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An Asymptotic Solution to a Two-Dimensional Exit Problem Arising in Population Dynamics

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This paper deals with a two-dimensional stochastic system, which is the diffusion approximation to a birth-death process. At the boundary of the state space, the diffusion matrix becomes singular. The stochastic fluctuations are assumed to be small. By an asymptotic analysis, expressions are derived that determine the probability of exit at each of the two boundaries and the expectation and variance of the exit time. These expressions contain constants that can be computed numerically.

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1. Introduction

Consider a two-dimensional stochastic system that has a stable deterministic equilibrium and in which the stochastic fluctuations are small. Various systems of this type have been studied in literature, see Matkowsky and Schuss [1,2], Matkowsky, Schuss and Tier [3], Hanson and Tier [4]. With respect to the behaviour of the deterministic system at the boundary of the region under consideration, different cases can be distinguished: the deterministic vector field enters the region [1], or it is tangent to the boundary of the region, without [2] or with [3] critical points on the boundary. It is assumed that the diffusion tensor is nonsingular. The asymptotic theories for small stochastic fluctuations lead to expressions for the exit distribution and the (lowest) statistical moments of the exit time. An asymptotic analysis of a one-dimensional stochastic system in which the diffusion coefficient becomes singular at the boundary is given in [4]. In this system, both the drift and the diffusion coefficients vanish, linearly with the distance to the boundary.

The two-dimensional stochastic system treated in this paper arises as the diffusion approximation of a birth-death process with two populations having large equilibrium values of equal order [5]. The diffusion matrix is diagonal and becomes singular at the boundary. There, the normal components of both the drift and the diffusion vanish linearly with the distance to the boundary. This system differs from the system treated in [3] in that the diffusion tensor becomes singular at the boundary, and from the system in [4] in the dimension. Extending the methods presented in [3,4], asymptotic expressions are derived for the probabilities of exit at the two boundaries as well as the expectation and variance of the exit time.

Section 2 describes the stochastic model and formulates the boundary value problems with respect to exit boundary and exit time. In section 3 we find asymptotic expressions for the probability of exit at each of the boundaries, valid uniformly outside an asymptotic small neighbourhood of the origin. In section 4, a derivation largely analogous to that in section 3 leads to asymptotic expressions for the expectation and variance of the exit time that are uniformly valid. Section 5 is concerned with the numerical determination of constants that appear in the formulas obtained in sections 3 and 4. As an example, section 6 treats a predator-prey system. Section 7 contains some remarks on the diffusion approximation to a birth-death process and possible extensions of the approach presented in this

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paper.

2. The stochastic model and the boundary value problems

Consider a system of two populations with birth and death rates B_i,D_i of the form:

$$B_i(N_1,N_2) = N_i(\lambda_{i0}+\lambda_{i1}N_1+\lambda_{i2}N_2)\,, \tag{2.1}$$
$$D_i(N_1,N_2) = N_i(\mu_{i0}+\mu_{i1}N_1+\mu_{i2}N_2)\,,$$

in which λ_{ij}, μ_{ij} are positive constants and N_i denotes the number of individuals in population i, $i,j=1,2$. By assumption, B_i-D_i vanishes at the equilibrium (N_1^e,N_2^e) with:

$$N_i^e = \frac{1}{\kappa_i\epsilon}>0, \quad 0<\epsilon<<1, \quad \kappa_i = O(1)\,, \tag{2.2}$$

the values of κ_i depending on the choice of ϵ. The birth-death process (2.1) is defined on the discrete state space N:

$$N = \{(N_1,N_2)\,|\,N_1,N_2\in\mathbb{N}\}\,. \tag{2.3}$$

It can be approximated by the diffusion process described by the forward Fokker-Planck (or forward Kolmogorov) equation:

$$\frac{\partial v(x,t)}{\partial t} = M_\epsilon v \equiv \sum_{i=1}^{2}\left[-\frac{\partial}{\partial x_i}[b_i(x)v(x,t)] + \frac{\epsilon}{2}\frac{\partial^2}{\partial x_i^2}[a_i(x)v(x,t)]\right]\,, \tag{2.4}$$

in which v is the probability density function and x_i represents N_i/N_i^e, defined on the continuous state space $\overline{R}$,

$$R = \{(x_1,x_2)\,|\,x_1,x_2\in\mathbb{R} \text{ and } x_1,x_2>0\}\,, \tag{2.5}$$

see [5]. The diffusion matrix is diagonal with elements:

$$a_1(x) = x_1(a_{10}+a_{11}x_1+a_{12}x_2)\,, \tag{2.6}$$
$$a_2(x) = x_2(a_{20}+a_{21}x_1+a_{22}x_2)\,,$$

in which the a_{ij} are the positive numbers:

$$a_{i0} = \kappa_i(\lambda_{i0}+\mu_{i0})\,, \tag{2.7}$$
$$a_{ij} = \kappa_i(\lambda_{ij}+\mu_{ij})N_j^e\,,$$

$i,j=1,2$. This diffusion matrix is singular at $x_1=0$ and $x_2=0$. The drift vector is given by

$$b_1(x) = x_1(b_{10}+b_{11}x_1+b_{12}x_2)\,, \tag{2.8}$$
$$b_2(x) = x_2(b_{20}+b_{21}x_1+b_{22}x_2)\,,$$

with

$$b_{i0} = \lambda_{i0}-\mu_{i0}\,, \tag{2.9}$$
$$b_{ij} = (\lambda_{ij}-\mu_{ij})N_j^e\,,$$

$i,j=1,2$. Thus, $x_1=0$ and $x_2=0$ are characteristic boundaries.

In this paper we investigate the stochastic system described by (2.4), with a diffusion matrix of the form (2.6), a_{ij} positive, and a drift vector of the form (2.8), in which the b_{ij} are restricted by assumptions made below.

The deterministic system

$$\frac{dx_1}{dt} = b_1(x)\,, \tag{2.10}$$

$$\frac{dx_2}{dt} = b_2(x),$$

associated with the stochastic system (2.4) has the equilibria:

$$(0,0), \tag{2.11a}$$

$$(0, -\frac{b_{20}}{b_{22}}), \tag{2.11b}$$

$$(-\frac{b_{10}}{b_{11}}, 0), \tag{2.11c}$$

$$x^e = (x_1^e, x_2^e) \equiv \left[\frac{b_{22}b_{10} - b_{12}b_{20}}{b_{21}b_{12} - b_{11}b_{22}}, \frac{b_{11}b_{20} - b_{21}b_{10}}{b_{21}b_{12} - b_{11}b_{22}} \right]. \tag{2.11d}$$

By assumption the critical points (2.11b, c) lie on the positive x_2-axis, x_1-axis respectively, with order $O(1)$ distance from the origin:

$$-\frac{b_{20}}{b_{22}} > 0, \quad -\frac{b_{20}}{b_{22}} = O(1), \quad -\frac{b_{10}}{b_{11}} > 0, \quad -\frac{b_{10}}{b_{11}} = O(1), \qquad \text{(assumption 1)}$$

and are attracting along the x_2-axis, x_1-axis, respectively:

$$b_{20} > 0, \quad b_{10} > 0. \qquad \text{(assumption 2)}$$

The deterministic system has an equilibrium in R with coordinates of order $O(1)$:

$$x_1^e > 0, \quad x_1^e = O(1), \quad x_2^e > 0, \quad x_2^e = O(1). \qquad \text{(assumption 3)}$$

The following assumption is made with respect to the stability of the deterministic system at x^e. In the neighbourhood of x^e we have by linearization of the deterministic vector field:

$$b \equiv (b_1(x), b_2(x))^t \approx B(x - x^e), \tag{2.12}$$

where the matrix B is given by:

$$B = (B_{ij}) \equiv (\frac{\partial b_i}{\partial x_j}|x^e) = (b_{ij}x_i^e). \tag{2.13}$$

The eigenvalues of B are

$$\lambda_{1,2} = \frac{1}{2}[b_{11}x_1^e + b_{22}x_2^e \pm \sqrt{(b_{11}x_1^e + b_{22}x_2^e)^2 - 4(b_{11}b_{22} - b_{12}b_{21})x_1^e x_2^e}\,]. \tag{2.14}$$

The condition for stability of the deterministic system at x^e is that the real parts of λ_1 and λ_2 are negative. Using the assumptions (1-3) this condition results in:

$$b_{11}b_{22} > b_{12}b_{21}. \qquad \text{(assumption 4)}$$

By the assumptions (1-4) the equilibria (2.11b,c) are saddle points. The equilibrium (2.11a) is an unstable node.

At the boundary $x_i = 0$ we have:

$$J_i(x,t) \equiv b_i(x)v(x,t) - \frac{\epsilon}{2}\frac{\partial}{\partial x_i}[a_i(x)v(x,t)] < 0 \tag{2.15a}$$

$$b_i(x) = 0, \quad a_i(x) = 0, \tag{2.15b}$$

$i = 1,2$. By (2.15a) the probability current J_i at $x_i = 0$ is negative, indicating that $x_i = 0$ can be reached from R. Once $x_i = 0$ has been reached, by (2.15b) it cannot be left. Thus, $x_1 = 0$ and $x_2 = 0$ are exit boundaries.

Starting away from $x_1=0$ and $x_2=0$, the stochastic system described above will likely remain in the neighbourhood of the stable equilibrium x^e of the deterministic system for a long time. With small probabilities large excursions from x^e occur. In such an excursion the system may exit at $x_1=0$ or $x_2=0$. This will happen within a finite time with probability one.

The boundary value problems describing exit are commonly defined on a bounded region. However, for the asymptotic analysis held in this paper, the use of the unbounded region $\bar{R}$ and the boundary ∂R defined by

$$\partial R \equiv \bar{R} \setminus R = \{(x_1,x_2) | x_1 x_2 = 0 \text{ and } x_1 + x_2 \geq 0\} \tag{2.16}$$

will not lead to any difficulty.

In order to determine the probabilities of exit at $x_1=0$ and $x_2=0$, a study is made of the stationary backward Fokker-Planck (or backward Kolmogorov) equation:

$$0 = L_\epsilon u \equiv \sum_{i=1}^{2} [b_i(x)\frac{\partial u}{\partial x_i} + \frac{\epsilon}{2} a_i(x) \frac{\partial^2 u}{\partial x_i^2}] \qquad \text{in } \bar{R}\ , \tag{2.17a}$$

with the boundary condition:

$$u = f(x) \qquad \text{on } \partial R\ , \tag{2.17b}$$

in which

$$u(x) = \int_{\partial R} f(x')P(x,x')ds_{x'}\ , \tag{2.17c}$$

where $P(x,x')$ is the probability of exit at $x' \in \partial R$, starting from $x \in \bar{R}$. By defining f as:

$$f(x) = \begin{cases} 1, & x_i=0, \\ 0, & \text{else}, \end{cases} \tag{2.18}$$

the function $u(x)$ is the probability of exit at $x_i=0$, starting from $x \in \bar{R}$, $i=1,2$. In this paper only boundary conditions of the form

$$f(x) = \begin{cases} C_{b1}, & x_1=0, \\ C_{b2}, & x_2=0, \end{cases} \tag{2.19}$$

are considered, with C_{b1}, C_{b2} constants that are either zero or one.

Another point of interest is the determination of the expectation $E_T(x)$ and variance $Var_T(x)$ of the exit time $T(x)$, starting from $x \in \bar{R}$. By

$$E_T(x) = T_1\ , \tag{2.20a}$$

$$Var_T(x) = T_2 - T_1^2\ , \tag{2.20b}$$

the expectation and variance of T are expressed in the moments

$$T_i(x) = \langle T^i \rangle \tag{2.21}$$

of T, which satisfy the equations:

$$L_\epsilon T_i = g_i(x) \qquad \text{in } \bar{R}\ , \tag{2.22a}$$

and conditions

$$T_i = 0 \qquad \text{on } \partial R\ , \tag{2.22b}$$

$i=1,2$, with

$$g_1(x) \equiv -1\ , \tag{2.22c}$$

$$g_2(x) \equiv -2T_1(x)\ . \tag{2.22d}$$

Equation (2.22a) with $i = 1$ is the Dynkin equation. Although higher moments can be determined as well, the analysis of the exit time in this paper is restricted to its expectation and variance. For a derivation of the boundary value problems (2.17, 2.22) the reader is referred to [6,7].

In biological terms, exit means extinction of a species. The expected exit time is a measure of stochastic persistence of the ecosystem. The type of interaction between the two populations with densities x_1 and x_2 is mutualism for $b_{12}>0, b_{21}>0$, competition for $b_{12}<0, b_{21}<0$ and predation-prey in the other cases.

3. The exit boundary

In this section the exit problem (2.17) with f as in (2.19) is solved asymptotically for small ϵ. The solution contains an unknown constant. To obtain an expression for this constant, we use an integral formula which results from the divergence theorem. In the integral formula, a formal solution of the forward equation adjoint to (2.17a) is needed. This adjoint equation is solved by the WKB-method. Near the boundaries $x_1=0$ and $x_2=0$, the solution of the adjoint equation is peaked at the critical points $(0, -b_{20}/b_{22})$ and $(-b_{10}/b_{11}, 0)$ respectively. Neighbourhoods of these critical points play an important role in the subsequent analysis.

3.1. The backward equation

An asymptotic analysis of the boundary value problem (2.17) reveals the existence of an outer solution, valid away from $x_1=0$ and $x_2=0$. Near these boundaries, an examination of different stretchings of the normal coordinate shows the presence of a boundary layer of width $O(\epsilon)$. Inside the boundary layers, the diffusion parallel to the boundary is negligible, except near critical points of the deterministic system. Thus, the following regions are distinguished:

$$\text{region } A: \quad x_1 = O(\epsilon), \quad x_2 + \frac{b_{20}}{b_{22}} > O(\sqrt{\epsilon}), \tag{3.1a}$$

$$\text{region } B: \quad x_1 = O(\epsilon), \quad \left| x_2 + \frac{b_{20}}{b_{22}} \right| = O(\sqrt{\epsilon}), \tag{3.1b}$$

$$\text{region } C: \quad x_1 = O(\epsilon), \quad O(\epsilon) < x_2 < -\frac{b_{20}}{b_{22}} - O(\sqrt{\epsilon}), \tag{3.1c}$$

$$\text{region } D: \quad x_1 = O(\epsilon), \quad x_2 = O(\epsilon), \tag{3.1d}$$

$$\text{region } A': \quad x_1 + \frac{b_{10}}{b_{11}} > O(\sqrt{\epsilon}), \quad x_2 = O(\epsilon), \tag{3.1e}$$

$$\text{region } B': \quad \left| x_1 + \frac{b_{10}}{b_{11}} \right| = O(\sqrt{\epsilon}), \quad x_2 = O(\epsilon), \tag{3.1f}$$

$$\text{region } C': \quad O(\epsilon) < x_1 < -\frac{b_{10}}{b_{11}} - O(\sqrt{\epsilon}), \quad x_2 = O(\epsilon), \tag{3.1g}$$

see fig. 1.

3.1.1 The outer solution

The reduced equation corresponding to (2.17a) reads:

$$\sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i} = 0, \tag{3.2}$$

which has the solution:

$$u = C_b, \tag{3.3}$$

with C_b a constant with respect to x, which is yet undetermined. An expression for C_b will be found in subsection 3.3. The solution (3.3) is valid in R except near $x_1=0$ and $x_2=0$ because the boundary condition (2.19) cannot be satisfied.

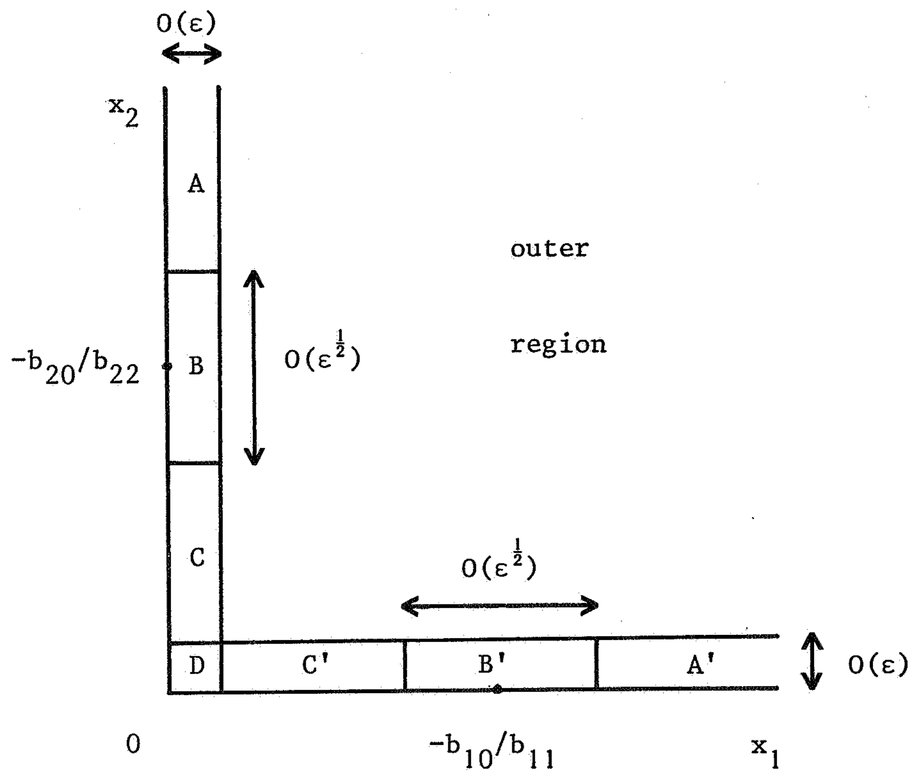


FIGURE 1. The outer region and the boundary layer regions. In the regions A, A', C, C' the diffusion parallel to the boundary is negligible, while this is not the case in the regions B, B' and D.

3.1.2 The boundary layer solution in the regions B and B'

Near the critical point $(0,-b_{20}/b_{22})$ of the deterministic system, we introduce the stretched coordinates:

$$\bar{x}_1 = \frac{x_1}{\epsilon}, \tag{3.4a}$$

$$\bar{x}_2 = \frac{x_2+\frac{b_{20}}{b_{22}}}{\sqrt{\epsilon}}, \tag{3.4b}$$

and the boundary layer function

$$U(\bar{x}_1,\bar{x}_2) = u(\epsilon\bar{x}_1,-\frac{b_{20}}{b_{22}}+\sqrt{\epsilon}\bar{x}_2). \tag{3.5}$$

Substitution into (2.17a) leads to the boundary layer equation:

$$k_1\bar{x}_1\frac{\partial U}{\partial \bar{x}_1} + k_2\bar{x}_1\frac{\partial^2 U}{\partial \bar{x}_1^2} - k_3\bar{x}_2\frac{\partial U}{\partial \bar{x}_2} + k_4\frac{\partial^2 U}{\partial \bar{x}_2^2} = 0, \tag{3.6}$$

in which

$$k_1 = b_{10} - \frac{b_{12}b_{20}}{b_{22}}, \tag{3.7a}$$

$$k_2 = \frac{1}{2}(a_{10}-a_{12}\frac{b_{20}}{b_{22}}), \tag{3.7b}$$

$$k_3 = b_{20}, \tag{3.7c}$$

$$k_4 = -\frac{1}{2}\frac{b_{20}}{b_{22}}(a_{20}-\frac{a_{22}b_{20}}{b_{22}}). \tag{3.7d}$$

From the assumptions (1-4) and the positivity of the a_{ij} it follows that the constants k_i are positive. By separation of variables:

$$U(\bar{x}_1,\bar{x}_2) = w(\bar{x}_1)z(\bar{x}_2), \tag{3.8}$$

equation (3.6) leads to the ordinary differential equations:

$$k_2\bar{x}_1\frac{d^2w}{d\bar{x}_1^2} + k_1\bar{x}_1\frac{dw}{d\bar{x}_1} - \lambda w = 0, \tag{3.9a}$$

$$k_4\frac{d^2z}{d\bar{x}_2^2} - k_3\bar{x}_2\frac{dz}{d\bar{x}_2} + \lambda z = 0, \tag{3.9b}$$

in which λ is the separation constant. The general solution of (3.9a) is:

$$w(\bar{x}_1) = e^{-\frac{\tilde{x}_1}{2}}[c_1W_{\lambda_1,\frac{1}{2}}(-\tilde{x}_1) + c_2W_{-\lambda_1,\frac{1}{2}}(\tilde{x}_1)], \tag{3.10}$$

in which

$$\tilde{x}_1 = \frac{k_1}{k_2}\bar{x}_1, \tag{3.11a}$$

$$\lambda_1 = \frac{\lambda}{k_1}, \tag{3.11b}$$

$W_{\lambda_1,\frac{1}{2}}$ and $W_{-\lambda_1,\frac{1}{2}}$ are Whittaker functions [8] and c_1, c_2 are arbitrary constants. The general solution of (3.9b) is:

$$z(\bar{x}_2) = e^{\frac{\tilde{x}_2^2}{4}}[c_3 D_{\lambda_2}(\tilde{x}_2)+c_4 D_{-\lambda_2-1}(i\tilde{x}_2)]\,, \tag{3.12}$$

in which

$$\tilde{x}_2 = \sqrt{\frac{k_3}{k_4}}\bar{x}_2\,, \tag{3.13a}$$

$$\lambda_2 = \frac{\lambda}{k_3}\,, \tag{3.13b}$$

D_{λ_2} and $D_{-\lambda_2-1}$ are parabolic cylinder functions [9] and c_3, c_4 are arbitrary constants. At $\bar{x}_1=0$ we have the boundary condition

$$U(0,\bar{x}_2) = C_{b1}\,. \tag{3.14}$$

This condition can be satisfied only if $\lambda=0$. The matching condition with the outer solution (3.3) is stated as:

$$\lim_{\bar{x}_1\to\infty} U(\bar{x}_1,\bar{x}_2) = C_b\,. \tag{3.15}$$

The boundary layer solution satisfying both conditions (3.14) and (3.15) is given by

$$U(\bar{x}_1,\bar{x}_2) = C_b+(C_{b1}-C_b)e^{-\bar{x}_1} \tag{3.16a}$$

or, in the original notation:

$$u(x_1,x_2) = C_b+(C_{b1}-C_b)e^{-\frac{k_1}{k_2}\frac{x_1}{\epsilon}}\,. \tag{3.16b}$$

The boundary layer region B' around the critical point $(-b_{10}/b_{11},0)$ yields a similar result:

$$u(x_1,x_2) = C_b+(C_{b2}-C_b)e^{-\frac{k_1'}{k_2'}\frac{x_2}{\epsilon}}\,, \tag{3.17}$$

in which

$$k_1' = b_{20} - \frac{b_{21}b_{10}}{b_{11}}\,, \tag{3.18a}$$

$$k_2' = \frac{1}{2}(a_{20}-a_{21}\frac{b_{10}}{b_{11}})\,. \tag{3.18b}$$

3.1.3 The boundary layer solution in the regions A,A′ and C,C′

Introduction of the stretched coordinate (3.4a) and the boundary layer function

$$U(\bar{x}_1,x_2) = u(\epsilon\bar{x}_1,x_2) \tag{3.19}$$

into equation (2.17a) leads to the boundary layer equation:

$$\bar{x}_1(b_{10}+b_{12}x_2)\frac{\partial U}{\partial \bar{x}_1} + x_2(b_{20}+b_{22}x_2)\frac{\partial U}{\partial x_2} + \bar{x}_1\frac{1}{2}(a_{10}+a_{12}x_2)\frac{\partial^2 U}{\partial \bar{x}_1^2} = 0\,. \tag{3.20}$$

In order to make this equation separable, the variable $\bar{x}_1$, is replaced by the new variable

$$y = \bar{x}_1\gamma(x_2)\,, \tag{3.21}$$

with the function γ still to be determined. Equation (3.20) becomes:

$$y\frac{\partial^2 W}{\partial y^2} + \Gamma(x_2)y\frac{\partial W}{\partial y} + \frac{x_2(b_{20}+b_{22}x_2)}{\frac{1}{2}(a_{10}+a_{12}x_2)\gamma}\frac{\partial W}{\partial x_2} = 0\,, \tag{3.22}$$

where

$$W(y,x_2) = U(\bar{x}_1,x_2)\,, \tag{3.23}$$

and

$$\Gamma(x_2) \equiv \frac{1}{\frac{1}{2}(a_{10}+a_{12}x_2)\gamma}[b_{10}+b_{12}x_2+x_2(b_{20}+b_{22}x_2)\frac{\gamma'}{\gamma}]\,. \tag{3.24}$$

The function γ is chosen such that

$$\Gamma(x_2) \equiv 1\,. \tag{3.25}$$

Then (3.24) is a Bernouilli equation, the integration of which is discussed in appendix A. The corresponding integration constant follows from a matching condition, see below. The partial differential equation (3.22) with (3.25) can be solved by separation of variables:

$$W(y,x_2) = w(y)z(x_2)\,, \tag{3.26}$$

leading to the ordinary differential equations:

$$y\frac{d^2 w}{dy^2} + y\frac{dw}{dy} - \lambda w = 0\,, \tag{3.27a}$$

$$\frac{x_2(b_{20}+b_{22}x_2)}{\frac{1}{2}(a_{10}+a_{12}x_2)\gamma}\frac{dz}{dx_2} + \lambda z = 0\,, \tag{3.27b}$$

in which λ is the separation constant. To satisfy the matching conditions

$$U(0,x_2) = C_{b1}\,, \qquad \lim_{\bar{x}_1\to\infty} U(\bar{x}_1,x_2) = C_b\,, \tag{3.28}$$

λ must equal zero and the solution of (3.20), (3.28) is obtained as:

$$U(\bar{x}_1,x_2) = C_b+(C_{b1}-C_b)e^{-\gamma(x_2)\bar{x}_1}\,, \tag{3.29a}$$

or, in the original notation:

$$u(x_1,x_2) = C_b+(C_{b1}-C_b)e^{-\gamma(x_2)\frac{x_1}{\epsilon}}\,. \tag{3.29b}$$

The integration constant in the problem (3.24), (3.25) for γ is chosen such that (3.29) matches the solution (3..16), that is, by the condition:

$$\lim_{x_2\to -\frac{b_{20}}{b_{22}}} \gamma(x_2) = \frac{k_1}{k_2}\,. \tag{3.30}$$

The boundary layer regions A' and C' along the x_1-axis are treated similarly. There, the solution is given by

$$u(x_1,x_2) = C_b+(C_{b2}-C_b)e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}\,, \tag{3.31}$$

in which $\tilde{\gamma}(x_1)$ solves a Bernoulli problem analogous to (3.24), (3.25), (3.30).

The treatment above in the direction along the boundary leads to a correct result only for constant boundary conditions. Readers interested in boundary conditions (2.17b) with nonconstant f are referred to the approach in [3].

3.1.4 The boundary layer solution in the region D

We introduce the stretched coordinates (3.4a) and

$$\bar{x}_2 = \frac{x_2}{\epsilon} \tag{3.32}$$

and the boundary layer function

$$U(\bar{x}_1,\bar{x}_2) = u(\epsilon\bar{x}_1,\epsilon\bar{x}_2)\,. \tag{3.33}$$

Substitution into equation (2.17a) leads to the boundary layer equation:

$$b_{10}\bar{x}_1\frac{\partial U}{\partial \bar{x}_1} + \tfrac{1}{2}a_{10}\bar{x}_1\frac{\partial^2 U}{\partial \bar{x}_1^2} + b_{20}\bar{x}_2\frac{\partial U}{\partial \bar{x}_2} + \tfrac{1}{2}a_{20}\bar{x}_2\frac{\partial^2 U}{\partial \bar{x}_2^2} = 0\,. \tag{3.34}$$

This equation is solved by separation of variables. By the assumption

$$U(\bar{x}_1,\bar{x}_2) = w(\bar{x}_1)z(\bar{x}_2) \tag{3.35}$$

the ordinary differential equations

$$b_{10}\bar{x}_1\frac{dw}{d\bar{x}_1} + \tfrac{1}{2}a_{10}\bar{x}_1\frac{d^2 w}{d\bar{x}_1^2} - \lambda w = 0\,, \tag{3.36a}$$

$$b_{20}\bar{x}_2\frac{dz}{d\bar{x}_2} + \tfrac{1}{2}a_{20}\bar{x}_2\frac{d^2 z}{d\bar{x}_2^2} + \lambda z = 0\,, \tag{3.36b}$$

are obtained, in which λ is the separation constant. The general solution of (3.36a) is:

$$w(\bar{x}_1) = e^{\frac{-\overset{\approx}{x}_1}{2}}\,[c_1 W_{\lambda_1,-\frac{1}{2}}(-\overset{\approx}{x}_1)+c_2 W_{-\lambda_1,-\frac{1}{2}}(\overset{\approx}{x}_1)]\,, \tag{3.37}$$

in which

$$\overset{\approx}{x}_1 = \frac{b_{10}}{\frac{1}{2}a_{10}}\bar{x}_1\,, \tag{3.38a}$$

$$\lambda_1 = \frac{\lambda}{b_{10}}\,. \tag{3.38b}$$

The general solution of (3.36b) is:

$$z(\bar{x}_2) = e^{\frac{-\overset{\approx}{x}_2}{2}}\,[c_3 W_{\lambda_2,-\frac{1}{2}}(-\overset{\approx}{x}_2)+c_4 W_{-\lambda_2,-\frac{1}{2}}(\overset{\approx}{x}_2)]\,, \tag{3.39}$$

in which:

$$\overset{\approx}{x}_2 = \frac{b_{20}}{\frac{1}{2}a_{20}}\,\bar{x}_2\,, \tag{3.40a}$$

$$\lambda_2 = -\frac{\lambda}{b_{20}}\,. \tag{3.40b}$$

The solution of (3.34) must obey the boundary conditions:

$$U(0,\bar{x}_2) = C_{b1}\,, \tag{3.41a}$$

$$U(\bar{x}_1,0) = C_{b2}\,, \tag{3.41b}$$

and the matching conditions:

$$\lim_{\bar{x}_2\to\infty} U(\bar{x}_1,\bar{x}_2) = C_b+(C_{b1}-C_b)e^{-\frac{b_{10}}{\frac{1}{2}a_{10}}\bar{x}_1}, \tag{3.42a}$$

$$\lim_{\bar{x}_1\to\infty} U(\bar{x}_1,\bar{x}_2) = C_b+(C_{b2}-C_b)e^{-\frac{b_{20}}{\frac{1}{2}a_{20}}\bar{x}_2}, \tag{3.42b}$$

with respect to the solution in the regions C and C' respectively. The conditions (3.42) follow from

$$\lim_{x_2\to 0} \gamma(x_2) = \frac{b_{10}}{\frac{1}{2}a_{10}}, \tag{3.43a}$$

$$\lim_{x_1\to 0} \tilde{\gamma}(x_1) = \frac{b_{20}}{\frac{1}{2}a_{20}}, \tag{3.43b}$$

see appendix A. The desired solution is a rather complicated expression, which will not be given here. Instead, we remark that

$$U(\bar{x}_1,\bar{x}_2) = \frac{1}{C_b}\,[C_b+(C_{b1}-C_b)e^{-\tilde{x}_1}][C_b+(C_{b2}-C_b)e^{-\tilde{x}_2}], \tag{3.44}$$

obtained with $\lambda=0$, solves the differential equation (3.34), satisfies the matching conditions (3.42) and has the boundary values:

$$U(0,\bar{x}_2) = C_{b1}[1+(\frac{C_{b2}}{C_b}-1)e^{-\tilde{x}_2}], \tag{3.45a}$$

$$U(\bar{x}_1,0) = C_{b2}[1+(\frac{C_{b1}}{C_b}-1)e^{-\tilde{x}_1}], \tag{3.45b}$$

which are different from the boundary conditions (3.41).

3.1.5 Summary

It is easily verified that the results of subsection 3.1 can be summarized as follows. The uniform asymptotic expansion for small ϵ in $\bar{R}\setminus\bar{D}$, D an $O(\epsilon)$ neighbourhood of the origin, of the boundary value problem (2.17, 2.19) is given by:

$$u(x) = \frac{1}{C_b}[C_b+(C_{b1}-C_b)e^{-\gamma(x_2)\frac{x_1}{\epsilon}}][C_b+(C_{b2}-C_b)e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}], \tag{3.46}$$

in which $\gamma,\tilde{\gamma}$ solve Bernoulli problems as discussed in 3.1.3. Expression (3.46) is the uniform asymptotic expansion in $\bar{R}$ of the boundary value problem (2.17) with boundary conditions (3.45). The remainder of section 3 concerns the determination of C_b, which is yet unknown.

3.2 The adjoint equation

The forward equation adjoint to (2.17a) is given by

$$M_\epsilon v = 0, \tag{3.47}$$

with the operator M_ϵ defined in (2.4). The function $v(x)$ describes the probability density, corresponding to the (quasi-) stationary state of the system (2.4). The solution of equation (3.47) is needed in section 3.3.

3.2.1 The WKB-approximation
A solution of (3.47) is sought in the form of the WKB-Ansatz [10]:

$$v(x_1,x_2) = w(x_1,x_2)\, e^{\frac{-Q(x_1,x_2)}{\epsilon}} \quad , \quad \epsilon \to 0 \,, \tag{3.48a}$$

where

$$Q(x_1^e,x_2^e) = 0 \,, \tag{3.48b}$$

$$w(x_1^e,x_2^e) = 1 \,, \qquad \text{(normalization)}. \tag{3.48c}$$

Substitution of this form into (3.47) leads to leading order $O(\epsilon^{-1})$ to the eikonal equation:

$$\sum_{i=1}^{2} [b_i \frac{\partial Q}{\partial x_i} + \tfrac{1}{2} a_i (\frac{\partial Q}{\partial x_i})^2] = 0 \,, \tag{3.49}$$

and to order $O(\epsilon^0)$ to the transport equation:

$$\sum_{i=1}^{2} [\frac{\partial}{\partial x_i}(b_i w) + \frac{\partial Q}{\partial x_i}\frac{\partial}{\partial x_i}(a_i w) + \tfrac{1}{2} a_i w \frac{\partial^2 Q}{\partial x_i^2}] = 0 \,. \tag{3.50}$$

The numerical computation of the functions Q and w subject to the conditions (3.48b, c) is treated in section 5.

3.2.2 Behaviour near the boundary
To investigate the asymptotic behaviour of Q in the x_2-direction for small x_1, the expansion

$$Q(x_1,x_2) = \overline{Q}_0(x_2) + \overline{Q}_1(x_2)x_1 + \tfrac{1}{2}\overline{Q}_2(x_2)x_1^2 + \cdots \tag{3.51}$$

is substituted into (3.49). Terms of order $O(x_1^0)$ are collected, resulting in:

$$\frac{d\overline{Q}_0}{dx_2} = -\frac{b_{20}+b_{22}x_2}{\frac{1}{2}(a_{20}+a_{22}x_2)} \,. \tag{3.52}$$

This expression indicates that inside the interval $x_2 \in [0,\infty)$, the only extremum of $\overline{Q}_0$ is a minimum, situated at the critical point $x_2 = -b_{20}/b_{22}$. By (3.48a) the probability density function v is sharply peaked at this critical point. Therefore, the probability of meeting the stochastic system in the boundary layer $x_1 = O(\epsilon)$, asymptotically equals the probability of meeting the system in the boundary layer region *B*.

To study the *WKB*-solution in the region B the new variable

$$\hat{x}_2 = x_2 + \frac{b_{20}}{b_{22}} \tag{3.53}$$

is introduced and Q is approximated by the Taylor series expansion:

$$Q(x_1,x_2) = Q_0 + Q_2\hat{x}_2 + Q_1x_1 + \tfrac{1}{2}Q_3\hat{x}_2^2 + \cdots \tag{3.54}$$

Note that in the region B, x_1 is of the order $O(\hat{x}_2^2)$. Substitution of (3.54) into the eikonal equation (3.49) determines the constants

$$Q_2 = 0 \,, \qquad Q_1 = -\frac{k_1}{k_2} \,, \qquad Q_3 = \frac{k_3}{k_4} \,, \tag{3.55}$$

and leaves the constant Q_0 undetermined. The value of Q_0 is obtained by solving the problem (3.47, 3.48) numerically.

A boundary layer analysis is carried out to reveal the behaviour of the transport function w in the region B. The result (B9) for the one-dimensional exit problem, given in appendix B, indicates a singular behaviour. The stretched coordinates (3.4) and the boundary layer function

$$V(\bar{x}_1,\bar{x}_2) = v(\epsilon\bar{x}_1, -\frac{b_{20}}{b_{22}} + \sqrt{\epsilon}\bar{x}_2) \tag{3.56}$$

are introduced. Substitution into (3.47) leads to the boundary layer equation:

$$-k_1\frac{\partial}{\partial\bar{x}_1}(\bar{x}_1V) + k_2\frac{\partial^2}{\partial\bar{x}_1^2}(\bar{x}_1V) + k_3\frac{\partial}{\partial\bar{x}_2}(\bar{x}_2V) + k_4\frac{\partial^2V}{\partial\bar{x}_2^2} = 0\,, \tag{3.57}$$

with the k_i defined by (3.7). By the separation assumption

$$V(\bar{x}_1,\bar{x}_2) = r(\bar{x}_1)s(\bar{x}_2)\,, \tag{3.58}$$

(3.57) leads to the ordinary differential equations:

$$k_2\frac{d^2}{d\bar{x}_1^2}(\bar{x}_1r) - k_1\frac{d}{d\bar{x}_1}(\bar{x}_1r) - \mu r = 0\,, \tag{3.59a}$$

$$k_4\frac{d^2s}{d\bar{x}_2^2} + k_3\frac{d}{d\bar{x}_2}(\bar{x}_2s) + \mu s = 0\,, \tag{3.59b}$$

in which μ is the separation constant. The general solution of (3.59a) is:

$$r(\bar{x}_1) = \tilde{x}_1^{-1}e^{\frac{\tilde{x}_1}{2}}[c_1W_{\mu_1,\frac{1}{2}}(-\tilde{x}_1)+c_2W_{-\mu_1,\frac{1}{2}}(\tilde{x}_1)]\,, \tag{3.60}$$

with

$$\mu_1 = \frac{\mu}{k_1}\,, \tag{3.61}$$

c_1,c_2 arbitrary constants and $\tilde{x}_1$ defined by (3.11a). The general solution of (3.59b) is:

$$s(\bar{x}_2) = e^{-\frac{\tilde{x}_2^2}{4}}[c_3D_{\mu_2}(\tilde{x}_2)+c_4D_{-\mu_2-1}(i\tilde{x}_2)]\,, \tag{3.62}$$

with

$$\mu_2 = \frac{\mu}{k_3}\,, \tag{3.63}$$

c_3, c_4 arbitrary constants and $\tilde{x}_2$ defined by (3.13a). Putting

$$\mu = 0\,,\quad c_2 = 0\,,\quad c_4 = 0\,, \tag{3.64}$$

the boundary layer solution

$$V(\bar{x}_1,\bar{x}_2) = \text{const.}\ \tilde{x}_1^{-1}e^{\tilde{x}_1-\frac{\tilde{x}_2^2}{2}} \tag{3.65}$$

is obtained. The leading order part of the WKB-solution (3.48a) with Q given by (3.54), (3.55) agrees with the exponential function in the boundary layer solution (3.65). The solution (3.65) indicates that the transport function w behaves according to

$$w \sim \frac{1}{x_1} \tag{3.66}$$

in the region B. Substitution of the expansion

$$w(x_1,x_2) = \frac{1}{x_1}(w_0+w_2\hat{x}_2+w_1x_1+\frac{1}{2}w_3\hat{x}_2^2+\cdots) \tag{3.67}$$

into the transport equation (3.50) and using (3.55), leaves the constant w_0 undetermined. Its value is obtained by solving the problem (3.47), (3.48) numerically.

As a conclusion, in the boundary layer region B the WKB solution (3.48) behaves as:

$$v(x_1,x_2) = \frac{C_1(\epsilon)}{x_1} e^{\frac{1}{\epsilon}(\frac{k_1}{k_2}x_1 - \frac{k_3}{k_4}\frac{\hat{x}_2^2}{2})} \tag{3.68}$$

in which

$$C_1(\epsilon) = w_0 e^{-\frac{Q_0}{\epsilon}}, \tag{3.69}$$

where w_0, Q_0 as in (3.67), (3.54) respectively, have to be determined numerically. A similar result can be derived in the boundary layer region B'. There, the WKB-solution is:

$$v(x_1,x_2) = \frac{C_2(\epsilon)}{x_2} e^{\frac{1}{\epsilon}(\frac{k_1'}{k_2'}x_2 - \frac{k_3'}{k_4'}\frac{\hat{x}_1^2}{2})}, \tag{3.70}$$

where the constants in C_2, which is the analogue of C_1, have to be determined numerically. The constants k_3', k_4' are given by

$$k_3' = b_{10}, \tag{3.71a}$$

$$k_4' = -\frac{1}{2}\frac{b_{10}}{b_{11}}(a_{10} - \frac{a_{11}b_{10}}{b_{11}}), \tag{3.71b}$$

and $\hat{x}_1$ by:

$$\hat{x}_1 = x_1 + \frac{b_{10}}{b_{11}}. \tag{3.72}$$

3.3 Application of the divergence theorem

Using the divergence theorem the following integral relation can be derived:

$$\int_{R'}(vL_\epsilon u - uM_\epsilon v)dR' = \int_{\partial R'} \sum_{i=1}^{2} \nu_i \left[\frac{\epsilon}{2}a_i(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}) + (b_i - \frac{\epsilon}{2}\frac{\partial a_i}{\partial x_i})uv\right] ds, \tag{3.73}$$

where R' is a region with boundary $\partial R'$ on which the operators L_ϵ, M_ϵ are defined, and ν denotes the outward normal on $\partial R'$. In the right side of (3.73) v and its conormal derivative need to be evaluated at the boundary S'. By (3.68), (3.70) these functions become singular at $x_1 = 0$ and $x_2 = 0$. To avoid singular functions, R' is chosen as a slight modification of the region R:

$$R' = \{(x_1,x_2)|x_1>\delta, x_2>\delta\}, \tag{3.74}$$

and

$$\partial R' \equiv \bar{R}' \setminus R' = \{(x_1,x_2)|(x_1-\delta)(x_2-\delta)=0 \text{ and } x_1+x_2 \geqslant 2\delta\} \tag{3.75}$$

with δ a small number:

$$0<\delta \ll \epsilon. \tag{3.76}$$

By (2.17a) and (3.47) the left side of (3.73) equals zero. First the boundary $x_1 = \delta$ of R' is considered. There, the right side of (3.73) is written as:

$$\int_{\delta}^{\infty} \left[\frac{\epsilon}{2} a_1 (-v \frac{\partial u}{\partial x_1} + u \frac{\partial v}{\partial x_1}) - (b_1 - \frac{\epsilon}{2} \frac{\partial a_1}{\partial x_1}) \, uv \right]_{x_1 = \delta} dx_2 \, . \tag{3.77}$$

The only significant contribution to this integral comes from the boundary layer region B. Using the expression (3.16b) for u and (3.68) for v, the integrand in (3.77) is evaluated. Subsequently the limit $\delta \to 0$ is taken and asymptotically for small ϵ the following result is obtained:

$$\int_0^{\infty} C_1(\epsilon) \frac{k_1}{k_2} e^{-\frac{1}{\epsilon} \frac{k_3}{k_4} \frac{x_2^2}{2}} \left[(a_{10} - b_{10} \frac{k_2}{k_1}) C_{b1} - \frac{1}{2} a_{10} C_b + \left\{ (a_{12} - b_{12} \frac{k_2}{k_1}) C_{b1} - \frac{1}{2} a_{12} C_b \right\} x_2 \right] dx_2 \, . \tag{3.78}$$

This integral is evaluated by the method of Laplace [11]. Expression (3.78) becomes:

$$C_1(\epsilon) \frac{k_1}{k_2} \sqrt{2\pi\epsilon \frac{k_4}{k_3}} \left[(a_{10} - b_{10} \frac{k_2}{k_1}) C_{b1} - \frac{1}{2} a_{10} C_b - \frac{b_{20}}{b_{22}} \left\{ (a_{12} - b_{12} \frac{k_2}{k_1}) C_{b1} - \frac{1}{2} a_{12} C_b \right\} \right] . \tag{3.79}$$

Analogous to the derivation above the boundary $x_2 = \delta$ is treated. The following expression is found:

$$C_2(\epsilon) \frac{k_1'}{k_2'} \sqrt{2\pi\epsilon \frac{k_4'}{k_3'}} \left[(a_{20} - b_{20} \frac{k_2'}{k_1'}) C_{b2} - \frac{1}{2} a_{20} C_b - \frac{b_{10}}{b_{11}} \left\{ (a_{21} - b_{21} \frac{k_2'}{k_1'}) C_{b2} - \frac{1}{2} a_{21} C_b \right\} \right] . \tag{3.80}$$

Combining the results (3.79), (3.80) in the divergence theorem formula (3.73), an expression is obtained for C_b:

$$C_b = \frac{C_{b1} C_1(\epsilon) K_1 + C_{b2} C_2(\epsilon) K_2}{C_1(\epsilon) K_1 + C_2(\epsilon) K_2} \, , \tag{3.81}$$

with

$$K_1 = k_1 \sqrt{\frac{k_4}{k_3}} \, ,$$

$$K_2 = k_1' \sqrt{\frac{k_4'}{k_3'}} \, . \tag{3.82}$$

Expression (3.81) completes the analysis of this section. With (3.46) the following result is obtained. Denoting the probability of exit at boundary $x_i = 0$, starting at x, by $u_i(x)$, we have :

$$u_1(x) = \frac{C_1(\epsilon) K_1 + C_2(\epsilon) K_2 e^{-\gamma(x_2) \frac{x_1}{\epsilon}}}{C_1(\epsilon) K_1 + C_2(\epsilon) K_2} \left[1 - e^{-\tilde{\gamma}(x_1) \frac{x_2}{\epsilon}} \right] , \tag{3.83a}$$

$$u_2(x) = \frac{C_1(\epsilon) K_1 e^{-\tilde{\gamma}(x_1) \frac{x_2}{\epsilon}} + C_2(\epsilon) K_2}{C_1(\epsilon) K_1 + C_2(\epsilon) K_2} \left[1 - e^{-\gamma(x_2) \frac{x_1}{\epsilon}} \right] , \tag{3.83b}$$

asymptotically for small ϵ in $\overline{R} \setminus \overline{D}$, D and $O(\epsilon)$ neighbourhood of the origin. It is easily verified that in the region $\overline{R} \setminus \overline{D}$, expressions (3.83a) and (3.83b) add up to one. Rewriting (3.69) and the analog expression for C_2 as:

$$C_i(\epsilon) = w_{0i} \, e^{\frac{-Q_{0i}}{\epsilon}} \, , \quad (i = 1,2) \tag{3.84}$$

and using the fact that w_{oi}, K_i are order $O(1)$ constants, (3.83) can be simplified. In the case $Q_{01} < Q_{02}$ we find:

$$u_1(x) \sim 1 - e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}},$$

$$u_2(x) \sim e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}, \tag{3.85a}$$

and in the case $Q_{01} > Q_{02}$:

$$u_1(x) \sim e^{-\gamma(x_2)\frac{x_1}{\epsilon}},$$

$$u_2(x) \sim 1 - e^{-\gamma(x_2)\frac{x_1}{\epsilon}}. \tag{3.85b}$$

4. The expectation and variance of the exit time

In this section the boundary value problems (2.22) are solved asymptotically for small ϵ. Assume that $T_i(x)$ is of the form:

$$T_i(x) = C_{T,i}(\epsilon)\tau_i(x), \tag{4.1}$$

in which

$$C_{T,i}^{-1}(\epsilon)g_i(x) = o(\epsilon), \quad \epsilon \to 0. \tag{4.2}$$

Substitution of (4.1) into (2.22a) yields to leading order the reduced equation:

$$\sum_{j=1}^{2} b_j(x)\frac{\partial \tau_i}{\partial x_j} = 0, \tag{4.3}$$

which is solved by a constant that is taken 1 without loss of generality (any other constant can be incorporated in $C_{T,i}$):

$$\tau_i(x) = 1. \tag{4.4}$$

This is the outer solution, valid away from $O(\epsilon)$ boundary layers along $x_1 = 0$ and $x_2 = 0$. A boundary layer analysis can be held as in section 3, with u replaced by τ_i. The only difference is in the boundary condition, for this case stated by (2.22b). The following uniform asymptotic expansion for τ_i is obtained:

$$\tau_i(x_1, x_2) = \left[1 - e^{-\gamma(x_2)\frac{x_1}{\epsilon}}\right]\left[1 - e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}\right], \tag{4.5}$$

valid in R (the region D included). The unknown $C_{T,i}(\epsilon)$ are determined using the integral relation (3.73) with u replaced by T_i. After some calculus, this integral relation reduces to:

$$-\frac{2}{\epsilon}\int_{\delta}^{\infty}\int_{\delta}^{\infty} v g_i \, dx_1 dx_2 = \int_{\delta}^{\infty}\left[a_1 v \frac{\partial T_i}{\partial x_1}\right]_{x_1 = \delta} dx_2 + \int_{\delta}^{\infty}\left[a_2 v \frac{\partial T_i}{\partial x_2}\right]_{x_2 = \delta} dx_1 . \tag{4.6}$$

On the right side, the largest contributions to the integrals are from the boundary layer regions B and B'. These integrals are evaluated by the method of Laplace, using expressions (4.1), (4.5) with

$$\gamma(x_2) = \frac{k_1}{k_2}, \tag{4.7}$$

$$\tilde{\gamma}(x_1) = \frac{k_1'}{k_2'},$$

for T and expressions (3.68), (3.70) for v. The left side of (4.6) is evaluated using the WKB-expression (3.48a) for v and the method of Laplace for double integrals. Letting $\delta \to 0$, the following expressions are found for $C_{T,i}(\epsilon)$:

$$C_{T,1}(\epsilon) = \frac{\sqrt{\dfrac{2\pi\epsilon}{H(x^e)}}}{C_1(\epsilon)K_1 + C_2(\epsilon)K_2}\,, \tag{4.8a}$$

$$C_{T,2}(\epsilon) = 2C_{T,1}^2(\epsilon)\,, \tag{4.8b}$$

in which $H(x^e)$ is the determinant of the Hessian matrix of Q at x^e. With $Q^* = \min(Q_{01}, Q_{02})$, see (3.84), and using the fact that w_{oi}, K_i, $H(x^e)$ are of order $O(1)$, $C_{T,1}(\epsilon)$ is of the order

$$C_{T,1}(\epsilon) \sim \sqrt{\epsilon}\, e^{\frac{Q^*}{\epsilon}}\,. \tag{4.9}$$

In the evaluation of the left side of (4.6) we let $\delta \to 0$, while the WKB-expression (3.48a) for v fails to be integrable in this limit. This procedure was proposed by LUDWIG [10]. Its correctness has not been proven. See also the remarks in [4] at this point. By (2.20) the resulting uniform asymptotic expansions in $\bar{R}$ of the expectation and variance of the exit time are given as:

$$E_T(x) = C_{T,1}(\epsilon)\left[1 - e^{-\gamma(x_2)\frac{x_1}{\epsilon}}\right]\left[1 - e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}\right], \tag{4.10a}$$

$$Var_T(x) = C_{T,1}^2(\epsilon)\left[1 - \left[e^{-\gamma(x_2)\frac{x_1}{\epsilon}} + e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}} - e^{-\gamma(x_2)\frac{x_1}{\epsilon} - \tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}\right]^2\right], \tag{4.10b}$$

respectively.

5. NUMERICAL DETERMINATION OF THE WKB-SOLUTION.

To obtain the constants $Q_{01}, w_{01}, Q_{02}, w_{02}$ in (3.84), the WKB-solution (3.48) of the adjoint equation (3.47) is determined numerically. By the Hamilton-Jacobi theory [12], the eikonal equation (3.49) is written in terms of the Hamiltonian H:

$$H(x,p) \equiv \sum_{i=1}^{2}[b_i p_i + \tfrac{1}{2}a_i p_i^2] = 0, \tag{5.1a}$$

where

$$p_i = \frac{\partial Q}{\partial x_i}\,. \tag{5.1b}$$

The corresponding system of bicharacteristics reads:

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i} = b_i + a_i p_i\,, \qquad (i=1,2) \tag{5.2a}$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{2}\left[\frac{\partial b_j}{\partial x_i}p_j + \frac{1}{2}\frac{\partial a_j}{\partial x_i}p_j^2\right], \qquad (i=1,2) \tag{5.2b}$$

with s a parameter along the characteristics. The rate of change of Q with s is given by:

$$\frac{dQ}{ds} = -H + \sum_{i=1}^{2}\frac{dx_i}{ds}p_i = \sum_{i=1}^{2}\frac{1}{2}a_i p_i^2\,, \quad \text{(which is } \geq 0). \tag{5.2c}$$

At $s=0$ all characteristics start in a neighbourhood of the equilibrium

$$x = x^e\,,$$
$$p = 0\,, \qquad (5.3)$$
$$Q = 0\,,$$

of the system (5.2). The initial position of a characteristic is specified on a circle around x^e with radius $r \ll 1$, by the variable θ:

$$x_1 = x_1^e + r\cos\theta\,, \qquad (5.4)$$
$$x_2 = x_2^e + r\sin\theta\,.$$

The corresponding initial values of p_1, p_2, Q are obtained by the following local analysis.

In the neighbourhood of (5.3), Q is approximated by the quadratic form:

$$Q \approx \frac{1}{2}(x - x^e)^t P (x - x^e), \qquad (5.5)$$

in which P is a symmetric matrix and t denotes the transpose. It follows that:

$$p = \frac{dQ}{dx} \approx P\,(x - x^e)\,. \qquad (5.6)$$

Substitution of the approximations (5.6), (2.12) and

$$\begin{bmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{bmatrix} \approx \begin{bmatrix} a_1(x^e) & 0 \\ 0 & a_2(x^e) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \equiv A \qquad (5.7)$$

into the eikonal equation (3.49) leads to the matrix equation:

$$PAP + PB + B^t P = 0\,, \qquad (5.8)$$

which is solved to give:

$$P = \frac{-2(B_{11} + B_{22})}{(B_{21}A_1 - B_{12}A_2)^2 + A_1A_2(B_{11} + B_{22})^2} \cdot \qquad (5.9)$$

$$\cdot \begin{bmatrix} B_{21}^2 A_1 + [B_{11}B_{22} - B_{12}B_{21} + B_{11}^2]A_2 & B_{22}B_{21}A_1 + B_{11}B_{12}A_2 \\ B_{22}B_{21}A_1 + B_{11}B_{12}A_2 & B_{12}^2 A_2 + [B_{11}B_{22} - B_{12}B_{21} + B_{22}^2]A_1 \end{bmatrix}.$$

The initial values of p_1, p_2, Q are determined by (5.6), (5.5), (5.9). Notice that (5.9) also determines the determinant

$$H(x^e) = P_{11}P_{22} - P_{12}P_{21} \qquad (5.10)$$

in (4.8a).

Next we consider the transport equation (3.50). With (5.2a) and

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{dx_i}{ds}\right) = \frac{d}{ds}\ln|J|, \qquad (5.11a)$$

see [10], in which J is the Jacobian

$$J = \begin{vmatrix} \dfrac{dx_1}{ds} & \dfrac{\partial x_1}{\partial \theta} \\ \dfrac{dx_2}{ds} & \dfrac{\partial x_2}{\partial \theta} \end{vmatrix}, \qquad (5.11b)$$

equation (3.50) is rewritten as:

$$\frac{d}{ds}(\ln w^2|J|) = -\sum_{i=1}^{2}\left[\frac{\partial b_i}{\partial x_i} + p_i\frac{\partial a_i}{\partial x_i}\right]. \tag{5.12}$$

Differentiation of (5.2a,b) with respect to θ leads to the equations

$$\frac{d}{ds}(\frac{\partial x_i}{\partial \theta}) = \frac{\partial b_i}{\partial \theta} + \frac{\partial a_i}{\partial \theta}p_i + a_i\frac{\partial p_i}{\partial \theta}, \tag{5.13a}$$

$$\frac{d}{ds}(\frac{\partial p_i}{\partial \theta}) = -\sum_{j=1}^{2}\left[\frac{\partial^2 b_j}{\partial x_i \partial \theta}p_j + \frac{\partial b_j}{\partial x_i}\frac{\partial p_j}{\partial \theta} + \frac{1}{2}\frac{\partial^2 a_j}{\partial x_i \partial \theta}p_j^2 + \frac{\partial a_j}{\partial x_i}\frac{\partial p_j}{\partial \theta}p_j\right], \tag{5.13b}$$

$i=1,2$, which describe the rate of change with s of $\partial x_i/\partial\theta$ and, using (5.2a), of J. The initial value at $s=0$ of w is chosen according to (3.48c). The initial values of $\partial x_i/\partial\theta$, $\partial p_i/\partial\theta$ are obtained by differentiation of the initial expressions (5.4) for x_i and (5.6) for p_i with respect to θ.

To obtain Q_{01}, w_{01}, the system of 10 ordinary differential equations (5.2), (5.12), (5.13) is integrated. By trial and error the angle θ of the initial point is manipulated in order to obtain a characteristic containing points close to $(0, -b_{20}/b_{22})$. Once a (Δ^2,Δ)-neighbourhood of $(0,-b_{20}/b_{22})$ is reached, $\Delta \ll 1$, the integration is terminated. Using the values of Q,w obtained numerically at the end point(s) of the characteristic and the formulas (3.54), (3.55) for Q and (3.67) for w, valid in the case $x_1=O(\hat{x}_2^2)$, $\hat{x}_2$ small, we approximate the values of Q_{01}, w_{01}.

The solutions of Q and w obtained by the numerical method described above are not always unique functions of x. By assumption, the solution is unique along the characteristic which starts at the initial point with $r=0$ and ends at $(0,-b_{20}/b_{22})$. In numerical computations, this characteristic cannot be followed exactly. Near $(0,-b_{20}/b_{22})$ the characteristics curve upward or downward along $x_1=0$ and get into caustic surfaces, as indicated by a change of sign in the determinant (5.11b). There, the solution is not a unique function of x. The numerical integration must be terminated before the determinant vanishes. The boundary $x_1=0$ cannot be approached too close. Consequently, Δ cannot be taken arbitrary small, which limits the accuracy of the computed values of Q_{01}, w_{01}. In the subsequent example, the numerical computation was stopped at $x_1 \approx 0.02$.

6. An example

Consider the predator-prey system defined by the diffusion

$$a_1(x) = x_1(1.28 + 0.80x_1 + 0.32x_2), \tag{6.1a}$$
$$a_2(x) = x_2(1.08 + 0.28x_1 + 0.40x_2),$$

and the drift:

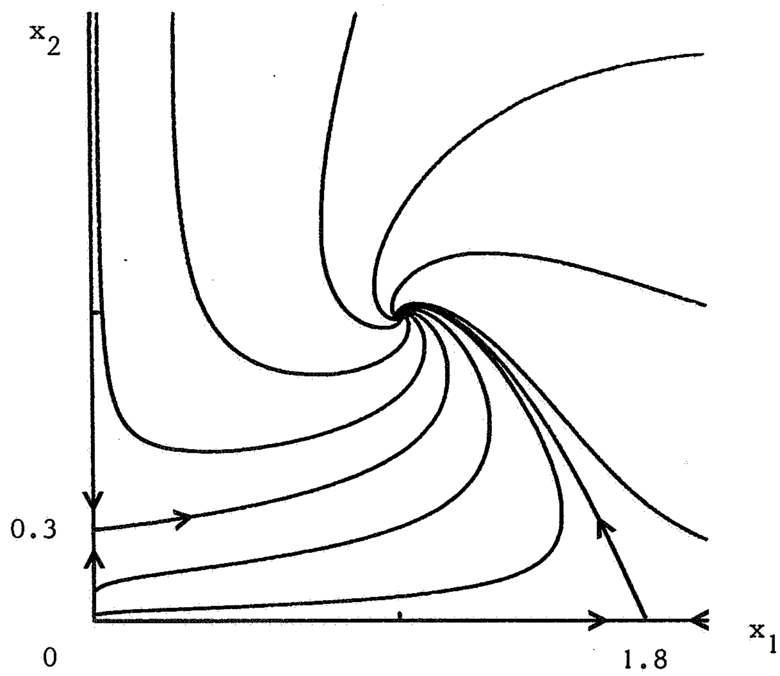
$$b_1(x) = x_1(0.72 - 0.40x_1 - 0.32x_2), \tag{6.1b}$$
$$b_2(x) = x_2(0.12 + 0.28x_1 - 0.40x_2),$$

in which x_1 and x_2 denote the prey and predator density, respectively. The stochastic system defined by (6.1) is the diffusion approximation to a birth-death process treated in [5]. Some trajectories of the deterministic system are depicted in fig.2. The numerical computation described in section 5 produces the values:

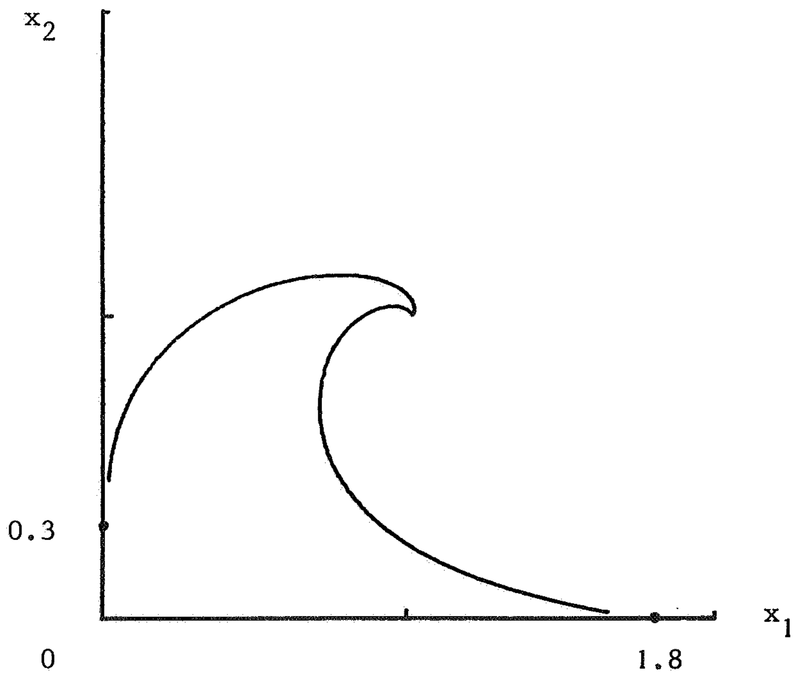
$$Q_{01} = 0.26,\ w_{01} = 1.1,\ Q_{02} = 0.30,\ w_{02} = 1.5. \tag{6.2}$$

The projection on the x-plane of the characteristics (called rays), used in this computation, are depicted in fig. 3. Outside the region D, the probability of exit at boundary $x_i=0$ is given by u_i:

2



3



4

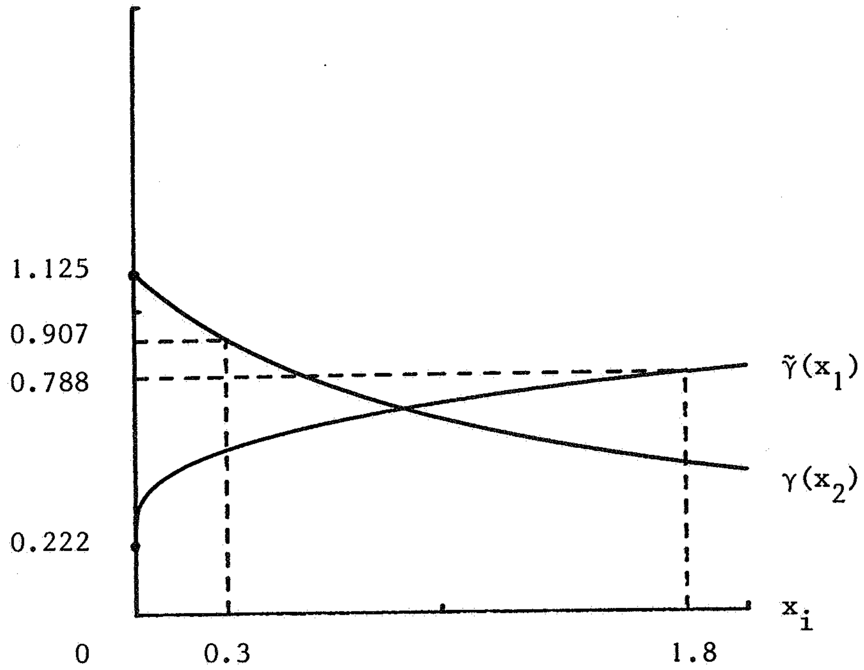


FIGURE 2. Trajectories of the deterministic system associated with the stochastic system (6.1). The critical points are (0,0), (0,.3), (1.8,0), (1,1).

FIGURE 3. The rays used in the numerical computation of Q_{01}, w_{01} and Q_{02}, w_{02}.

FIGURE 4. The functions $\gamma, \tilde{\gamma}$. The critical points, at which the initial condition for the Bernoulli differential equation is specified, are indicated. The values denoted along the vertical axis follow from (3.30), a similar formula for the other boundary, and (3.43a,b).

$$u_1(x) = \frac{.69e^{-\frac{.26}{\epsilon}} + 1.03e^{-\frac{1}{\epsilon}[.30 + x_1\gamma(x_2)]}}{.69e^{-\frac{.26}{\epsilon}} + 1.03e^{-\frac{.30}{\epsilon}}} \left[1 - e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}\right] \sim 1 - e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}, \tag{6.3a}$$

$$u_2(x) = \frac{.69e^{-\frac{1}{\epsilon}[.26 + x_2\tilde{\gamma}(x_1)]} + 1.03e^{-\frac{.30}{\epsilon}}}{.69e^{-\frac{.26}{\epsilon}} + 1.03e^{-\frac{.30}{\epsilon}}} \left[1 - e^{-\gamma(x_2)\frac{x_1}{\epsilon}}\right] \sim e^{-\tilde{\gamma}(x_1)\frac{x_2}{\epsilon}}, \tag{6.3b}$$

according to (3.83), (3.85). The expectation and variance of the exit time satisfy equations (4.10), uniformly in R, with

$$C_{T,1}(\epsilon) = \frac{\sqrt{\epsilon}}{0.11e^{-\frac{.26}{\epsilon}} + 0.16e^{-\frac{.30}{\epsilon}}} \sim 9.1\sqrt{\epsilon}\, e^{\frac{.26}{\epsilon}}. \tag{6.4}$$

The functions $\gamma, \tilde{\gamma}$ are computed numerically by the method described in appendix A. Their graphs are shown in fig. 4. In the boundary layers along $x_1 = 0$ and $x_2 = 0$ a small (large) value of $\gamma, \tilde{\gamma}$ respectively, may be interpreted as a relatively weak (strong) stochastic stability. From fig. 4 we conclude that for low prey density the stochastic stability of the system (6.1) decreases with increasing predator density; for low predator density the stochastic stability increases with increasing prey density.

7. Some remarks

Problems with diffusion and drift coefficients vanishing nonlinearly at the boundary require a different asymptotic analysis. In such a case the boundaries may be not of exit type. It is noted that for two-dimensional stochastic systems no complete classification of boundaries exists as for one- dimensional systems [13,14].

In a diffusion approximation to a birth-death process, only the first and second jump moments are incorporated. An approximation to birth-death processes taking into account higher jump moments can be found in [15].

Various extensions of the theory presented in this paper are conceivable. Investigations can be directed towards a two-dimensional model, more general than treated in this paper, with the same qualitative behaviour of drift and diffusion at the boundary and the same type of deterministic critical points. The boundary condition (2.17b) can be used with general f. With respect to these points, see the approach in [1,2,3]. The theory may be extended to higher dimensions.

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APPENDIX A: SOLUTION OF THE BERNOULLI-PROBLEM

The Bernoulli-equation

$$(b_{10} + b_{12}x_2)\gamma + x_2(b_{20} + b_{22}x_2)\gamma' = \frac{1}{2}(a_{10} + a_{12}x_2)\gamma^2, \tag{A1a}$$

must be solved with the condition:

$$\gamma(-\frac{b_{20}}{b_{22}}) = \frac{k_1}{k_2}. \tag{A1b}$$

By the substitution

$$\gamma = g^{-1} \tag{A2}$$

equation (A1a) changes into the linear differential equation:

$$x_2(b_{20} + b_{22}x_2)g' - (b_{10} + b_{12}x_2)g = -\frac{1}{2}(a_{10} + a_{12}x_2), \tag{A3a}$$

with the condition

$$g(-\frac{b_{20}}{b_{22}}) = \frac{k_2}{k_1}. \tag{A3b}$$

The general solution of equation (A3a) is given by the variation of constants formula:

$$g(x_2) = e^{\int^{x_2} \frac{b_{10}+b_{12}x}{x(b_{20}+b_{22}x)}dx} \left[C - \int^{x_2} \frac{\frac{1}{2}(a_{10}+a_{12}x)}{x(b_{20}+b_{22}x)} e^{-\int^{x} \frac{b_{10}+b_{12}x'}{x'(b_{20}+b_{22}x')}dx'} dx \right]. \tag{A4}$$

Only in special cases this formula can be worked out analytically. Next it is shown that

$$g(0) = \frac{\frac{1}{2}a_{10}}{b_{10}}. \tag{A5}$$

The homogeneous equation corresponding to the inhomogeneous equation (A3a) is solved by:

$$g_h(x_2) = e^{\int^{x_2} \frac{b_{10}+b_{12}x}{x(b_{20}+b_{22}x)}dx} = C(b_{20} + b_{22}x_2)^{\frac{b_{12}}{b_{22}} - \frac{b_{10}}{b_{20}}} x_2^{\frac{b_{10}}{b_{20}}}. \tag{A6}$$

By the assumptions made in section 2 with respect to the b_{ij} we have:

$$\frac{b_{12}}{b_{22}} - \frac{b_{10}}{b_{20}} < 0, \tag{A7a}$$

$$\frac{b_{10}}{b_{20}} > 0, \tag{A7b}$$

so that

$$\lim_{x_2 \uparrow -\frac{b_{20}}{b_{22}}} g_h(x_2) = \begin{cases} +\infty, & C>0 \\ -\infty, & C<0 \end{cases} \tag{A8a}$$

$$\lim_{x_2 \downarrow 0} g_h(x_2) = 0. \tag{A8b}$$

A particular solution of the inhomogeneous equation (A3a) is obtained by the Taylor series expansion around the regular singular point $x_2 = -b_{20}/b_{22}$:

$$g_{p,1}(x_2) = \frac{k_2}{k_1} + \frac{a_{12}b_{10} - a_{10}b_{12}}{2k_1(b_{20}+k_1)}(x_2 + \frac{b_{20}}{b_{22}}) + \cdots, \tag{A9}$$

which converges on at least $(0, -b_{20}/b_{22}]$. Since $g_{p,1}$ satisfies the condition (A3b) and, by (A8a), g_h becomes infinite at $x_2 = -b_{20}/b_{22}$, the solution of (A3) on $(0, -b_{20}/b_{22}]$ is given by:

$$g(x_2) = g_{p,1}(x_2). \tag{A10}$$

Another particular solution of the inhomogeneous equation (A3a) is obtained by the Taylor series expansion around the regular singular point $x_2 = 0$:

$$g_{p,2}(x_2) = \frac{\frac{1}{2}a_{10}}{b_{10}} - \frac{a_{12}b_{10} - a_{10}b_{12}}{2b_{10}(b_{20} - b_{10})}x_2 + \cdots, \tag{A11}$$

which converges on at least $[0, -b_{20}/b_{22})$. The general solution of equation (A3a) on $[0, -b_{20}/b_{22})$ may be written as:

$$\bar{g}(x_2) = g_h(x_2) + g_{p,2}(x_2). \tag{A12}$$

By (A8b) and (A11) we have

$$\lim_{x_2 \downarrow 0} \bar{g}(x_2) = \frac{\frac{1}{2}a_{10}}{b_{10}}, \tag{A13}$$

independent of C. Remark that since the regions of convergence of (A9) and (A12) overlap, the constant C is determined by the condition:

$$\bar{g}(x_2) \equiv g_{p,1}(x_2), \quad x_2 \in (0, -\frac{b_{20}}{b_{22}}). \tag{A14}$$

Generally, γ has to be determined numerically. Expressions (A2), (A9) supply the starting values $\gamma(-b_{20}/b_{22} \pm \Delta)$, Δ small, and a forward (backward) finite difference scheme base on (A1a) is used to obtain $\gamma(x_2)$ for $x_2 > -b_{20}/b_{22} + \Delta$ $(,x_2 < -b_{20}/b_{22} - \Delta)$.

Appendix B: The one-dimensional exit problem

Consider the one-dimensional stochastic system defined by the diffusion coefficient

$$a(x) = x(a_0 + a_1 x), \quad a_0 > 0,\ a_1 > 0, \tag{B1}$$

and the drift coefficient:

$$b(x) = x(b_0 + b_1 x). \tag{B2}$$

The deterministic system has the equilibria

$$x = 0, \tag{B3a}$$

$$x = x^e \equiv -\frac{b_0}{b_1}. \tag{B3b}$$

By the assumptions

$$x^e > 0\,, \quad x^e = O(1), \tag{B4a}$$

$$b_0 > 0, \tag{B4b}$$

the equilibrium $x = 0$ is instable and the equilibrium x^e has a positive coordinate of order one and is stable. The determination of the probability of exit at $x = 0$ is a trivial problem since only one boundary is involved. To avoid a trivial problem we assume a second boundary $x = l$,

$$l >> x^e, \tag{B5}$$

and study exit at either $x = 0$ or $x = l$. Using the methods of section 3 the uniform expansion of u in

$$\bar{R} = \{x|0 \leqslant x \leqslant l\} \tag{B6}$$

is determined:

$$u(x) = C_b + (C_{b0} - C_b)e^{-\frac{2b_0}{a_0}\frac{x}{\epsilon}} + (C_{bl} - C_b)e^{-\frac{2(b_0+b_1 l)}{a_0+a_1 l}\frac{x-l}{\epsilon}}, \tag{B7}$$

with the constant C_b given by:

$$C_b = \frac{[C_{bx}v(x)b(x)]_{x=l}^{x=0}}{[v(x)b(x)]_{x=l}^{x=0}}, \tag{B8}$$

in which v is the WKB-solution of the adjoint forward problem, found analytically as:

$$v(x) = \frac{-\frac{b_0}{b_1}(a_0 - a_1\frac{b_0}{b_1})}{x(a_0 + a_1 x)}\, e^{-\frac{1}{\epsilon}\left[-2\frac{b_1}{a_1}(x+\frac{b_0}{b_1})-2\frac{b_0 a_1 - a_0 b_1}{a_1^2}\ln(\frac{a_0+a_1 x}{a_0 - a_1 b_0/b_1})\right]}, \tag{B9}$$

which is singular at $x=0$. In the limit $l \rightarrow \infty$ the trivial result

$$u(x) = C_{b0} \tag{B10}$$

is obtained. The asymptotic expansion for the expectation of the exit time yields:

$$E_T(x) = \frac{\sqrt{\frac{2\pi\epsilon}{Q''(x^e)}}}{[v(x)b(x)]_{x=l}^{x=0}}\left[1 - e^{-\frac{2b_0}{a_0}\frac{x}{\epsilon}} - e^{-\frac{2(b_0+b_1 l)}{a_0+a_1 l}\frac{x-l}{\epsilon}}\right], \tag{B11}$$

with

$$Q''(x^e) = \frac{-2b_1^2}{a_0 b_1 - a_1 b_0}. \tag{B12}$$

In the limit $l \rightarrow \infty$ we obtain:

$$E_T(x) = \frac{\sqrt{\frac{2\pi\epsilon}{Q''(x^e)}}}{[v(x)b(x)]^{x=0}}\left[1 - e^{-\frac{2b_0}{a_0}\frac{x}{\epsilon}}\right]. \tag{B13}$$

The variance of the exit time is given asymptotically by:

$$\mathrm{Var}_T(x) = \frac{\frac{2\pi\epsilon}{Q''(x^e)}}{\{[v(x)b(x)]_{x=l}^{x=0}\}^2}\left[1 - \left[e^{-\frac{2b_0}{a_0}\frac{x}{\epsilon}} + e^{-\frac{2(b_0+b_1 l)}{a_0+a_1 l}\frac{x-l}{\epsilon}}\right]^2\right], \tag{B14}$$

which, in the limit $l \rightarrow \infty$, reduces to:

$$\mathrm{Var}_T(x) = \frac{\frac{2\pi\epsilon}{Q''(x^e)}}{\{[v(x)b(x)]^{x=0}\}^2}\left[1 - e^{-\frac{4b_0}{a_0}\frac{x}{\epsilon}}\right]. \tag{B15}$$

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