n} \lambda_i \beta_i(q,u,\dot{u},t) + \sum_{j=1}^{m} \mu_j \beta_{n+j}(q,u,\dot{u},t)$$

for β_i given by (2.16) $(i=1,\ldots,n+m)$, is represented by coordinates $(q,u,\dot{u},\lambda,\mu,t)$. In the sequel we call such coordinates on M and E canonical coordinates for given coordinates on B. Define

$$(2.22) \qquad H(q,u,\dot{u},\lambda,\mu,t) = G(q,u,f(q,u),\dot{u},t) + \lambda^T f(q,u) + \mu^T \dot{u}.$$

Then

$$(2.23) \qquad \theta_G = H\,dt - \sum_{i=1}^{n} \lambda_i dq_i - \sum_{j=1}^{m} \mu_j du_j.$$

Working out (2.19) for $\eta(t) = (q(t),u(t),\dot{u}(t),\lambda(t),\mu(t),t)$ stationary we obtain

$$(2.24) \qquad \dot{q} = \frac{dH}{d\lambda}(q,u,\dot{u},\lambda,\mu,t) \qquad (=f(q,u)),$$

$$(2.25) \qquad \dot{\lambda} = \frac{-dH}{\partial q}(q,u,\dot{u},\lambda,\mu,t),$$

$$(2.26) \qquad \dot{u} = \frac{d}{dt}\,u(t),$$

$$(2.27) \qquad \dot{\mu} = \frac{-\partial H}{\partial u}(q,u,\dot{u},\lambda,\mu,t),$$

$$(2.28) \qquad \mu = \frac{-\partial G_M}{\partial \dot{u}}(q,u,\dot{u},t).$$

Note that substitution of (2.28) in (2.27) yields an implicit system of nonlinear differential equations in u.

In many applications G does not dependend on $\dot{u}$. Then (2.28) yields $\mu = 0$ and H does not depend on $\dot{u}$ or μ

$$(2.29) \qquad H(q,u,\lambda,t) = G(q,u,f(q,u),t) + \lambda^T f(q,u).$$

Restricting the equations to the submanifold in E given by $\mu = 0$ we obtain the well-known equations following from Pontryagin's maximum principle (the smooth case):

$$(2.30) \qquad \dot{q} = \frac{\partial H}{\partial \lambda}(q,u,\lambda,t),$$

$$(2.31) \qquad \dot{\lambda} = \frac{-\partial H}{\partial q}(q,u,\lambda,t),$$

$$(2.32) \qquad 0 = \frac{\partial H}{\partial u}(q,u,\lambda,t).$$

The transversality condition (2.20) becomes, in the general case,

$$(2.33) \qquad \lambda(T) = \frac{dh}{dq}(q(T)),$$

$$(2.34) \qquad \mu(T) = 0.$$

Note that the last equation is consistent with $\mu = 0$ for the case that G is independent of $\dot{u}$.

3. THE INFINITE HORIZON PROBLEM

In this section we shall extend the results of section 2 to the IHVP and IHP. We have

(3.1) $$I = [0,\infty), \quad h \equiv 0.$$

With an IH-variation of $\phi\colon I \to M$ we mean a FE-variation of ϕ according to Definition 2.1 with $S = M$ (i.e. no condition on the end point). To assure finiteness of the cost integral we have to adapt the definition of stationarity in the following sense

DEFINITION 3.1. An injective curve $\phi\colon I \to M$ is

(i) *stationary* w.r.t. the IHVP with action form α iff

(3.2) $$\int_I |\phi^*\alpha| < \infty$$

(note that $\phi^*\alpha$ is of the form $\psi(t)dt$, so that $|\phi^*\alpha|$ here means $|\psi(t)|dt$) and for all IH-variations ϕ_ε of ϕ:

(3.3) $$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0;$$

(ii) *formally stationary* w.r.t. the restricted IHVP with action form α and restriction codistribution E iff (3.2) is satisfied and for all IH-variations ϕ_ε of ϕ:

$$\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon^*\beta = 0 \;\; \forall\beta\in E\right) \Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0.$$

In extending the results of section 2 to the infinite horizon case the following lemma is crucial.

LEMMA 3.2. *An injective curve* $\phi\colon I \to M$ *is stationary for the IHVP if and only if (3.2) is satisfied and for all* $t \in I$:

(3.4) $$\phi_*\left(\frac{\partial}{\partial t}\Big|_t\right) \in \ker d\alpha.$$

PROOF. Let ϕ be stationary. Then (3.2) is satisfied by hypothesis. Define, for arbitrary $T \in \mathbb{R}_+$, $\bar\phi = \phi|_{[0,T]}$ and suppose $\bar\phi$ is not stationary for the CEFHVP with end point $\phi(T)$ ($=\bar\phi(T)$). Then there exists a CE-variation ϕ_ε of $\bar\phi$ on $[0,T]$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^T \bar\phi_\varepsilon^*\alpha \neq 0.$$

However, we can construct a smooth CE-variation ϕ_ε of ϕ on I which is arbitrarily close to the (non-smooth) curve $\hat\phi_\varepsilon$ defined by:

$$\begin{aligned} \hat\phi_\varepsilon(t) &= \bar\phi_\varepsilon(t) \quad t \in [0,T],\\ &= \phi(t) \quad t > T. \end{aligned}$$

Hence

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha \neq 0.$$

This contradicts the stationarity for the IHVP. Hence $\bar\phi$ is stationary for the CEFHVP for every finite T. Then Proposition 2.3 yields the necessity.

To prove sufficiency, define a strictly monotonously increasing sequence $\{T_i\}_{i=0}^{\infty}$, with $\lim_{i\to\infty} T_i = \infty$. Denote

$$I_i = [0,T_i], \quad \phi^i = \phi|_{I_i}.$$

We know by hypothesis (3.2)

$$\lim_{i\to\infty} \int_{I_i} |(\phi^i)^*\alpha| = \lim_{i\to\infty} \int_{I_i} |\phi^*\alpha| = \int_I |\phi^*\alpha| < \infty.$$

Now let ϕ_ε be an arbitrary IH-variation of ϕ on I and define

$$\phi^i_\varepsilon(t) = \begin{cases} \phi_\varepsilon(t) & t \in I_{i-1}, \\ \sigma^i_\varepsilon(t) & t \in [T_{i-1}, T_i], \\ \phi(t) & t \geq T_i, \end{cases}$$

where σ^i_ε is chosen such that $\phi^i_\varepsilon(t)$ is smooth in t and ε, $\sigma^i_0(t) = \phi(t)$ and $|(\sigma^i_\varepsilon - \phi)^*\alpha|$ is bounded on $[T_{i-1}, T_i]$ for all i. Then $\phi^i_\varepsilon|_{I_i} \overset{\nabla}{=} \bar{\phi}^i_\varepsilon$ is a CE-variation of ϕ^i. By assumption (3.4) and Proposition 2.3 ϕ^i is stationary:

$$(3.5) \qquad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{I_i} (\bar{\phi}^i_\varepsilon)^*\alpha = 0 \qquad i = 1,2,\ldots .$$

Now define

$$f_i(\varepsilon) = \int_{I_i} (\phi^i_\varepsilon)^*\alpha, \qquad g_i(\varepsilon) = \frac{d}{d\varepsilon} f_i(\varepsilon).$$

Then $\lim_{i\to\infty} g_i(0) = 0$ and by smoothness of f and g w.r.t. ε we see that $g_i(\varepsilon)$ converges uniformly on some ε-interval. Moreover

$$\lim_{i\to\infty} f_i(0) = \int_I \phi^*\alpha.$$

Using DIEUDONNÉ [1969, thm. 8.6.4] we obtain

$$\frac{d}{d\varepsilon} \lim_{i\to\infty} f_i(\varepsilon) = \lim_{i\to\infty} g_i(\varepsilon).$$

Substituting $\varepsilon = 0$ yields

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi^*_\varepsilon\alpha = 0.$$

This proves the stationarity of ϕ. □

In order to prove a generalization of the Lagrange multiplier rule (thm. 2.5) we need Ascoli's theorem. We shall reproduce here the formulation of this theorem given in ROYDEN [1968]. The proof can be found in this reference. It has been given for continuous functions but can easily be extended to smooth functions.

THEOREM 3.3. *(Ascoli) Let* F *be a family of functions from a separable space* X *to a metric space* Y *which is equicontinuous with equicontinuous families of derivatives. Let* $\{f_n\}$ *be a sequence in* F *such that for each* $x \in X$ *the set* $c\ell\{f_n(x) \mid 0 \leq n < \infty\}$ *(*$c\ell$ *denotes closure) is compact. Then, there is a subsequence* $\{f_{n_k}\}$ *which converges pointwise to a smooth function, the convergence being uniform on each compact subset of* X.

THEOREM 3.4. *(Lagrange multiplier rule). Let* $\phi: I = [0,\infty) \to M$ *be an injective curve. Then* ϕ *is formally stationary for the restricted IHVP with action form* α *and restriction codistribution* E, *if and only if there exists an injective curve* $\eta: I \to E$ *with* $\pi_E \circ \eta = \phi$, *such that* η *is stationary w.r.t. the Cartan form* $\theta_\alpha = \pi_E^* \alpha + \theta_E$ *(*$\pi_E: E \to M$ *natural projection and* θ_E *the restriction to* E *of the canonical 1-form).*

PROOF. Define a sequence of strictly monotonously increasing time intervals $I_i = [0,T_i]$ with $\lim_{i\to\infty} T_i = \infty$. Let ϕ be formally stationary and define $\phi^i = \phi|_{I_i}$. With the same arguments as in the necessity part of Lemma 3.2 we may conclude that ϕ^i is formally stationary for the CEFHVP for every i. Using theorem 2.5 we know that there exist $\eta^i: I_i \to E$ $(i=1,2,\ldots)$ such that $\pi_E \circ \eta^i = \phi^i$ and η^i is stationary w.r.t. θ_α. From the proof of theorem 2.5 (see BUS [1982, thm. 3.8]) we see that we can choose η^i uniquely $(i=1,2,\ldots)$ such that $\eta^i(t) = \eta^j(t)$ for $i \leq j$ and $t \in I_i$. Indeed, the vector field Z defining F_1 and F_2 in that proof can be chosen along ϕ (not just ϕ^i). Then, if $\eta^j|_{I_i} \neq \eta^i$ we know that $\eta^j|_{I_i}$ is also stationary on I_i. Therefore, the Lagrange multipliers for both η^i and $\eta^j|_{I_i}$ satisfy the same differential equation and end point condition ((3.21) in that proof). So they are equal. We assume such a choice for η^i and define curves $\bar{\eta}^i: I \to E$ $(i=1,2,\ldots)$ smooth satisfying:

$$\bar{\eta}^i(t) = \begin{cases} \eta^i(t), & t \in I_i, \\ \eta^i(T_i), & t \geq T_{i+1}. \end{cases} \tag{3.6}$$

Clearly $\{\bar{\eta}^i\}_{i=1}^\infty$ is an equicontinuous family on I and their derivatives are equicontinuous families too. Moreover, we have for fixed $t \in I$:

$$c\ell\{\bar{\eta}^i(t) \mid 1 \le i < \infty\} = \{\eta^{N+1}(t), \bar{\eta}^N(t), \eta^k(T_k), k=1,\dots,N\},$$

for N chosen such that $t \in [T_N, T_{N+1}]$. As this is a finite set of points in E it is compact in E and we can use Ascoli's theorem yielding convergence of $\bar{\eta}^i$ to a smooth curve η: $I \to E$. Moreover, $\eta^i = \eta|_{I_i}$ is stationary w.r.t. θ_α for all i. Finally, if $\beta_1,\dots,\beta_m$ span E locally then

$$\text{(3.7)} \qquad \begin{aligned} \int_I |\eta^*\theta_\alpha| &= \int_I |\eta^*\pi_E^*\alpha + \sum_{j=1}^m \eta^*(\lambda_j\pi_E^*\beta_j)| \\ &= \int_I |\phi^*\alpha + \sum_{j=1}^m \lambda_j\phi^*\beta_j| = \int_I |\phi^*\alpha| < \infty, \end{aligned}$$

by admissibility and stationarity of ϕ. This also holds globally as we may split up a global problem in a sequence of local problems. The boundedness of the integral together with the stationarity and use of proposition 2.3 and lemma 3.2 yields the necessity.

To prove the sufficiency, let η: $I \to E$ be stationary w.r.t. θ_α. Then

$$\int_I |\eta^*\theta_\alpha| < \infty$$

and $\eta^i \overset{\nabla}{=} \eta|_{I_i}$ (i=1,2,...) is stationary for the CEFHVP with 1-form θ_α. Use of theorem 2.5 gives formal stationarity of $\phi^i = \pi_E \circ \eta^i$ under restriction distribution E for i = 1,2,... . Choose $\bar{\phi}^i$: $I \to M$ (i=1,2,...) smooth such that

$$\text{(3.8)} \qquad \bar{\phi}^i(t) = \begin{cases} \pi_E \circ \eta^i(t), & t \in I_i, \\ \pi_E \circ \eta^i(T_i), & t \ge T_{i+1}. \end{cases}$$

Then use of Ascoli's theorem again gives convergence of $\bar{\phi}^i$ to ϕ and it follows that $\phi = \pi_E \circ \eta$. Moreover (3.7) and (3.8) yield

$$\int_I |\phi^*\alpha| < \infty.$$

Therefore, the formal stationarity of ϕ^i for the CEFHVP yields the formal stationarity of ϕ using the same arguments as at the end of the proof of lemma 3.2. This completes the proof of the theorem. □

We can apply this theorem to the IHP given by a nonlinear control system $\Sigma(Q,B,f)$, time interval $I = [0,\infty)$, cost function G: $TB \times I \to \mathbb{R}$ and initial point $q_0 \in Q$. First we recall the following proposition from BUS [1982]. Although it is formulated there for the FEFHP the proof is the same for the IHP as we here also have a free end point in Q.

PROPOSITION 3.6. *Let be given an IHP with affine analytic control system* $\Sigma(Q,B,f)$. *Then, for the restricted IHVP assiciated with it we have equivalence between formal stationarity and stationarity of curves.*

COROLLARY 3.7. *Let* $\Sigma(Q,B,f)$ *be an affine analytic nonlinear system. Let* G: $TB \times I \to \mathbb{R}$ *be a given cost function. Suppose* ζ: $I \to B$ *is a trajectory-input of* Σ *satisfying* $\tau(\zeta(0)) = q_0$ *and*

$$(3.9) \qquad \int_I |G(\zeta_*(\tfrac{\partial}{\partial t}|_t),t)|dt < \infty.$$

Let $M = P \times I \subset TB \times I$, *with* P *defined by (2.13). Then* ζ *is stationary for this IHP if and only if there exists an* η: $I \to E$ *(*E *defined by (2.15), interpreted as a subbundle of* T^*M*) such that:*

$$(3.10) \qquad i \circ \pi_E \circ \eta = (\zeta_*(\tfrac{\partial}{\partial t}|_t),t),$$

with π_E: $E \to M$ *natural projection and* i: $M \to TB \times I$ *the embedding, and*

$$(3.11) \qquad \eta_*(\tfrac{\partial}{\partial t}) \in \ker d\theta_G \;\; \forall t \in I,$$

with $\theta_G = \pi_E^*(G|_M dt) + \theta_E$ *the Cartan form on* E *associated with* $G|_M dt$.

If we would work out these results in coordinates we would obtain formulas (2.24) up to (2.28) for the unbounded time domain.

In the next section we shall shed some more light on this result and try to motivate its importance.

4. MOTIVATION

Based on the foregoing theory we shall define some important concepts. We shall see that applying this theory to the linear-quadratic optimal control problem leads to an elegant representation of the solution in terms of the stabilizing solution of the algebraic Riccati equation. Moreover, it sheds some light on along which lines a nonlinear geometrical approach to infinite horizon optimal control might go.

We consider the IHP and look to the set of curves in E which are candidates for the stationary curves we search for. This is called the *set of extremals* Γ of the IHP and is defined by

$$(4.1) \qquad \Gamma = \{\eta\colon I \to E \mid \pi_E(\eta(t)) \in P \times \{t\}, \quad \forall t \in I; \quad \eta_*(\tfrac{\partial}{\partial t}) \in \ker d\theta_G, \quad \forall t \in I\}.$$

(For the definition of P see (2.13).) As $M = P \times I$ and E is a subbundle of $TM = TP \times TI$ we can write

$$(4.2) \qquad E = \bar{E} \times I.$$

Note that curves $\eta \in \Gamma$ are in fact graphs of curves $\gamma\colon I \to \bar{E}$. We denote

$$(4.3) \qquad \bar{\Gamma} = \{\gamma\colon I \to \bar{E} \mid \eta\colon t \to (\gamma(t),t) \text{ is in } \Gamma\},$$

and often call $\bar{\Gamma}$ also the set of extremals. One might ask whether all curves in $\bar{\Gamma}$ are integral curves of a certain vector field on $\bar{E}$. In fact, one can prove that, under certain regularity conditions, there exist a submanifold of $\bar{E}$ and a vector field on this submanifold such that its integral curves are extremals as curves in $\bar{E}$. Then we would be interested in the equilibrium points and associated stable manifolds of this vector field. However we shall not go this way as it requires a priori regularity conditions. We shall define the concept equilibrium point directly.

DEFINITION 4.1. An element $\bar{e} \in \bar{E}$ is called an *equilibrium point* for Γ (or $\bar{\Gamma}$) iff the curve $\gamma: I \to E$, defined by $\gamma(t) \equiv \bar{e}\ \forall t$, is in $\bar{\Gamma}$.

Before we discuss the question of existence of equilibrium points for Γ and their computation, we introduce one more important concept.

DEFINITION 4.2. The *basin* of the set of extremals $\bar{\Gamma}$ w.r.t. the equilibrium point $\bar{e} \in \bar{E}$ for $\bar{\Gamma}$ is the set $\mathcal{B} \subset E$ defined by

$$(4.4) \qquad \mathcal{B} = \{x \in \bar{E} \mid \exists\gamma \in \bar{\Gamma}(\gamma(0) = x, \lim_{t\to\infty} \gamma(t) = \bar{e})\}.$$

In order to compute an equilibrium point we note that such a point $\bar{e} \in \bar{E}$ should satisfy by definition the following equation:

$$(4.5) \qquad (\iota_{\frac{\partial}{\partial t}} d\theta_{G})(\bar{e},t) \equiv 0.$$

In canonical coordinates $(q,u,\dot{u},\lambda,\mu)$ for $\bar{E}$ we obtain that $\bar{e} = (q,u,\dot{u},\lambda,\mu)$ should satisfy

$$\frac{\partial G}{\partial q_i} + \sum_k \frac{\partial G}{\partial \dot{q}_k}\frac{\partial f_k}{\partial q_i} + \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial q_i} = 0,$$
$$\frac{\partial G}{\partial u_j} + \sum_k \frac{\partial G}{\partial \dot{q}_k}\frac{\partial f_k}{\partial u_j} + \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial u_j} = 0,$$
$$\frac{\partial G}{\partial \dot{u}_j} + \mu_j = 0,$$
$$f_i(q,u) = 0,$$
$$\dot{u}_j = 0,$$

for $i = 1,\dots,n$, $j = 1,\dots,m$. With definition of H by (2.22), these equations reduce to

(4.6) $$\frac{\partial H}{\partial q} = 0, \quad \frac{\partial H}{\partial \lambda} = 0, \quad \frac{\partial H}{\partial u} = 0,$$

(4.7) $$\mu = \frac{\partial G}{\partial \dot{u}}, \quad \dot{u} = 0.$$

If G is independent of t, then so is H and restricting to the submanifold with $\dot{u} = 0$ yields H dependent on q, u and λ only. Then an equilibrium point can be obtained as solution of (4.6) (gives q,u and λ coordinates) together with (4.7) (gives μ and $\dot{u}$).
We see that a solution (q^*,u^*) of the nonlinear programming problem:

$$\min_{q,u} G(q,u,f(q,u),0)$$
$$\text{subject to } f(q,u) = 0,$$

together with the values of the Lagrange multipliers λ^* at the solution, yields the equilibrium point

$$(q^*,u^*,0,\lambda^*,\frac{\partial G}{\partial \dot{u}}(q^*,u^*,f(q^*,u^*),0)).$$

The solution (q^*,u^*) is also called the *steady state* solution of the optimal control system. Clearly, the existence and uniqueness of an equilibrium point (locally) can be analyzed by studying the equations (4.6).
Once we have an equilibrium point for $\bar{\Gamma}$ we have to analyse its basin. The following questions are relevant:

1. Does the basin *cover* the manifold Q, i.e. is the projection of $\mathcal{B} \subset E$ on Q the whole of Q? In that case, for every starting point $q_0 \in Q$ we can find an extremal in $\bar{\Gamma}$ which projects on a trajectory-input of the system that starts in q_0 and ends in the equilibrium point.
2. Is the covering *simple*, i.e. is there for each $q_0 \in Q$ a unique extremal in Γ as under 1?

Conditions for existence and uniqueness of stationary solutions might be derived from studying these questions. For further motivation one might for instance look up the book of YOUNG [1969]. We shall here illustrate the relevance of the approach by working it out for the linear-quadratic optimal control problem. We have

$$f(q,u) = Aq + Bu,$$
$$G(q,u) = q^T Mq + u^T Ru,$$

with A, B, M and R matrices of appropriate dimension, M, R symmetric and R positive definite. Then the equations for an equilibrium point (4.6) and (4.7) yield

$$(4.8)\qquad \begin{aligned} &Mq + A^T\lambda = 0,\\ &Aq + Bu = 0,\\ &Ru + B^T\lambda = 0,\\ &\mu = 0,\ \dot{u} = 0. \end{aligned}$$

By positive definiteness of R we may solve the third equation for u and substitute, yielding

$$(4.9)\qquad H\binom{q}{\lambda} = 0,$$

$$(4.10)\qquad u = -R^{-1}B^T\lambda,\ \mu = 0,\ \dot{u} = 0,$$

with

$$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -M & -A^T \end{pmatrix}.$$

If we assume that the pair $(A,BR^{-1}B^T)$ is *stabilizable* (i.e. there exists a matrix F such that the spectrum of $A - BR^{-1}B^T F$ lies in the negative complex half plane) then H is nonsingular and a unique solution $(q,\lambda) = (0,0)$ of (4.9) exists. Therefore the equilibrium point in $\bar{E}$ is the origin (in the given canonical coordinates).
To see how $\bar{\Gamma}$-curves are like we note that they satisfy the equations (2.24) up to (2.28) which become:

$$(4.11)\qquad \begin{cases} \dot{q} = Aq + Bu, \\ \dot{\lambda} = -Mq - A^T\lambda, \\ \dot{u} = \frac{d}{dt}u, \\ 0 = Ru + B^T\lambda, \\ \mu = 0. \end{cases}$$

Again we can solve for u yielding $u = -R^{-1}B^T\lambda$. Substitution of u yields a differential equation in (q,λ):

$$(4.12)\qquad \begin{pmatrix}\dot{q}\\ \dot{\lambda}\end{pmatrix} = H\begin{pmatrix}q\\ \lambda\end{pmatrix}.$$

Solutions of (4.12) define the extremals. We know (see VAN SWIETEN [1977]) that, if $(A,BR^{-1}B^T)$ is stabilizable, then there exists a stabilizing solution K^- of the algebraic Riccati equation

$$(4.13)\qquad A^TK + KA - KBR^{-1}B^T + M = 0,$$

such that columns of the matrix $[\frac{I}{K^-}]$ span an H-invariant subspace in (q,λ)-space and curves of the form

$$\begin{pmatrix}q(t)\\ \lambda(t)\end{pmatrix} = \begin{pmatrix}q(t)\\ K^-q(t)\end{pmatrix}$$

are stable, with q(t) a solution of

$$(4.14)\qquad \dot{q} = (A-BR^{-1}B^TK^-)q, \quad q(0) = q_0.$$

So the basin $\mathcal{B} \subset \bar{E}$ is the set

$$(4.15)\qquad \mathcal{B} = \{(q,-R^{-1}B^TK^-q,-R^{-1}B^TK^-(A-BR^{-1}B^TK^-)q,K^-q,0),q \in Q\}.$$

Clearly the basin $\mathcal{B}$ covers Q and the covering is simple as extremals are defined by (4.14). So, for every $q_0 \in Q$ we can take the corresponding point in the basin $\mathcal{B}$ and the unique extremal in Γ emanating from this point and going to the equilibrium point the origin. This completes the picture for the linear-quadratic optimal control problem as an example for possible treatment of the nonlinear case.

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