stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 174/79 JUNI
(DEPARTMENT OF APPLIED MATHEMATICS)

O. DIEKMANN, D. HILHORST & L.A. PELETIER

A SINGULAR BOUNDARY VALUE PROBLEM ARISING IN A PRE-BREAKDOWN GAS DISCHARGE

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1980): 34B15; 34E15; 34E05; 34C35

A singular boundary value problem arising in a pre-breakdown gas discharge *)

by

O. Diekmann, D. Hilhorst & L.A. Peletier

ABSTRACT

We consider the nonlinear two-point boundary value problem $\varepsilon x y'' + (g(x)-y)y' = 0$, $y(0) = 0$, $y(R) = k$, where g is a given function. We prove that the problem has a unique solution and we study the limiting behaviour of this solution as $R \to \infty$ and as $\varepsilon \downarrow 0$.

Furthermore, we show how a so-called pre-breakdown discharge in an ionized gas between two electrodes can be described by an equation of this form, and we interpret the results physically.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem; pre-breakdown discharge in an ionized gas between two electrodes.*

---

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we study the two-point boundary value problem

(1.1) $\varepsilon x y'' + (g(x)-y)y' = 0, \quad x \in (0,R),$

in which R is a positive number, which may be infinite, and g a given function, which satisfies the hypotheses

H_g: $g \in C^2(\mathbb{R}_+)$; $g(0) = 0$; $g'(x) > 0$ and $g''(x) < 0$ for all $x \geq 0$.

We are interested in solutions of (1.1) which satisfy the boundary conditions

(1.2) $y(0) = 0$

(1.3) $y(R) = k$

in which $k \in (0, g(\infty))$ and $R > x_0$, x_0 being the (unique) root of the equation $g(x) = k$.

In section 2 we shall sketch how problem (1.1)-(1.3) arises in the study of electrical discharges in an ionized gas. It will appear that y' and g' are measures for, respectively, the electron- and ion densities, and that the parameter ε is proportional to the temperature of the gas.

In section 3 we begin the mathematical analysis of problem (1.1)-(1.3). We derive some a priori estimates and then prove the existence of a solution. Subsequently, in section 4 we prove that the solution is unique.

The main objective of this paper is the study of the dependence of the solution on the parameters ε and R. In section 4 we prove that the solution is a monotone function of ε and R. From the physical point of view the interesting regions of the parameters are small ε and large R. In section 5 we analyze the limiting behaviour of the solution when R tends to infinity and ε is kept fixed. It turns out that the solution converges uniformly in x to a function $\bar{y}$ which satisfies (1.1)-(1.2) and the limiting form of (1.3), i.e. $\bar{y}(\infty) = k$, if and only if $\varepsilon \leq g(\infty)-k$. If on the other hand, this inequality is violated, then the solution converges uniformly on compact subsets to a function $\bar{y}$ which satisfies (1.1)-(1.2) and $\bar{y}(\infty) = \max\{g(\infty) - \varepsilon, 0\}$. In particular this implies that $\bar{y}$ is identically zero if $\varepsilon \geq g(\infty)$.

In section 6 we analyze the limiting behaviour of the solution when ε tends to zero and R is kept fixed. It turns out that the solution y converges uniformly for $x \in [0,R]$ to the function $\tilde{y}(x) = \min\{g(x),k\}$, but that its derivative y' converges uniformly to $\tilde{y}'$ only on compact subsets of [0,R] which do not contain the transition pont x_0.

In section 7 we discuss in greater detail the behaviour of y' near the point x_0 as $\varepsilon \downarrow 0$. By the standard method of matched asymptotic expansions we formally obtain in section 8 an approximation y_a. In section 9 we prove that for each n > 1

$$y - y_a = O(\varepsilon^{n+\frac{1}{2}}), \qquad y' - y_a' = O(\varepsilon^{n-\frac{1}{2}}), \qquad \text{as } \varepsilon \downarrow 0,$$

uniformly on [0,R], where n counts the number of terms included in the approximation. In this part of our treatment of the singular perturbation problem we derived much inspiration from reading bits and pieces of van Harten's thesis [9].

Since the limits $\varepsilon \downarrow 0$ and $R \to \infty$ (for $\varepsilon \leq g(\infty)-k$) are interchangeable, the two separate limits give a complete picture of the limiting behaviour with respect to both parameters.

Finally, in section 10, we consider problem (1.1)-(1.3) under the much weaker condition on g:

$$\tilde{H}_g\colon\ g \in C^1([0,R]); \quad g(0) = 0; \quad g(R) \geq k;$$
$$g \text{ has only finitely many local extrema on } [0,R].$$

Again, the existence and uniqueness of a solution $y(x;\varepsilon)$ is established and it is shown that y' > 0. In addition

$$y(x;\varepsilon) \to u(x) \quad \text{as} \quad \varepsilon \downarrow 0,$$

uniformly on [0,R], where the function u, which is continuous, consists of pieces where $u(x) = g(x)$ and pieces where $u(x)$ is a constant. The arguments we employ here are borrowed from the theory of dynamical systems and are somewhat unusual in this context.

Problems like the one treated in this paper have also been considered by HALLAM & LOPER [8], HOWES & PARTER [11] (also see HOWES [10]), CLÉMENT & EMMERTH [4] and CLÉMENT & PELETIER [5]. Both of the first two papers deal with one particular equation and the second two papers deal with concave solutions y_ε of a general class of equations. In all of these $\lim_{\varepsilon\downarrow 0} y_\varepsilon$ is determined. In this paper we do the same by the method of upper and lower solutions, which was also used by HOWES & PARTER, and in addition we give precise estimates of the behaviour of y_ε and y'_ε as $\varepsilon \downarrow 0$.

2. PHYSICAL BACKGROUND

2.1. An electrical discharge

MARODE et al. [14] consider an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(r)$ and $n_e(r)$ respectively, where $r = (x_1,x_2,x_3)$. The ions are heavy and slow, and the density $n_i(r)$ may therefore be regarded as fixed. The electrons are highly mobile and assume a spatial distribution in thermal equilibrium with the ions. The problem is then to find $n_e(r)$ for given $n_i(r)$.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. GALLIMBERTI [7] and MARODE [13]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $\rho := (x_1^2+x_2^2)^{\frac{1}{2}}$ only. Using Coulomb's law and a constitutive equation for the electric current, which contains both a diffusion and a conduction term, MARODE et al. [14] derived that the electron density $n_e(\rho)$ should satisfy the equation

$$(2.1)\qquad -\frac{\varepsilon}{2}\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho}{n_e(\rho)}\frac{d}{d\rho}n_e(\rho)\right) = n_i(\rho) - n_e(\rho),$$

where ε is a combination of physical constants which is proportional to the temperature. In addition n_e has to satisfy the boundary condition

$$(2.2) \qquad \frac{dn_e}{d\rho}(0) = 0$$

and the condition

$$(2.3) \qquad \int_0^\infty \{n_i(\rho) - n_e(\rho)\}\rho d\rho = N > 0,$$

where N is a measure for the excess of ions.

In the experiment the ions are concentrated near the center of the discharge. Hence we shall take for n_i a function which decreases monotonically to zero as ρ tends to infinity. In this paper we study the solution n_e of (2.1)-(2.3) and in particular its behaviour as $\varepsilon \downarrow 0$.

In order to cast (2.1) in a more convenient form, we make the change of variable

$$(2.4) \qquad x = \rho^2$$

and we define the new dependent variable

$$(2.5) \qquad y(x) = \int_0^{x^{\frac{1}{2}}} n_e(s)sds.$$

Thus, y(x) represents the number of electrons contained in a cylinder of unit height and radius $x^{\frac{1}{2}}$. Analogously, we define

$$(2.6) \qquad g(x) = \int_0^{x^{\frac{1}{2}}} n_i(s)sds.$$

If we now multiply (2.1) by ρ, integrate from $\rho = 0$ to $\rho = x^{\frac{1}{2}}$ and use (2.4)-(2.6) we obtain (1.1). The boundary condition (1.2) is implied by (2.5) and the boundary condition (1.3), with $R = \infty$, follows from (2.3):

$$y(\infty) = k := g(\infty) - N,$$

where clearly $k \in (0,g(\infty))$.

2.2. The two-dimensional Coulomb gas

Equation (1.1) describes the equilibrium distribution of electrons interacting, via the Coulomb potential, with themselves and with a fixed positive background in a two-dimensional geometry. Theoretically one can generalize Coulomb's law to a space of arbitrary dimension d and then the corresponding equation would become

$$(2.7) \qquad \varepsilon x^{2\frac{d-1}{d}} y'' + (g(x)-y)y' = 0$$

in which ε is again a positive constant which is proportional to the temperature.

The behaviour of an assembly of charges depends on the competition between the electrostatic forces, which tend to bind positive and negative charges together, and the thermal motion which drives them apart. By physical arguments one can show that for $d > 2$ the thermal motion wins: at no non-zero temperature are the electrons bound to the ions. For $d < 2$, the electrostatic forces win, and whatever the temperature the charges are bound together (see CHUI & WEEKS [3]).

For the model problem consisting of equation (2.7) supplemented with the boundary conditions (1.2) and (1.3), with $R = \infty$, we find these matters reflected in the fact that for arbitrary positive ε, no solution exists when $d > 2$ whereas, on the contrary, a unique solution exists when $d < 2$. One can prove this along the lines indicated in section 5.

The marginal case $d = 2$ is of greatest interest. Presumably there is a critical value of the temperature at which a transition occurs from bound to unbound charges and recently there has been much interest in the precise nature of this transition (see KOSTERLITZ & THOULESS [12]).

In our study of the two-dimensional case we find indeed, in section 5, a critical value of ε (and hence of the temperature)

$$\varepsilon_1 = g(\infty) - k = N$$

at which the nature of the solution n_e changes, corresponding to the loss (towards infinity) of part of the negative charge. Beyond a still higher value of ε:

$$\varepsilon_2 = g(\infty)$$

there appears to be no solution, indicating that the negative charge is no longer bound to the positive background.

2.3. Low temperatures

We also have studied the equations in the low temperature regime, i.e. for $\varepsilon \downarrow 0$. Physically one then expects all the electrons to gather in the region of lowest energy, that is in the center of the ion distribution. Indeed we have found that for $\varepsilon \downarrow 0$ the solution of equation (2.1) exhibits transition behaviour

$$\lim_{\varepsilon \downarrow 0} n_e(\rho) = \begin{cases} n_i(\rho) & \rho < \rho_0 \\ 0 & \rho > \rho_0 \end{cases}$$

where ρ_0 is determined by the boundary condition (2.3). There appears to be a transition layer of width of order $\varepsilon^{\frac{1}{2}}$ which, according to MARODE et al. [14], has the form of a Debye shielding length.

3. A PRIORI ESTIMATES AND THE EXISTENCE OF A SOLUTION

In this section we consider the problem (1.1)-(1.3) for fixed values of the parameters ε and R. By a solution we shall mean a function $y \in C^2([0,R])$ which satisfies (1.1)-(1.3). We first derive some a priori estimates for a solution and its first two derivatives. Subsequently we prove that a solution actually exists by constructing an upper and lower solution and by verifying the appropriate Nagumo condition.

<u>THEOREM 3.1</u>. *Let* y *be a solution, then for all* $x \in (0,R)$

(i) $0 < y(x) < \min\{g(x),k\}$;

(ii) $0 < y'(x) < g'(0)$;

(iii) $-\frac{(g'(0))^2}{\varepsilon} < y''(x) < 0$.

PROOF. Let us first prove that $y'(x) > 0$ for all $x \in (0,R)$. Suppose that $y'(x_1) = 0$ for some $x_1 > 0$, then the standard uniqueness theorem for ordinary differential equations implies that $y(x) = y(x_1)$ for all x. Since this is not compatible with the two boundary conditions we conclude that y' is sign-definite. Invoking the boundary conditions once more, we see that the sign has to be positive.

The positivity of y' implies that $0 < y(x) < k$ for $x \in (0,R)$. Next we shall prove that $y(x) < g(x)$. We begin by observing that this inequality holds for $x \geq x_0$. Suppose there is an interval $[x_1,x_2] \subset [0,x_0]$ such that $y - g$ is strictly positive in the interior of $[x_1,x_2]$ and $y(x_1) - g(x_1) = y(x_2) - g(x_2) = 0$. Then $y'(x_2) \leq g'(x_2) < g'(x_1) \leq y'(x_1)$. On the other hand the equation (1.1) implies that $y''(x) > 0$ for $x \in (x_1,x_2)$ and hence $y'(x_2) = y'(x_1) + \int_{x_1}^{x_2} y''(\xi)d\xi > y'(x_1)$. So our assumption must be false since it leads to a contradiction. Thus, $y(x) \leq g(x)$. Now, let us suppose that $y(x_1) = g(x_1)$ for some $x_1 > 0$, then necessarily $y'(x_1) = g'(x_1)$. However, because $y''(x_1) = 0$ (by (1.1)) and $g''(x_1) < 0$, this would imply that $y(x) > g(x)$ in a right-hand neighbourhood of x_1, which is impossible. Hence the inequality is strict for $x \in (0,R]$, and this completes the proof of (i).

From (i), $y'(x) > 0$ and equation (1.1) we deduce that $y''(x) < 0$ for $x \in (0,R)$. Hence $y'(x) < y'(0) \leq g'(0)$ for $x \in (0,R)$ which completes the proof of (ii).

Finally, we note that H_g implies that $g(x) \leq g'(0)x$ and hence that $y''(x) = (\varepsilon x)^{-1}(y(x) - g(x))y'(x) > -(\varepsilon x)^{-1}g(x)g'(0) \geq -\varepsilon^{-1}(g'(0))^2$. This proves property (iii). □

THEOREM 3.2. *There exists a function* $y \in C^2([0,R])$ *which satisfies* (1.1)-(1.3).

PROOF. We define two functions α and β by $\alpha(x) := 0$ and $\beta(x) := g(x)$ for $x \in [0,R]$. Moreover, we define a function f by $f(x,y,y') := (\varepsilon x)^{-1}(y-g(x))y'$. Then $\alpha''(x) = 0 \geq 0 = f(x,\alpha(x),\alpha'(x))$ and $\beta''(x) = g''(x) < 0 = f(x,\beta(x),\beta'(x))$ for $x \in (0,R)$. Hence α and β are, respectively, a lower and an upper solution of (1.1). The existence of a solution now follows from [1, Theorem 1. 5.1] if we can show that f satisfies a Nagumo condition with respect to the pair α,β. This amounts to finding a positive continuous function h on $[0,\infty)$ such that $|f(x,y,y')| \le h(|y'|)$ for all $x \in [0,R]$, $\alpha(x) \le y \le \beta(x)$ and $y' \in \mathbb{R}$ and, furthermore, such that

$$\int_{R^{-1}\beta(R)}^{\infty} \frac{s}{h(s)}\, ds > \beta(R),$$

cf. [1, Definition 1.4.1]. The function h defined by $h(s) := \varepsilon^{-1}g'(0)(s+1)$ satisfies all these conditions. □

4. A COMPARISON THEOREM

In order to emphasize that we are going to study the dependence of a solution on the parameters ε and R, we introduce the notation $P(\varepsilon,R)$ for the problem (1.1)-(1.3). The main result of this section is a comparison theorem which is proved by standard maximum principle arguments. As corollaries we obtain that the solution is unique and that it depends in a monotone fashion on both ε and R.

<u>THEOREM 4.1</u>. *Let y_i be a solution of $P(\varepsilon_i,R_i)$ for $i = 1,2$ and suppose that $R_2 \ge R_1 > x_0$ and $\varepsilon_2 \ge \varepsilon_1$. Then $y_1(x) \ge y_2(x)$ for $0 < x < R_1$. Moreover, if one of the inequalities for the parameters is strict, then so is the inequality for the solutions.*

<u>PROOF</u>. Let the function m be defined by $m(x) := y_1(x) - y_2(x)$. Suppose that m achieves a nonpositive minimum on $(0,R_1)$, i.e. suppose that for some $x_1 \in (0,R_1)$, $m(x_1) \le 0$, $m'(x_1) = 0$ and $m''(x_1) \ge 0$. By subtracting the equation for y_2 from the one for y_1 we obtain

$$\varepsilon_1 x_1 m''(x_1) - (\varepsilon_2-\varepsilon_1)x_1 y_2''(x_1) - y_1'(x_1)m(x_1) = 0.$$

However, all the terms on the left-hand side of this equality are nonnegative and if either $\varepsilon_2 > \varepsilon_1$ or $m(x_1) < 0$ at least one of them is positive. If $\varepsilon_1 = \varepsilon_2$ and $m(x_1) = 0$ then the uniqueness theorem for ordinary differential equations implies that $m(x) = 0$ for all $x \in [0,R_1]$, which cannot be true if $R_2 > R_1$. So we see that m cannot achieve a negative minimum and that m cannot become zero on $(0,R_1)$ if one of the inequalities for the parameters is strict. Since $m(0) = 0$ and $m(R_1) \ge 0$ this proves the theorem. □

COROLLARY 4.2. *The problem* $P(\varepsilon,R)$ *has one and only one solution.*

PROOF. We know that at least one solution exists (Theorem 3.2). Let both y_1 and y_2 satisfy $P(\varepsilon,R)$, then Theorem 4.1 implies that $y_1(x) \geq y_2(x)$ but likewise that $y_2(x) \geq y_1(x)$. Hence, $y_1(x) = y_2(x)$ for $x \in [0,R]$. □

COROLLARY 4.3. *Let* $y = y(x;\varepsilon,R)$ *be the solution of* $P(\varepsilon,R)$. *Then* y *is a monotone decreasing function of* ε *for each* $R > x_0$ *and each* $x \in (0,R)$, *and* y *is a monotone decreasing function of* R *for each* $\varepsilon > 0$ *and each* $x \in (0,R)$.

5. THE LIMITING BEHAVIOUR AS $R \to \infty$

In this section we study the limiting behaviour as $R \to \infty$ of the solution $y = y(x;\varepsilon,R)$ of the problem $P(\varepsilon,R)$. Since y is a bounded and monotone function of R, the definition $\bar{y}(x;\varepsilon) := \lim_{R\to\infty} y(x;\varepsilon,R)$ makes sense for all $x,\varepsilon > 0$. This definition implies at once that $\bar{y}(0;\varepsilon) = 0$ and that $\bar{y}$ is a nondecreasing function of x and a nonincreasing function of ε.

From the estimates in Theorem 3.1 we obtain, via the Arzela-Ascoli theorem, that both $y(\cdot;\varepsilon,R)$ and $y'(\cdot;\varepsilon,R)$ converge uniformly on compact subsets. Invoking equation (1.1) we see that the same must be true for $y''(\cdot;\varepsilon,R)$. It follows that $\bar{y}(\cdot;\varepsilon)$ belongs to $C^2(\mathbb{R}_+)$ and satisfies equation (1.1). Now it remains to determine $\bar{y}(\infty;\varepsilon)$. We will estimate $\bar{y}(\infty;\varepsilon)$ from below by constructing a more subtle lower solution for y. But first we prove a result which can be used to estimate $\bar{y}(\infty;\varepsilon)$ from above.

LEMMA 5.1. *Let* $z \in C^2(\mathbb{R}_+)$ *satisfy equation* (1.1) *and* $z(0) = 0$. *Suppose that* $z(\infty) := \lim_{x\to\infty} z(x)$ *exists and satisfies* $0 < z(\infty) < \infty$. *Then* $z(\infty) \leq$ $\leq g(\infty) - \varepsilon$.

PROOF. Both z and z' are positive on $(0,\infty)$ (cf. the proof of Theorem 3.1). For the purpose of contradiction, let us suppose that $z(\infty) > g(\infty) - \varepsilon$. Let x_1 be such that $\beta := \varepsilon^{-1}(z(x_1) - g(\infty)) > -1$. Then $z(x) - g(x) \geq z(x_1) - g(\infty) =$ $= \varepsilon\beta$ for all $x \geq x_1$. Integrating equation (1.1) twice from x_1 to x we obtain

$$z(x) = z(x_1) + z'(x_1) \int_{x_1}^{x} \exp\left(\int_{x_1}^{\xi} \frac{z(\eta) - g(\eta)}{\varepsilon\eta}\, d\eta \right) d\xi.$$

Thus, for $x \geq x_1$,

$$z(x) \geq z'(x_1) \int_{x_1}^{x} \exp\left(\beta \ln \frac{\xi}{x_1}\right) d\xi = \frac{x_1 z'(x_1)}{\beta+1} \left(\left(\frac{x}{x_1}\right)^{\beta+1} - 1\right).$$

Since $\beta + 1 > 0$ this would imply that $z(x) \to \infty$ as $x \to \infty$. Hence the assumption that $z(\infty) > g(\infty) - \varepsilon$ must be false. □

We define a function $s = s(x;\lambda,x_1,\nu)$ by

$$s(x;\lambda,x_1,\nu) := \lambda\left(1 - \left(\frac{x}{x_1}\right)^{-\nu}\right) \tag{5.1}$$

and we investigate which conditions for the parameters λ, x_1 and ν guarantee that $s'' \geq f(x,s,s')$ for $x \geq x_1$ (recall that $f(x,y,y') = (\varepsilon x)^{-1}(y-g(x))y')$. A simple computation shows that this inequality holds indeed for all $x \geq x_1$ if and only if $g(x_1) - \lambda - \varepsilon\nu - \varepsilon \geq 0$, or equivalently, $\nu \leq \varepsilon^{-1}(g(x_1)-\lambda) - 1$. The latter inequality can be satisfied for some *positive* value of ν if and only if $\lambda < g(x_1) - \varepsilon$. In its turn this inequality can be satisfied for sufficiently large x_1 and some *positive* value of λ if and only if $g(\infty)-\varepsilon > 0$.

We now have all the ingredients at hand to prove the following theorem.

<u>THEOREM 5.2.</u>

(i) *If* $\varepsilon \leq g(\infty) - k$ *then* $\bar{y}(\infty;\varepsilon) = k$ *and* $\lim_{R\to\infty} \sup_{0\leq x\leq R} |y(x;\varepsilon,R)-\bar{y}(x;\varepsilon)| = 0$;

(ii) *if* $g(\infty) - k < \varepsilon < g(\infty)$ *then* $\bar{y}(\infty;\varepsilon) = g(\infty) - \varepsilon$;

(iii) *if* $\varepsilon \geq g(\infty)$ *then* $\bar{y}(x;\varepsilon) = 0$ *for all* $x \geq 0$.

<u>PROOF</u>. (i) For any $\lambda < k$ we can choose x_1 such that $\lambda < g(x_1) - \varepsilon$ and subsequently ν such that $0 < \nu \leq \varepsilon^{-1}(g(x_1)-\lambda) - 1$. For these values of the parameters, s is a lower solution on the interval $[x_1,R]$. The function t defined by $t(x) := k$ is an upper solution and f satisfies a Nagumo condition with respect to the pair s,t and the interval $[x_1,R]$. It follows that the inequality

$$s(x;\lambda,x_1,\nu) \leq y(x;\varepsilon,R) \leq k,$$

which holds for $x = x_1$ and for $x = R$, actually is satisfied for all

$x \in [x_1,R]$. By taking first the limit $R \to \infty$ and then the limit $x \to \infty$ we obtain

$$\lambda \le \bar{y}(\infty;\varepsilon) \le k.$$

Since this inequality holds for $\lambda < k$, necessarily $\bar{y}(\infty;\varepsilon) = k$. This result and the monotonicity of y with respect to x together imply that the convergence of y to $\bar{y}$ is in fact uniform in x (we refer to [6, Lemma 2.4] for the proof of this statement).

(ii) If $g(\infty) - k < \varepsilon < g(\infty)$, we can make s into a lower solution by a suitable choice of x_1 and ν if and only if $\lambda < g(\infty) - \varepsilon$. The argument we used in the proof of (i) now shows that $\bar{y}(\infty;\varepsilon) \ge g(\infty) - \varepsilon$. On the other hand, Lemma 5.1 implies that $\bar{y}(\infty;\varepsilon) \le g(\infty) - \varepsilon$. So $\bar{y}(\infty;\varepsilon) = g(\infty) - \varepsilon$.
(iii) From Lemma 5.1 we deduce that no solution of (1.1) with a positive limit at infinity can exist if $\varepsilon \ge g(\infty)$. Hence $\bar{y}(\infty;\varepsilon) = 0$ and consequently $\bar{y}(x;\varepsilon) = 0$ for all $x \ge 0$. □

The results of this section are at the same time results concerning the existence and non-existence of a solution of the problem $P(\varepsilon,\infty)$ defined by (1.1), (1.2) and $\lim_{x\to\infty} y(x) = k$. By exactly the same arguments which we used before one can derive the bounds of Theorem 3.1 and one can show that there exists at most one solution of $P(\varepsilon,\infty)$. For convenience we formulate this result in the following theorem.

THEOREM 5.3. *There exists a function* $y \in C^2(\mathbb{R}_+)$ *which satisfies* (1.1), (1.2) *and the condition* $\lim_{x\to\infty} y(x) = k$ *if and only if* $\varepsilon \le g(\infty) - k$. *If it exists, it is unique and it satisfies the inequalities given in Theorem* 3.1.

6. THE LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

Throughout this section $R > x_0$ will be fixed and we will suppress the dependence on R in the notation, because it is inessential. The solution y of (1.1) - (1.3) is a bounded and monotone function of ε and we define $\tilde{y}(x) := \lim_{\varepsilon\downarrow 0} y(x;\varepsilon)$. From Theorem 3.1(i) and (ii) and the Arzela-Ascoli theorem we deduce that $\tilde{y}$ is continuous and that in fact

$\lim_{\varepsilon\downarrow 0} \sup_{0\le x\le R} |\tilde{y}(x)-y(x;\varepsilon)| = 0.$

<u>THEOREM 6.1</u>. $\tilde{y}(x) = \min\{g(x),k\}$.

<u>PROOF</u>. From Theorem 3.1(i) we know that $\tilde{y}(x) \le \min\{g(x),k\}$. Take any $x < x_0$, then $y(x) < k$. We claim that this implies that $\liminf_{\varepsilon\downarrow 0} y'(x;\varepsilon) > 0$. Indeed, suppose that the sequence $\{\varepsilon_i\}$ is such that $\varepsilon_i \downarrow 0$ and $y'(x;\varepsilon_i) \downarrow 0$ as $i \to \infty$, then by taking the limit $i \to \infty$ in the relation

$$k = y(R;\varepsilon_i) = y(x;\varepsilon_i) + \int_x^R y'(\xi;\varepsilon_i)d\xi \le y(x;\varepsilon_i) + (R-x)y'(x;\varepsilon_i),$$

we arrive at the conclusion that $\tilde{y}(x) \ge k$, which is impossible.

Integrating equation (1.1) from 0 to x we obtain

$$\varepsilon(y'(x;\varepsilon)-y'(0;\varepsilon)) = \int_0^x \frac{y(\xi;\varepsilon)-g(\xi)}{\xi}\, y'(\xi;\varepsilon)d\xi. \tag{6.1}$$

Suppose that $x < x_0$ and $\max_{0\le\xi\le x} |\tilde{y}(\xi)-g(\xi)| > 0$ then, since $g'(0) > y'(\xi;\varepsilon) \ge y'(x;\varepsilon)$ for $0 < \xi \le x$ and $\liminf_{\varepsilon\downarrow 0} y'(x;\varepsilon) > 0$, the right-hand side of (6.1) is bounded away from zero as $\varepsilon \downarrow 0$. However, this is impossible since the left-hand side tends to zero as $\varepsilon \downarrow 0$. So $\tilde{y}(x) = g(x)$ for all $x < x_0$, and by continuity $\tilde{y}(x_0) = k$. The function $\tilde{y}$, being the limit of monotone functions, is monotone nondecreasing. Hence $\tilde{y}(x) \ge k$ for $x > x_0$ and consequently $\tilde{y}(x) = k$ for $x > x_0$. □

By taking $\varepsilon = 0$ in (1.1) we obtain the reduced equation

$$(g(x)-y)y' = 0. \tag{6.2}$$

The limiting function $\tilde{y}$ satisfies the boundary conditions (1.2) and (1.3) and the equation (6.2) except at the point $x = x_0$, where $\tilde{y}'$ is not defined. Motivated in part by the physical application (cf. section 2) we shall now investigate the limiting behaviour of $y'(x;\varepsilon)$ as $\varepsilon \downarrow 0$. It will then become even more apparent that $x = x_0$ is an exceptional point. The following lemma is needed in the proof of Theorem 6.3, but it is of some interest in itself.

<u>LEMMA 6.2</u>. *Let* $\delta > 0$ *be arbitrary. For any* $\varepsilon_0 > 0$ *there exists an* $M > 0$

such that $0 < g(x) - y(x;\varepsilon) < M\varepsilon x$ *for all* $x \in [0,x_0-\delta]$ *and all* $\varepsilon \in (0,\varepsilon_0)$.

PROOF. Let $\delta > 0$ and $\varepsilon_0 > 0$ arbitrary. We define

$$m(\varepsilon) := \min_{x_0-\delta\le x\le x_0-\frac{1}{2}\delta} \{g(x)-y(x;\varepsilon)\}.$$

Then there exist positive constants C_i, $i = 1,2,3$, such that for $\varepsilon \in (0,\varepsilon_0)$

$$m(\varepsilon) \le C_1 \int_{x_0-\delta}^{x_0-\delta/2} (g(\xi)-y(\xi;\varepsilon))d\xi$$

$$\le C_2 \int_{x_0-\delta}^{x_0-\delta/2} \frac{g(\xi)-y(\xi;\varepsilon)}{\xi} y'(\xi;\varepsilon)d\xi \le C_3\varepsilon$$

(see the proof of Theorem 6.1 and in particular formula (6.1)). Let the function $v = v(x;\varepsilon)$ be defined by $v(x;\varepsilon) := g(x)-y(x;\varepsilon) - M\varepsilon x$, where the constant $M > 0$ is still at our disposal. Then v satisfies the equation

$$\varepsilon x v'' - y'(x;\varepsilon)v = \varepsilon x(g''(x) + My'(x;\varepsilon))$$

and consequently $\varepsilon x v'' - \mu v > 0$ if $M > \gamma\mu^{-1}$, $\varepsilon \in (0,\varepsilon_0)$ and $x \in (0,x_0-\frac{1}{2}\delta]$, where the positive numbers γ and μ are defined by

$$\gamma := -\inf_{0<x\le x_0-\frac{1}{2}\delta} g''(x)$$

and

$$\mu := \inf_{0<\varepsilon<\varepsilon_0} y'(x_0-\frac{\delta}{2};\varepsilon).$$

So if $M > \gamma\mu^{-1}$ and $\varepsilon \in (0,\varepsilon_0)$, then v cannot assume a nonnegative maximum on $(0,x_0-\frac{1}{2}\delta)$. Let $x(\varepsilon)$ be such that $g(x) - y(x;\varepsilon)$ achieves its minimum on the set $[x_0-\delta,x_0-\frac{1}{2}\delta]$ in the point $x = x(\varepsilon)$. Then $v(x(\varepsilon);\varepsilon) = m(\varepsilon)-M\varepsilon x(\varepsilon) < 0$ if $M > (x_0-\delta)^{-1} C_3$. Since $v(0;\varepsilon) = 0$, this implies that for $M > \max\{\gamma\mu^{-1},(x_0-\delta)^{-1} C_3\}$, $v(x;\varepsilon) < 0$ for $x \in (0,x(\varepsilon))$ and a fortiori for $x \in (0,x_0-\delta)$. □

THEOREM 6.3. *Let* $\delta > 0$ *be arbitrary. Then*

(i) $\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le x_0 - \delta} |g'(x) - y'(x;\varepsilon)| = 0;$

(ii) $\lim_{\varepsilon \downarrow 0} \sup_{x_0 + \delta \le x \le R} |y'(x;\varepsilon)| = 0.$

PROOF. (i) From the equation (1.1), Theorem 3.1(ii) and Lemma 6.2 we deduce that $-g'(0)M < y''(x;\varepsilon) < 0$ for $x \in [0,x_0-\delta]$ and $\varepsilon \in (0,\varepsilon_0)$. By the Arzela-Ascoli theorem this implies that the limit set of $\{y'(\cdot;\varepsilon) \mid \varepsilon > 0\}$ as $\varepsilon \downarrow 0$ is nonempty in $C([0,x_0-\delta])$. The result now follows from the fact that y tends to g on $[0,x_0-\delta]$ as $\varepsilon \downarrow 0$.

(ii) Integrating equation (1.1) from $x_0 + \frac{1}{2}\delta$ to x we obtain

$$\varepsilon(y'(x;\varepsilon) - y'(x_0+\tfrac{1}{2}\delta;\varepsilon)) = \int_{x_0+\frac{1}{2}\delta}^{x} \frac{y(\xi;\varepsilon)-g(\xi)}{\xi}\, y'(\xi;\varepsilon)d\xi.$$

For $x \in [x_0+\delta,R]$ the right-hand side is smaller than $\frac{1}{2}\delta R^{-1}(k-g(x_0+\frac{\delta}{2}))y'(x;\varepsilon)$. Consequently $0 < y'(x;\varepsilon) < 2g'(0)\varepsilon R\delta^{-1}(g(x_0+\frac{\delta}{2})-k)^{-1}$. □

In the next section we shall concentrate on a formal approximation for y and y' in the neighbourhood of $x = x_0$.

In section 5 it was shown that the problem $P(\varepsilon,\infty)$ has a unique solution for ε sufficiently small. The analysis of this section can be repeated, mutatis mutandis, to derive the analogous results concerning the limiting behaviour of this solution as $\varepsilon \downarrow 0$. In particular this implies that the limits $\varepsilon \downarrow 0$ and $R \to \infty$ are interchangeable.

7. THE TRANSITION LAYER

In Theorem 6.3 we have shown that y' converges nonuniformly on the interval $[0,R]$ as $\varepsilon \downarrow 0$. This feature is typical for a singular perturbation problem. In this section we use the standard method of the stretching of a variable to obtain more information about the behaviour of y' near the transition point $x = x_0$.

By the stretching of the variable x near x_0 we mean the introduction of a local coordinate ξ according to $x = x_0 + \varepsilon^{\alpha}\xi$. At the same time we introduce a local dependent variable η according to

$$y(x) = g(x_0) + \varepsilon^{\beta}\eta(\xi).$$

If we make these substitutions in the equation, and subsequently only retain the terms of lowest order in ε, it depends on the values of α and β what the resulting equation will be. One easily verifies that the choice $\alpha = \beta = \frac{1}{2}$ leads to a significant equation, namely to

$$(7.1) \qquad x_0\eta_1'' + (\xi g'(x_0) - \eta_1)\eta_1' = 0,$$

where we have introduced the subscript 1 to indicate that we consider in fact a first approximation. To this equation we add the condition that its solution should match the limits of y to the left and to the right of x_0, respectively, up to the appropriate order in $\sqrt{\varepsilon}$. This amounts to the conditions

$$(7.2) \qquad \begin{cases} \eta_1(\xi) = g'(x_0)\xi + o(1), & \text{as } \xi \to -\infty \\ \eta_1(\xi) = o(1), & \text{as } \xi \to +\infty . \end{cases}$$

A straightforward application of the maximum principle (see Theorem 4.1) shows that the problem (7.1)-(7.2), which we shall denote by Π_1, admits at most one solution.

The problem Π_1 is nonautonomous. However, if we set $\eta_1' = z_1$, divide the equation by z_1 and then differentiate it, we formally obtain an autonomous problem, which we denote by $\tilde{\Pi}_1$:

$$(7.3) \qquad x_0\left(\frac{z_1'}{z_1}\right)' + g'(x_0) - z_1 = 0$$

$$(7.4) \qquad \begin{cases} z_1(\xi) = g'(x_0) + o(1), & \text{as } \xi \to -\infty , \\ z_1(\xi) = o(1), & \text{as } \xi \to +\infty. \end{cases}$$

One should note that, at least formally up to first order in $\sqrt{\varepsilon}$, z_1 describes the shape of y' in the neighbourhood of x_0. In the remainder of this section we shall discuss the existence of a family of solutions of problem $\tilde{\Pi}_1$, and we shall show how this family can be used to obtain the solution of problem Π_1.

One way to handle problem $\tilde{\Pi}_1$ is to write (7.3) as a two-dimensional first order system and analyze the trajectories in the phase plane. It turns out that the singular point $(z_1, z_1') = (g'(x_0), 0)$ is a saddle point and that one branch of the unstable manifold lies in the half-plane $z_1' < 0$ and enters the (singular) singular point (0,0). Hence $\tilde{\Pi}_1$ has a one-parameter family of strictly decreasing solutions, where the parameter describes simply the translation of one particular solution.

However, it so happens that $\tilde{\Pi}_1$ can be solved explicitly for ξ in terms of z_1. To this end we put

$$z_1 = g'(x_0)e^v \quad \text{and} \quad \xi' = \sqrt{\frac{2g'(x_0)}{x_0}}\,\xi.$$

Then $v = v(\xi')$ has to satisfy

$$\begin{cases} 2v'' + 1 - e^v = 0 & , \\ v(-\infty) = 0 \qquad v(+\infty) = -\infty & , \end{cases}$$

and we obtain, after multiplication by v' and one integration,

$$(v')^2 + v - e^v = -1$$

and finally

$$(7.5) \qquad \xi' = \int_v^C \frac{dw}{\sqrt{e^w - w - 1}} \quad ,$$

where the parameter C corresponds to the free translation parameter. From this expression we easily obtain the asymptotic behaviour of the solutions:

$$z_1(\xi) \sim g'(x_0) + e^{\sqrt{\frac{g'(x_0)}{x_0}}(\xi - C)} \quad , \qquad \xi \to -\infty,$$

$$z_1(\xi) \sim g'(x_0) e^{-\frac{g'(x_0)}{2x_0}(\xi-C)^2}, \qquad \xi \to +\infty.$$

As candidates for a solution of Π_1 we take the functions

$$\psi(\xi,C) = \int_{\infty}^{\xi} \tilde{z}_1(\tau+C)d\tau = \int_{\infty}^{\xi+C} \tilde{z}_1(\tau)d\tau,$$

where $\tilde{z}_1$ is the particular solution of $\tilde{\Pi}_1$ which satisfies $\tilde{z}_1(0) = \frac{1}{2}g'(x_0)$ (or, in other words, which corresponds with $C = \frac{1}{2}g'(x_0)$ in (7.5)). Using equation (7.3) we obtain after some manipulation

$$(x_0\psi''+(\xi g'(x_0)-\psi)\psi')' = \frac{\psi''}{\psi'}(x_0\psi''+(\xi g'(x_0)-\psi)\psi'),$$

where primes denote differentiation with respect to ξ and where we have suppressed the dependence on C in the notation. Hence

$$x_0\psi'' + (\xi g'(x_0)-\psi)\psi' = K_1\psi'.$$

Furthermore, we deduce from $\tilde{\Pi}_1$ that

$$\psi(\xi;C) = g'(x_0)\xi + K_2 + o(1), \qquad \xi \to -\infty.$$

Since ψ''/ψ' tends to zero as $\xi \to -\infty$ it follows that $K_2 = -K_1$.

Of course the constants K_1 and K_2 depend on C and it remains to show that we can choose C in such a way that they both become zero. We observe that

$$K_1(C) = x_0\frac{\psi''(0;C)}{\psi'(0;C)} - \psi(0;C)$$
$$= x_0\frac{\tilde{z}_1'(C)}{\tilde{z}_1(C)} - \int_{\infty}^{C}\tilde{z}_1(\tau)d\tau.$$

From the known asymptotic behaviour of $\tilde{z}_1$ we deduce that K_1 tends to $\pm\infty$ as C tends to $\mp\infty$. Moreover

$$\frac{dK_1}{dC}(C) = x_0\left(\frac{\tilde{z}_1'}{\tilde{z}_1}\right)'(C) - \tilde{z}_1(C) = -g'(x_0) < 0.$$

Thus, K_1 is a strictly decreasing function with range $(-\infty,\infty)$ and we conclude that there exists a unique value of C, C_1 say, such that $K_1(C) = 0$. Consequently $\eta_1 := \psi(\cdot;C_1)$ is the solution of problem Π_1. Furthermore, the properties of $\tilde{z}_1$ imply that (i) η_1 is negative, strictly increasing and concave, (ii) $\eta_1(\xi) \to 0$ faster than exponentially as $\xi \to +\infty$, (iii) the function $\eta_1(\xi) - g'(x_0)\xi$, as well as all its derivatives, converge exponentially to zero as $\xi \to -\infty$.

The idea of singular perturbation theory is that $\tilde{z}_1(\cdot+C_1)$ describes the transition of y' near $x = x_0$ for small values of ε, and that one can approximate y' uniformly on $[0,R]$ by using the building-stones $\tilde{z}_1(\cdot+C_1)$ and $\tilde{y}'$. In the following sections we shall elaborate this idea and we shall prove its correctness. It turns out that this will require the construction of at least five terms in a uniform asymptotic expansion. Since for us, as for many mathematicians, five is almost equal to infinity we shall first discuss the construction of a complete asymptotic expansion.

8. MATCHED ASYMPTOTIC EXPANSIONS

Throughout this and the next section we shall assume that $g \in C^\infty([0,R])$.

On the interval $[0,x_0-\delta]$ we look for an asymptotic expansion of the form

$$(8.1)\qquad y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x).$$

We find that $y_0(x) = g(x)$ and that y_n is defined recursively by

$$(8.2)\qquad y_n(x) = (y_0'(x))^{-1}\{xy''_{n-1}(x) - \sum_{k=1}^{n-1} y_k(x)y'_{n-k}(x)\},\qquad n \geq 1.$$

In order to calculate the matching conditions for the transition layer expansion, we expand each y_n in a Taylor series

$$y_n(x) = \sum_{k=0}^{\infty} (\sqrt{\varepsilon})^k \frac{y_n^{(k)}(x_0)}{k!} \xi^k$$

where, as before, $\xi = \frac{x-x_0}{\sqrt{\varepsilon}}$. If we substitute this in the expansion for y and rearrange the resulting expression by collecting terms with like powers of $\sqrt{\varepsilon}$, we obtain

(8.3) $$y(x) = \sum_{m=0}^{\infty} (\sqrt{\varepsilon})^m u_m(\xi)$$

where, by definition,

(8.4) $$u_m(\xi) = \sum_{n=0}^{[\frac{m}{2}]} \frac{y_n^{(m-2n)}(x_0)}{(m-2n)!} \xi^{m-2n}.$$

On the interval $[x_0+\delta,R]$ one can also introduce a series expansion in powers of ε, but it will quickly turn out that all the terms, except the one of zero'th order which is k, are zero.

Next we introduce the transition layer expansion

(8.5) $$y(x) = \sum_{n=0}^{\infty} (\sqrt{\varepsilon})^n \eta_n(\xi)$$

where $\eta_0(\xi) = g(x_0)$ and η_1 is the solution of the problem Π_1 discussed in section 7. Substitution in the equation yields an equation for each η_n. Together with the matching condition which is obtained by formal identification of (8.5), as $\xi \to -\infty$, with (8.3), this yields for $n \geq 2$ a *linear* problem Π_n defined recursively by

(8.6) $$\begin{cases} x_0\eta_n'' + (g'(x_0)\xi-\eta_1)\eta_n' - \eta_1'\eta_n = q_n, \\ \eta_n(\xi) = u_n(\xi) + o(1), \quad \text{as } \xi \to -\infty, \\ \eta_n(\xi) = o(1), \quad \text{as } \xi \to +\infty, \end{cases}$$

where

(8.7) $$q_n(\xi) := -\frac{g^{(n)}(x_0)}{n!}\xi^n\eta_1' - \xi\eta_{n-1}'' - \sum_{k=2}^{n-1} \eta_{n+1-k}'\left(\frac{g^{(k)}(x_0)}{k!}\xi^k - \eta_k\right).$$

As before the maximum principle implies that problem Π_n can have at most one solution. In order to discuss the existence of a solution we first rewrite the equation by making use of the equation (7.1) for η_1:

$$x_0\left(\frac{\eta_n'}{\eta_1'}\right)' - \eta_n = \frac{q_n}{\eta_1'} .$$

Introducing $z_1 := \eta_1'$, $\zeta_n := (z_1)^{-1}\eta_n'$ and $h_n := ((z_1)^{-1}q_n)'$, we obtain by differentiation

$$(8.8) \qquad x_0\zeta_n'' - z_1\zeta_n = h_n .$$

At this point it is important to observe that we know a particular solution of the homogeneous equation $x_0\phi'' - z_1\phi = 0$, namely

$$(8.9) \qquad \phi(\xi) := \frac{z_1'(\xi)}{z_1(\xi)}$$

(one can verify this by differentiation of equation (7.3)).
Hence we can construct solutions of (8.8) through the method of variation of constants, and we find

$$(8.10) \qquad \zeta_n(\xi;C) = \frac{\phi(\xi)}{x_0}\int_0^{\xi} \phi^{-2}(\tau)\int_{-\infty}^{\tau} \phi(\sigma)h_n(\sigma)\,d\sigma\,d\tau + C\phi(\xi)$$

(note that we do not consider the general solution of the homogeneous equation since only ϕ has the right asymptotic behaviour as $\xi \to -\infty$). For any C, the function defined in (8.10) is of polynomial growth as $\xi \to +\infty$ and behaves like $g'(x_0)u_n'$ as $\xi \to -\infty$. The last statement can be verified by working out the consistency relations between q_n and u_n which follow from the identity

$$x_0u_n'' - g'(x_0)u_n = -\frac{g^{(n)}(x_0)}{n!}\,\xi^n g'(x_0) - \xi u_{n-1}'' - \sum_{k=2}^{n-1} u_{n+1-k}'\left(\frac{g^{(k)}(x_0)}{k!}\xi^k - u_k\right)$$

and by making use of the known asymptotic behaviour of ϕ.

Finally, we define

$$(8.11) \qquad \eta_n(\xi;C) = \int_{\infty}^{\xi} z_1(\tau)\zeta_n(\tau;C)\,d\tau = \eta_n(\xi;0) + C\eta_1'(\xi) .$$

Then $\eta_n(\xi;C) = u_n(\xi) + B_n + g'(x_0)C + o(1), \xi \to -\infty$, where B_n is some number, which does not depend on C. It follows that there exists a unique constant, say C_n, for which the matching condition is satisfied and consequently $\eta_n(\xi;C_n)$ is the unique solution of the problem Π_n. This completes the construction of the transition layer expansion.

To conclude this section we construct a uniform approximation of formal order 2n+1 in $\sqrt{\varepsilon}$. We introduce two C^∞-functions H and J defined on $\mathbb{R}$ (so-called cut-off functions) with the following properties

$$H(x) = \begin{cases} 0 & \text{if} \quad |x-x_0| \geq \delta_1 \\ 1 & \text{if} \quad |x-x_0| \leq \frac{\delta_1}{2} \end{cases}$$

$$J(x) = \begin{cases} 0 & \text{if} \quad |x| \leq \delta_2 \\ 1 & \text{if} \quad |x| \geq 2\delta_2 \end{cases}$$

where δ_1 and δ_2 are suitable constants which do not depend on ε. Then the formal approximation $y_a(x)$ is defined by

$$(8.12) \qquad y_a(x) = \begin{cases} J\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) \sum_{m=1}^{n} \varepsilon^m y_m(x) + H(x) \sum_{m=1}^{2n+1} (\sqrt{\varepsilon})^m \left(\eta_m\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) - J\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) u_m\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)\right) & \text{for } x \leq x_0 \\ J\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) k(1-H(x)) + H(x) \sum_{m=1}^{2n+1} (\sqrt{\varepsilon})^m \eta_m\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) & \text{for } x \geq x_0. \end{cases}$$

Apart from the cut-off functions this formula is the usual one, expressing a uniform approximation as the sum of approximations in the different regions minus the matching terms, which are contained in two approximations and hence should be subtracted in order to avoid double counting. The cut-off functions are used to achieve two ends: the approximation should satisfy the boundary conditions and it should be smooth at $x = x_0$. Moreover, the cut-off functions are harmless in the sense that they are multiplied by factors which are small (if ε is small) in regions where the cut-off functions are different from one. In the next section we shall prove that y_a

and y_a' are indeed uniform approximations of y and y' up to the order $\varepsilon^{n+\frac{1}{2}}$ and $\varepsilon^{n+3/2}$, respectively.

9. A PROOF OF THE VALIDITY OF THE FORMAL CONSTRUCTION

We begin by deriving an estimate for the difference

$$z(x) := y(x) - y_a(x). \tag{9.1}$$

It follows from the equation for y and from the construction of y_a that z satisfies

$$\begin{cases} \varepsilon\, x\, z'' + (g-y)z' - y'z + zz' = r \\ z(0) = 0, \qquad z(R) = 0 \end{cases} \tag{9.2}$$

where the remainder term r, defined by

$$r(x) := -(\varepsilon x y_a'' + (g-y_a)y_a'), \tag{9.3}$$

can be shown, after an elaborate computation, to satisfy

$$r(x) = O(x\varepsilon^n) \quad \text{as } \varepsilon \downarrow 0 \text{ and/or } x \downarrow 0. \tag{9.4}$$

If we multiply the equation for z and integrate from 0 to R we obtain after some integrations by parts and an application of the Cauchy-Schwarz inequality

$$\varepsilon \int_0^R x(z'(x))^2 dx + \tfrac{1}{2} \int_0^R (g'(x)+y'(x))z^2(x)dx \le \|z\|\,\|r\|,$$

where $\|\cdot\|$ denotes the L_2-norm. Since $g'(x) + y'(x) \ge g'(R)$ this implies, first of all, that

$$\|z\| \le \frac{2}{g'(R)}\,\|r\|$$

and hence that

$$\varepsilon \int_0^R x(z'(x))^2 dx + \frac{g'(R)}{2} \|z\|^2 \le \frac{2}{g'(R)} \|r\|^2 .$$

Now, fix $\delta \in (0,x_0)$. The estimate above is easily translated into an estimate for the $H^1(\delta,R)$-norm of z, where H^1 denotes the usual Sobolev space of L_2-functions which have a generalized derivative belonging to L_2. Thus, by the continuous imbedding of H^1 into the space of continuous functions we obtain

$$|z(x)| \le C(\varepsilon^{-1}\|r\|^2)^{\frac{1}{2}} \le C\varepsilon^{n-\frac{1}{2}}, \delta \le x \le R,$$

where C depends on δ. Having established this estimate on the interval $[\delta,R]$, we can extend it to the interval $[0,R]$ by means of the maximum principle in exactly the same way as we proved Lemma 6.2.

Next, it is advantageous to take explicitly into account the dependence on the parameter n, which counts the number of terms included in the approximation. So putting $z = z_n$ we write the estimate obtained so far as

$$|z_n(x)| \le Cx\varepsilon^{n-\frac{1}{2}}, \qquad 0 \le x \le R, \qquad n \in \mathbb{N} .$$

Then, observing that

$$|z_{n+1}(x) - z_n(x)| \le Cx\varepsilon^{n+1},$$

we deduce the sharper estimate

$$|z_n(x)| \le |z_n(x)-z_{n+1}(x)| + |z_{n+1}(x)| \le Cx\varepsilon^{n+\frac{1}{2}}.$$

(This is the familiar "throwing away" of terms which are needed in the proof, but do not contribute to the result.) We state this as a theorem.

<u>THEOREM 9.1</u>. *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that*

$$|y(x)-y_a(x)| \le Cx\varepsilon^{n+\frac{1}{2}}$$

for $0 < \varepsilon < \varepsilon_0$ *and* $0 \le x \le R$.

Our next objective is to show that the derivative of y_a is a good approximation for the derivative of y (recall that y_a is more or less constructed through the integration of its derivative, and that in our application the derivative is the function which has a direct physical meaning). Our proof will be based on the following interpolation inequality.

LEMMA 9.2. *There exist constants* $\mu_0 > 0$ *and* $D > 0$ *such that for any* $\phi \in C^2([0,R])$ *and each* $\mu \in (0,\mu_0)$

$$\sup|\phi'(x)| \le D\{\mu \sup|\phi''(x)| + \mu^{-1}\sup|\phi(x)|\},$$

where the suprema are taken over the interval $[0,R]$.

PROOF. See BESJES [2]. The proof is based on a result to be found in MIRANDA [15, 33,III,p.149]. □

THEOREM 9.3. *There exist constants* $\varepsilon_0 > 0$ *and* $C > 0$ *such that*

$$|y'(x)-y_a'(x)| \le C\varepsilon^{n-\frac{1}{2}}$$

for $0 < \varepsilon < \varepsilon_0$ *and* $0 \le x \le R$.

PROOF. From the equation for z (see (9.2)) we deduce that

$$|z''(x)| \le \varepsilon^{-1}\{|\frac{r(x)}{x}| + C_1|z'(x)| + C_2|\frac{z(x)}{x}|\}$$

where

$$C_1 := \sup_{0\le x\le R} \frac{g(x)-y(x)}{x}, \qquad C_2 := \sup_{0\le x\le R} |y_a'(x)|.$$

Next we apply Lemma 9.2 with $\mu = \varepsilon(2C_1D)^{-1}$ to obtain

$$\sup|z''(x)| \le 2\varepsilon^{-1}\{\sup|\frac{r(x)}{x}| + 2(C_1D)^2\varepsilon^{-1} \sup|z(x)| + C_2 \sup|\frac{z(x)}{x}|\}.$$

By Theorem 9.1 and the estimate (9.4) this implies that

$$\sup|z''(x)| = O(\varepsilon^{n-3/2}).$$

Then a second application of Lemma 9.2, this time with $\mu = \varepsilon$, leads to the desired result. □

10. SOME REMARKS ABOUT THE CASE WHERE g IS NEITHER EVERYWHERE INCREASING NOR EVERYWHERE CONCAVE

In this section we shall discuss some extensions of our results to equations in which the conditions on the function g are considerably relaxed. In fact we shall merely assume that g satisfies the following hypotheses

$$\tilde{H}_g: \; g \in C^1([0,R]); \quad g(0) = 0, \quad g(R) \geq k;$$

g has only finitely many local extrema or [0,R].

Thus, in particular the sign conditions on g' and g" are dropped.

First of all we observe that the existence of a solution of (1.1)-(1.3) can be proved as in Theorem 3.2 by using zero as a lower solution and G as an upper solution, where G is any increasing, concave and smooth function such that $G(0) = 0$ and $G(x) \geq g(x)$ or [0,R].

As before we find that if $y = y(x;\varepsilon)$ is a solution then $y' > 0$ and hence sign $y'' =$ sign $(y-g)$; subsequently, reasoning along the lines indicated in the proofs of Theorem 3.1 one can show that for any $\varepsilon > 0$

$$(10.1) \qquad 0 < y'(x;\varepsilon) \leq \sup_{0\leq\xi\leq R} g'(\xi).$$

This in turn enables one to prove by means of the maximum principle that (1.1)-(1.3) can have at most one solution, and that the mapping $\varepsilon \mapsto y(\cdot;\varepsilon)$ is continuous from $\mathbb{R}_+$ into $C = C([0,R])$.

By (10.1) the set $\{y(\cdot;\varepsilon) \mid \varepsilon > 0\}$ is a precompact subset of C. Let X denote its limit set, as $\varepsilon \downarrow 0$, in C. Taking into account the continuity with respect to ε, we conclude that X is a nonempty, compact and connected subset of C (see SELL [16,p.20]).

Any element u of X is a nondecreasing function with $u(0) = 0$ and $u(R) = k$. Our first objective is to give further characteristics of the elements of X.

LEMMA 10.1. *Let* $u \in X$. *Then there exist a nonempty, open set* A *and a closed set* B *such that*

(i) $u(x) = g(x)$ if $x \in A$,

(ii) u *is constant on each connected component of* B,

(iii) $A \cap B = \emptyset$, $A \cup B = [0,R]$.

PROOF. Since $u \in X$, there exists a sequence $\{\varepsilon_n\}$ such that as $n \to \infty$, $\varepsilon_n \downarrow 0$ and $y(\cdot;\varepsilon_n) \to u$ strongly in C. By (10.1) $\{y(\cdot;\varepsilon_n)\}$ is bounded in $H^1 = H^1(0,R)$ and hence it is possible to pick a subsequence, again denoted by $\{\varepsilon_n\}$, such that as $n \to \infty$, $y(\cdot;\varepsilon_n) \to u$ weakly in H^1.

Next, we multiply equation (1.1) by an arbitrary function $\phi \in H^1$, integrate from 0 to R, integrate the first term by parts and let n tend to infinity. This yields the identity

$$\int_0^R (g(x)-u(x))u'(x)\phi(x)dx = 0,$$

whence

$$(g(x)-u(x))u'(x) = 0 \quad \text{a.e. on } [0,R]. \tag{10.2}$$

Define the sets A and B by

$$A = \{x \in [0,R] \mid u = g \text{ in a neighbourhood of } x\}, \quad B = [0,R]\backslash A,$$

then clearly $u'(x) = 0$ a.e. on B. In view of the continuity of g and u the sets A and B have all the properties listed in the lemma. □

LEMMA 10.2. *Let* $u \in X$ *and let* I *be a connected component of* B *such that* $I \subset (0,R)$. *Then*

$$(10.3) \qquad \int_I \frac{u(x)-g(x)}{x}\,dx = 0.$$

Before proving this lemma, we prove an auxiliary result.

LEMMA 10.3. *Suppose that, as* $n \to \infty$, $\varepsilon_n \downarrow 0$ *and* $y(x;\varepsilon_n) \to g(x)$ *uniformly on* $[a,b] \subset [0,R]$. *Then*

$$\varepsilon_n \log y'(x;\varepsilon_n) \to 0 \qquad \textit{as } n \to \infty$$

uniformly on $[a,b]$.

PROOF. Choose a subinterval $[c,d]$ of $[a,b]$ and a positive constant $\delta > 0$ such that $g'(x) \geq \delta$ on $[c,d]$. Define for each $n \geq 1$, a point $\xi_n \in [c,d]$ such that

$$y'(\xi_n;\varepsilon_n) = \max\{y'(x;\varepsilon_n) \mid c \leq x \leq d\}.$$

Then it follows that there exists an $N_1 \geq 1$ such that

$$y'(\xi_n;\varepsilon_n) \geq \tfrac{1}{2}\delta \qquad \text{for } n \geq N_1.$$

If we divide equation (1.1) by xy' and integrate from ξ_n to x we obtain

$$\varepsilon_n \ln y'(x;\varepsilon_n) = \varepsilon_n \ln y'(\xi_n;\varepsilon_n) + \int_{\xi_n}^{x} \frac{y(\tau;\varepsilon_n)-g(\tau)}{\tau}\,d\tau.$$

Since the right-hand side tends to zero as $n \to \infty$, the same must be true for the left-hand side and the result follows. □

PROOF OF LEMMA 10.2. Let $I = (e,f)$, where, by assumption, $0 < e < f < R$. Manipulating as above we obtain

$$\varepsilon_n \ln\ y'(e;\varepsilon_n) - \varepsilon_n \ln\ y'(f;\varepsilon_n) = \int_e^f \frac{y(\tau,\varepsilon_n)-g(\tau)}{\tau}\,d\tau.$$

Applying Lemma 10.3 to a left-hand neighbourhood of e and to a right-hand neighbourhood of f, we deduce that the left-hand side of this identity tends to zero as $n \to \infty$. So taking the limit $n \to \infty$ leads to the desired result. □

We now collect the information we have obtained about an arbitrary element u of X: u is a continuous, nondecreasing function with $u(0) = 0$ and $u(R) = k$, which is composed out of pieces where $u(x) = g(x)$ and pieces where $u(x)$ is constant. Moreover, if I is a maximal interval on which u is constant, and I does not contain 0 or R, then (10.3) has to be satisfied. For convenience of formulation we shall call the set of functions having all these characteristics Y.

Our next objective is to show that Y is finite. First we shall illustrate our approach by discussing one example in full detail.

Consider a function g satisfying $\tilde{H}_g$ and such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0 < b < c < R$ and $0 < g(c) < g(b) < k$. Let g_1^{-1} denote the inverse of g on $[0,b]$ and

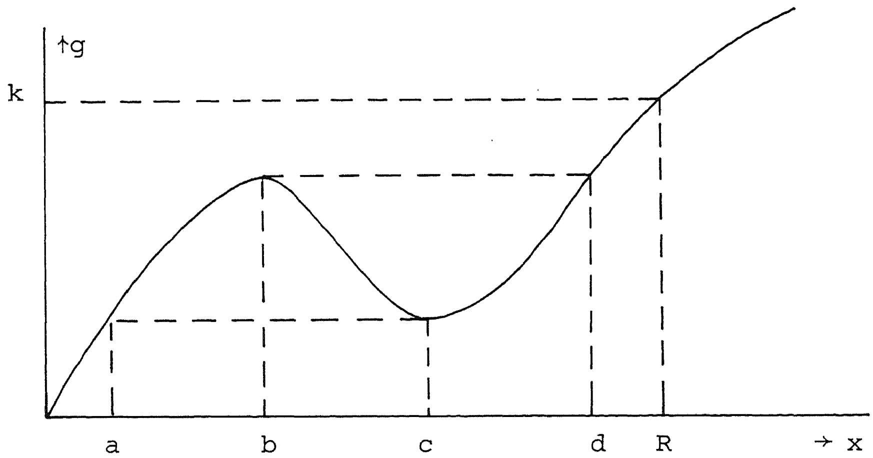

Figure 1

g_2^{-1} the inverse of g on $[c,R]$. Define two points a and d by

$$a = g_1^{-1}(g(c)), \qquad d = g_2^{-1}(g(b)).$$

Then $g([a,b]) = g([c,d])$. (See Figure 1.)

On $[a,b]$ we define a mapping F by

$$F(x) = \int_x^{g_2^{-1}(g(x))} \frac{g(x)-g(\tau)}{\tau}\, d\tau.$$

Then on (a,b)

$$F'(x) = g'(x) \int_x^{g_2^{-1}(g(x))} \frac{d\tau}{\tau} > 0$$

and $F(a) < 0$, $F(b) > 0$. Consequently F has a unique zero on [a,b].

Let w be an arbitrary element of Y. Then w has to coincide with g on $[0,a]$ and $[d,g_2^{-1}(k)]$ and it has to be equal to k on $[g_2^{-1}(k),R]$. Since w is nondecreasing the inverse function of w must "jump" from a point on [a,b] to a point on [c,d]. In view of (10.3) this jump can only take place at the *unique* zero of F. Thus Y consists of one and only one element.

Returning to a general function g satisfying $\tilde{H}_g$ we define E to be the set of local maxima and minima of g and D to be the closure of the set $\{x \mid g \text{ is increasing in a neighbourhood of } x\}$. Let D_c be one of the finitely many connected components of D. The set $g^{-1}(E) \cap D_c$ is finite. Take two successive points α_0 and β_0 in this set. To $[\alpha_0,\beta_0]$ there correspond finitely many disjunct intervals $[\alpha_i,\beta_i] \subset D$ such that $\alpha_i > \alpha_0$ and $g([\alpha_0,\beta_0]) = g([\alpha_i,\beta_i])$. Define g_i^{-1} on $[g(\alpha_0),g(\beta_0)]$ as the inverse of g with range in $[\alpha_i,\beta_i]$. On $[\alpha_0,\beta_0]$ we define mappings F_i by

$$F_i(x) = \int_x^{g_i^{-1}(g(x))} \frac{g(x)-g(\tau)}{\tau} \, d\tau .$$

Since F_i is monotone, it has at most one zero.

As already noted above the condition (10.3) implies that a point where the inverse function of an element of Y makes a jump should be a zero of some F_i for some connected component D_c of D and some pair of points α_0,β_0. Hence the set of possible "jump" points is finite and likewise the set Y is finite.

Thus X, being a subset of Y, must be discrete. Because it is also connected it can only consist of a single element. Consequently $y(\cdot;\varepsilon)$ converges in C to this function as $\varepsilon \downarrow 0$. We summarize the results in the following theorem.

<u>THEOREM 10.4</u>. *There exists a function* $u \in Y$ *such that*

$$\lim_{\varepsilon \downarrow 0} y(x;\varepsilon) = u(x), \quad \textit{uniformly on } [0,R].$$

In some cases the conditions determine the limit uniquely. For instance, this happens in the example we discussed at length and, more generally if the local extrema are ordered in such a way that with each connected component of D there corresponds precisely one possible "jump" point. In other cases our analysis is not constructive in the sense that, although we have shown that convergence occurs as $\varepsilon \downarrow 0$, we are not able to describe the limit completely. (See Figure 2.) We intend to investigate whether this ambiguity can be resolved by using variational principles

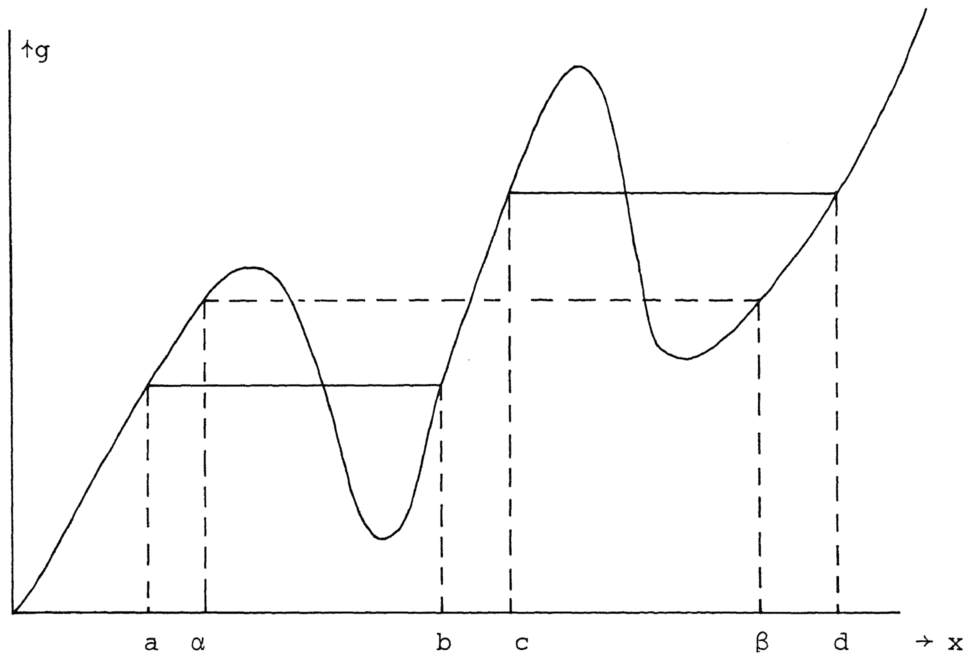

Figure 2

Two possible configurations: separate jumps (a-b,c-d) or a two-in-one jump (α-β).

In conclusion we remark that the hypothesis $g(R) \geq k$ was made in order to obtain the uniform convergence on $[0,R]$. If $g(R) < k$ the solution will exhibit boundary layer behaviour near the right endpoint. However, outside a small neighbourhood of this endpoint, the solution will behave in exactly the same way as we have shown for the case $g(R) > k$.

ACKNOWLEDGEMENT

This problem was suggested to us by E. Marode and I. Gallimberti. H.J. Hilhorst patiently explained us many physical aspects and suggested important improvements in the presentation. The comments and suggestions of M. Bakker, J. Grasman and E.J.M. Veling helped us to overcome several difficulties. The critical remarks of the referees have led us to a fairly substantial extension of the first version.

REFERENCES

[1] BERNFELD, S.R. & V. LAKSHMIKANTHAM, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[2] BESJES, J.G., *Singular Perturbation Problems for Partial Differential Equations*, Thesis Delft Technological University.

[3] CHUI, S.T. & J.D. WEEKS, *Phase transition in the two-dimensional Coulomb gas, and the interfacial roughening transition*, Phys. Rev. B 14 (1976) 4978-4982.

[4] CLÉMENT, Ph. & I.B. EMMERTH, *On the structure of continua of positive and concave solutions for two point nonlinear eigenvalue problems*, MRC Report 1766, 1977.

[5] CLÉMENT, Ph. & L.A. PELETIER, *On positive, concave solutions of two point nonlinear eigenvalue problems*, J. Math. Anal. Appl., in press.

[6] DIEKMANN, O., *Limiting behaviour in an epidemic model*, Nonlinear Analysis TMA 1 (1977) 459-470.

[7] GALLIMBERTI, I., *A computer model for streamer propagation*, J. Phys. D. Appl. Phys. 5 (1972) 2179-2189.

[8] HALLAM, T.G. & D.E. LOPER, *Singular boundary value problems arising in a rotating fluid flow*, Arch. Rat. Mech. Anal. 60 (1976) 355-369.

[9] HARTEN, A. van, *Singularly Perturbed Nonlinear Second Order Elliptic Boundary Value Problems*, Thesis University of Utrecht.

[10] HOWES, F.A., *Boundary-interior layer interactions in nonlinear singular perturbation theory,* AMS Memoir 203, 1978.

[11] HOWES, F.A. & S.V. PARTER, *A model nonlinear problem having a continuous locus of singular points,* Stud. in Appl. Math. LVIII (1978) 249-262.

[12] KOSTERLITZ, J.M. & D.J. THOULESS, *Ordering, metastability and phase transitions in two-dimensional systems,* J. Phys. C 6 (1973) 1181-1203.

[13] MARODE, E., *The mechanism of spark breakdown in air at atmospheric pressure between a positive point and a plane;* I: *Experimental: nature of the streamer track;* II: *Theoretical: Computer simulation of the streamer track,* J. Appl. Phys. 46 (1975) 2005-2015, 2016-2020.

[14] MARODE, E. et al., in preparation.

[15] MIRANDA, C., *Partial Differential Equations of Elliptic Type,* Springer, Berlin, 1970.

[16] SELL, G.R., *Topological Dynamics and Ordinary Differential Equations,* Van Nostrand Reinhold, London, 1971.