stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 148/75 MARCH

J.W. de ROEVER

FOURIER TRANSFORMS OF HOLOMORPHIC FUNCTIONS AND APPLICATION TO NEWTON INTERPOLATION SERIES, II

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 32A07, 32C35, 41A05, 42A68 46E10, 46F15, 65D05

FOURIER TRANSFORMS OF HOLOMORPHIC FUNCTIONS AND APPLICATION TO NEWTON INTERPOLATION SERIES, II

by

J.W. de Roever.

ABSTRACT

This paper treats a generalization of the Martineau-Ehrenpreis theorem and applies it to the derivation of the Newton interpolation series for the largest possible class of functions. By means of Fourier transformation the Martineau-Ehrenpreis theorem establishes the isomorphism between analytic functionals with compact carrier and some space of entire functions. In this paper the analytic functionals are carried by unbounded convex sets with respect to some class of weightfunctions and its Fourier transforms are no longer entire functions, but they are holomorphic in cones in $\mathbb{C}^n$.

KEY WORDS & PHRASES: *Fourier transformation; analytic functionals carried by unbounded convex sets; holomorphic functions of several complex variables; cohomology with bounds; the Martineau-Ehrenpreis theorem on Fouriertransforms of analytic functionals; Newton interpolation series in several variables for non-entire functions.*

CONTENTS

1. INTRODUCTION

This paper is the last of two papers dealing with Fourier transforms of holomorphic functions and the Newton interpolation series.

In [10] KIOUSTELIDIS derived the Newton series with the aid of Fourier transformation. The advantage of this method against the classical one (Cauchy's integral formula, NÖRLUND [13], GELFORD [5]) is that it treats the case of several variables as well. However, his treatment is valid for entire functions only. The aim of this paper is to show that this restriction is not due to the method, but that the method (namely the formalism of Fourier transformation) can be extended so as to include all possible non-entire functions for which the Newton series is valid.

In the first paper [14] the Newton series has been derived for functions, holomorphic in tubular radial domains, of polynomial growth in $|\mathrm{Re}\, z|$ and of exponential growth in $|\mathrm{Im}\, z|$. Such functions are the Fourier transforms of tempered distributions with support in unbounded convex sets according to a well known theorem (see [16]) generalizing the theorem of PALEY-WIENER-SCHWARTZ. In this case, however, one only uses the real part of the domain of convergence of the Fourier transformed Newton series. In [10] KIOUSTELIDIS has considered complex compact subsets of this domain using a Paley-Wiener type theorem, namely the theorem of EHRENPREIS [2] and MARTINEAU [12] dealing with Fourier transforms of analytic functionals with compact carrier. These Fourier transforms are entire functions of exponential type in $|z|$ and for such functions the Newton series is derived.

Generalizing the Ehrenpreis-Martineau theorem the main theorem of this paper states that holomorphic functions of exponential type in cones are the Fourier transforms of analytic functionals carried by unbounded convex sets with respect to some class of weightfunctions. One can formulate two versions of this theorem (based on formula (5.5)(i) and (5.5)(ii) respectively) and surprisingly it turns out that the apparently weaker version (i) equals the stronger version (ii). A particular case of version (i) has already been proved by KAWAI in [9]. However, this case cannot be handled very well in the derivation of the Newton series. Therefore, it still has sense to present the theorem as it is done here.

The proof of the main theorem is very different from the proof in [16] of the similar theorem in part I [14]. In fact the last theorem in 2n variables is used in proving the former in the n-dimensional case. The pattern of the proof is actually the same as that of Ehrenpreis' fundamental principle [3], only here one deals with non-entire functions. While in the Ehrenpreis-Martineau theorem the injectivity of the map F (Fourier transformation) presents no problem, it seems to be the most difficult part of the generalization given here. For this part and for the transition from version (i) to version (ii) cohomology with bounds is used.

Together with the theorem on Fourier transforms some other theorems are given dealing with estimates for products of a polynomial matrix with a holomorphic non-entire vectorfunction similar to the case of entire functions given by HÖRMANDER in [7]. These theorems as well as the main theorem itself may be useful in other applications, for example if one is interested in solutions of systems of partial differential equations that can be written as boundary values of functions holomorphic in tubular radial domains.

The main theorem yields all the tools for deriving the Newton series for non-entire functions in several variables. Now the domain of convergence in $\mathbb{C}^n$ is used completely, so that the most general class of functions is obtained for which the Newton series holds. This generalizes the case of one variable in NÖRLUND [13].

In section 2 the Ehrenpreis-Martineau theorem is discussed, and section 3 describes how the Newton series can be derived from this theorem as it is done by KIOUSTELIDIS in [10]. Section 4 deals with some properties of unbounded convex sets. Section 5 gives the space of holomorphic functions in cones in $\mathbb{C}^n$ of exponential type and the space of their Fourier transforms, which turns out to be the dual of some other space of holomorphic functions. These spaces are topologized in such a way that they are reflexive and that Fourier transformation is an isomorphism. A part of the version (i) of this isomorphism is also proved. In section 6 the main theorem of this paper, i.e. version (ii) of this isomorphism, is stated and the problems used to prove the main theorem are formulated. In section 7 these problems are solved, formulated so as to make them useful in other applications too. Here cohomology with bounds is derived and used. Section 8 gives some corollaries and particular cases. Especially those concerned with functions holomorphic in tubular radial domains prepare section 9, where the Newton series is derived for these functions. The appendix deals with the problem how to extend local relations between holomorphic functions to global relations. It uses cohomology as derived from the existence theorems for the $\bar{\partial}$-operator given by HÖRMANDER in [7]. Furthermore special coverings of open sets in $\mathbb{C}^n$ are constructed, adapted to the case of non-entire functions. Finally a short description of Ehrenpreis' fundamental principle is given.

2. ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

This section deals with the relation between an entire function of exponential type and the carrier of its Fourier transform. It contains nothing new, but it is merely a rearrangement of some theorems of [7], stated in the appendix, in a way to make it suitable for generalization in section 5.

Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A(\Omega)$ be the space of in Ω holomorphic functions with the topology of uniform convergence on compact subsets K of Ω. Elements μ of the strong dual $A'(\Omega)$ of $A(\Omega)$ are called *analytic functionals in* Ω. $A(\Omega)$ with the norm

$$(2.1) \qquad \|f\|_K = \sup_{\zeta\in K} |f(\zeta)|, \qquad K \subset\subset \Omega$$

is a linear subspace of C(K), the space of continuous functions on the compact set K. Therefore, in view of the Hahn-Banach theorem and the theorem of Riesz, each analytic functional in Ω can be represented as a measure in a compact set K of Ω. We say that an analytic functional μ in Ω is *concentrated on the compact set* K of Ω, when for all $f \in A(\Omega)$

$$|\langle\mu,f\rangle| \le M\|f\|_K$$

with some positive constant M. In that case μ can be represented as a measure in K. Thus every analytic functional μ in $\mathbb{C}^n$ can be considered as an analytic functional in Ω, where Ω is an arbitrary open neighborhood of the compact set K μ is concentrated on. We denote the space of analytic functionals in $\mathbb{C}^n$ concentrated on the compact sets of Ω as

$$A'_\Omega(\mathbb{C}^n).$$

Conversely, analytic functionals in Ω are analytic functionals in $\mathbb{C}^n$ too by means of their action on the restrictions to Ω of entire functions. This correspondence is 1-1, when Ω is a Runge domain (see def. A5), for then $A(\mathbb{C}^n)$ is dense in $A(\Omega)$. Hence there is a map of $A'(\Omega)$ onto $A'_\Omega(\mathbb{C}^n)$, which is 1-1 when Ω is a Runge domain. For example, when n = 1 one can think of $\Omega = \mathbb{C} \setminus \{0\}$; then the map

$$A'(\Omega) \twoheadrightarrow A'_\Omega(\mathbb{C})$$

is surjective, but not injective. Here $A'_\Omega(\mathbb{C}) = A'(\mathbb{C})$, since by the maximum principle for every compact neighborhood $\hat{K}$ of 0 with boundary K in $\mathbb{C} \setminus \{0\}$

$$\|f\|_K = \|f\|_{\hat{K}}, \tag{2.2}$$

when f is entire.

We now give a more rigorous exposition of the foregoing. Let Ω be an open set in $\mathbb{C}^n$ and K a compact subset of Ω. Denote by

$$A(\bar{K})$$

the space of functions holomorphic in a neighborhood of K with the norm (2.1) and by

$$A_K(\Omega)$$

the space of functions holomorphic in Ω with the same norm. It is clear that $A_K(\Omega)$ is a linear subspace of $A(\bar{K})$ and that both spaces are not Banach spaces. Since we are only interested in their duals, it doesn't matter if we consider these spaces or their completions, the Banach spaces $\bar{A}(\bar{K})$ and

$\bar{A}_K(\Omega)$, respectively, consisting of functions continuous on K and holomorphic in the interior of K if this is not empty. We denote by

$$K \subset\subset\!\!\to \Omega$$

a sequence $\{K_m\}_{m=1}^{\infty}$ of compact subsets of Ω with $\text{int}\, K_m \subset K_m \subset \text{int}\, K_{m+1} \subset \subset K_{m+1} \subset \Omega$ and with $\bigcup_{m=1}^{\infty} K_m = \Omega$. Then we have the following characterization of the space $A(\Omega)$

$$A(\Omega) = \underset{K\subset\subset\to\Omega}{\text{proj lim}}\ A_K(\Omega) = \underset{K\subset\subset\to\Omega}{\text{proj lim}}\ \bar{A}_K(\Omega) = \underset{K\subset\subset\to\Omega}{\text{proj lim}}\ A(\bar{K}) = \underset{K\subset\subset\to\Omega}{\text{proj lim}}\ \bar{A}(\bar{K}).$$

Both $\bar{A}_K(\Omega)$ and $\bar{A}(\bar{K})$ are closed linear subspaces of $A_\infty(1;K)$, see [14] B.4 or [18], so that according to [14] C.6 and C.7 the maps $\bar{A}_K(\Omega) \to \bar{A}_S(\Omega)$ and $\bar{A}(\bar{K}) \to \bar{A}(\bar{S})$ are compact, $S \subset\subset K$. Thus $A(\Omega)$ is an $\overline{FS}$-space (see [14] F.8), which is nuclear according to [14] G.7. Since $\bar{A}_K(\Omega)$ is dense in $\bar{A}_S(\Omega)$, the dual can be represented as

$$(2.3) \qquad A'(\Omega) = \underset{K\subset\subset\to\Omega}{\text{ind lim}}\ A'_K(\Omega)$$

according to [14] F.12. However, in general an element of $A'_K(\Omega)$ does not uniquely determine an analytic functional in any neighborhood $\Omega' \subset \Omega$ of K. This is true for distributions: distributions in O with support in K are also distributions in O', $K \subset\subset O' \subset O$. Only when $A(\Omega)$ is dense in $A_K(\Omega')$, representation (2.3) of $A'(\Omega)$ is the inductive limit of all analytic functionals in any open neighborhood Ω' of K concentrated on K. So we must find a sequence $K \subset\subset\to \Omega$ for which $A(\bar{K}_{m+1})$ is dense in $A(\bar{K}_m)$, for then $A(\Omega)$ is dense in $A(\bar{K})$ (see [14] lit.[2], §26,2.5), thus also in $A_K(\Omega') \subset A(\bar{K})$. In that case (2.3) can be written as

$$(2.4) \qquad A'(\Omega) = \underset{K\subset\subset\to\Omega}{\text{ind lim}}\ A'(\bar{K})$$

the inductive limit of analytic functionals concentrated on K. It is not possible to find such a sequence $\{K_m\}_{m=1}^{\infty}$ for all domains Ω. Only for pseudoconvex domains Ω we will find one or, actually for domains of holomorphy, but according to theorem A.3 the domains of holomorphy are just the open pseudoconvex domains.

We define for any compact subset K of Ω the set

$$\hat{K}_\Omega = \{\zeta \mid \zeta \in \Omega,\ |f(\zeta)| \le \|f\|_K \text{ for all } f \in A(\Omega)\} \overset{\text{not}}{=\!=} \hat{K} \subset \Omega,$$

see (A1). Hence for $f \in A(\Omega)$ we have $\|f\|_K = \|f\|_{\hat{K}}$, thus $A_K(\Omega) = A_{\hat{K}}(\Omega)$. Ω is a domain of holomorphy, if

$$(2.5) \qquad \hat{K}_\Omega \subset\subset \Omega, \text{ whenever } K \subset\subset \Omega,$$

according to theorem A.1. In the sequel we will assume that Ω is pseudoconvex expressing that (2.5) is satisfied. Then the restriction map from $A_{\hat{K}}(\Omega)$ into $A(\bar{\hat{K}})$ exists. According to theorem A.4 $A_{\hat{K}}(\Omega)$ is a dense linear subspace of $A(\bar{\hat{K}})$. Hence

$$(2.6) \qquad \bar{A}_K(\Omega) = \bar{A}(\bar{\hat{K}}_\Omega) \qquad (\Omega \text{ pseudoconvex}).$$

Thus for any sequence $K \subset\subset\!\!\rightarrow \Omega$, $\hat{K}_\Omega$ is the desired sequence satisfying (2.4).

We have obtained that the closure of the space

$$A_{\Omega'}(\Omega) \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{K \subset\subset\rightarrow \Omega'} A_K(\Omega)$$

equals

$$\bar{A}_{\Omega'}(\Omega) = \operatorname*{proj\,lim}_{K \subset\subset\rightarrow \Omega'} \bar{A}_K(\Omega) = \operatorname*{proj\,lim}_{K \subset\subset\rightarrow \Omega'} \bar{A}(\bar{\hat{K}}_\Omega).$$

$A_{\Omega'}(\Omega)$ is a pré-F$\bar{S}$-space, that means its closure is an F$\bar{S}$-space. If Ω' is such that $K \subset\subset\!\!\rightarrow \Omega'$ implies $\hat{K}_\Omega \subset\subset\!\!\rightarrow \Omega$ we get $A_{\Omega'}(\Omega) = A(\Omega)$; for example $A_\Omega(\Omega) = A(\Omega)$; another example is (2.2). However, we will consider cases where $A_{\Omega'}(\Omega) \neq A(\Omega)$.

The strong dual of $A_{\Omega'}(\Omega)$ is the LS-space

$$(2.7) \qquad A'_{\Omega'}(\Omega) = \operatorname*{ind\,lim}_{K \subset\subset\rightarrow \Omega'} A'(\bar{\hat{K}}_\Omega),$$

which yields (2.4) when $\Omega' = \Omega$.

Let $\Omega_1 \subset \Omega_2$ be both pseudoconvex open sets in $\mathbb{C}^n$; then the restriction maps

$$A(\Omega_2) \twoheadrightarrow A_{\Omega_1}(\Omega_2) \hookrightarrow A(\Omega_1)$$

are continuous. The first map is a surjection from a Fréchet space onto a pré-Fréchet space and since $A_K(\Omega_2)$ is a linear subspace of $A_K(\Omega_1)$, the second map is in fact the embedding of the linear subspace $A_{\Omega_1}(\Omega_2)$ into $A(\Omega_1)$. The transposed maps are the continuous maps

$$A'(\Omega_1) \twoheadrightarrow A'_{\Omega_1}(\Omega_2) \hookrightarrow A'(\Omega_2),$$

where the first map is surjective according to the Hahn-Banach theorem and the second map is injective. We always have $\hat{K}_{\Omega_1} \subset \hat{K}_{\Omega_2}$, but if

$$(2.8) \qquad \hat{K}_{\Omega_1} = \hat{K}_{\Omega_2},$$

then in view of (2.3), (2.6) and (2.7) we have (see theorem A.7)

$$(2.9) \qquad A'(\Omega_1) = A'_{\Omega_1}(\Omega_2).$$

When each component of Ω_2 contains a component of Ω_1, for example when both are connected, $A'(\Omega_1)$ is dense in $A'(\Omega_2)$, for then $A(\Omega_2) = A''(\Omega_2)$ is mapped injectively into $A(\Omega_1) = A''(\Omega_1)$. We do not have this in the case of distributions: $E'(0_1)$ is not dense in $E'(0_2)$, $0_1 \subset 0_2$; $E'(\bar{0}_1)$ is even a closed linear subspace of $E'(0_2)$, $\bar{0}_1 \subset 0_2$ and 0_1 convex (see [14], G.5).

The linear hull L of the following set of entire functions in ζ

$$\{e^{iz\cdot\zeta}\}_{z\in\mathbb{C}^n}$$

is dense in $A(\Omega_2)$, when Ω_2 is a Runge domain, so this set is dense in $A_{\Omega_1}(\Omega_2)$ too. Indeed, differentiating $e^{iz\cdot\zeta}$ with respect to z and setting $z = 0$, we get $i\zeta$, so that we can approximate the polynomials by elements of L. Therefore, the map

$$F:\ \mu \in A'_{\Omega_1}(\Omega_2) \rightsquigarrow f(z) = \langle \mu_\zeta, e^{iz\cdot\zeta}\rangle$$

is an injective map from $A'_{\Omega_1}(\Omega_2)$ into some set of entire functions f.

Let H_K be the function from $\mathbb{C}^n$ into $\mathbb{R}$

$$H_K(z) = \sup_{\zeta\in K} \operatorname{Im}(-z\cdot\zeta), \qquad K \subset\subset \mathbb{C}^n.$$

We have $H_K = H_{ch(K)}$, where ch(K) is the convex hull of K, see section 4. When μ is concentrated on K, $f = F(\mu)$ satisfies

$$(2.10) \qquad |f(z)| \le M \exp H_K(z).$$

Hence we define the Banach space (see [14] B.4)

$$\operatorname{Exp}(K) = A_\infty(\exp - H_K(z); \mathbb{C}^n)$$

and the LS-space

$$\widetilde{\operatorname{Exp}}(\Omega) = \operatorname*{ind\,lim}_{K\subset\subset\Omega} \operatorname{Exp}(K)$$

We have $\widetilde{\operatorname{Exp}}(\Omega) = \widetilde{\operatorname{Exp}}(ch(\Omega))$ and according to [14] G.7 $\widetilde{\operatorname{Exp}}(\Omega)$ is nuclear.

Hence F is an injective map from $A'_{\Omega_1}(\Omega_2)$ into $\widetilde{\operatorname{Exp}}(\Omega_1)$ when Ω_2 is a Runge domain. Also F is a bounded map, which follows from (2.10) and the fact that $A'_{\Omega_1}(\Omega_2)$ being an LS-space is regular, see [14] F.15 and F.16. Since $A'_{\Omega_1}(\Omega_2)$ is bornological, F is continuous. We will see that if Ω_1 is convex, F is surjective and its inverse is continuous too. Convex sets are Runge domains (see [16] 16.11), hence with Ω_1 convex (2.8) is satisfied according to theorem A.6, so that then (2.9) holds.

<u>THEOREM 2.1</u>. *Let Ω be a convex domain in $\mathbb{C}^n$. The map F from* $A'(\Omega)$ *into* $\widetilde{\operatorname{Exp}}(\Omega)$ *given by*

$$F(\mu)(z) = \langle \mu_\zeta, e^{iz\cdot\zeta}\rangle, \qquad \mu \in A'(\Omega),$$

is an isomorphism.

Before proving this theorem we write $A'_{\Omega_1}(\Omega_2)$ in a different way. We have introduced the notion of an analytic functional in Ω concentrated on the compact set $K \subset\subset \Omega$ and $A_K(\Omega)$ was a linear subspace of $C(K)$. However, $A(\Omega)$ also is a linear subspace of $\mathcal{E}(\Omega)$, the space of all C^∞-functions in $\Omega \subset \mathbb{C}^n = \mathbb{R}^{2n}$ with the topology of uniform convergence of all derivatives on compact subsets. Indeed, all the derivatives of holomorphic functions converge on compact sets of Ω when the functions converge. So we can give $A(\Omega)$ the topology induced by $\mathcal{E}(\Omega)$ and each $\mu \in A'(\Omega)$ can be extended to an element of some $\mathcal{E}'(K)$. Then μ is a distribution with compact support K and for $f \in A(\Omega)$ we get

$$|\langle\mu,f\rangle| \le M \sup_{|\alpha|\le k} \|D^\alpha f\|_K \; ;$$

so Cauchy's formula yields for all $\varepsilon > 0$

$$(2.11) \qquad |\langle\mu,f\rangle| \le M_\varepsilon \|f\|_{K_\varepsilon} ,$$

where K_ε is a closed ε-neighborhood of K in $\mathbb{C}^n$ with $K_\varepsilon \subset\subset \Omega$ and M_ε is a positive constant depending on ε. When (2.11) holds we say that the analytic functional μ in Ω is *carried by* K. Thus an analytic functional in Ω carried by K is concentrated on any neighborhood of K in Ω. Sometimes it is said that μ is carried by such a neighborhood, see [15]. An analytic functional can be carried by several compact sets, but it is not true that it is carried by the intersection of all carriers, unlike the notion of support of a distribution, see [7] 4.5.

We will now describe the topology of $A(\Omega)$ using the concept of carrier, although this makes the description more complicated. Analytic functionals concentrated on compact sets are easier to describe, but analytic functionals carried by compact sets are easier to handle and are more natural as we will see.

Let $K \subset\subset \Omega$ be a compact subset of Ω. We define the pré-LS-space (this means that its closure is an LS-space)

(2.12) $$A_{\bar{K}}(\Omega) = \operatorname{ind}\lim_{\varepsilon\downarrow 0} A_{K_\varepsilon}(\Omega).$$

The closure $\bar{A}_K(\Omega)$ of $A_K(\Omega)$ does not consist of holomorphic functions in K, but the closure $\bar{A}_{\bar{K}}(\Omega)$ of $A_{\bar{K}}(\Omega)$ consists of functions each holomorphic in a neighborhood of K. $\bar{A}_{\bar{K}}(\Omega)$ consists of all holomorphic functions in a neighborhood of K when Ω is pseudoconvex and $K = \hat{K}_\Omega$ according to (2.6), for example when K is convex. The dual of $A_{\bar{K}}(\Omega)$ is the $F\bar{S}$-space

(2.13) $$A'_{\bar{K}}(\Omega) = \operatorname{proj}\lim_{\varepsilon\downarrow 0} A'_{K_\varepsilon}(\Omega) ,$$

the space of analytic functionals in Ω carried by K. Now $A'_{\Omega_1}(\Omega_2)$ is the inductive limit of the spaces $A'_{\bar{K}}(\Omega)$, namely

$$\operatorname{ind}\lim_{K\subset\subset\to\Omega_1} A'_{\bar{K}}(\Omega_2) = A'_{\Omega_1}(\Omega_2) = \operatorname{ind}\lim_{K\subset\subset\to\Omega_1} A'_K(\Omega_2).$$

Indeed, each $A'_{\bar{K}}(\Omega_2)$ can be mapped continuously into $A'_{K_\varepsilon}(\Omega_2)$ and into $A'_{\Omega_1}(\Omega_2)$ successively and conversely for all $\varepsilon > 0$ each $A'_K(\Omega_2)$ can be mapped continuously into $A'_{K_\varepsilon}(\Omega_2)$, thus into $A'_{\bar{K}}(\Omega_2)$, see [14] F.6. In this representation $A'_{\Omega_1}(\Omega_2)$ is an LS-space too: a neighborhood of zero in $A'_{\bar{K}}(\Omega_2)$, that is a neighborhood of zero in some $A'_{K_\varepsilon}(\Omega_2)$, is mapped into a relatively compact set of $A'_{S_\eta}(\Omega_2)$ for any $\eta > 0$, $K \subset\subset S$ and ε small enough, thus into a relatively compact set of $A'_{\bar{S}}(\Omega_2)$. This is in contrast with distributions, where the inductive limit $\mathcal{E}'(O) = \operatorname{ind}\lim \mathcal{E}'(K)$, $K \subset\subset\to O$, is strict, when O and K are convex, see [14] G.5.

Along the same lines one can see that $\widetilde{\operatorname{Exp}}(\Omega)$ can be represented as the LS-space

$$\widetilde{\operatorname{Exp}}(\Omega) = \operatorname{ind}\lim_{K\subset\subset\to\Omega} \operatorname{Exp}(K_0)$$

with

$$\operatorname{Exp}(K_0) \overset{\text{def}}{=\!=\!=} \operatorname{proj}\lim_{\varepsilon\downarrow 0} A_\infty(\exp(-H_K(z)-\varepsilon\|z\|;\mathbb{C}^n) = \operatorname{proj}\lim_{\varepsilon\downarrow 0} \operatorname{Exp}(K_\varepsilon).$$

We will now prove theorem 2.1.

<u>PROOF OF THEOREM 2.1</u>. It is clear that F maps $A'_{\overline{K}}(\Omega_2)$ continuously and injectively into $\mathrm{Exp}(K_0)$. It is sufficient to prove that F is a surjective map between the Fréchet-spaces $A'_{\overline{K}}(\Omega_2)$ and $\mathrm{Exp}(K_0)$, for then F^{-1} is continuous according to the open mapping theorem. When K is convex, this is exactly theorem 4.5.3 of [7] with $\Omega_2 = \mathbb{C}^n$. Thus F is an isomorphism between $A'_{\Omega}(\mathbb{C}^n) = A'(\Omega)$ and $\widetilde{\mathrm{Exp}}(\Omega)$, when Ω is convex. □

Theorem 2.1 is due to EHRENPREIS [2] and MARTINEAU [12]; for polydiscs this theorem can also be found in [15] and [18], where analytic functionals are used concentrated on compact sets. The notion of carrier of an analytic functional enables us to prove the continuity of F^{-1} by the open mapping theorem. In [7], [15] and [18] $e^{z\cdot\zeta}$ is used instead of $e^{iz\cdot\zeta}$, but we use $e^{iz\cdot\zeta}$ in view of the the generalization in section 5.

In the sequel we will start with a space of holomorphic functions of exponential growth. Let a(y,x) be a continuous function of z = x+iy on the unit sphere of $\mathbb{C}^n = \mathbb{R}^{2n}$, such that the following function, which is homogeneous of degree one, is convex

$$\tilde{a}(z) \overset{\text{def}}{=\!=} a\left(\frac{y}{\|z\|},\frac{x}{\|z\|}\right) \|z\| .$$

In that case we call a(y,x) itself convex, see section 4. This function determines a convex compact set $K \subset \mathbb{C}^n$ by

$$K = \{\zeta \mid \zeta = \xi+i\eta, -y\cdot\xi-x\cdot\eta \le \tilde{a}(z),\ z = x+iy \in C^n\},$$

see section 4. With this compact set K we denote the space Exp(K) also as

$$\mathrm{Exp}(a) \overset{\text{def}}{=\!=} \mathrm{Exp}(K).$$

The function $a(y,x)+\varepsilon$ on the unit sphere determines the function $\tilde{a}(z)+\varepsilon\|z\|$ on $\mathbb{C}^n$ and we have

$$\mathrm{Exp}(a+\varepsilon) = \mathrm{Exp}(K_\varepsilon).$$

Similarly we denote

$$\mathrm{Exp}(a+0) \overset{\mathrm{def}}{=\!=} \mathrm{Exp}(K_0).$$

We conclude this section with a corollary about the difference between analytic functionals and distributions expressed in properties of the spaces of their Fourier transforms.

Let a and b be two convex functions with $\tilde{a}(y,x) \le \tilde{b}(y,x)$ on $\mathbb{C}^n$ and let they determine the compact sets K and S, respectively; then $K \subset S$. The LS-space $\bar{A}_{\bar{S}}(\Omega)$ is reflexive and it is mapped injectively into $\bar{A}_{\bar{K}}(\Omega)$:

$$A''_{\bar{S}}(\Omega) \subset A''_{\bar{K}}(\Omega).$$

Hence (see [15], corollary 5 to th.18.1) $A'_{\bar{K}}(\Omega)$ is dense in $A'_{\bar{S}}(\Omega)$ and since F is an isomorphism Exp(a+0) is dense in Exp(b+0).

In [14] section 2 we have seen that the space of Fourier transforms of distributions with support in some compact set K_1 in $\mathbb{R}^n$ is a closed linear subspace of the space of Fourier transforms of distributions with support in a compact set S_1 with $K_1 \subset S_1$. Let us take the example when $K_1 = K$ and $S_1 = S$: let K and S be balls in the real part of $\mathbb{C}^n$ with radius a and b respectively ($a < b$). Then a(y,x) becomes $a\|y\|$ and $\tilde{a}(z) = a\|y\|$, so that we get

$$\operatorname*{proj\,lim}_{\varepsilon \downarrow 0} A_\infty(e^{-(a+\varepsilon)\|y\| - \varepsilon\|x\|};\mathbb{C}^n) = \mathrm{Exp}(a+0) \xrightarrow{\text{dense}} \mathrm{Exp}(b+0)$$

$$\operatorname*{ind\,lim}_{m\to\infty} A_\infty\left(\frac{e^{-a\|y\|}}{(1+\|z\|)^m};\mathbb{C}^n\right) = H(a;C^n) \xrightarrow[\text{linear subspace}]{\text{closed}} H(b;C^n).$$

The difference between these maps is not a consequence of the different structure of the spaces Exp and H. For we can make them look similar without changing them. Namely, it follows from the second map that

$$\operatorname*{proj\,lim}_{\varepsilon \downarrow 0} \operatorname*{ind\,lim}_{m\to\infty} A_\infty\left(\frac{e^{-(a+\varepsilon)\|y\|}}{(1+\|z\|)^m};\mathbb{C}^n\right) = H(a;\mathbb{C}^n)$$

and since the following injections are continuous

$$A_\infty(e^{-(a+\varepsilon)\|y\|-\varepsilon\|x\|};\mathbb{C}^n) \longrightarrow \underset{m\to\infty}{\text{ind lim}}\, A_\infty\left(\frac{e^{-(a+\varepsilon)\|y\|-\varepsilon\|x\|}}{(1+\|z\|)^m};\mathbb{C}^n\right) \longrightarrow$$

$$\longrightarrow A_\infty\,(e^{-(a+\eta)\|y\|-\eta\|x\|};\mathbb{C}^n)$$

with $\varepsilon < \eta$, we get, according to [14] F.6,

$$\text{(2.14)} \qquad \underset{\varepsilon\downarrow 0}{\text{proj lim}}\ \underset{m\to\infty}{\text{ind lim}}\, A_\infty\left(\frac{e^{-(a+\varepsilon)\|y\|-\varepsilon\|x\|}}{(1+\|z\|)^m};\mathbb{C}^n\right) = \text{Exp}(a+0)\ .$$

It follows from [14] C.3 that $H(a;\mathbb{C}^n)$ can be mapped continuously into Exp(a+0). Hence the transposed map between the inverse Fourier transforms of these spaces, which are reflexive, is continuous:

$$A_{\overline{K}}(\mathbb{C}^n) \quad \to \mathcal{E}(K).$$

$A_{\overline{K}}(\mathbb{C}^n)$ is the space of all real analytic functions on $K \subset\subset \mathbb{R}^n$ in view of (2.6) and the fact that $K = \hat{K}_{\mathbb{C}^n}$, because K is convex. Actually, every compact set in $\mathbb{R}^n$ is polynomially convex in $\mathbb{C}^n$, see [11]. Since an analytic function, that vanishes in an open set in $\mathbb{R}^n$, vanishes, the above map is injective. Therefore, the map from $H(a;C^n)$ into Exp(a+0) is dense and this implies that the distributions with support in a compact set K in $\mathbb{R}^n$ are dense in the space of analytic functionals carried by K. Even since Exp(a+0) is dense in Exp(b+0), the distributions with support in K are dense in the space of analytic functionals carried by the connected compact set S in $\mathbb{C}^n$, where K is any compact subset of S; for example K may consist of only one point of S.

3. NEWTON SERIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

In this section we derive the Newton interpolation series (see [14]) for entire functions of exponential type. The same is treated by KIOUSTELIDIS in [10]. However, the form given here yields a stronger result on the convergence and serves as a good introduction to the generalization in section 9.

Each vector h in $\mathbb{C}^n$ determines a convex open set Ω_h in $\mathbb{C}^n$ by

$$\Omega_h = \{\zeta \mid \zeta \in \mathbb{C}^n,\ |e^{-h\cdot\zeta}-1| < 1\}. \tag{3.1}$$

LEMMA 3.1. *For all* $z \in \mathbb{C}^n$ *and* $s \in \mathbb{C}$ *the sequence*

$$e^{iz\cdot\zeta} \sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\zeta}-1)^k$$

converges for $N \to \infty$ *to* $e^{i(z+ish)\cdot\zeta}$ *in the space* $A(\Omega_h)$ *regarded as functions of* ζ.

PROOF. The series converges uniformly on compact subsets of Ω_h, which is the convergence of the space $A(\Omega_h)$. □

For $h \in \mathbb{C}^n$ let f be a function in $\widetilde{\mathrm{Exp}}(\Omega_h)$ and let $s \in \mathbb{C}$. Then using theorem 2.1 and lemma 3.1 we derive the Newton series

$$\begin{aligned} f(z+ish) &= \langle\mu_\zeta, e^{i(z+ish)\cdot\zeta}\rangle = \langle\mu_\zeta, e^{iz\cdot\zeta} \lim_{N\to\infty} \sum_{k=0}^{N} \binom{s}{k}(e^{-h\cdot\zeta}-1)^k\rangle = \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} \binom{s}{k} \langle\mu_\zeta, e^{iz\cdot\zeta}(e^{-h\cdot\zeta}-1)^k\rangle = \\ &= \sum_{k=0}^{\infty} \binom{s}{k} \langle\mu_\zeta, \sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} e^{i(z+imh)\cdot\zeta}\rangle = \\ &= \sum_{k=0}^{\infty} \binom{s}{k} \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} f(z+imh) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta_{ih}^k f(z), \end{aligned} \tag{3.2}$$

where $\Delta_{ih} f(z) \overset{\text{def}}{=} f(z+ih)-f(z)$, so that

$$\Delta_{ih}^k f(z) = \sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} f(z+imh).$$

If μ belongs to $A'_{\bar{K}}(\Omega_h)$, $K \subset\subset \Omega_h$, the sequence

$$\mu_\zeta \sum_{k=0}^{N} \binom{s}{k}(e^{-h\cdot\zeta}-1)^k$$

converges weakly in each $A'_{K_{\varepsilon/2}}(\Omega_h)$, hence it converges strongly in each $A'_{K_\varepsilon}(\Omega_h)$, thus in $A'_{\bar{K}}(\Omega_h)$. Therefore (3.2) with $f(z) = \langle\mu_\zeta, e^{iz\cdot\zeta}\rangle$ converges in the topology of $\mathrm{Exp}(K_0)$. So we have found that if f satisfies for some $K \subset\subset \Omega_h$

(3.3) $\quad \forall\delta > 0: |f(z)| \le M_\delta \exp(H_K(z)+\delta\|z\|), \qquad z \in \mathbb{C}^n,$

the series (3.2) converges according to:

$$\text{(3.4)} \quad \begin{array}{l} \forall\varepsilon > 0, \quad \forall\delta > 0, \quad \exists N_0(\varepsilon,\delta) \ge N_1(s), \quad \forall N \ge N_0, \quad \forall z \in \mathbb{C}^n \\ \displaystyle |f(z+ish) - \sum_{k=0}^{N} \binom{s}{k} \Delta_{ih}^{k} f(z)| < \varepsilon\, A(s) \exp(H_K(z)+\delta\|z\|), \end{array}$$

where $N_1(s)$ is determined by (5.1) in [14] and A(s) by (5.4) in [14]. Thus there is certainly uniform convergence on compact subsets of $\mathbb{C}^n$, which is the convergence given in [10].

There exists a $\rho < 1$ with for $\zeta \in K_\varepsilon$ $|e^{-h\cdot\zeta}-1| \le \rho$, so that

$$|\langle\mu_\zeta, e^{iz\cdot\zeta} \sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\zeta}-1)^k\rangle| \le C \exp(H_K(z)+\varepsilon\|z\|) \sum_{k=0}^{N} |\binom{s}{k}| \rho^k.$$

Hence the series (3.2) converges absolutely.

We restate the results in

<u>THEOREM 3.1</u>. *For* $h \in \mathbb{C}^n$, $s \in \mathbb{C}$ *and* $f \in \widetilde{\mathrm{Exp}}(\Omega_h)$ *with* Ω_h *given by* (3.1) *the Newton series* (3.2) *is valid; the series converges absolutely in the topology of* $\widetilde{\mathrm{Exp}}(\Omega_h)$ *or more precisely* (3.2) *converges according to* (3.4) *when* f *satisfies* (3.3)*; the series* (3.2) *converges uniformly in* s *on compact subsets of* $\mathbb{C}$.

For a more detailed description of the function $H_K(z)$ when $K \subset\subset \Omega_h$, see KIOUSTELIDIS [10] Satz 9, when K is given by (40) and (41).

4. CONVEX SETS

In this section we describe how a closed convex set O in $\mathbb{R}^n$ determines an open convex cone C in $\mathbb{R}^{n*}$ and a homogeneous convex function $\tilde{a}$ on C and that, conversely, C and $\tilde{a}$ determine a closed convex set $\mathcal{O}$ in $\mathbb{R}^n$.

A closed convex set $\mathcal{O}$ in $\mathbb{R}^n$ is the intersection of closed halfspaces. Let $\mathcal{H}$ be the largest collection of halfspaces in $\mathbb{R}^n$ such that $\mathcal{O}$ is the intersection of halfspaces $H \in \mathcal{H}$. Let y be the unit vector perpendicular to the hyperplane ∂H bordering a halfspace $H \in \mathcal{H}$, in other words $y \in \mathbb{R}^{n*}$ is the linear functional which vanishes on the translation of ∂H to the origin. We identify the action of a linear functional y on $\xi \in \mathbb{R}^n$ with the inner-product: $\langle y,\xi\rangle = y\cdot\xi$. Then the halfspace H can be written as

$$H_y(a) = \{\xi \mid -y\cdot\xi \le a\}$$

with $y \in \mathbb{R}^{n*}$ and a a real number. Thus we have

$$\mathcal{O} \subset H_y(a) \subset H_y(b) \qquad \text{when } b \ge a,$$

that is $H_y(a) \in \mathcal{H}$ implies $H_y(b) \in \mathcal{H}$.

The normals y to ∂H vary in a set pr C on the unit sphere S of $\mathbb{R}^{n*}$, when H varies in $\mathcal{H}$. For each $y \in \text{pr}\, C$ let a(y) be the smallest of all the numbers a with $\mathcal{O} \subset H_y(a)$. Thus for each $y_0 \in \text{pr}\, C$ and each sequence $a_k \uparrow a(y_0)$, there is a sequence $\xi_k \in \mathcal{O}$ with

$$(4.1) \qquad -y_0\cdot\xi_k = a_k \le a(y_0).$$

Let C be the cone in $\mathbb{R}^{n*}$ determined by pr C

$$C = \{y \mid y \ne 0,\ \tilde{y} \in \text{pr}\, C\}$$

with the notation $\tilde{y} = y/\|y\|$. Hence any closed convex set $\mathcal{O}$ in $\mathbb{R}^n$ determines a cone C in $\mathbb{R}^{n*}$ and a function a(y) on pr C such that

$$(4.2) \qquad \mathcal{O} = \{\xi \mid -y\cdot\xi \le \tilde{a}(y) \overset{\text{def}}{=\!=} a(\tilde{y})\|y\|,\ y \in C\}.$$

It is clear that for $y \in C$ the function

$$(4.3) \qquad I_{\mathcal{O}}(y) \overset{\text{def}}{=\!=} \sup_{\xi\in\mathcal{O}} -y\cdot\xi$$

satisfies $I_0(y) \le \tilde{a}(y)$ and that $0 = \{\xi \mid -y\cdot\xi \le I_0(y),\ y \in C\}$. Since a(y) is the smallest possible function determining the set 0, we have

$$(4.4) \qquad \tilde{a}(y) = I_0(y), \qquad y \in C.$$

The cone C is convex, for

$$-(ty_1+(1-t)y_2)\cdot\xi = -ty_1\cdot\xi-(1-t)y_2\cdot\xi \le tI_0(y_1)+(1-t)I_0(y_2)$$

with $\xi \in 0$, $0 \le t \le 1$ and $y_1, y_2 \in C$, hence

$$0 \subset H_{ty_1+(1-t)y_2}(tI_0(y_1)+(1-t)I_0(y_2)).$$

From this it also follows that the function $I_0(y)$ is convex, that is

$$I_0(ty_1+(1-t)y_2) \le tI_0(y_1)+(1-t)I_0(y_2), \qquad y_1,y_2 \in C.$$

Taking into account (4.4) we find that $\tilde{a}(y)$ is convex, hence continuous and a(y) is bounded from below on pr C. We say that the continuous function a(y) on pr C is convex, when the function $\tilde{a}(y)$, which is homogeneous of degree one, is convex on C.

It is possible that the cone C is contained in a linear subspace of $\mathbb{R}^{n*}$ of lower dimension. Therefore, we consider C in the lowest dimensional space containing it. Then we speak of the interior int C of C and we show that the open cone int C determines the same convex set 0 as C. We denote the closed convex set 0 determined by a cone C and a convex function a(y) on pr C according to (4.2) by 0(a;C).

It is clear that $0(a;C) \subset 0(a;\text{int } C)$. Now let ξ_0 be a point outside 0(a;C), then there is a vector $y_0 \in$ pr C such that

$$-\xi_0\cdot y_0 > (y_0).$$

Hence there is an $\varepsilon > 0$ with

$$-\xi_0\cdot y_0 > a(y_0)+\varepsilon.$$

Since a(y) is continuous on pr C, there is a $y \in$ pr int C with

$$|a(y)-a(y_0)| < \varepsilon/2 \quad \text{and} \quad \|y-y_0\| < \varepsilon/(2\|\xi_0\|).$$

Hence

$$-\xi_0 \cdot y = -\xi_0 \cdot y_0 - \xi_0 \cdot (y-y_0) > a(y_0)+\varepsilon-\varepsilon/2 > a(y),$$

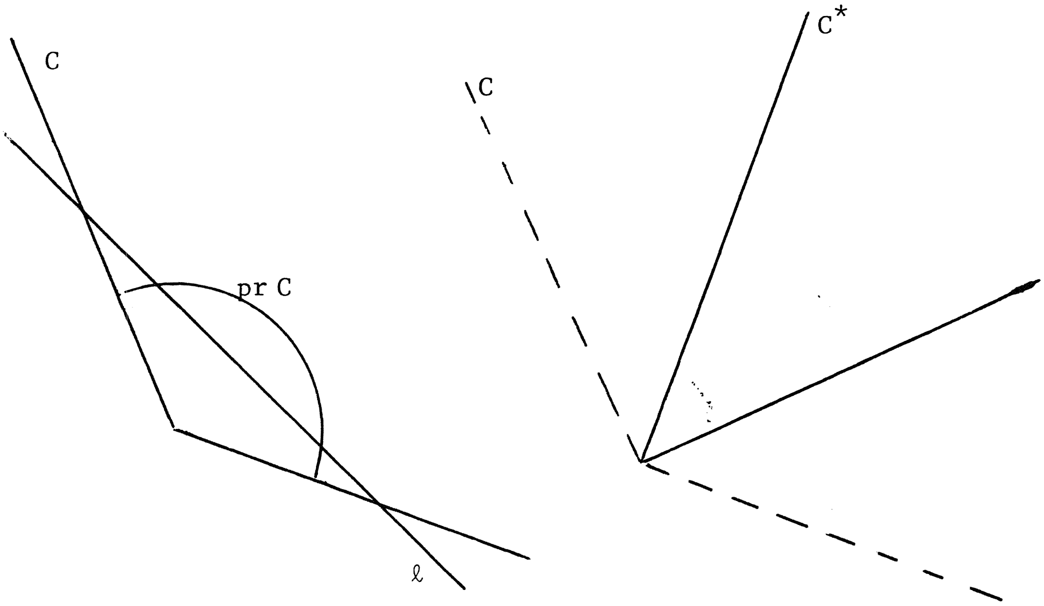
thus $\xi_0 \notin O(a;\text{int } C)$ by (4.2).

So we have found that each closed convex set in $\mathbb{R}^n$ determines an open convex cone C in $\mathbb{R}^{n*}$ (open relatively to a linear subspace of $\mathbb{R}^{n*}$) and a convex function a(y) on pr C. Now we will prove that, conversely, each open convex cone C in $\mathbb{R}^{n*}$ and each convex function a on pr C determine a closed convex set O in $\mathbb{R}^n$ by (4.2) that satisfies (4.4).

Indeed, O is convex and closed being the intersection of closed half-spaces and we only have to prove (4.1). Since $\tilde{a}$ is convex and C is open, we can find for each $y_0 \in$ pr C a linear function on C, say $\alpha \cdot y$ for some vector α, with $\alpha \cdot y \le \tilde{a}(y)$, $y \in C$ and $\alpha \cdot y_0 = a(y_0)$. Then the point $\xi_0 = -\alpha \in \mathbb{R}^n$ satisfies $-\xi_0 \cdot y = \alpha \cdot y \le \tilde{a}(y)$ for all $y \in C$, thus $\xi_0 \in O$. Furthermore, $-\xi_0 \cdot y_0 = \alpha \cdot y_0 = a(y_0)$, hence (4.1) holds. We have also obtained that in (4.1) we may take $a_k = a(y_0)$ and $\xi_k = \xi_0 \in O$ for all k, when $y_0 \in$ pr int C.

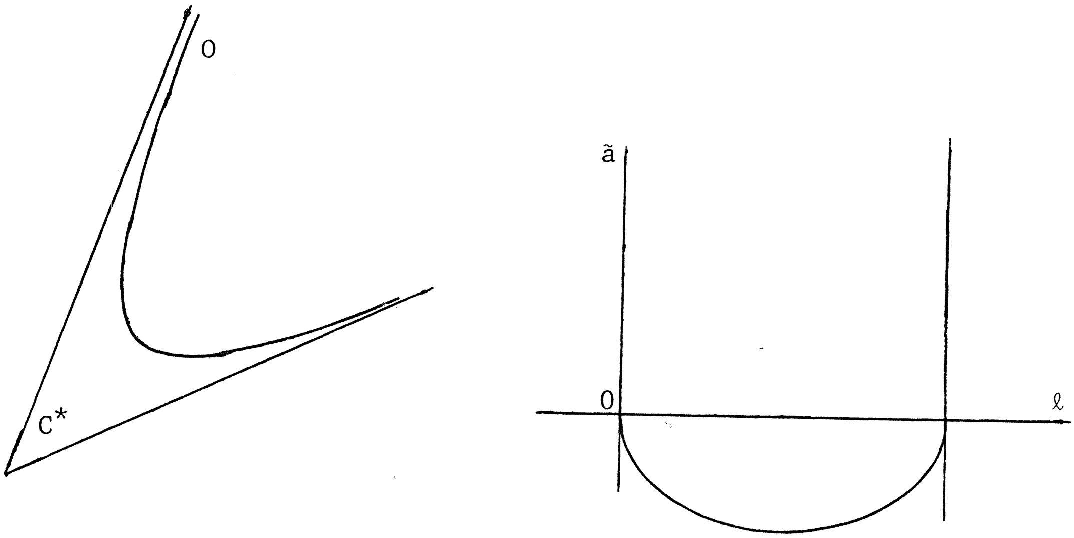
COROLLARY 4.1. *Each closed convex set* O *in* $\mathbb{R}^n$ *determines an open (with respect to some linear subspace of* $\mathbb{R}^{n*}$*) convex cone* C *in* $\mathbb{R}^{n*}$ *and a continuous convex function* a *on* pr C *by* (4.3) *such that* (4.2) *holds. Conversely, each open convex cone* C *in* $\mathbb{R}^{n*}$ *and each convex function* a *on* pr C *determine a closed convex set* O(a;C) *in* $\mathbb{R}^n$ *by* (4.2), *such that* (4.4) *is satisfied.*

We give some examples. Let C be an open cone in the first quadrant of $\mathbb{R}^{2*}$. We consider the function $\tilde{a}$ on some straight line $\ell \cap C \subset \mathbb{R}^{2*}$.



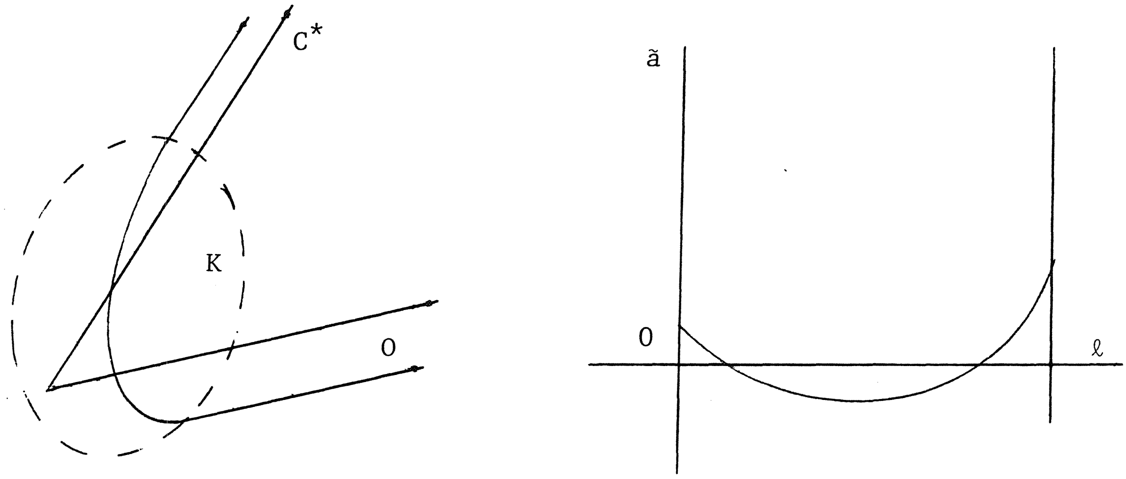
Then the dual cone C^* is $C^* = \{\xi \mid -y\cdot\xi \le 0,\ y \in C\} \subset \mathbb{R}^2$. We have the following cases with different behaviour of the convex function ã near the boundary of $\ell \cap C$:

I



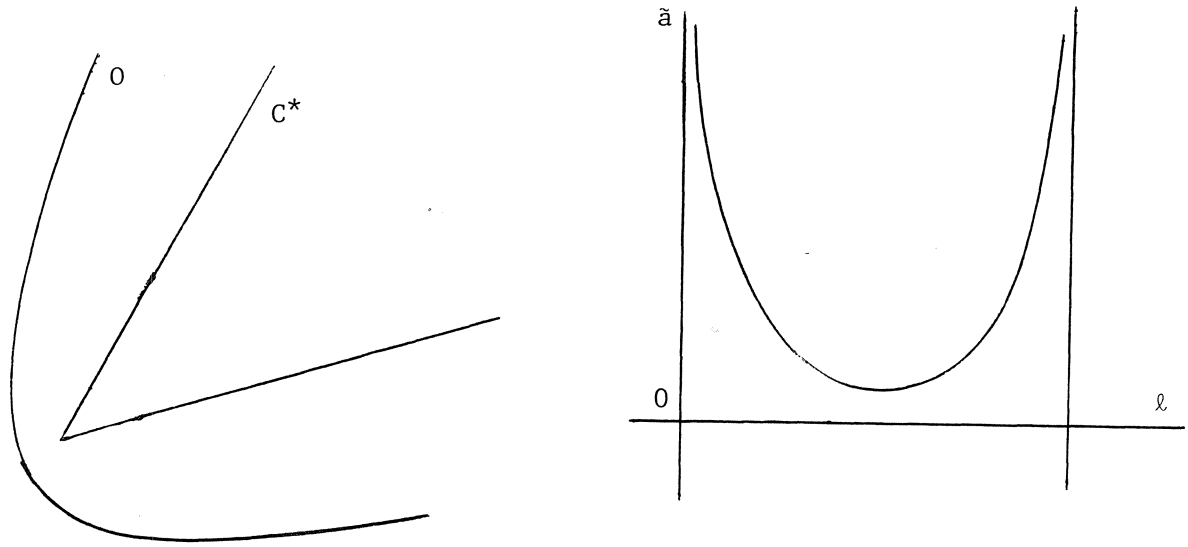
ã(y) is vertical at the boundary of C and the boundary of O is asymptotically parallel to the boundary of C^*.

II



ã(y) is not vertical at the boundary of C and outside some compact set K the boundary of O is parallel to the boundary of C^*.

III



ã(y) tends to infinity when y approaches the boundary of C and the distance between the boundaries of O and C^* increases to infinity.

In cases I and III we say that the function a is vertical at the boundary of C. In case II, when a is not vertical at the boundary, we may consider the closed convex set $O' = O \cap K'$, where K' is a compact set the interior of which contains K. Then O' is compact, so the cone C' determined by it is $\mathbb{R}^{n*}$ and the convex function a' on $\mathrm{pr}\, \mathbb{R}^{n*} = S$ determined by O' coincides with a on $\mathrm{pr}\, C$, that is

$$(4.5) \qquad a'(y) = a(y), \qquad y \in \mathrm{pr}\, C.$$

Thus when $\tilde{a}$ is not vertical at the boundary of C, it can be extended to a convex homogeneous function $\tilde{a}'$ on the whole of $\mathbb{R}^{n*}$.

Finally we describe how we can exhaust O by closed convex sets O_m not touching the boundary of O, namely $O = \bigcup_{m=1}^{\infty} O_m$ with O_m closed convex sets satisfying $O_m \subset \mathrm{int}\, O_{m+1} \subset O_{m+1} \subset \mathrm{int}\, O \subset \mathbb{R}^n$, $m = 1,2,\ldots$. Let $\{a_m\}_{m=1}^{\infty}$ be an increasing sequence of convex functions on $\mathrm{pr}\, C$ with $a_m(y) < a_{m+1}(y) < a(y)$ for $y \in \mathrm{pr}\, C$ and $\lim_{m\to\infty} a_m(y) = a(y)$, $y \in \mathrm{pr}\, C$ and moreover, either there are positive numbers ε_m with $a(y) - a_m(y) \geq \varepsilon_m$, $y \in \mathrm{pr}\, C$, or all the functions a_m are vertical at the boundary of C. Then the sets $O_m = O(a_m;C)$ satisfy the conditions. When the functions a_m are vertical at the boundary of C, the boundary of each O_m approaches the boundary of O. Otherwise the boundaries of O_m and O have a distance greater than ε_m.

When C does not contain a straight line (then C^* is not contained in a proper linear subspace of $\mathbb{R}^n$ and conversely) let C_k be a sequence of open convex subcones of C with $\mathrm{pr}\, C_k \subset \mathrm{pr}\, \bar{C}_k \subset \mathrm{pr}\, C_{k+1} \subset \mathrm{pr}\, \bar{C}_{k+1} \subset C \subset \mathbb{R}^{n*}$ and $\bigcup_{k=1}^{\infty} C_k = C$. We call C_k a relatively compact open subcone of C and we write $C_k \subset\subset C$. When C^* is contained in a linear subspace of $\mathbb{R}^n$ we take open cones C_k^* in this subspace with $\mathrm{pr}\, C_k^* \supset \overline{\mathrm{pr}\, C_{k+1}^*} \supset \mathrm{pr}\, C_{k+1}^* \supset C^*$ and the cones C_k are defined by the interior of the duals of C_k^*: $C_k = \mathrm{int}(C_k^*)^*$. Also in this case we call C_k a relatively compact subcone of C.

The functions $a+1/k$ on $\mathrm{pr}\, C$ are also convex and the sets $O(a+1/k;C_k)$ are of type II. Then

$$O(a;C) = \bigcap_{k=1}^{\infty} O(a+1/k;C_k),$$

but none of the sets $O(a+1/k;C_k)$ is contained in $O(a+1;C)$.

In the next sections we will regard $\mathbb{C}^n$ as a 2n-dimensional real space

$\mathbb{R}^{2n}$ by the identification

$$\zeta = \xi + i\eta \in \mathbb{C}^n \leftrightarrow (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

We identify the action of an element z in the complex dual space $\mathbb{C}^{n*}$ with the ordinary product of complex numbers

$$\langle z,\zeta\rangle = z\cdot\zeta = z_1\zeta_1 + \ldots + z_n\zeta_n$$

and we identify $\mathbb{C}^{n*}$ with $\mathbb{R}^{2n}$ by

$$z = x+iy \in \mathbb{C}^{n*} \leftrightarrow (y,x) \in \mathbb{R}^{n*} \times \mathbb{R}^{n*} = \mathbb{R}^{2n*}.$$

Then regarded as 2n-dimensional real vectors the action of $z \in \mathbb{R}^{2n*}$ on $\zeta \in \mathbb{R}^{2n}$ is

$$\operatorname{Im} z\cdot\zeta = (y,x)\cdot(\xi,\eta) = y\cdot\xi + x\cdot\eta.$$

When C is a cone in $\mathbb{R}^{n*}$ the set $T^C = \mathbb{R}^n + iC$ is a cone in $\mathbb{C}^{n*}$ containing a straight line. The dual cone $(T^C)^*$ is the cone C^* contained in the imaginary subspace of $\mathbb{C}^n$. Relatively compact subcones, constructed in the above way, are $\mathbb{R}^{n*} + iC_k$, where $C_k \subset\subset C$.

5. FUNCTIONS OF EXPONENTIAL TYPE HOLOMORPHIC IN CONES

In this section we discuss the space of functions of exponential type, holomorphic in cones in $\mathbb{C}^n$ and the space of their Fourier transforms (sometimes called Fourier-Borel transforms or Fourier-Laplace transforms).

Let C be an open convex cone in $\mathbb{C}^n$, which is identified with $\mathbb{R}^{2n}$ by $z = x+iy \in \mathbb{C}^n \leftrightarrow (y,x) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Let a(z), regarded as a function of (y,x), be continuous on pr C, such that the function

$$\tilde{a}(z) = \|z\|\, a\left(\frac{z}{\|z\|}\right) = \|(y,x)\|\, a\left(\frac{y}{\|(y,x)\|}, \frac{x}{\|(y,x)\|}\right)$$

is convex in C. A function f holomorphic in $C \subset \mathbb{C}^n$ is of exponential type a, when for all $\varepsilon > 0$ and $\delta > 0$ and for all open relatively compact subcones $C' \subset\subset C$ constants $M(\varepsilon,\delta,C')$ exist such that

$$|f(z)| \le M(\varepsilon,\delta,C')e^{\tilde{a}(z)+\varepsilon\|z\|}, \qquad z \in C',\ \|z\| > \delta.$$

We denote the space of all such functions f by Exp(a+0;C) or sometimes by Exp and we give this space a topology of an $F\bar{S}$-space by means of

$$\mathrm{Exp}(a+0;C) = \operatorname*{proj\,lim}_{k\to\infty} A_\infty(e^{-\tilde{a}(z)+1/k\|z\|};C(k)),$$

where $C(k) = C_k \cap \{z \mid \|z\| > 1/k\}$ and $\{C_k\}_{k=1}^{\infty}$ is an increasing sequence of open relatively compact subcones of C exhausting C (see section 4). According to [14] G.7 this space can also be written as a projective limit of Hilbert spaces (Exp(a+0;C) is nuclear).

We will construct a reflexive space A', which is the dual of some space A of holomorphic functions, such that Exp is the Fourier transform of A'. Assume that there is a continuous map $\mathcal{F}^t$ from Exp' into the completion $\bar{A}$ of A

(5.1) $\qquad \mathcal{F}^t\colon \mathrm{Exp}' \to \bar{A}$

then the transposed map $\mathcal{F}$ is a continuous map between the duals. So, since Exp is reflexive we get

(5.2) $\qquad \mathcal{F}\colon A' \to \mathrm{Exp}$

and since A' is reflexive, $\mathcal{F}^t$ is the transposed map of $\mathcal{F}$.

In order to get information about $\bar{A}$ we investigate Exp'. According to [14] C.3 and F.6 we can write Exp also as the FS-space

$$\mathrm{Exp}(a+0;C) = \operatorname*{proj\,lim}_{k\to\infty} A_{\infty,0}(\exp(-\tilde{a}(z)-1/k\|z\|);\overline{C(k)}),$$

where $A_{\infty,0}(M;\bar{\Omega})$ consists of functions holomorphic in Ω and continuous on $\bar{\Omega}$

with $M|f| < \infty$ on $\bar{\Omega}$ and with $M|f| = 0$ at infinity. Hence by [14] B.5 Exp' is the inductive limit of spaces, whose elements σ can be represented as bounded measures $\sigma(z)$ in $C(k)$, namely for $f \in$ Exp

$$(5.3) \qquad \langle\sigma,f\rangle = \int_{\overline{C(k)}} f(z)\,\exp(-\tilde{a}(z)-1/k\|z\|)\,d\sigma(z)$$

and

$$\int_{\overline{C(k)}} |d\sigma(z)| < \infty.$$

Next we define the map F^t. Therefore we regard $\mathbb{C}^n$ with elements $z = (y,x)$ as the dual $\mathbb{R}^{2n*}$ of some other space $\mathbb{R}^{2n}$, whose elements are denoted by (ξ,η) and which is identified with $\mathbb{C}^n$ by $\zeta = \xi+i\eta$. Then $\operatorname{Im} z\cdot\zeta = \langle(y,x),(\xi,\eta)\rangle$. The cone $C \subset \mathbb{R}^{2n*}$ and the convex function $a(z)$ on $\operatorname{pr} C$ determine a closed convex set Ω in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by (4.2)

$$(5.4) \qquad \Omega \overset{\text{not}}{=\!=} \Omega(a;C) \overset{\text{def}}{=\!=} \{\zeta \mid \operatorname{Im} z\cdot\zeta \le a(z),\ z \in \operatorname{pr} C\}.$$

Furthermore let us introduce the closed convex sets Ω_k either by

$$(5.5)(i) \qquad \Omega_k = \Omega(a+1/k;C_{k+2})$$

or by

$$(5.5)(ii) \qquad \Omega_k = \Omega(a+1/k;C).$$

In both cases $\Omega = \bigcap_{k=1}^{\infty} \Omega_k$. It is easy to see that $e^{iz\cdot\zeta}$ belongs to Exp if ζ belongs to Ω. Therefore, we can define the map F^t (5.1) by

$$(5.6) \qquad \sigma \in \text{Exp}':\ F^t(\sigma)(\zeta) = \langle\sigma_z,e^{iz\cdot\zeta}\rangle \qquad \text{for } \zeta \in \Omega.$$

The representation (5.3) yields for some k

$$(5.7) \qquad \phi(\zeta) = F^t(\sigma)(\zeta) = \int_{\overline{C(k)}} e^{-\tilde{a}(z)-1/k\|z\|}\, e^{iz\cdot\zeta}\, d\sigma(z).$$

In both cases (5.5)(i) and (5.5)(ii) ϕ is holomorphic in int Ω_k and satisfies there for some $K > 0$

$$(5.8) \qquad |\phi(\zeta)| \le K \exp(-\delta_k 1/k\|\zeta\|), \qquad \zeta \in \Omega_k,$$

where $\delta_k = \sin \alpha_k$ the minimum distance in radials between pr C_{k+1} and pr C_k, see [14] proof of lemma 6.3. Indeed, for

$$\zeta \in U = C^*_{k+1} \cap \{\zeta \mid \|\zeta\| \ge -a/\delta_k, \quad a = \min_{z \in \mathrm{pr}\, C_k} (a(z)+1/k)\}$$

we have

$$(5.9) \qquad -\tilde{a}(z)-1/k\|z\|+\mathrm{Im}-z\cdot\zeta \le -(a+\delta_k\|\zeta\|)\|z\| \le -a/k-\delta_k 1/k\|\zeta\|$$

when $z \in \overline{C(k)}$ and the set $\Omega_k \cap \overline{U^c}$ is compact in both cases (5.5)(i) and (5.5)(ii), so that (5.8) follows.

Therefore, we introduce the weightfunctions

$$M_k(\zeta) = \exp \delta_k \frac{1}{k} \|\zeta\| .$$

Then it follows from (5.9) that the map F^t given by (5.6) is a bounded, hence continuous, map from Exp' into the LS-space

$$(5.10) \qquad \operatorname*{ind\,lim}_{k\to\infty} A_\infty(M_k; \mathrm{int}\, \Omega_k)$$

in both cases (5.5)(i) and (5.5)(ii), see [14] F.11, F.16 and C.7.

Our aim is to choose such a space A that the map F (5.2) is an isomorphism. In [14] we have seen that, when the support of a distribution is contained in all the sets Ω_k, it is contained in Ω. But when the analytic functionals in A' are concentrated on all the sets Ω_k, we cannot immediately conclude that they are concentrated on Ω. Therefore, we do not yet know which of the alternatives (5.5)(i) or (5.5)(ii) we should take.

Now we will define linear subspaces A_i and A_{ii} of (5.10), depending on (5.5)(i) or (5.5)(ii) respectively, such that the map F (5.2) is injective. In fact we give the linear hull L of the set $\{e^{iz\cdot\zeta}\}_{z\in C}$ the topology of the space (5.10).

Let L_k be the linear hull of the set

$$\{e^{iz\cdot\zeta}\}_{\overline{z\in C(k)}}$$

of functions in ζ. We provide L_k with the norm

$$\|\cdot\|_k = \sup_{\zeta\in\Omega_k} M_k(\zeta)|\cdot(\zeta)|$$

and denote it by

$$A_{M_k;\Omega_k}(L_k).$$

Then A_i and A_{ii} are defined as

$$A \overset{\text{not}}{=\!=} A_{M;\bar{\Omega}}(L) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{k\to\infty} A_{M_k;\Omega_k}(L_k),$$

where Ω_k is given by (5.5)(i) or (5.5)(ii), respectively. The closure $\bar{A}$ of both spaces in an LS-space, namely

$$(5.11)\qquad \bar{A} = \operatorname{ind}\lim_{k\to\infty} \bar{A}_{M_k;\Omega_k}(L_k),$$

since LS-spaces are complete (see [14] F.14). $\bar{A}$ consists of functions each holomorphic in a neighborhood int Ω_k of Ω (compare 2.12). The duals A_i' of A_i and A_{ii}' of A_{ii} are F$\bar{\text{S}}$-spaces

$$(5.12)\qquad A' \overset{\text{not}}{=\!=} A'_{M;\bar{\Omega}}(L) = \operatorname{proj}\lim_{k\to\infty} A'_{M_k;\Omega_k}(L_k)$$

(compare 2.13). We only have to check that A is not too small, in other words that (5.1) still holds. By letting $\sigma(z)$ be δ-functions we see that L is contained in the range of F^t and when we write (5.7) as a defining "Lebesgue sum", it follows from (5.9) that this sum converges in the topology of $\bar{A}_{M_{k+1};\Omega_{k+1}}(L_{k+1})$ to $\phi(\zeta)$. Hence F^t (5.1) has dense image, so that F (5.2) is an injective map from A_i' and from A_{ii}', given by (5.12), into Exp.

The definition of F (5.2) as the transposed map of (5.1) yields for $\mu \in A'$

$$\forall\sigma \in \mathrm{Exp}': \langle\sigma_z, F(\mu_\zeta)(z)\rangle \overset{\mathrm{def}}{=\!=} \langle\mu_\zeta, F^t(\sigma_z)(\zeta)\rangle = \langle\mu_\zeta, \langle\sigma_z, e^{iz\cdot\zeta}\rangle\rangle =$$

$$= \langle\sigma_z, \langle\mu_\zeta, e^{iz\cdot\zeta}\rangle\rangle$$

by Fubini's theorem, so that F (5.2) may as well be defined as

$$(5.13) \qquad F(\mu)(z) = \langle\mu_\zeta, e^{iz\cdot\zeta}\rangle, \qquad z \in C$$

like (5.6).

Now F is a continuous injective map from A_i' and from A_{ii}' into Exp. Since A_i can be continuously embedded into A_{ii}, A_i' is a priori larger than A_{ii}'. So it is easier to prove that F is also surjective from A_i' onto Exp. In that case the inverse map would be continuous according to the open mapping theorem, because A_i' and Exp are $F\bar{S}$-spaces and F would be an isomorphism between A_i' and Exp. If we can also prove that F is a surjective map from A_{ii}' onto Exp, then A_{ii}' too would be isomorphic to Exp, so that $A_{ii}' = A_i'$ and $\bar{A}_i = \bar{A}_{ii}$. First we will prove the apparently weaker version (i), theorem 5.1, of the main result of this paper, namely that F is an isomorphism between A_i' and Exp. Then in a next section we investigate the spaces $\bar{A}_i$ and $\bar{A}_{ii}$ and finally in section 7 we will show that $F(A_{ii}') = \mathrm{Exp}$, which is the stronger version (ii) of the main theorem of this paper, theorem 6.1.

<u>THEOREM 5.1</u>. *Let* a *be a convex function on* pr C *for some open convex cone* C *in* $\mathbb{C}^n$ *and let* Ω *and* Ω_k *be the closed convex sets in* $\mathbb{C}^n$ *determined by* (5.4) *and* (5.5)(i) *respectively. Then the map* F *from* $A'_{M;\bar{\Omega}}(L)$ (5.12) *into* Exp(a+0;C) *given by* (5.13) *is an isomorphism.*

<u>PROOF</u>. We only have to prove the surjectivity of the map F. So given an $f \in \mathrm{Exp}$, we have to find for each $k = 1,2,\ldots$ a linear functional μ^k on L_k with

$$f(z) = \langle\mu^k_\zeta, e^{iz\cdot\zeta}\rangle, \qquad z \in \overline{C(k)}$$

and with

$$|<\mu^k_\zeta,\phi(\zeta)>| \le K_k \sup_{\zeta\in\Omega_k} \exp(\delta_k \frac{1}{k}\|\zeta\|)\ |\phi(\zeta)|$$

for $\phi \in A_{M_k;\Omega_k}(L_k)$.

Like in the proof of the theorem for entire functions (see HÖRMANDER [7], EHRENPREIS [3]) we try to extend f as a holomorphic function F in 2n complex variables ν satisfying a certain bound and apply a Paley-Wiener-Schwartz type theorem. Precisely, choose an integer k and assume that we have found a function F_k of the complex variables $\nu \overset{\text{not}}{=\!=} (\nu^1,\nu^2)\in\mathbb{C}^n\times\mathbb{C}^n = \mathbb{C}^{2n}$ holomorphic in $\mathbb{R}^{2n}+iC(q)$ with $q > \max(k+2,(k+1)/\delta_{k+1})$ that satisfies for some m

$$(5.14) \qquad |F_k(\nu^1,\nu^2)| \le M_k(1+\|\nu\|)^m \exp(\tilde{a}(\mathrm{Im}\ \nu)+1/q\|\mathrm{Im}\ \nu\|)$$

for $\mathrm{Im}\ \nu \in C(q)$ and

$$(5.15) \qquad F_k(z,iz) = f(z) \qquad \text{for } z \in C(q).$$

Then we can apply theorem 9.1 of [14] (remark 9.1 and formula (9.5) or in fact the 5^{th} line from below on page 61, since $1/q < \delta_{k+1}/(k+1)$), which says that F_k can be written as

$$F_k(\nu) = <\mu^k_{\xi,\eta}, e^{i\nu^1\cdot\xi+i\nu^2\cdot\eta}>, \qquad \mathrm{Im}\ \nu \in C(k+1)$$

with $\mu^k \in (S^{k+1^*}_{m+n+1}(a+1/q;C_q))'$. This means that for $\phi \in S^{k+1^*}(a+1/q;C_q)$

$$|<\mu^k_{\xi,\eta},\phi(\xi,\eta)>| \le K'_k \sup_{\substack{|p|\le m+n+1\\ (\xi,\eta)\in O(a+1/q;C_q)}}$$

$$(1+\|(\xi,\eta)\|)^{m+n+1} \exp(\delta_{k+1}\frac{1}{k+1}\|(\xi,\eta)\|)\ |D^p\phi(\xi,\eta)|.$$

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, $(\xi,\eta) \leftrightarrow \zeta = \xi+i\eta$, we get, because $C(k) \subset C(k+1)$,

$$f(z) = \langle \mu^k_\zeta, e^{iz\cdot\zeta} \rangle, \qquad z \in \overline{C(k)}$$

with

$$|\langle \mu^k_\zeta, \phi(\zeta)\rangle| \le K'_k \sup_{\substack{|p|\le m+n+1 \\ \zeta\in\Omega(a+1/q;C_q)}}$$

$$(1+\|\zeta\|)^{m+n+1} \exp(\delta_{k+1} \frac{1}{k+1} \|\zeta\|) \; |D^p\phi(\zeta)| \le$$

$$\le K_k \sup_{\zeta\in\Omega_k} \exp(\delta_k \frac{1}{k} \|\zeta\|) \; |\phi(\zeta)|$$

for any $\phi \in A_{M_k;\Omega_k}(L_k) \subset S^{k+1^*}(a+1/q;C_q)$, since int $\Omega(a+1/q;C_q) \subset \Omega_k$ because $q > k+2$.

Now we have to find an F_k satisfying (5.14) and (5.15). For an arbitrary ℓ we will construct a function $F_{k,\ell}$ that satisfies

$$(5.16) \qquad |F_{k,\ell}(\nu^1,\nu^2)| \le M_{k,\ell}(1+\|\nu\|)^m \exp(\tilde{a}(\text{Im } \nu)+1/\ell\|\text{Im } \nu\|), \text{Im } \nu \in C(q).$$

The construction follows the same pattern of the proof of theorem 4.4.3 of [7], only here we have to be careful near the boundary of $\mathbb{R}^n+iC$.

Since an open domain $\mathbb{R}^n+iB$ in $\mathbb{C}^n$ is pseudoconvex (domain of holomorphy) only if B is convex (theorem A.2), we will use domains of the form $\mathbb{R}^n+i\,\text{ch}(C(q))$, where ch(C(q)) is the convex hull of C(q). This does not change anything, because for all q there is a p with $C(q) \subset \text{ch}(C(q)) \subset C(p)$.

Let

$$(5.17) \qquad C(q+1)_{\delta,j} = \{(y,x) \mid x_1 = x^0_1,\dots,x_j = x^0_j, \quad |x_k-x^0_k| < \delta$$
$$\text{for } k = j+1,\dots,n \text{ and } (y,x^0) \in \text{ch}(C(q+1))\} \subset \mathbb{R}^{2n}.$$

Then $\text{ch}(C(q+1)) = C(q+1)_{\delta,n} \subset \dots \subset C(q+1)_{\delta,j} \subset \dots \subset C(q+1)_{\delta,0}$. We can choose $\delta > 0$ so small that there exists an integer $p > q+1$, such that

$$(5.18) \qquad C(q+1)_{\delta,0} \subset C(p).$$

Let ψ_k be a C^2-function in $\mathbb{C}$ between 0 and 1 which is equal to 1 in the disc with radius 1/2 δ and vanishes outside the disc with radius δ. We write the coordinates in $\mathbb{C}$ as $w = u+iv$. Then there is a constant K_k with

$$\left|\frac{\partial\psi}{\partial\bar{w}}(w)\right| \leq K_k, \qquad w \in \mathbb{C}.$$

Let us define the (0,1)-form (see appendix section II) $\psi'(w) = \partial\psi/\partial\bar{w}(w)\, d\bar{w}$ and let $w_j = i\nu_j^1 - \nu_j^2$, then $d\bar{w}_j = -id\bar{\nu}_j^1 - d\bar{\nu}_j^2$. When f is regarded as an element of $A_\infty(\exp(-\tilde{a}(z)-1/\ell\|z\|);C(p))$ we define the function $F_{k,\ell}$ as follows:

$$F_{k,\ell}(\nu^1,\nu^2) = F_{k,\ell}(\nu) =$$

$$= \prod_{j=1}^{n} \psi_k(w_j)\, f(\nu^1) - \sum_{j=1}^{n}\left[\prod_{m=j+1}^{n} \psi_k(w_m)\right] w_j\, U_j^\ell(\nu^1;\nu_1^2,\ldots,\nu_j^2)$$

for certain functions U_j^ℓ in n+j complex variables. When $\mathrm{Im}\,\nu \in C(q+1)$, $\Pi_{j=1}^n \psi_k(w_j)$ vanishes for $\nu^1 = (\mathrm{Im}\,\nu^1, \mathrm{Re}\,\nu^1) \notin C(p)$ according to (5.17) and (5.18), thus $F_{k,\ell}$ is defined for $\nu \in \mathbb{R}^{2n}+i\,\mathrm{ch}(C(q+1))$. When $\nu^2 = i\nu^1$, that is $w_j = 0$ for $j = 1,\ldots,n$, we get

$$F_{k,\ell}(\nu^1,i\nu^1) = f(\nu^1) \qquad \text{for } \nu^1 \in \mathrm{ch}(C(q+1)),$$

so that (5.14) is certainly satisfied.

Now we choose the functions U_j^ℓ with a suitable bound such that $F_{k,\ell}$ is holomorphic, that is such that $\bar{\partial}F_{k,\ell} = 0$. We can write $F_{k,\ell}$ in a different way, namely let $H_0^\ell(\nu^1) = f(\nu^1)$ and let

$$H_j^\ell(\nu^1;\nu_1^2,\ldots,\nu_j^2) =$$

$$= \psi_k(w_j)\, H_{j-1}^\ell(\nu^1;\nu_1^2,\ldots,\nu_{j-1}^2) - w_j\, U_j^\ell(\nu^1;\nu_1^2,\ldots,\nu_j^2)$$

for $j = 1,\ldots,n$ successively, then $H_n^\ell = F_{k,\ell}$. If H_{j-1}^ℓ is holomorphic for $(\mathrm{Im}\,\nu_1^1,\ldots,\mathrm{Im}\,\nu_n^1, \mathrm{Im}\,\nu_1^2,\ldots,\mathrm{Im}\,\nu_{j-1}^2, \mathrm{Re}\,\nu_j^1,\ldots,\mathrm{Re}\,\nu_n^1; \mathrm{Re}\,\nu_1^1,\ldots,\mathrm{Re}\,\nu_{j-1}^1, \mathrm{Re}\,\nu_1^2,\ldots,\mathrm{Re}\,\nu_{j-1}^2) \in C(q+1)_{\delta,j-1} \times \mathbb{R}^{2(j-1)} \overset{\mathrm{def}}{=\!=} B_{j-1} \subset \mathbb{C}^{n+j-1}$, which is true for $j = 1$ by (5.18), then H_j^ℓ is holomorphic in B_j when U_j^ℓ satisfies

(5.19) $\bar{\partial} U_j^\ell = H_{j-1}^\ell(\nu^1;\nu_1^2,\ldots,\nu_{j-1}^2)\ \psi'(w_j)/w_j \overset{\text{not}}{=\!=} g_j^\ell.$

It follows when $j = n$, that $F_{k,\ell}$ is holomorphic in $B_n = \mathbb{R}^{2n}+i\,\mathrm{ch}(C(q-1)) \subset \subset \mathbb{C}^{2n}$. Since by assumption H_{j-1}^ℓ is holomorphic in B_{j-1}, $1/w_j$ is holomorphic outside any neighborhood of zero, $\psi'(w_j) = 0$ in a neighborhood of zero and since

$$\bar{\partial}\ \psi'(w_j) = \bar{\partial}\ \frac{\partial\psi}{\partial\bar{w}_j}(i\nu_j^1-\nu_j^2)[-id\bar{\nu}_j^1-d\bar{\nu}_j^2] =$$

$$= \left(i\ \frac{\partial^2\psi}{\partial\bar{\nu}_j\ \partial\bar{w}_j} - \frac{\partial^2\psi}{\partial\bar{\nu}_j^1\ \partial\bar{w}_j}\right) d\bar{\nu}_j^1 \wedge d\bar{\nu}_j^2 = \left(-i\ \frac{\partial^2\psi}{\partial\bar{w}_j^2} + i\ \frac{\partial^2\psi}{\partial\bar{w}_j^2}\right) d\bar{\nu}_j^1 \wedge d\bar{\nu}_j^2 = 0,$$

we get $\bar{\partial} g_j^\ell = 0$. Furthermore the domain B_j is convex, thus pseudoconvex. Therefore we can apply theorem A.10 in order to solve (5.19). As a weight-function we may take $(1+\|z\|^2)^{-3(j-1)} \exp(-2\tilde{a}(z)-2/\ell\|z\|)$, since $\tilde{a}(z)+1/\ell\|z\|$ is a convex function and $\log(1+\|z\|^2)$ is plurisubharmonic. Write $z^j = (\mathrm{Im}\ \nu_1^1,\ldots,\mathrm{Im}\ \nu_n^1,\ \mathrm{Im}\ \nu_1^2,\ldots,\mathrm{Im}\ \nu_j^2,\ \mathrm{Re}\ \nu_{j+1}^1,\ldots,\mathrm{Re}\ \nu_n^1) \in \mathbb{R}^{2n}$ and $\nu[j] = (\nu^1;\nu_1^2,\ldots,\nu_j^2) \in \mathbb{C}^{n+j}$ and let $\lambda(\nu[j])$ be the Lebesgue measure in $\mathbb{C}^{n+j}$. Then by theorem A.10 there exists a solution U_j^ℓ of (5.19) with

$$\int_{B_j} |U_j^\ell(\nu[j])|^2\ \frac{\exp(-2\tilde{a}(z^j)-2/\ell\|z^j\|)}{(1+\|\nu[j]\|^2)^{3j-1}}\ d\lambda(\nu[j]) \le$$

$$\le \int_{B_j} |g_j^\ell(\nu[j])|^2\ \frac{\exp(-2\tilde{a}(z^j)-2/\ell\|z^j\|)}{(1+\|\nu[j]\|^2)^{3(j-1)}}\ d\lambda(\nu[j]).$$

Since $\tilde{a}|_{C_p}$ can be extended to a convex homogeneous function $\tilde{a}'$ on $\mathbb{R}^{2n}$, see (4.5), we get for $x,y \in C_p$

$$\tilde{a}(x)-\tilde{a}(y) = 2\tilde{a}(\tfrac{y}{2}+\tfrac{x-y}{2})-\tilde{a}(y) \le \tilde{a}(y)+\tilde{a}'(x-y)-\tilde{a}(y) =$$

$$= \|x-y\|\,a'(\widehat{x-y}) \le A\|x-y\|$$

and also

$$\tilde{a}(y)-\tilde{a}(x) \le \|x-y\| a'(\widetilde{y-x}) \le A\|x-y\|$$

for some constant A. We set $M = \exp(2\delta A+2\delta/\ell)$ and $C_k = 2(\delta^2+16K_k^2)\pi M$, then we can estimate H_j^ℓ in terms of H_{j-1}^ℓ using $(a+b)^2 \le 2(a^2+b^2)$, hence $|w_j|^2/(1+\|\nu[j]\|^2) \le 2$,

$$\int_{B_j} |H_j^\ell(\nu[j])|^2 \frac{\exp(-2\tilde{a}(z^j)-2/\ell\|z^j\|)}{(1+\|\nu[j]\|^2)^{3j}} d\lambda(\nu[j]) \le$$

$$\le 2\left\{\pi\delta^2 M \int_{B_{j-1}} |H_{j-1}^\ell(\nu[j-1])|^2 \frac{\exp(-2\tilde{a}(z^{j-1})-2/\ell\|z^{j-1}\|)}{(1+\|\nu[j-1]\|^2)^{3(j-1)}} d\lambda(\nu[j-1]) + \right.$$

$$\left. + \int_{B_j} |g_j^\ell(\nu[j])|^2 \frac{\exp(-2\tilde{a}(z^j)-2/\ell\|z^j\|)}{(1+\|\nu[j]\|^2)^{3(j-1)}} d\lambda(\nu[j])\right\} \le$$

$$\le C_k \int_{B_{j-1}} |H_{j-1}^\ell(\nu[j-1])|^2 \frac{\exp(-2\tilde{a}(z^{j-1})-2/\ell\|z^{j-1}\|)}{(1+\|\nu[j-1]\|^2)^{3(j-1)}} d\lambda(\nu[j-1]).$$

Since for $j=n$ $B_n = \mathbb{R}^{2n}+i\,\mathrm{ch}(C(q+1))$, $H_n^\ell = F_{k,\ell}$, $\nu[n] = (\nu^1;\nu^2) = \nu$, $z^n = \mathrm{Im}\,\nu$ and for $j=0$ $B_0 \subset C(p)$, $H_0^\ell = f$, $\nu[0] = \nu^1 = z \in \mathbb{C}^n$, $z^0 = (y,x) = z$, it follows that

$$\int_{\mathbb{R}^{2n}+iC(q+1)} |F_{k,\ell}(\nu)|^2 \frac{\exp(-2\tilde{a}(\mathrm{Im}\,\nu)-2/\ell\|\mathrm{Im}\,\nu\|)}{(1+\|\nu\|^2)^{3n}} d\lambda(\nu) \le$$

$$\le C_k^n \int_{C(p)} |f(z)|^2 \exp(-2\tilde{a}(z)-2/\ell\|z\|)\, d\lambda(z).$$

According to condition HS_2 (see [14] G.7) we can estimate the sup-norm by the L^2-norm and we find that (5.16) is satisfied with $m = 3n$, since also Exp can be written as projective limit of Hilbert spaces. □

<u>REMARK 5.1</u>. If we could choose for all k the functions $F_{k,\ell}$ satisfying (5.15) and (5.16) such that $F_{k+1,\ell}(\nu) = F_{k,\ell}(\nu)$ for $\nu \in \mathbb{R}^{2n}+i\,C(k)$, thus if there is one function F_ℓ holomorphic in $\mathbb{R}^{2n}+i\,C$ satisfying for all k

$$(5.20)\qquad |F_\ell(\nu)| \le M_k(1+\|\nu\|)^m \exp(\tilde{a}(\mathrm{Im}\,\nu)+1/(\ell+1)\|\mathrm{Im}\,\nu\|), \qquad \mathrm{Im}\,\nu \in C(k),$$

for some m and

$$(5.21)\qquad F_\ell(z,iz) = f(z) \qquad \text{for } z \in C,$$

then F_ℓ would belong to $H^*(a+1/(\ell+1);C)$ and we would have according to theorem 9.1 [14]

$$F_\ell(\nu) = \langle\mu^\ell_{\xi,\eta}, e^{i\nu^1\cdot\xi+i\nu^2\cdot\eta}\rangle$$

with $\mu^\ell \in S^*(a+1/(\ell+1);C)'$. In that case

$$f(z) = \langle\mu^\ell_\zeta, e^{iz\cdot\zeta}\rangle \qquad \text{for } z \in C$$

and since $\delta_\ell/\ell\|\zeta\|$ is uniformly continuous on Ω_ℓ, we get for $\phi \in A_{M_\ell;\Omega_\ell}(L_\ell)$

$$|\langle\mu^\ell_\zeta,\phi(\zeta)\rangle| \le K'_\ell \sup_{\substack{|p|\le m+n+1\\ \zeta\in\Omega(a+1/(\ell+1);C)}} (1+\|\zeta\|)^{m+n+1} \exp(\delta_{\ell+1}/(\ell+1)\|\zeta\|)\ |D^p\phi(\zeta)| \le$$

$$\le K_\ell \sup_{\zeta\in\Omega_\ell} \exp(\delta_\ell/\ell\|\zeta\|)\ |\phi(\zeta)|,$$

where Ω_ℓ is now given by (5.5)(ii). Hence $\mathcal{F}$ would be a surjective map from A'_{ii} onto Exp.

6. FORMULATION OF THE PROBLEMS AND STATEMENT OF THE MAIN RESULT

In this section we reconsider the procedure followed in the last section and we formulate the problems to be solved in order that $\bar{A}_i$ and $\bar{A}_{ii}$ given in (5.11) are indeed also given by (5.10), that is the space of *all* holomorphic functions satisfying the growth conditions in the sets int Ω_k, where Ω_k is given by (5.5)(i) or (5.5)(ii), respectively. Theorem 5.1 says that F is surjective from A_i' onto Exp, from which it follows that F^t (5.1) defined by (5.6) is injective. Since $\bar{A}_i$ is reflexive (see (5.11)) the linear hull of the set $\{e^{iz\cdot\zeta}\}_{\zeta\in\Omega}$ is dense in Exp(a+0;C). On the other hand when the above problems are solved, we see that the linear hull of the set $\{e^{iz\cdot\zeta}\}_{z\in C}$ is dense in the space (5.10) and the main result (theorem 6.1) follows.

Anticipating on the results we will get we mean in this section by A_i' and A_{ii}' the dual A' of the spaces $\bar{A}_i$ and $\bar{A}_{ii}$, respectively, given by (5.10)

$$(6.1) \qquad \bar{A} = \operatorname{ind}\lim_{k\to\infty} A_\infty(M_k;\Omega_k),$$

where Ω_k is given in (5.5)(i) or (5.5)(ii), respectively. Then $\bar{A}$ is an LS-space. There are also other possibilities of writing $\bar{A}_i$ as an inductive limit of spaces A_i^k, or $\bar{A}_{ii}$ as inductive limit of spaces A_ℓ. We will choose appropriate spaces A_i^k and A_ℓ. In the above $A_i^k = A_\infty(M_k;\Omega_k)$.

In the last section we have embedded (a linear subspace, namely (5.11), of) A_i^k into the space $S^{*k+1}(\Omega_{k+1})$. Roughly we can say that A_i^k consists of those elements ϕ in $S^{*k+1}(\Omega_{k+1})$ with $\bar{\partial}\phi = 0$ and that any element μ of $S^*(\Omega)'$ that satisfies $\mu = \bar{\partial}^t\sigma_k$ for some $\sigma_k \in (S^{k*}(\Omega_k)')^n$, is zero when restricted to A_i^k. Hence the elements of A_i' can be identified with the equivalence classes of the elements in $S^*(\Omega)'$, when two elements in $S^*(\Omega)'$ are equivalent if their difference μ can be written as $\mu = \bar{\partial}^t\sigma_k$ in each $S^{*k}(\Omega_k)'$ for some $\sigma_k \in (S^{*k}(\Omega_k)')^n$. Now we investigate this more precisely.

First we write $\bar{A}_i$ as inductive limit of spaces having the topology of $S^{k*}(\Omega_k)$, that is A_i^k now is the closed linear subspace of

$$S^k \overset{\text{not}}{=\!=} S^{k*}(a+1/k;C_{k+2}) \overset{\text{def}}{=\!=} \operatorname{proj}\lim_{m\to\infty} W^m_{\infty,0}\ ((1+\|\zeta\|)^m \exp(\delta_k \tfrac{1}{k}\|\zeta\|);\Omega_k)$$

consisting of the functions holomorphic in int Ω_k and C^∞ on Ω_k with the topology heredited from S^k. Therefore, according to [15] prop. 35.5(a) the following sets can be identified

$$(A_i^k)' = (S^k)' \Big/ (A_i^k)^0$$

and according to [15] prop. 35.6 this is also true for the topologies, when we provide these spaces with the weak* topology and the quotient topology with respect to the weak* topology, respectively:

$$(A_i^k)'_\sigma \cong (S^k)'_\sigma \Big/ (A_i^k)^0 \quad .$$

On the other hand A_i^k is the kernel of the map $\bar\partial = (\partial/\partial\bar\zeta_1,\ldots,\partial/\partial\bar\zeta_n)$

$$\bar\partial : S^k \longrightarrow (S^k)^n \ ,$$

so that according to [15] prop. 35.4 $(A_i^k)^0$ is the weak* closure in $(S^k)'_\sigma$ of $\operatorname{Im}\bar\partial^t$. Since S^k is reflective (it is an $F\bar{S}$-space), the weak* closure of $\operatorname{Im}\bar\partial^t$ is equal to the closure in the strong topology in $(S^k)'$, [15] prop. 35.2. We denote the closure in $(S^k)'$ of the range of the map

$$\bar\partial^t : ((S^k)')^n \longrightarrow (S^k)'$$

by $\overline{\operatorname{Im}\bar\partial_k^t}$. So we get

$$(A_i^k)'_\sigma \cong (S^k)'_\sigma \Big/ \overline{\operatorname{Im}\bar\partial_k^t} \quad .$$

Finally we will obtain isomorphisms also for the strong topologies.

Therefore, we consider this spaces only with the topology of weakly* converging sequences denoted by $(A_i^k)'_{\sigma,s}$ and $(S^k)'_{\sigma,s}$. Since S^k is a Montel space we get

$$(S^k)'_{\sigma,s} = (S^k)'_{b,s}$$

where $(S^k)'_b$ is the dual of S^k provided with the strong topology. According to [14] theorem 9.1 ((9.6) and page 61 5^{th} line from below) the Fourier transformation F maps $(S^{k+1})'_b$ continuously into

$$H^k = \operatorname*{ind\,lim}_{m\to\infty} H^{m*}(a+1/(k+1);C_k,k)$$

with

$$H^{m*}(a+1/k;C_k,k) \stackrel{\text{def}}{=\!=} A_\infty((1+\|\nu\|)^{-m}\exp(-\tilde{a}(\mathrm{Im}\ \nu)-1/\|\mathrm{Im}\ \nu\|);\ \mathbb{R}^{2n}+i\,C(k))$$

and F^{-1} maps H^k continuously into $S^{p*}(a+1/(k+1);C_k)'$, when $1/k < \delta_p/p$, hence into $(S^p)_b'$.

Let $W = (w_1,\ldots,w_n)$, where $w_j = i\nu_j^1-\nu_j^2$, $j = 1,\ldots,n$ and let $W\cdot\vec{H}^k$ be the subspace of H^k consisting of functions $f(\nu)$ that can be written as

$$f(\nu) = \sum_{j=1}^{n} w_j\, g_j(\nu)$$

with $g_j \in H^k$, $j = 1,\ldots,n$. Then

$$F\ \overline{\mathrm{Im}\ \bar{\partial}_{k+1}^t} \subset \overline{W\cdot\vec{H}^k} \quad \text{and} \quad F^{-1}\ W\cdot\vec{H}^{k-1} \subset \overline{\mathrm{Im}\ \bar{\partial}_p^t}$$

when $1/(k-1) < \delta_p/p$. Furthermore

$$\overline{W\cdot\vec{H}^k} \subset W\cdot\vec{H}^{k-1}\ ,$$

for, let $f_\alpha \in W\cdot\vec{H}^k$ be a Cauchy net converging to $f \in H^k$. Then $f_\alpha = W\cdot\vec{g}_\alpha$ with $\vec{g}_\alpha \in (H^k)^n$, so that f_α and hence f vanishes on

$$V_k = \{\mathbb{R}^{2n}+i\,C(k)\} \cap \{\nu \mid i\nu_j^1-\nu_j^2 = 0,\ j = 1,\ldots,n\}.$$

The inclusion follows if we have shown

<u>PROBLEM 6.1</u>. *A function* $f \in H^k$ *vanishing on* V_k *can be written as*

$$f(\nu) = W\cdot\vec{g}^{k-1}(\nu), \qquad \nu \in \mathbb{R}^{2n}+i\,C(k-1)$$

with $\vec{g}^{k-1} \in (H^{k-1})^n$*; in particular there is a positive* N *such that* $\vec{g}^{k-1} \in H^{m+N*}(a+1/k;C_{k-1},k-1)$*, when* $f \in H^{m*}(a+1/k;C_k,k)$.

In that case we have

$$\operatorname*{proj\,lim}_{k\to\infty} (A_i^k)'_{\sigma,s} \cong \operatorname*{proj\,lim}_{k\to\infty} (S^k)'_{b,s} \Big/ \operatorname{Im} \bar{\partial}_k^t \overset{F}{\hookrightarrow\!\!\!\to} \operatorname*{proj\,lim}_{k\to\infty} (H^k)_s \Big/ W\cdot\vec{H}^k ,$$

where $(H^k)_s$ means that H^k is provided with the topology of convergent sequences.

Furthermore weakly* converging sequences in $(A_i^{k+1})'$ converge strongly in $(A_i^k)'$ because A_i^k is an LS-space, so that

$$\operatorname*{proj\,lim}_{k\to\infty} (A_i^k)'_{\sigma,s} = \operatorname*{proj\,lim}_{k\to\infty} (A_i^k)'_{b,s} = (A_i)'_{b,s} = A_i',$$

where the last equality follows from the fact that the topology in the metric space (namely the $F\bar{S}$-space) A_i' is determined by convergent sequences. Thus F is an isomorphism between

$$(6.2) \qquad F\colon A_i' \hookrightarrow\!\!\!\to \operatorname*{proj\,lim}_{k\to\infty} (H^k)_s \Big/ W\cdot\vec{H}^k .$$

In case (ii) when Ω_k is defined by (5.5)(ii) we define S-spaces with L^2-norms rather than with sup-norms. For, a continuous map from one Hilbert space into another is weakly compact, so that projective and inductive limits of sequences of Hilbert spaces (called FS*-spaces and DFS*-spaces, respectively) are reflexive and they are dual to each other (see [19]). Also we apply th. 15 of [19], where the isomorphism holds for the strong topologies and not only for the weak* topologies as in [15] prop. 35.6. Hence we do not have to restrict ourselves to the topology of convergent sequences (this also applies to case (i)).

Let

$$S_\ell^m = \operatorname*{ind\,lim}_{k\to\infty} W_2^m((1+\|\zeta\|)^m \exp \delta_k\ 1/k\|\zeta\|;\Omega_\ell)$$

and let A_ℓ^m be the closed linear subspace $S_\ell^m \cap A(\Omega_\ell)$. In virtue of [19] th. 7 the topology of $\operatorname*{ind\,lim}_{k\to\infty} A_\ell^m(k)$ with

$$A_\ell^m(k) \overset{\text{def}}{=\!=} A(\Omega_\ell) \cap W_2^m((1+\|\zeta\|)^m \exp \delta_k\ 1/k\|\zeta\|;\Omega_\ell)$$

if finer than the topology in A_ℓ^m induced by S_ℓ^m, but one easily sees that it is also less fine, hence A_ℓ^m is a DFS*-space. Since $\bar{A}_{ii}$ can also be described by L^2-norms ([14] G.7) we get

$$\bar{A}_{ii} = \operatorname{ind}\lim_{\ell\to\infty} A_\ell,$$

where $A_\ell = \operatorname{proj}\lim_{m\to\infty} A_\ell^m$. Although $\bar{A}_{ii}$ is an LS-space, this fact is not expressed by the above inductive limit, which is only a weakly compact sequence. Indeed, a neighborhood of zero in A_ℓ is bounded in $A_{\ell+1}$, hence relatively weakly compact in each $A_{\ell+1}^m$ $(m = 0,1,2,\ldots)$, since A_ℓ^m is reflexive ([15] prop. 36.3), and thus relatively weakly compact in $A_{\ell+1}$. Therefore, $(A_{ii})_0' = \operatorname{proj}\lim_{\ell\to\infty} (A_\ell)_b'$. However, the projective limit in A_ℓ has no nice properties, so we are forced to consider the weak* topology in $(A_{ii})'$. The topology of $(A_{ii})'$ is also determined by weakly* converging sequences, hence $(A_{ii})_b' = (A_{ii})_{\sigma,s}' = \operatorname{proj}\lim_{\ell\to\infty} (A_\ell)_{\sigma,s}'$. Any weakly* converging sequence in $(A_\ell)_{\sigma,s}'$ converges weakly* in $(A_\ell^m)_{\sigma,s}'$ for some m and thus it converges weakly* in each $A_\ell^m(k)$, $k = 1,2,\ldots$. Since the embedding map from $A_\ell^m(k)$ into $A_{\ell+1}^m(k+1)$ is compact according to [14] G.7, the sequence converges strongly in $(A_{\ell-1}^m)'$ ([14] E.2), thus in $\operatorname{ind}\lim_{m\to\infty} (A_{\ell-1}^m)_{b,s}'$. Since $(A_\ell^m)'$ is a Fréchet space, namely an FS*-space, $(A_{\ell-1}^m)_{b,s}' = (A_{\ell-1}^m)_b'$, so

$$(A_{ii})_b' = \operatorname{proj}\lim_{\ell\to\infty}\ \operatorname{ind}\lim_{m\to\infty} (A_\ell^m)_b' .$$

Now A_ℓ^{m+1} is the kernel of the continuous map

$$\bar{\partial}_m : S_\ell^{m+1} \to (S_\ell^m)^n.$$

In virtue of [19] th.15 and [15] prop.35.4 we get

$$(A_\ell^{m+1})' \cong (S_\ell^{m+1})' \Big/ \overline{\operatorname{Im} \bar{\partial}_m^t},$$

where the closure in $(S_\ell^{m+1})'$ of $\operatorname{Im} \bar{\partial}_m^t$ equals the weak* closure, since S_ℓ^{m+1} is reflexive.

Let

$$H_\ell^m = \text{proj}\lim_{k\to\infty} H^{m*}(a+1/\ell;C_k,k)$$

and let

$$H_\ell^* = \text{ind}\lim_{m\to\infty} H_\ell^m .$$

Then it follows from [14] th.9.1 (using D.2 instead of G.5) and G.3 that

$$\mathcal{F}(S_\ell^m)' \subset H_\ell^{m+n+1} \quad \text{and} \quad \mathcal{F}^{-1}\, H_{\ell+1}^m \subset (S_\ell^{m+2n+2})' .$$

As in case (i) problem 6.1 and the following problem imply that

$$\overline{W\cdot\vec{H}_\ell^m} \subset \bigcap_{k=1}^{\infty} \overline{W\cdot\vec{H}^{m*}}(a+1/\ell;C_{k+1},k+1) \subset \bigcap_{k=1}^{\infty} W\cdot\vec{H}^{m+1+N*}(a+1/\ell;C_k,k) \subset$$

$$\subset W\cdot\vec{H}_\ell^M ,$$

where the closures are taken in the corresponding spaces with m+1 instead of m and where $M > m+1+N$.

<u>PROBLEM 6.2</u>. *When a function* $f \in A(\mathbb{R}^{2n}+i\,C)$ *in each* $\mathbb{R}^{2n}+i\,C(k)$ *can be written as*

$$f = W\cdot\vec{g}_k \quad \textit{for some } \vec{g}_k \in (H^{m*}(a+1/\ell;C_k,k))^n,$$

(obviously in that case $f \in H_\ell^*$*), then* f *can be written in* $\mathbb{R}^{2n}+i\,C$ *as*

$$f = W\cdot\vec{g} \quad \textit{with } \vec{g} \in (H_\ell^*)^n;$$

in particular $\vec{g} \in (H_\ell^{m+N})^n$ *for some positive* N *independent of* f.

Hence $H_\ell^{m+1}/\overline{W\cdot\vec{H}_\ell^m} \longrightarrow H_\ell^M/W\cdot\vec{H}_\ell^M \longrightarrow H_\ell^{M+1}/\overline{W\cdot\vec{H}_\ell^M}$ and thus $\mathcal{F}$ is an isomorphism between

(6.3) $$\mathcal{F}\colon A_{ii}^! \hookrightarrow\!\!\!\!\rightarrow \text{proj}\lim_{\ell\to\infty}\ \text{ind}\lim_{m\to\infty} H_\ell^m \Big/ W\cdot\vec{H}_\ell^m .$$

Let $H^*_{ii} = \text{proj}\lim_{\ell\to\infty} H^*_\ell$ and let $H^*_i = \text{proj}\lim_{k\to\infty} H^k$. Then the right hand side in (6.2) is equal to the equivalence classes of elements in H^*_i, when two elements in H^*_i are equivalent if their difference f can be written in each $\mathbb{R}^{2n}+i\,C(k)$ as $f = W\cdot\vec{g}_k$ with $\vec{g}_k \in (H^k)^n$. The right hand side in (6.3) is equal to the equivalence classes in H^*_{ii}, when two elements in H^*_{ii} are equivalent of their difference f for each ℓ can be written in $\mathbb{R}^{2n}+i\,C$ as $f = W\cdot\vec{g}_\ell$ with $\vec{g}_\ell \in (H^*_\ell)^n$.

Next we consider the set in $\mathbb{R}^{2n}+i\,C$ where $W = (w_1,\ldots,w_n)$ vanishes, namely

$$V = \{\nu \mid \nu \in \mathbb{R}^{2n}+i\,C,\ i\nu^1_j-\nu^2_j=0,\ j=1,\ldots,n\}$$

and

$$V_k = V \cap \{\mathbb{R}^{2n}+i\,C(k)\}.$$

Since $W\cdot\vec{H}^k$ vanishes on V_k and $W\cdot\vec{H}^*_\ell$ vanishes on V, we can define the continuous restriction maps I

$$I_i:\ H^k\big/_{W\cdot\vec{H}^k} \longrightarrow\!\!\!\!\rightarrow H^k\Big|_{V_k}$$

and

$$I_{ii}:\ H^*_\ell\big/_{W\cdot\vec{H}^*_\ell} \longrightarrow\!\!\!\!\rightarrow H^*_\ell\Big|_{V}$$

by $If(\nu) = f(\nu^1,i\nu^1)$. Here $H^k|_{V_k}$ and $H^*_\ell|_V$ are the spaces of restrictions of functions in H^k or H^*_ℓ to V_k or V with the topology induced by H^k or H^*_ℓ, respectively. Then the maps I are surjective. When we regard $z = \nu^1$ as the variable in V there are natural continuous injections J

$$J_i:\ H^k\Big|_{V_k} \hookrightarrow A_\infty(\exp(-\tilde{a}(z)-1/k\|z\|);C(k))$$

and

$$J_{ii}:\ H^*_\ell\Big|_{V} \hookrightarrow \text{proj}\lim_{k\to\infty}\ A_\infty(\exp(-\tilde{a}(z)-1/(\ell-1)\|z\|);C(k)).$$

Now the topologies of

$$\operatorname*{proj\,lim}_{k\to\infty} H^k\Big|_{V_k} = H_i^*\big|_V \quad\text{and}\quad \operatorname*{proj\,lim}_{\ell\to\infty} H_\ell^*\big|_V = H_{ii}^*\big|_V$$

become extremely simple, as they both coincide with the one induced by Exp(a+0;C), compare (2.14), thus $H_i^*|_V = H_{ii}^*|_V$ is an $F\bar{S}$-space. Finally we have obtained

$$(6.4)\quad A_i^! \overset{F}{\hookrightarrow} \operatorname*{proj\,lim}_{k\to\infty} H^k \Big/ W\cdot\vec{H}^k \overset{I_i}{\twoheadrightarrow} \operatorname*{proj\,lim}_{k\to\infty} H^k\Big|_{V_k} \overset{J_i}{\hookrightarrow} \mathrm{Exp}(a+0;C)$$

and

$$(6.5)\quad A_{ii}^! \overset{F}{\hookrightarrow} \operatorname*{proj\,lim}_{\ell\to\infty}\ \operatorname*{ind\,lim}_{m\to\infty} H_\ell^m \Big/ W\cdot\vec{H}_\ell^m \overset{I_{ii}}{\twoheadrightarrow} \operatorname*{proj\,lim}_{\ell\to\infty}\ \operatorname*{ind\,lim}_{m\to\infty} H_\ell^m\Big|_V$$

$$\overset{J_{ii}}{\hookrightarrow} \mathrm{Exp}(a+0;C).$$

In theorem 5.1 we have proved that the map $J_i \circ I_i$ is surjective, hence J_i is surjective. Remark 5.1 is concerned with the question whether $J_{ii} \circ I_{ii}$ is surjective. Using the proof of theorem 5.1 we see that indeed $J_{ii} \circ I_{ii}$ and hence J_{ii} are surjective, if the following problem is solved.

<u>PROBLEM 6.3</u>. *Let the function* $f_k \in H^{m*}(a+1/\ell;C_k,k)$ *satisfy for all* $k = 1,2,\ldots$

$$(f_{k+1}-f_k)\Big|_{V_k} = 0.$$

Then there exists a function $f \in H_\ell^*$ *with for all* $k = 1,2,\ldots$

$$(f-f_k)\Big|_{V_k} = 0.$$

For, in that case (5.20) and (5.21) can be satisfied.

Problem 6.1 says that the map I_i is injective. Problem 6.1 as well as problem 6.2 follow from problem 6.4, which says that the map I_{ii} is injective.

PROBLEM 6.4. *A function* $f \in H^*_\ell$ *vanishing on* V *can be written in* $\mathbb{R}^{2n}+iC$ *as*

$$f = W\cdot\vec{g} \quad \textit{with} \quad \vec{g} \in (H^*_\ell)^n.$$

The next step is to investigate holomorphic functions vanishing on the set V, but before doing this we give an intuitive interpretation of the isomorphisms (6.4) and (6.5) in terms of the last section revealing the a priori difference between the spaces (5.11) and (5.10) in terms of this section. In section 5 we have shown that Exp is isomorphic to the dual of the closure (given in (5.11) and here denoted by $\tilde{A}_i$) of the linear hull of $\{e^{iz\cdot\zeta}\}_{z\in C}$ in $\bar{A}_i$ and in this section Exp is isomorphic to $H^*_i|_V$. Hence $(H^*_i|_V)'$ is isomorphic to $\tilde{A}_i$ and problem 6.3 implies the same for $\tilde{A}_{ii}$. Indeed, let us examine what elements of $(H^*)'$ yield $\tilde{A}$ or $\bar{A}$ under F^t defined in (5.6) (we do not distinguish between cases (i) and (ii) here). Let $\phi \in \tilde{A}$, thus

$$\phi(\zeta) = \sum_{k=1}^{\infty} c_k\, e^{iz_k\cdot\zeta}$$

with $z_k \in C$ and with some constants c_k. If for some $\sigma \in (H^*)'$

$$<\sigma_\nu, e^{i\nu^1\cdot\xi+i\nu^2\cdot\eta}> = \phi(\zeta)$$

then

$$\sigma_\nu = \sum_{k=1}^{\infty} c_k\, \delta(\nu^1-z_k)\delta(\nu^2-iz_k) = \sum_{k=1}^{\infty} c_k\, \delta(\nu^1-z_k)\delta(\nu^2-i\nu^1) =$$
$$= \delta(i\nu^1-\nu^2) \sum_{k=1}^{\infty} c_k\, \delta(\nu^1-z_k) ,$$

thus σ acts on the restrictions of functions in H^* to V, that is $\sigma \in (H^*|_V)'$. Now consider an element $\phi \in \bar{A}$. If for some $\sigma \in (H^*)'$

$$<\sigma_\nu, e^{i\nu^1\cdot\xi+i\nu^2\cdot\eta}> = \phi(\zeta) ,$$

then we only know that

$$<\sigma_\nu, w_j e^{i\nu^1\cdot\xi+i\nu^2\cdot\eta}> = 0, \qquad j = 1,\dots,n ,$$

since $\bar{\partial}_j\phi = 0$. The exponentials are dense in H^*, so that $\langle\sigma, W\cdot\vec{H}^*\rangle = 0$, thus

$$\sigma \in \left(H^* \big/ W\cdot\vec{H}^*\right)' ,$$

see [15] prop.35.5(b). When we have shown that the map I is injective (problem 6.4), the spaces $\tilde{A}$ (5.11) and $\bar{A}$ (5.10) coincide and we obtain a theorem similar to theorem 2.1.

V is defined as the simultaneous zero-set of the polynomials $w_1 = i\nu_1^1 - \nu_1^2, \ldots, w_n = i\nu_n^1 - \nu_n^2$. These polynomials generate a prime ideal in any point of a pseudoconvex set $\Omega \subset \mathbb{C}^{2n}$. Therefore, according to Hilbert's Nullstellensatz all holomorphic functions f in Ω vanishing on V can locally, that is in a neighborhood ω of any point in Ω, be written as

$$(6.6) \qquad f = W\cdot\vec{g}_\omega \quad \text{with } \vec{g}_\omega \in A(\omega)^n ,$$

see appendix (A.18). With the aid of Cartan's theorem B (theorem A 14) it is shown in the appendix that $f \in A(\Omega)$ satisfying (6.6), satisfies (6.6) globally, that is f can be written as

$$f = W\cdot\vec{g} \quad \text{with } \vec{g} \in A(\Omega)^n .$$

Problem 6.4 asks for functions $g \in H_\ell^*$, so it is the analogue with estimates of the problem treated in the appendix. By (6.6) we can reformulate problem 6.4:

<u>PROBLEM 6.5</u>. *If* $f \in H_\ell^*$ *can locally (that is in a neighborhood* ω *of any point in* T^C*) be written as*

$$f = W\cdot\vec{g}_\omega, \quad \vec{g}_\omega \in A(\omega)^n ,$$

then there exists $\vec{g} \in (H_\ell^*)^n$ *with* $f = W\cdot\vec{g}$.

In the next section we will solve this problem for general polynomial systems P instead of W. Also in that case, a set V can be so defined (see EHRENPREIS [3]) that a function f vanishing on V can locally be written as

$f = P\cdot\vec{g}$, see theorem A 17. Provided that J is surjective, the isomorphism I in (6.4) and (6.5) is the analogue with estimates of the isomorphism (A 19).

Using (6.6) and the above mentioned problem of the appendix (theorem A 15) we can reformulate problem 6.3:

PROBLEM 6.6. *Let the functions* $f_k \in H^{m*}(a+1/\ell;C_k,k)$ *satisfy for all* $k = 1,2,\ldots f_{k+1}-f_k = W\cdot\vec{g}_k$ *in* $\mathbb{R}^{2n}+i\,C(k)$, $\vec{g}_k \in A(\mathbb{R}^{2n}+i\,C(k))^n$, *then there exists a function* $f \in H^*_\ell$ *with for all* $k = 1,2,\ldots$ $f-f_k = W\cdot\vec{\tilde{g}}_k$ *in* $\mathbb{R}^{2n}+i\,C(k)$ *for some* $\vec{\tilde{g}}_k \in A(\mathbb{R}^{2n}+i\,C(k))^n$.

Also this problem will be solved in the next section for general polynomial systems P instead of W. Therefore, J_{ii} is surjective and we have proved the main theorem of sections 5, 6 and 7, namely

THEOREM 6.1. *Let* a *be a convex function on* pr C *for some open convex cone* C *in* $\mathbb{C}^n$ *and let* Ω *and* Ω_k *be the closed convex sets in* $\mathbb{C}^n$ *determined by* (5.4) *and* (5.5)(ii), *respectively. Then the map* F *from* A', *the dual of the space* $\bar{A}$ (6.1), *into* Exp(a+0;C) *given by*

$$F(\mu)(z) = \langle\mu_\zeta, e^{iz\cdot\zeta}\rangle, \qquad \mu \in A'$$

is an isomorphism.

We have also shown that Exp is isomorphic to A'_i, hence $\bar{A}_i = \bar{A}_{ii}$. Taking into account theorem 5.1 we can conclude that the linear hull of the set $\{e^{iz\cdot\zeta}\}_{z\in C}$ is dense in $\bar{A}$ (6.1) in both cases (i) and (ii).

Theorem 6.1 is a generalization to non-entire functions of the theorem of EHRENPREIS [2] and MARTINEAU [12] of section 2 which deals with entire functions as Fourier transforms. A particular case of this theorem with (5.5)(i) instead of (5.5)(ii) has already been proved by KAWAI in [9].

7. COMPLETION OF THE PROOFS

In this section we solve problems 6.5 and 6.6. For that purpose cohomology with bounds is introduced. The solution requires estimates in the steps of the proof of a similar statement without bounds in the appendix. We formulate the theorems in a more general way making them useful in other applications too.

Let Ω be an open pseudoconvex set in $\mathbb{C}^n$ such that there is an increasing sequence of open pseudoconvex subsets Ω_k with union Ω and with

$$\forall k,\ \exists \varepsilon = \varepsilon(k):\quad \Omega_k(\varepsilon) \subset \Omega_{k+1}$$

where $\Omega_k(\varepsilon)$ is the ε-neighborhood of Ω_k. Moreover, in some theorems we require that there is a continuous plurisubharmonic function σ in Ω with

$$(7.1) \qquad \Omega_k = \{z \mid \sigma(z) < k\}.$$

This is only a special condition on Ω ([7], th.2.6.7.ii), if the sets Ω_k are unbounded.

For example, we may take for Ω_k suitable ε-neighborhoods of each other, since the function $d(z) = -\log \delta(z,\Omega^c)$ (here $\delta(z,\Omega^c)$ is the distance between $z \in \Omega$ and the complement of Ω) is plurisubharmonic when Ω is pseudoconvex. We show that in some sense also the sets $\mathbb{R}^{2n}+i\,C(k) \subset \mathbb{C}^{2n}$ of the last section are an example. Therefore, we say that two increasing sequences $\{\Omega_k\}_{k=1}^{\infty}$ and $\{\Omega'_k\}_{k=1}^{\infty}$ exhausting Ω are equivalent if for every k there is an ℓ with $\Omega_k \subset \Omega'_\ell$ and $\Omega'_k \subset \Omega_\ell$. Then it is clear that any function on Ω that is bounded in some norm on all subsets Ω_k is also bounded on the subsets Ω'_k and conversely.

<u>LEMMA 7.1</u>. *The increasing sequence* $\{\mathbb{R}^{2n}+i\,C(k)\}_{k=1}^{\infty}$ *exhausting* $\Omega = \mathbb{R}^{2n}+ iC \subset \subset \mathbb{C}^{2n}$ *is equivalent to an increasing sequence* $\{\Omega_k\}_{k=1}^{\infty}$ *satisfying* (7.1).

<u>PROOF</u>. Choose a vector α in $C \subset \mathbb{R}^{2n}$ and a number $c > 1$ and consider the hyperplane $H = \{y \mid \alpha\cdot y = c\} \subset \mathbb{R}^{2n}$. Let for each $y \in C$

$$y^* = \frac{c}{\alpha\cdot y}\, y$$

be the intersection of the vector y with H. We define a plurisubharmonic (even convex) function χ in C by $\chi(y) = d(y^*) = -\log \delta(y^*,C^c)$. Then the sets $C'_k = \{y \mid \chi(y) < k\}$, $k = 1,2,\ldots$, are relatively compact subcones of C exhausting C. Now we set for $z = x+iy$

$$\sigma(z) = \max(d(y),\chi(y)),$$

which is plurisubharmonic (even convex) and we have

$$\Omega_k = \{z \mid \sigma(z) < k\} = \{\mathbb{R}^{2n}+i\,C_k'\} \cap \{z \in \Omega \mid \delta(z,\Omega^c) > e^{-k}\}$$

Hence the sequence $\{\Omega_k\}_{k=1}^{\infty}$ is obviously equivalent to $\{\mathbb{R}^{2n}+i\,C(k)\}_{k=1}^{\infty}$. □

Let ϕ be a plurisubharmonic function in Ω. In some theorems ϕ will be such that for every $z \in \Omega_k$ and $\|z'-z\| < \varepsilon(k)$

$$\phi(z')-\phi(z) \le K_k, \tag{7.2}$$

where the constant K_k does not depend on z and z', but may depend on k. For example the function $m \log(1+\|z\|^2) + 2\tilde{a}(y) + 2/\ell\|y\|$ is plurisubharmonic in $\mathbb{R}^{2n}+i\,C$ and satisfies (7.2) for every sequence $\{\Omega_k\}_{k=1}^{\infty}$ equivalent to $\{\mathbb{R}^{2n}+i\,C(k)\}_{k=1}^{\infty}$. Finally let $P = (P_{jk})$, $j = 1,\dots,p$, $k = 1,\dots,q$, be a matrix of polynomials.

Then problems 6.5 and 6.6 follow from lemma 7.1 and the next two theorems, theorem 7.1(a) and theorem 7.2, respectively. In theorem 7.1 we formulate a part (b) with uniform bounds, which we do not need here, but which may be useful in other purposes. Part (b) is derived in the same way as part (a).

THEOREM 7.1. *If* $f \in A(\Omega)^p$ *can locally (that is in a neighborhood* ω *of any point in* Ω*) be written as*

$$f = Pg_\omega, \quad g_\omega \in A(\omega)^q ,$$

then there is a number N, *such that*

(a) *there is a function* $v \in A(\Omega)^q$ *with* $f = Pv$ *and with*

$$\int_{\Omega_k} |v(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) < \infty, \quad k = 1,2,\dots ,$$

when $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *satisfies* (7.1) *and* ϕ *is a plurisubharmonic function in*

Ω such that

$$\int_{\Omega_k} |f(z)|^2 \exp-\phi(z)\, d\lambda(z) < \infty, \qquad k = 1,2,\dots .$$

(b) *For all* $k = 1,2,\dots$ *there are constants* K_k*, integers* $\ell_k \geq k$ *and functions* $v_k \in A(\Omega_k)^q$ *with* $f = Pv_k$ *in* Ω_k *and with*

$$\int_{\Omega_k} |v_k(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) \leq K_k \int_{\Omega_{\ell_k}} |f(z)|^2 \exp-\phi(z)\, d\lambda(z),$$

when the right hand side is finite for some plurisubharmonic function ϕ.

In part (b) the pseudoconvex subsets Ω_k of Ω do not have to satisfy (7.1).

<u>THEOREM 7.2</u>. *If* $f_k \in A(\Omega_k)^p$, $k = 1,2,\dots,$ *are functions with* $f_{k+1}-f_k = Pg_k$ *in* Ω_k, $g_k \in A(\Omega_k)^q$*, then there are a number* N *and a function* $f \in A(\Omega)^p$ *with* $f-f_k = P\tilde{g}_k$ *in* Ω_k, $\tilde{g}_k \in A(\Omega_k)^q$*, and with*

$$\int_{\Omega_k} |f(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) < \infty, \qquad k = 1,2,\dots ,$$

when $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *satisfies* (7.1) *and* ϕ *is a plurisubharmonic function satisfying* (7.2) *such that*

$$\int_{\Omega_k} |f_k(z)|^2 \exp-\phi(z)\, d\lambda(z) < \infty, \qquad k = 1,2,\dots .$$

Here $|f(z)|^2$ means $|f_1(z)|^2 + \dots + |f_p(z)|^2$ when $f = (f_1,\dots,f_p) \in A(\Omega)^p$ and $\lambda(z)$ denotes the Lebesgue measure in $\mathbb{C}^n$.

First we need similar theorems as theorem A13 and Castan's theorem B, theorem A 14, but now with estimates. Let $\mathcal{U}^{(\lambda)} = \{U_i^{(\lambda)}\}_{i\in I_\lambda}$, $\lambda = 0,1,2,\dots$ be the coverings of Ω given in the appendix section V satisfying properties (A15)(i),(ii),(iii),(iv),(v) and (vi) and let for every k $\mathcal{U}_k^{(\lambda)} = \{U_i^{(\lambda)} \cap \Omega_k\}_{i\in I_\lambda}$ be the corresponding coverings of Ω_k. When $\mathcal{F}$ is an analytic sheaf on Ω, we denote by $C^p[\mathcal{U}^{(\lambda)},\mathcal{F},\phi]$ the set of alternating cochains $c = \{c_s\}$ in Ω, $s \in I_\lambda^{p+1}$, $c_s \in \Gamma(U_s^{(\lambda)},\mathcal{F})$, satisfying for all k

$$\|c\|^2_{\phi,k} \overset{\text{def}}{=\!=} \sum_{|s|=p+1} \int_{U_s^{(\lambda)}\cap\Omega_k} |c_s(z)|^2 \exp-\phi(z)\, d\lambda(z) < \infty$$

and by $C^p(\mathcal{U}_k^{(\lambda)},\mathcal{F},\phi)$ the set of all alternating cochains c in Ω_k with $\|c\|^2_{\phi,k} < \infty$. By ϕ_N we will mean the plurisubharmonic function $\phi_N(z) = \phi(z) + N\log(1+\|z\|^2)$.

Lemma 7.2 will be obtained in the same way as theorem A 13, only we write down explicitely the construction of the map δ^* (A10), so that we can bring estimates in the statements involving δ^*. We do not work with the sheaf $\mathcal{E}$ of germs of C^∞-functions, but rather with the sheaf $\mathcal{L}$ of germs of locally square integrable functions. Then we may use theorem A10 instead of theorem A 9. So let $\mathcal{L}_q$ be the sheaf of germs of (0,q)-forms with locally square integrable coefficients and let $\mathcal{Z}_q$ be the subsheaf of $\bar\partial$-closed forms of type (0,q). Again we have a part (a) with globally defined functions on Ω and a part (b) with functions in Ω_k and uniform bounds.

LEMMA 7.2.

(a) *Every cochain* c *in* $C^p[\mathcal{U}^{(\lambda)},\mathcal{A},\phi]$, $p \geq 1$, *with* $\delta c = 0$ *can be written as* $c = \delta c'$ *for a* $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},\mathcal{A},\phi_{2m}]$, *where* $m = \min(p,n)$, *when* $\{\Omega_k\}_{k=1}^\infty$ *satisfies* (7.1).

(b) *For all* k *every cochain* c *in* $C^p[\mathcal{U}^{(\lambda)},\mathcal{A},\phi]$, $p \geq 1$, *with* $\delta c = 0$ *can be written in* Ω_k *as* $c = \delta c_k'$ *for a* $c_k' \in C^{p-1}(\mathcal{U}_k^{(\lambda)},\mathcal{A},\phi_{2m})$ *such that for some constants* K_k

$$\|c_k'\|_{\phi_{2m},k} \leq K_k \|c\|_{\phi,k},$$

where $m = \min(p,n)$. *Also for fixed* k *every cocycle* $c \in C^p(\mathcal{U}_k^{(\lambda)},\mathcal{A},\phi)$ *satisfies the above property* (b) *for this* k.

PROOF. A section $c \in \Gamma(\Omega,\mathcal{L}_0)$ with $\bar\partial c = 0$ determines a holomorphic function $c \in A(\Omega)$ (this follows by repeated use of lemma 4.2.4 in [7]). For $c \in C^p[\mathcal{U}^{(\lambda)},\mathcal{Z}_q,\phi]$ with $\delta c = 0$ we want to find a $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},\mathcal{Z}_q,\phi_{2m}]$ such that $\delta c' = c$, when $q = 0$ or in part (b) cochains $c_k' \in C^{p-1}(\mathcal{U}_k^{(\lambda)},\mathcal{Z}_q,\phi_{2m})$ such that $\delta c_k' = c$ in Ω_k. We assume that this has already been proved for smaller values of p and all q, when $p > 1$, $m = p$ and when the constants K_k in part (b) depend on p.

First we give estimates in the construction of g in theorem A12. For each k we choose $\ell = \ell(k)$ such that, when $U_i^{(\lambda)} \cap \Omega_k \neq \emptyset$, $U_i^{(\lambda)} \subset \Omega_\ell$ according to property (A15)(iii). Since also all sets in $\mathcal{U}^{(\lambda)}$ contained in Ω_k have a minimum size (say they contain a ball with radius $\varepsilon_k(\lambda)$), we can construct a partition $\{\phi_\nu\}_{\nu=1}^{\infty}$ of unity subordinate to the covering $\mathcal{U}^{(\lambda)}$ of Ω (ϕ_ν has its support in $U_{i_\nu}^{(\lambda)}$), such that for all k

$$(7.3) \qquad \max_z |\bar{\partial}\phi_\nu(z)|^2 \le C_k$$

for those ν with $U_{i_\nu}^{(\lambda)} \cap \Omega_k \neq \emptyset$. Here

$$|\bar{\partial}\phi(z)|^2 = \sum_{j=1}^{n} |\bar{\partial}_j\phi(z)|^2.$$

For example let for each $\nu \in I_{\lambda+1}$ χ_ν be a C^∞-function equal to one in $U_\nu^{(\lambda+1)}$ and to zero outside the $\varepsilon_{\ell(k_\nu)}(\lambda+1)$-neighborhood of $U_\nu^{(\lambda+1)}$ (which

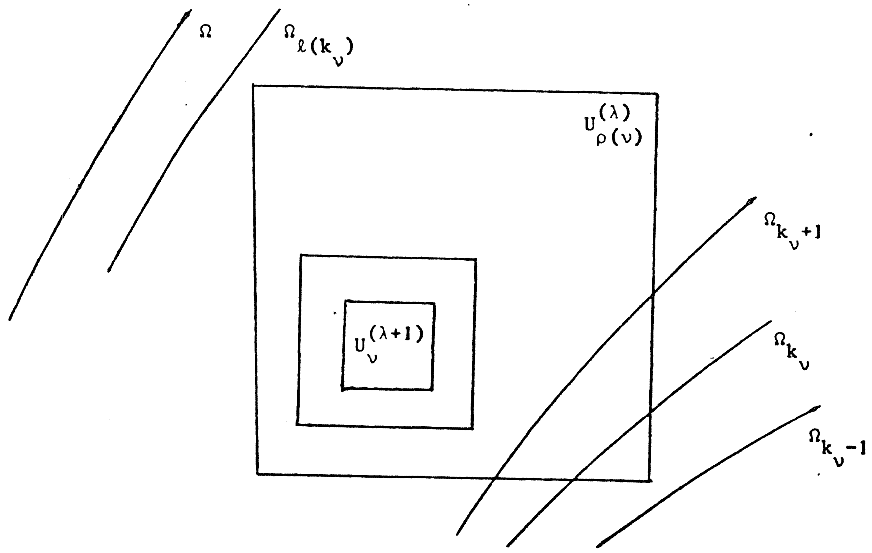


figure 7.1.

certainly is contained in $U_{\rho(\nu)}^{(\lambda)}$ with $\rho = \rho_{\lambda,\lambda+1}$, because of property (A15)(v)), where k_ν is the smallest integer k with $U_{\rho(\nu)}^{(\lambda)} \cap \Omega_k \neq \emptyset$, see figure 7.1. Then for those ν with $U_{\rho(\nu)}^{(\lambda)} \cap \Omega_k \neq \emptyset$ $\max_z |\bar{\partial}\chi_\nu(z)|$ depends on $\varepsilon_{\ell(1)}(\lambda+1),\ldots,\varepsilon_{\ell(k)}(\lambda+1)$. Since $\mathcal{U}^{(\lambda+1)}$ is a covering of Ω,

$$\phi_\nu(z) = \frac{\chi_\nu(z)}{\sum_{\mu=1}^{\infty} \chi_\mu(z)}\,, \qquad \nu = 1,2,\ldots$$

is a partition of unity subordinate to the covering $\mathcal{U}^{(\lambda)}$ with $i_\nu = \rho(\nu)$, that satisfies (7.3). Note that for each z not more than M terms in the denominator differ from zero because of property (A15)(ii).

As in the proof of theorem A12 we set for $s \in I^p_\lambda$

$$g_s = \sum_\nu \phi_\nu \, c_{i_\nu s}\,,$$

when $c \in C^p[\mathcal{U}^{(\lambda)}, Z_q, \phi]$. Then as in theorem A12 $g \in C^{p-1}(\mathcal{U}^{(\lambda)}, L_q)$ and $\delta g = c$, if $\delta c = 0$. Furthermore writing $\phi_\nu = \sqrt{\phi_\nu}\cdot\sqrt{\phi_\nu}$ and using $\sum_\nu \phi_\nu = 1$ we find

$$\int_{U_s^{(\lambda)} \cap \Omega_k} |g_s(z)|^2 \exp{-\psi(z)}\, d\lambda(z) \overset{\text{not}}{=\!=} \|g_s\|^2_{\psi,k} \le$$

$$\le \sum_\nu \int_{U_s^{(\lambda)} \cap \Omega_k} \phi_\nu(z)\, |c_{i_\nu s}(z)|^2 \exp{-\psi(z)}\, d\lambda(z) \le$$

$$\le \sum_\nu \|c_{i_\nu s}\|^2_{\psi,k}$$

for all plurisubharmonic functions ψ for which the right hand side is finite. Since not more than $M_{\lambda,\lambda+1}(k)$ different ν's are mapped by ρ onto the same i, when $U_i^{(\lambda)} \cap \Omega_k \neq \emptyset$, (property (A15)(vi), we get by summing up

$$\|g\|^2_{\psi,k} \le M_{\lambda,\lambda+1}(k)\, \|c\|^2_{\psi,k}.$$

Let $\bar\partial g = f$ be the cochain in $C^{p-1}(\mathcal{U}^{(\lambda)}, Z_{q+1})$ defined by

$$f_s = \bar\partial g_s = \sum_\nu \partial\phi_\nu \wedge c_{i_\nu s}\,.$$

Then

$$\| f_s \|^2_{\phi,k} \le \{\sum_\nu \| \partial\phi_\nu \wedge c_{i_\nu s} \|_{\phi,k}\}^2 \le N_k^{(\lambda)} \sum_\nu \| \bar\partial\phi_\nu \wedge c_{i_\nu s} \|^2_{\phi,k}$$

where at most $N_k^{(\lambda)}$ terms in the sum are different from zero, when $U_s^{(\lambda)} \cap \Omega_k \neq \emptyset$ according to property (A15)(iv). If $U_s^{(\lambda)} \cap \Omega_k \neq \emptyset$, then $U_s^{(\lambda)} \subset \Omega_{\ell(k)}$, so that for all $i \in I_\lambda$ with $U_i^{(\lambda)} \cap U_s^{(\lambda)} \neq \emptyset$

$$U_i^{(\lambda)} \cap \Omega_{\ell(k)} \neq \emptyset .$$

Hence using (7.3) in the above estimate we get with $K_k' = C_{\ell(k)} N_k^{(\lambda)} M_{\lambda,\lambda+1}(k)$

$$\| f \|^2_{\phi,k} \le K_k' \| c \|^2_{\phi,k}.$$

Now $\delta f = \bar\partial\delta g = \bar\partial c = 0$. If $p > 1$, by the inductive hypothesis of case (a) we can find a cochain $f' \in C^{p-2}[\mathcal{U}^{(\lambda)}, Z_{q+1}, \phi_{2p-2}]$ with $\delta f' = f$ and by the inductive hypothesis of case (b) we can find cochains $f_k' \in C^{p-2}(\mathcal{U}_k^{(\lambda)}, Z_{q+1}, \phi_{2p-2})$ with $\delta f_k' = f$ in Ω_k and with

$$\| f_k' \|^2_{\phi_{2p-2},k} \le K_k'' \| f \|^2_{\phi,k}$$

for some constants K_k'' depending on k. By theorem A10 and property (A15)(i) for every $s \in I_\lambda^{p-1}$ we can find $(g')_s \in \Gamma(U_s^{(\lambda)}, L_q)$ so that $\bar\partial(g')_s = (f')_s$ in $U_s^{(\lambda)}$ and

$$\| (g')_s \|^2_{\phi_{2p}} \le \| (f')_s \|^2_{\phi_{2p-2}}$$

and since the sets Ω_k are pseudoconvex, theorem A10 yields $(g_k')_s \in \Gamma(U_s^{(\lambda)} \cap \Omega_k, L_q)$, such that $\bar\partial(g_k')_s = (f_k')_s$ in $U_s^{(\lambda)} \cap \Omega_k$ and

$$\| (g_k')_s \|^2_{\phi_{2p}} \le \| (f_k')_s \|^2_{\phi_{2p-2}} .$$

Hence $\{(g')_s \mid s \in I_\lambda^{p-1}\} = g' \in C^{p-2}[\mathcal{U}^{(\lambda)}, L_q, \phi_{2p}]$ and $\{(g_k')_s \mid s \in I_\lambda^{p-1}\} = g_k' \in C^{p-2}(\mathcal{U}_k^{(\lambda)}, L_q, \phi_{2p})$.

Finally put $c' = g - \delta g'$ and $c_k' = g - \delta g_k'$, then for all k

$$\|c'\|^2_{\phi_{2p},k} \le \|g\|^2_{\phi_{2p},k} + pN_k^{(\lambda)}\|g'\|^2_{\phi_{2p},k} \le$$

$$\le M_{\lambda,\lambda+1}(k)\|c\|^2_{\phi_{2p},k} + pN_k^{(\lambda)}\|f'\|^2_{\phi_{2p-2},k} < \infty$$

and for some constants K_k

$$\|c_k'\|^2_{\phi_{2p},k} \le M_{\lambda,\lambda+1}(k)\|c\|^2_{\phi_{2p},k} + pN_k^{(\lambda)}\|f'\|^2_{\phi_{2p-2},k} \le K_k\|c\|^2_{\phi,k}.$$

Furthermore $\delta c' = \delta g = c$ and $\bar\partial c' = f-\delta\bar\partial g' = f-\delta f' = f-f = 0$, hence $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},Z_q,\phi_{2p}]$ and also $\delta c_k' = \delta g = c$ and $\bar\partial c_k' = f-\delta\bar\partial g_k' = f-\delta f_k' = f-f = 0$ in Ω_k, hence $c_k' \in C^{p-1}(\mathcal{U}_k^{(\lambda)},Z_q,\phi_{2p})$.

It remains to consider the case $p = 1$. The fact that $\delta f = 0$ then means that f defines uniquely a $(0,q+1)$-form f in Ω with $\bar\partial f = 0$. In case (a) we cannot immediately apply theorem A10, but we need a modification, where the integrals are performed in the sets Ω_k. Assume that this may be done. Then we can find $\tilde g \in \Gamma(\Omega,L_q)$ with $\bar\partial\tilde g = f$ and for all k

$$\int_{\Omega k} |\tilde g(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) < \infty .$$

In case (b) we use theorem A10 and obtain $(0,q)$-forms $\tilde g_k \in \Gamma(\Omega_k,L_q)$ in Ω_k with $\bar\partial\tilde g_k = f|_{\Omega_k}$ and

$$\int_{\Omega_k} |\tilde g_k(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) \le \int_{\Omega_k} |f(z)|^2 \exp-\phi(z)\, d\lambda(z).$$

Putting

$$(c')_s = g_s-\tilde g\Big|_{U_s^{(\lambda)}} \quad\text{and}\quad (c_k')_s = g_s-\tilde g_k\Big|_{U_s^{(\lambda)}\cap\Omega_k}$$

for $s \in I_\lambda$, we obtain cochains with the required properties (using (A15)(ii) in the estimates for $\tilde g$).

In fact we only have n induction steps, since all $(0,n)$-forms g satisfy $\bar\partial g = 0$. Therefore, the estimates hold already when p is replaced by $\min(p,n)$ and the constants K_k in part (b) may be taken independent of p.

We only have to prove the modification of theorem A10. □

LEMMA 7.3. *Let Ω be an open pseudoconvex set, let $\{\Omega_k\}_{k=1}^{\infty}$ be an increasing sequence of subsets of Ω satisfying* (7.1) *and let ϕ be a plurisubharmonic function in Ω. For every* (0,q+1)*-form* g *with locally square integrable coefficients and with $\bar\partial g = 0$, there is a* (0,q)*-form* u *in Ω with locally square integrable coefficients, such that $\bar\partial u = g$ and for all* k

$$\int_{\Omega_k} |u(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) < \infty$$

provided that for each k

$$\int_{\Omega_k} |g(z)|^2 \exp-\phi(z)\, d\lambda(z) < \infty.$$

PROOF. Let χ be a convex majorant of the nonnegative function $\tilde\chi$

$$\tilde\chi(a) = \begin{cases} 0 & \text{for } a < 1, \\ \max\{0, \log[2^k \int_{\Omega_{k+1}\setminus\Omega_k} |g(z)|^2 \exp-\phi(z)\, d\lambda(z)]\} & \text{for } k \le a < k+1, \end{cases}$$

$k = 1,2,\ldots$. Then $\psi(z) \overset{\text{def}}{=} \chi(\sigma(z))$ is plurisubharmonic in Ω, so that we may apply theorem A10 in the domain Ω with the plurisubharmonic function $\phi+\psi$. This yields a (0,q)-form u in Ω with $\bar\partial u = g$ and with for each k

$$\int_{\Omega_k} |u(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) \le e^{\chi(k)} \int_{\Omega_k} |u(z)|^2 \frac{\exp(-\phi(z)-\psi(z))}{(1+\|z\|^2)^2}\, d\lambda(z) \le$$

$$\le e^{\chi(k)} \int_{\Omega} |u(z)|^2 \frac{\exp(-\phi(z)-\psi(z))}{(1+\|z\|^2)^2}\, d\lambda(z) \le e^{\chi(k)} \int_{\Omega} |g(z)|^2 \exp(-\phi(z)-\psi(z))\, d\lambda(z) =$$

$$= e^{\chi(k)} \left(\int_{\Omega_1} + \sum_{k=1}^{\infty} \int_{\Omega_{k+1}\setminus\Omega_k} \right) |g(z)|^2 \exp(-\phi(z)-\psi(z))\, d\lambda(z) \le$$

$$\le e^{\chi(k)} \left[\int_{\Omega_1} |g(z)|^2 \exp-\phi(z)\, d\lambda(z) + \sum_{k=1}^{\infty} \frac{1}{2^k} \right] < \infty. \qquad \square$$

REMARK 7.1. In general lemma 7.3 is not true, if we consider different weightfunctions ϕ_k in the sets Ω_k, or in the same set Ω. For example assume that $\bar{\partial}g = 0$ and that for every k

$$(7.4) \qquad \int_{\Omega_k} |g(z)|^2 \exp(-\phi(z)-1/k\|z\|)\, d\lambda(z) < \infty,$$

where $\Omega_k \subset \Omega_{k+1} \subset \Omega$ or $\Omega_k = \Omega$ for all k. Then it is not true that there is a form u in Ω with $\bar{\partial}u = g$ and with for all k

$$(7.5) \qquad \int_{\Omega_k} |u(z)|^2 \frac{\exp(-\phi(z)-1/k\|z\|)}{(1+\|z\|^2)^2}\, d\lambda(z) < \infty.$$

For if this were true, using theorem 4.4.2 of [7] as in section 5, we could extend the entire function

$$f(z) = \oint e^{1/\zeta} e^{iz\cdot\zeta}\, d\zeta$$

in $\mathbb{C}^1$ satisfying

$$|f(z)| \le 2\varepsilon\, e^{1/\varepsilon} e^{\varepsilon\|z\|} \qquad \text{for all } \varepsilon > 0$$

to an entire function F in $\mathbb{C}^2$ satisfying

$$F(z,iz) = f(z)$$

$$|F(\nu_1,\nu_2)| \le K_\varepsilon (1+\|\nu\|)^m e^{\varepsilon\|\mathrm{Im}\,\nu\|} \qquad \text{for all } \varepsilon > 0.$$

But then according to VLADIMIROV [16] 29.1 F is a polynomial, hence f would be a polynomial, that is

$$f(z) = \sum_{j=0}^{k} \oint \frac{a_j}{\zeta^j} e^{iz\cdot\zeta}\, d\zeta$$

for some k and constants a_j contradicting the definition of f.

In [9] KAWAI has shown that for each (0,q+1)-form g with $\bar{\partial}g = 0$ satisfying (7.4) there does exist a (0,q)-form u with $\bar{\partial}u = g$ in Ω satisfying (7.5), when Ω satisfies

$$\sup_{z\in\Omega} \|\mathrm{Im}\, z\| \le K < \infty$$

for some constant K.

Next we derive Cartan's theorem B with bounds. Let $\mathcal{F}$ be either the sheaf of relations of P on Ω, thus $\mathcal{F} = \mathcal{R}_P$ or the image under P of the sheaf $\mathcal{A}^q$, thus $\mathcal{F} = P\mathcal{A}^q$, see (A5) and (A6).

LEMMA 7.4. *Let the plurisubharmonic function* ϕ *in the pseudoconvex open set* Ω *satisfy* (7.2). *There is a positive integer* N *(depending on* P*), such that for all* λ *there is a* $\mu > \lambda$ *with the following properties*

(a) *when moreover the subsets* Ω_k *of* Ω *satisfy* (7.1), *every cochain* $f \in C^p[\mathcal{U}^{(\lambda)},\mathcal{F},\phi]$ *with* $\delta f = 0$, $p \geq 1$, *can be written as* $\delta f' = \rho^*_{\lambda,\mu} f$ *for some* $f' \in C^{p-1}[\mathcal{U}^{(\mu)},\mathcal{F},\phi_N]$;

(b) *for all* k *there are integers* $\ell_k > k$ *and constants* $K_{\lambda,k}$, *such that every cochain* $f \in C^p[\mathcal{U}^{(\lambda)},\mathcal{F},\phi]$ *with* $\delta f = 0$, $p \geq 1$, *for all* k *can be written as* $\delta f'_k = \rho^*_{\lambda,\mu} f$ *in* Ω_k *with* $f'_k \in C^{p-1}(\mathcal{U}^{(\mu)}_k,\mathcal{F},\phi_N)$ *and with*

$$\| f'_k \|_{\phi_N,k} \leq K_{\lambda,k} \| f \|_{\phi,\ell_k}.$$

PROOF. First we change theorem A16 into a formulation with L^2-estimates. Let $K = \omega+z$ be so that $U \cap \Omega_k \neq \emptyset$ and $V = (t+1)\,\omega+z \subset \Omega_\ell$ for some $\ell \geq k$, where $t\omega$ is the enlargement of ω by a factor t with respect to some center in ω. Then V contains some ε-neighborhood of $t\omega+z$, where ε depends on the size of ω. The condition HS_1 ([14] G.7 with $M_p^2 = \exp-\phi_{N+m}$, $\Omega_p = \Omega_m = U$ and $M_m^2 = \exp-\phi_{N+m+(n+1)/2}$) and by (7.2) the condition HS_2 ([14] G.7 with $M_p^2 = \exp-\phi_m$, $\Omega_p = V$ and $M_m^2 = \exp-\phi_m$, $\Omega_m = t\omega+z$ and with $d_z = \varepsilon$) are satisfied. Hence instead of (A14) we get

$$\begin{aligned}
&\int_U |v(w)|^2 \frac{\exp-\phi(w)}{(1+\|w\|^2)^{N+m+(n+1)/2}}\, d\lambda(w) \leq C_1 \sup_{w\in U} |v(w)| \frac{\exp-\frac{1}{2}\phi(w)}{(1+\|w\|)^{N+m}} \leq \\
&\leq C_1 C \sup_{w\in t\omega+z} |P(w)u(w)| \frac{\exp-\frac{1}{2}\phi(w)}{(1+\|w\|)^m} \left(\frac{1+\|z\|}{1+\|w\|}\right)^N \leq \\
&\leq C_1 C\left(1+\sup_{\zeta\in t\omega}\|\zeta\|\right)^N \sup_{w\in t\omega+z} |P(w)u(w)| \frac{\exp-\frac{1}{2}\phi(w)}{(1+\|w\|)^m} \leq \\
&\leq C_2 \int_V |P(w)u(w)| \frac{\exp-\phi(w)}{(1+\|w\|^2)^m}\, d\lambda(w),
\end{aligned} \tag{7.6}$$

since

$$-\phi_{N+m}(w)+\phi_{N+m}(v) \le \text{some constant for } w \in U \text{ and } v \in t\omega+z$$

follows by repeated use of (7.2) and since the estimates with $(1+\|w\|^2)^m$ and $(1+\|w\|)^{2m}$ are equivalent and $|v_1(w)|+\dots+|v_q(w)| \le \sqrt{q}\ |v(w)|$ when $v = (v_1,\dots,v_q) \in A(U)^q$. The constants C_1, C and C_2 do not depend on z, C depends on the size of ω and C_2 depends moreover on ε, in fact $C_2 \sim \varepsilon^{-n}$ (see [18], proof of HS_2), but ε depends on the size of ω.

For $p \ge M$ (see (A15)(ii)) the theorem is true, since there are no non-zero cochains $f \in C^M[\mathcal{U}^{(\lambda)},\mathcal{F},\phi]$. Thus assume that the theorem has been proved for all P when p is replaced by p+1 and when the constants N and μ and in part (b) the constants ℓ and K depend on p.

In case $\mathcal{F} = \mathcal{R}_P$ there is a polynomial matrix Q, such that $\mathcal{F} = QA^r$ by (A6) and according to theorem A11 we can write $f \in C^p[\mathcal{U}^{(\mu)},\mathcal{F},\phi]$ as $f_s = Qg_s$ where $g \in C^p(\mathcal{U}^{(\mu)},A^r)$. In case $\mathcal{F} = PA^q$ we write $Q = P$ and $r = q$, then also $f = Qg$ with $g \in C^p(\mathcal{U}^{(\mu)},A^r)$ according to theorem A15. Let $\nu \gtrsim \mu+{}^2\log(t+1)$, then for every $i \in I_\nu$ (t+1) times $U_i^{(\nu)}$ is contained in $U^{(\mu)}_{\rho_{\mu,\nu}(i)}$, where t is such that (7.6) may be applied with $U = U_i^{(\nu)}$ and $V = U^{(\mu)}_{\rho_{\mu,\nu}(i)}$. From theorem A16 and (7.6) we obtain a cochain $\tilde g \in C^p(\mathcal{U}^{(\nu)},A^r)$ with $Q\tilde g_s = Qg_{s'} = f_{s'}$ where $s' = \rho_{\mu,\nu}(s)$, hence $\rho^*_{\mu,\nu}f = Q\tilde g$. When $U_s^{(\nu)} \cap \Omega_k = \emptyset$, then $U_{s'}^{(\mu)} \subset \Omega_{\ell(k)}$ for some $\ell(k)$ (property (A15)(iii)), so that (7.6) yields

$$\int_{U_s^{(\nu)}\cap\Omega_k} |\tilde g_s|^2 \exp-\phi_{N_1+m}\, d\lambda \le C_{k,\mu} \int_{U_{s'}^{(\mu)}} |f_{s'}|^2 \exp-\phi_m\, d\lambda$$

for some N_1 and all m. The constant $C_{k,\mu}$ depends on the smallest and the largest size of the sets $U_s^{(\nu)}$ with $U_s^{(\nu)} \cap \Omega_k \ne \emptyset$ and this depends on k and ν, but ν depends on μ; $C_{k,\mu}$ does not depend on s. Since not more than a finite number of different s are mapped by $\rho_{\mu,\nu}$ onto the same s' (property (A15)(vi)), we get by summing up

$$(7.7) \qquad \|\tilde g\|_{\phi_{N_1+m},k} \le C'_{k,\mu}\ \|f\|_{\phi_m,\ell(k)}, \qquad k = 1,2,\dots .$$

Thus $\tilde{g} \in C^p[\mathcal{U}^{(\nu)},\mathcal{A}^r,\phi_{N_1+m}]$. When $\delta f = 0$, $\delta Q\tilde{g} = Q\delta\tilde{g} = 0$, whence together with (A15)(iv) it follows that $\delta\tilde{g} = c$ is a cocycle in $C^{p+1}[\mathcal{U}^{(\nu)},\mathcal{R}_Q,\phi_{N_1+m}]$.

By the inductive hypothesis of case (a) we can find $\mu' > \nu$, N_2 and a cochain $c' \in C^p[\mathcal{U}^{(\mu')},\mathcal{R}_Q,\phi_{N_2+N_1+m}]$ with $\delta c' = \rho^*_{\nu,\mu'}c$ in Ω and by the inductive hypothesis of case (b) we can find moreover constants $\ell_k > k$, $K''_{\nu,k}$ and cochains $c'_k \in C^p(\mathcal{U}^{(\mu')}_k,\mathcal{R}_Q,\phi_{N_2+N_1+m})$ with $\delta c'_k = \rho^*_{\nu,\mu'}c$ in Ω_k and with

$$\|c'_k\|_{\phi_{N_2+N_1+m},k} \le K''_{\nu,k}\,\|c\|_{\phi_{N_1+m},\ell_k} .$$

We put $g_0 = \rho^*_{\nu,\mu'}\tilde{g} - c' \in C^p[\mathcal{U}^{(\mu')},\mathcal{A}^r,\phi_{N_2+N_1+m}]$, so that $\delta g_0 = = \rho^*_{\nu,\mu'}c-\rho^*_{\nu,\mu'}c = 0$, and $g_k = \rho^*_{\nu,\mu'}\tilde{g}-c'_k \in C^p(\mathcal{U}^{(\mu')}_k,\mathcal{A}^r,\phi_{N_2+N_1+m})$ so that $\delta g_k = \rho^*_{\nu,\mu'}c = 0$ in Ω_k. According to lemma 7.2 (a) and (b) there are $g' \in C^{p-1}[\mathcal{U}^{(\mu')},\mathcal{A}^r,\phi_N]$ with $\delta g' = g_0$ and $g'_k \in C^{p-1}(\mathcal{U}^{(\mu')}_k,\mathcal{A}^r,\phi_N)$ with $\delta g'_k = = g_k$ in Ω_k, respectively, where $N = N_2+N_1+m+2\min(n,p)$ and with moreover

$$\|g'_k\|_{\phi_N,k} \le K_k\,\|g_k\|_{\phi_{N_2+N_1+m},k}.$$

Finally we put $f' = Qg' \in C^{p-1}[\mathcal{U}^{(\mu')},\mathcal{F},\phi_{N_3+N}]$ and $f'_k = Qg'_k \in \in C^{p-1}(\mathcal{U}^{(\mu')}_k,\mathcal{F},\phi_{N_3+N})$ with N_3 depending on Q. Then

$$\delta f' = Q\delta g' = Qg_0 = \rho^*_{\nu,\mu'}Q\tilde{g} = \rho^*_{\nu,\mu'}\,\rho^*_{\mu,\nu}f = \rho^*_{\mu,\mu'}f$$

in Ω and similarly $\delta f'_k = \rho^*_{\mu,\mu'}f$ in Ω_k. Furthermore for all m and μ we get

$$\|f'_k\|_{\phi_{N_3+N},k} \le K'_k\,\|g_k\|_{\phi_{N_2+N_1+m},k} \le K'_k\{M_{\nu,\mu'}(k)^{p+1}\|\tilde{g}\|_{\phi_{N_2+N_1+m},k} +$$

$$+ K''_{\nu,k}\|c\|_{\phi_{N_1+m},\ell_k}\} \le$$

$$\le K'_k\{M_{\nu,\mu'}(k)^{p+1}+K''_{\nu,k}(p+2)\,N^{(\nu)}_{\ell_k}\}\,\|\tilde{g}\|_{\phi_{N_1+m},\ell_k} \le K_{\mu,k}\,\|f\|_{\phi_m,\ell} ,$$

where $\ell = \ell(\ell_k)$ depends on ℓ_k according to (7.7) and where $K_{\mu,k}$ is a constant depending on k,ν and μ', but μ' depends on ν, ℓ_k depends on k and ν depends on μ; N_3 depends on Q, N_2 on p by the inductive hypothesis and N_1 on P, but

Q depends on P.

Hence the lemma is proved when N, μ and in part (b) moreover ℓ and K depend on p. But there are only finitely many induction steps, so that we can take the largest N, μ, ℓ and K. We start the induction when p = M, $\mu = \lambda$ and m = 0. Therefore, the lemma is true for all p with constants N (depending on P), μ (depending on λ) and in part (b) ℓ (depending on k) and K (depending on λ and k). □

Now we are able to prove theorems 7.1 and 7.2.

PROOF OF THEOREM 7.1. It follows from theorem A15 that for all $s \in I_0$ we can take $f = Pg_s$ in $U_s^{(0)} \in \mathcal{U}^{(0)}$ with $g \in A(U_s^{(0)})^q$. As in the proof of lemma 7.4 we set $\nu \geq {}^2\log(t+1)$, so that (t+1) times $U_s^{(\nu)}$ is contained in $U_{s'}^{(0)}$ for all $s \in I_\nu$, where $s' = \rho_{0,\nu}\, s \in I_0$. As in (7.7) we can find $\tilde{g} \in$ $\in C^0[\mathcal{U}^{(\nu)}, A^q, \phi_{N_1}]$ with $P\tilde{g}_s = f$ in $U_s^{(\nu)}$ for all $s \in I_\nu$ and with (7.6) instead of (7.7)

$$(7.8)\qquad \int_{U_s^{(\nu)}\cap\Omega_m} |\tilde{g}_s^{(z)}|^2 \frac{\exp{-\phi(z)}}{(1+\|z\|^2)^{N_1}}\, d\lambda(z) \leq \int_{U_s^{(0)}} f(z)^2 \exp{-\phi(z)}\, d\lambda(z), \qquad m = 1,2,\ldots\ ,$$

where f is regarded as a cocycle in $C^0[\mathcal{U}^{(0)}, A^p, \phi]$. Consider the differences c of the functions $\tilde{g}_s$ in the overlaps of the sets $U_s^{(\nu)}$ for $s \in I_\nu$, that is $c = \delta\tilde{g}$. Since not more than a finite number of different s are mapped by $\rho_{0,\nu}$ onto the same s', there are constants C_m' with

$$\|c\|_{\phi_{N_1},m} \leq C_m' \int_{\Omega_{\ell(m)}} |f(z)|^2 \exp{-\phi(z)}\, d\lambda(z)\ .$$

Then $Pc = P\delta\tilde{g} = \delta f = 0$ and also $\delta c = 0$, hence c is a cocycle in $C^1[\mathcal{U}^{(\nu)}, \mathcal{R}_P, \phi_{N_1}]$. According to lemma 7.4(a) there are $\nu > \mu$, N_2 and

$$(7.9)\qquad c' \in C^0[\mathcal{U}^{(\mu)}, \mathcal{R}_P, \phi_N]\ ,$$

where $N = N_1+N_2$, with $\delta c' = \rho_{\nu,\mu}^{*} c$ in Ω and according to lemma 7.4(b) there are moreover constants K_k (also depending on ν), integers m > k (depending on k) and cochains $c_k' \in C^0(\mathcal{U}_k^{(\mu)}, \mathcal{R}_P, \phi_N)$ with $\delta c_k' = \rho_{\nu,\mu}^* c$ in Ω_k and with

$$(7.1)\qquad \|c_k'\|_{\phi_N,k} \le K_k \|c\|_{\phi_{N_1},m} \le K_k C_m' \int_{\Omega_\ell} |f(z)|^2 \exp-\phi(z)\, d\lambda(z) ,$$

where $\ell > m > k$ depends on k.

Finally for all $s \in I_\mu$ we put $v_s(z) = \tilde{g}_{s'}(z) - c_s'(z)$ for $z \in U_s^{(\mu)}$ with $s' = \rho_{\nu,\mu} s$ which by (A9) defines a function $v \in A(\Omega)^q$ because $\{v_s \mid s \in I_\mu\} \in C^0(\mathcal{U}^{(\mu)}, \mathcal{A}^q)$ and $\delta v = \rho_{\nu,\mu}^* \delta\tilde{g} - \rho_{\nu,\mu}^* c = 0$. Furthermore for all k

$$\int_{\Omega_k} |v(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) \le \|v\|_{\phi_N,k} < \infty$$

by (7.7) and (7.9). Similarly, for each k $(v_k)_s(z) = \tilde{g}_{s'}(z) - (c_k')_s(z)$ for $z \in U_s^{(\mu)} \cap \Omega_k$ defines a function $v_k \in A(\Omega_k)^q$ and there are constants K_k and $\ell_k > k$ with

$$\int_{\Omega_k} |v_k(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) \le K_k \int_{\Omega_{\ell_k}} |f(z)|^2 \exp-\phi(z)\, d\lambda(z)$$

by (7.8) and (7.19). Moreover, for all $s \in I_\mu$ in $U_s^{(\mu)}$ we have

$$Pv = Pv_s = P\tilde{g}_{s'} - Pc_s' = f ,$$

so that $Pv = f$ in Ω and similarly $Pv_k = f$ in Ω_k. □

<u>PROOF OF THEOREM 7.2</u>. Let $\mathcal{F}$ be the sheaf $P\mathcal{A}^q$. We construct a cochain $h \in C^0[\mathcal{U}^{(0)}, \mathcal{A}^p, \phi]$ as follows: for all $s \in I_0$, when $U_s^{(0)} \subset \Omega_1$ we define $h_s(z) = f_1(z)$ for $z \in U_s^{(0)}$; for k = 1,2,... successively, when $U_s^{(0)} \cap \Omega_k \ne \emptyset$, $U_s^{(0)} \not\subset \Omega_k$, let ℓ be the smallest integer with $U_s^{(0)} \subset \Omega_\ell$ or when $U_s^{(0)} \subset \Omega_{k+1} \cap \Omega_k^c$, let ℓ be $\ell = k+1$, then we define $h_s(z) = f_\ell(z)$ for $z \in U_s^{(0)}$. By (A15)(ii) we obtain for all k

$$\|h\|_{\phi,k} \le M \max_{1\le j\le \ell} \int_{\Omega_j} |f_j(z)|^2 \exp-\phi(z)\, d\lambda(z) < \infty ,$$

where $\ell = \ell(k)$ depends on k according to (A15)(iii).

Since $f_{k+m}-f_k = P(g_{k+m-1}+\dots+g_k)$ in Ω_k for all $m \geq 1$, the differences of the functions h_s in the overlaps $U^{(0)}_{s_1 s_2}$ of the sets $U^{(0)}_s$ are either zero or $Pg_{s_1 s_2}$ for some $g_{s_1 s_2} \in A(U^{(0)}_{s_1 s_2})^q$. Hence $\delta h \in C^1(\mathcal{U}^{(0)},\mathcal{F})$.

Now theorem 7.2 follows from the next theorem and theorem A15. □

THEOREM 7.3. *Let* $\mathcal{F}$ *be the sheaf* PA^q *in the pseudoconvex set* Ω*, where* Ω *is the union of the subsets* Ω_k *satisfying* (7.1) *and let* ϕ *be a plurisubharmonic function in* Ω *satisfying* (7.2). *If for some* λ $h \in C^0[\mathcal{U}^{(\lambda)},A^p,\phi]$ *with* $\delta h \in C^1(\mathcal{U}^{(\lambda)},\mathcal{F})$*, then there is a constant* N *and a function* $v \in A(\Omega)^p$ *with for all* $s \in I_\lambda$ $v(z)-h_s(z) = P(z)g_s(z)$ *for* $z \in U^{(\lambda)}_s$ *and for some* $g \in C^0(\mathcal{U}^{(\lambda)},A^q)$ *and with*

$$\int_{\Omega_k} |v(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) < \infty \qquad \text{for all } k = 1,2,\dots .$$

PROOF. We can estimate the cocycle $f = \delta h \in C^1(\mathcal{U}^{(\lambda)},\mathcal{F})$ in terms of h by use of (A15)(iv), hence $f \in C^1[\mathcal{U}^{(\lambda)},\mathcal{F},\phi]$ and $\delta f = 0$. According to lemma 7.4(a) there is a cochain $f' \in C^0[\mathcal{U}^{(\mu)},\mathcal{F},\phi_N]$ with $\delta f' = \rho^*_{\lambda,\mu} f$ in Ω for some integer N and $\mu > \lambda$.

Let for all $i \in I_\mu$ and $z \in U^{(\mu)}_i$

$$v_i(z) = h_{s'}(z) - f'_i(z)$$

where $s' = \rho_{\lambda,\mu}(i)$. Then $\delta v = \rho^*_{\lambda,\mu}\delta h - \delta f' = \rho^*_{\lambda,\mu} f - \rho^*_{\lambda,\mu} f = 0$ in Ω, thus $\{v_i \mid i \in I_\mu\}$ determines a function $v \in (\Omega)^p$. Moreover, using (A15)(vi) we obtain for all k

$$\int_{\Omega_k} |v(z)|^2 \frac{\exp-\phi(z)}{(1+\|z\|^2)^N}\, d\lambda(z) \leq \|v\|_{\phi_N,k} \leq M_{\lambda,\mu}(k)\|h\|_{\phi,k} + \|f'\|_{\phi_N,k} < \infty.$$

For $s \in I_\lambda$ let $I'(s) \subset I_\mu$ be the set of those $i \in I_\mu$ with $V_i \overset{\text{def}}{=\!=} U^{(\mu)}_i \cap U^{(\lambda)}_s \neq \emptyset$. For all $i \in I'(s)$ and $z \in V_i$ we have

$$v(z)-h_s(z) = h_{s'}(z)-f'_i(z)-h_s(z) .$$

Since $h_{s'}-h_s \in \Gamma(U^{(\lambda)}_{s'} \cap U^{(\lambda)}_s,\mathcal{F})$ and also $f'_i \in \Gamma(U^{(\mu)}_i,\mathcal{F})$, we obtain

$$v-h_s\Big|_{V_i} \in \Gamma(V_i,\mathcal{F}) .$$

As V_i is pseudoconvex, theorem A15 yields

$$v-h_s\Big|_{V_i} \in P\Gamma(V_i,\mathcal{A}^q)$$

and again by theorem A15 $v-h_s = Pg_s$ in $U_s^{(\lambda)}$ for some $g_s \in \Gamma(U_s^{(\lambda)},\mathcal{A}^q)$, because also $U_s^{(\lambda)}$ is pseudoconvex (property (A15)(i)). □

8. COROLLARIES AND EXAMPLES

In this section some corollaries and particular cases are given of the theorem on Fourier transforms in section 6, theorem 6.1. We can survey this theorem by: let

$$\text{(8.1)} \qquad \mathrm{Exp}(a+0;C) \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} A_\infty(\exp(-\tilde{a}(z)-1/k\|z\|);C(k))$$

and

$$\text{(8.2)} \qquad A(a+0;C) \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{k\to\infty} A_\infty(\exp 1/k\|\zeta\|;\Omega(a+1/k;C)) ,$$

then

$$\text{(8.3)} \qquad \mathcal{F}\, A(a+0;C)' = \mathrm{Exp}(a+0;C)$$

where $A(a+0;C)'$ is the strong dual of $A(a+0;C)$ and $\mathcal{F}$ is an isomorphism; Ω is given by formula (5.4). Here we have used the fact that the sequences of weightfunctions

$$\{\exp \delta_k/k\|\zeta\|\}_{k=1}^{\infty} \qquad \text{and} \qquad \{\exp 1/k\|\zeta\|\}_{k=1}^{\infty}$$

induce the same topology on the space $A(a+0;C)$.

Let $w \in \mathrm{pr}\, C$, then $w \in \mathrm{pr}\, C_k$ for some k; since $\Omega(a;C) \cap (C_{k+1}^*)^c$ is bounded and since for $\zeta \in C^*_{k+1}$

$$\delta\|\zeta\| \le \mathrm{Im}\, w\cdot\zeta \le \|\zeta\|$$

for some $\delta > 0$, also the system

$$(8.4) \qquad \{\exp \frac{1}{k}\, \mathrm{Im}\, w\cdot\zeta\}_{k=1}^{\infty}$$

induces the same topology on $A(a+0;C)$. These weightfunctions M_k satisfy $M_k = \exp-\phi_k$, where $\phi_k(\zeta) = -1/k\, \mathrm{Im}\, w\cdot\zeta$ is a plurisubharmonic function. Therefore, the theorems of the appendix and of section 7 may be applied to the space $A(a+0;C)$, because all the L^p-norms are equivalent, $p = 1,2,\ldots,\infty$ (the space $A(a+0;C)$ is nuclear (see [14] G.7)).

It follows from (8.3) and (8.4) that any $f \in \mathrm{Exp}(a+0;C)$ satisfies for all $\varepsilon > 0$ and $\delta > 0$

$$|f(z)| = |\langle\mu_\zeta, e^{iz\cdot\zeta}\rangle| \le M_{\varepsilon,\delta} \sup_{\zeta\in\Omega(a+\delta;C)} |e^{i(z-\varepsilon w)\cdot\zeta}| \le$$

$$\le M_{\varepsilon,\delta}\, e^{\tilde a(z-\varepsilon w)+\delta\|z-\varepsilon w\|}$$

when $z \in \varepsilon w+C$. Now let a be bounded on $\mathrm{pr}\, C$, then a can be continued as a continuous function to $\overline{\mathrm{pr}\, C}$ and thus a is uniformly continuous on $\mathrm{pr}\, C$. That means that for all $\delta > 0$, there is a $\varepsilon(\delta) > 0$ with for $\varepsilon \le \varepsilon(\delta)$

$$|a(\widetilde{z-\varepsilon w})-\tilde a(z)| < \delta.$$

Hence

$$\tilde a(z-\varepsilon w) \le a(\tilde z)\|z-\varepsilon w\|+\delta\|z-\varepsilon w\| \le \delta\varepsilon(\delta)+\varepsilon(\delta) \sup_{z\in\mathrm{pr}\, C} |a(z)|+\tilde a(z)+\delta\|z\| ,$$

so that f satisfies for $z \in \varepsilon w+C$, all $\varepsilon > 0$ and $\delta > 0$

$$|f(z)| \le M'_{\varepsilon,\delta}\, e^{\tilde a(z)+2\delta\|z\|}.$$

We can choose $w \in \mathrm{pr}\, C$ so, that the sets $\{\varepsilon w+C\}_{\varepsilon>0}$ are just the subsets of C consisting of all the points of C with distance larger than η to the boundary of C, $\eta > 0$ and $\eta \to 0$ if $\varepsilon \to 0$. Thus we have found that as sets

$$(8.5) \qquad \mathrm{Exp}(a+0;C) = \operatorname{proj}\lim_{k\to\infty} A_\infty(\exp(-\tilde{a}(z)-1/k\|z\|);1/k\, w+C),$$

when a is bounded on pr C. Since the topology defined by (8.1) is obviously weaker than the one defined by (8.5) and since both topologies turn Exp(a+0;C) into an $F\bar{S}$-space, both topologies coincide (see [15] corollary 2 to th.7.1), so that (8.5) also holds for the topologies. A similar property holds for the spaces H(a;C) and $H^*(a;C)$ of [14] provided that then $\tilde{a}$ is uniformly continuous on C, which is true when a is not vertical at the boundary of pr C (see section 4), for example when a is constant. This surprising property of functions of exponential type in cones is difficult to establish without Fourier transformation.

Another surprising corollary is that, as topological spaces, $\bar{A}_i = \bar{A}_{ii}$, as we have already seen. It means that it does not make a difference if we use $\Omega(a+1/k;C)$ or $\Omega(a+1/k;C_k)$ in the space A(a+0;C): any function ϕ holomorphic in int $\Omega(a+1/k;C)$ with

$$|\phi(\zeta)| \le M_1 \exp -1/k\|\zeta\|, \qquad \zeta \in \operatorname{int} \Omega(a+1/k;C)$$

is holomorphic in some larger set int $\Omega(a+1/m;C_m) \cup$ int $\Omega(a+1/k;C)$ and satisfies there

$$|\phi(\zeta)| \le M_2 \exp -1/\ell\|\zeta\|$$

for some $\ell \ge k$ depending on k and C_m and some M_2 depending on M_1, k and C_m, but not on ϕ.

Now we imagine an open convex set Ω in $\mathbb{C}^n$ being given or equivalently an open convex cone C in $\mathbb{C}^n$ and a convex homogeneous function a in G (Ω such that it does not contain a straight line, whence the cone C is open in $\mathbb{C}^n$). Let $\{\Omega_m\}_{m=1}^\infty$ be an increasing sequence of closed convex sets with union Ω and such that the points of Ω_m are those points in Ω with distance larger than

ε_m from $\partial\Omega$ (see section 4). The sets Ω_m determine convex homogeneous functions $\tilde{a}_m$ on C with for some $\eta_m \geq \varepsilon_m$

$$a(z)-\eta_m \leq a_m(z) \leq a(z)-\varepsilon_m, \qquad z \in \operatorname{pr} C$$

$\varepsilon_m > \varepsilon_{m+1} > 0$, $\varepsilon_m \to 0$ and $\eta_m > \eta_{m+1} > 0$, $\eta_m \to 0$ for $m \to \infty$. We define

$$(8.6) \qquad \widetilde{\mathrm{Exp}}(a;C) = \operatorname*{ind\,lim}_{m\to\infty} \mathrm{Exp}(a_m+0;C) \;,$$

where we may use (8.5) instead of (8.1) when a is bounded. An equivalent definition is

$$\widetilde{\mathrm{Exp}}(a;C) = \operatorname*{ind\,lim}_{m\to\infty} \operatorname*{proj\,lim}_{k\to\infty} A_\infty(\exp -a_m(z);C(k)).$$

We also define

$$\tilde{A}'(a;C) = \operatorname*{ind\,lim}_{m\to\infty} A(a_m+0;C)'$$

or equivalently

$$\tilde{A}'(a;C) = \operatorname*{ind\,lim}_{m\to\infty} \left[\operatorname*{ind\,lim}_{k\to\infty} A_\infty(\exp 1/k\|\zeta\|;\Omega(a_m;C))\right]' \;.$$

It easily follows that F is an isomorphism:

$$(8.7) \qquad \widetilde{\mathrm{Exp}}(a;C) = F\tilde{A}'(a;C).$$

$\widetilde{\mathrm{Exp}}$ and $\tilde{A}'$ are inductive limits of nuclear Fréchet spaces, so they are nuclear themselves.

In particular we may take for the cone in $\mathbb{C}^n$ a tubular radial domain $T^C \subset \mathbb{C}^n$, where $T^C = \mathbb{R}^n + i\,C$ with now C an open convex cone in $\mathbb{R}^n$. A relatively compact subcone of this domain is $\mathbb{R}^n + i\,C_k$ with $C_k \subset\subset C$ and the domains C(k) become

$$\{\mathbb{R}^n + i\,C_k\} \cap \{z \mid \|\mathrm{Im}\, z\| > 1/k\} \;,$$

see section 4. Let $a(y,x)$ be a convex homogeneous function on T^C which is bounded on each pr T^{C_k}. Then $a(0,x)$ exists and is finite; so $A = \max_{\|x\|=1} a(0,x) < \infty$. Then the domain $\Omega(a;T^C) \overset{\text{not}}{=\!=} \Omega(a;C)$ is bounded in the imaginary direction, that is $\Omega(a;C) \subset \mathbb{R}^n + i\,B_A$, where B_A is the ball with radius A in $\mathbb{R}^n$. This case will be used in the next section, where the Newton interpolation series will be derived.

We can consider boundary values of functions f holomorphic in $\mathbb{R}^n+i\,C$. When these are finite order distributions the function f satisfies

$$(8.8) \qquad |f(z)| \le M_k\,(1+\|y\|^{-m}), \qquad y \in C_k,\ \|y\| \le 1$$

for some m depending on f. When moreover $f \in \widetilde{\mathrm{Exp}}(a;T^C)$, f is the Fourier transform of an analytic functional in Z' (see [14] H.4) carried by $\Omega(a;C)$. Indeed, in the same way as theorem 6.1 was obtained, using polynomials as weightfunctions instead of (8.4) one can show

$$(8.9) \qquad \mathcal{D}'_F(a;C) = \mathcal{F}\, Z'(a;C)$$

with $Z'(a;C)$ the dual of

$$Z(a;C) = \operatorname*{proj\,lim}_{m\to\infty} A_\infty((1+\|\zeta\|)^m;\Omega(a_m;C))$$

and with

$$\mathcal{D}'_F(a;C) = \operatorname*{ind\,lim}_{m\to\infty}\ \operatorname*{proj\,lim}_{k\to\infty} A_\infty\left(\frac{\exp - a_m(z)}{1+\|y\|^{-m}};\ \mathbb{R}^n+i\,C_k\right),$$

where $\mathbb{R}^n+i\,C_k$ may be replaced by $\{\mathbb{R}^n+i\,C_k\} \cup \{\mathbb{R}^n+i(1/k\,y_0+C)\}$ with $y_0 \in \mathrm{pr}\,C$, when a is bounded on pr C. $Z'(a;C)$ and hence also $\mathcal{D}'_F(a;C)$ is a nuclear LS-space, so that for example they are reflexive (compare the spaces in section 6 [14]).

Finally we give three examples with Ω contained in $\mathbb{R}^n$ illustrating the differences between analytic functionals and distributions. For simplicity we assume that the function a is constant on pr C, so that (8.4) holds.

Firstly, let f be holomorphic in $\mathbb{R}^n+i\,C$ and satisfy for all $\varepsilon > 0$ and k

$$|f(z)| \leq K(\varepsilon,k)\; e^{(a+\varepsilon)\|y\|+\varepsilon\|x\|}, \qquad y \in C_k,\ \|y\| > \varepsilon,$$

then f satisfies these inequalities (with other constants $K(\varepsilon)$) also for $y \in \varepsilon y_0+C$, $y_0 \in \mathrm{pr}\, C$ and $f = \mathcal{F}(\sigma)$ with σ carried by

$$\Omega = \{\zeta \mid \eta = 0,\ -y\cdot\xi \leq a\|y\|,\ y \in C\} \subset \mathbb{C}^n ,$$

such that for all $\varepsilon > 0$ σ can be represented as a measure σ_ε in

$$\Omega_\varepsilon = \{\zeta \mid \|\eta\| \leq \varepsilon,\ -y\cdot\xi \leq (a+\varepsilon)\|y\|,\ y \in C\}$$

with

$$\int_{\Omega_\varepsilon} \exp-\varepsilon\|\zeta\| \; |d\sigma_\varepsilon(\zeta)| < \infty.$$

Secondly, let f satisfy for all $\varepsilon > 0$ and k and some m

$$|f(z)| \leq K(\varepsilon,k)\; e^{(a+\varepsilon)\|y\|+\varepsilon\|x\|}(a+\|y\|^{-m}), \qquad y \in C_k,$$

then f satisfies these inequalities (with other constants $K(\varepsilon,k)$) also for $y \in C_k \cup \{\varepsilon y_0+C\}$ and $f = \mathcal{F}(\mu)$ with $\mu \in Z'$ carried by Ω, that is for all $\varepsilon > 0$ μ can be represented as a measure μ_ε in Ω_ε with

$$\int_{\Omega_\varepsilon} (1+\|\zeta\|)^{-\ell} \; |d\mu_\varepsilon(\zeta)| < \infty$$

for some $\ell > m$ (actually $\ell = m+n+2$, see [14] (6.10)).

Finally, let f satisfy for all $\varepsilon > 0$ and k and some m

$$|f(z)| \leq K(\varepsilon,k)\; e^{(a+\varepsilon)\|y\|}\; (1+\|x\|)^m\; (1+\|y\|^{-m}), \qquad y \in C_k,$$

then f satisfies these inequalities (with other constants $K(\varepsilon,k)$ and $(1+\|x\|)^m$ replaced by $(1+\|x\|)^\ell$ for some $\ell > m$) also for $y \in C_k \cup \{\varepsilon y_0+C\}$ and $f = \mathcal{F}(g)$ with $g \in S'$ having its support contained in $0 = \{\xi \mid -y\cdot\xi \leq a\|y\|,\ y \in C\}$, so that g can be represented as a finite combination of derivations of measures g_j in O with

$$\int_O (1+\|\xi\|)^{-\ell} \, |dg_j(\xi)| < \infty$$

for $j = 0,1,\ldots,\ell$ ($\ell = m+n+2$, see [14] (6.10)).

As in section 2, in the first two examples we have when $b > a \geq 0$

$$\mathrm{Exp}(a+0;T^C) \xrightarrow{\text{dense}} \mathrm{Exp}(b+0;T^C),$$

while in the third example

$$H(a;C) \xrightarrow{\text{closed linear subspace}} H(b;C).$$

Even, since the restriction map from $A(b+0;C)$ into $S(a;C)$ is injective, we have

$$H(a;C) \xrightarrow{\text{dense}} \mathrm{Exp}(b+0;T^C).$$

9. NEWTON SERIES FOR FUNCTIONS HOLOMORPHIC IN TUBULAR RADIAL DOMAINS

In this section we derive the Newton interpolation series for functions in $\widetilde{\mathrm{Exp}}(a;T^C)$. We give the most general class of holomorphic functions for which the Newton series is valid for h in a convex cone C in $\mathbb{R}^n$. However, since the detailed description becomes quite complicated, we discuss a particular case, namely a class of holomorphic functions of constant exponential type and we give a uniform bound on the length of h. The bound for $\|h\|$ will not be the best possible, but still this case gives a good idea of the generalization of the validity of the Newton series discussed in this paper. Finally we make some general remarks on the validity of the Newton series.

In [10] KIOUSTELIDIS derived the Newton interpolation series (and similar series) with the aid of Fourier transformation. The adventage of this method against the classical one (Cauchy's integral formula, NÖRLUND

[13], GELFOND [5]) is that it treats the case of several variables as well. However, his treatment is valid only for entire functions. This is not a restriction due to the method, for, as we have shown here, one has to extend the method (namely the formalism of Fourier transformation) to non-entire functions. Then we are able to derive the Newton series (and the similar series of KIOUSTELIDIS [10] or [14] remark 10.1) in several variables for non-entire functions as well. Moreover, in some way we obtain the largest possible class, for which the formalism is valid, since we use the domain of convergence completely (that is we do not cut off a compact subset of this domain as it is done in [10]) and since outside this domain the formalism is not valid, see Satz 5 in [10].

As we have seen in [14], section 5, we have to restrict the vector h to a real open convex cone C in $\mathbb{R}^n$ in order to get the Newton series for non-entire functions. Moreover, let $\|h\|$ be bounded by a positive number b. Let the convex (unbounded) open set Ω in $\mathbb{C}^n$ be the interior of one of the components of

$$\{ \zeta \mid \zeta \in \mathbb{C}^n, \ |e^{-h\cdot\zeta}-1| < 1, \ \forall h \in C \text{ with } \|h\| \leq b\},$$

see figure 4.1 of [14]. Then Ω is bounded in the imaginary direction, because $|h\cdot\eta| \leq (2k+\frac{1}{2})\pi$ for some k and for all $h \in C$ with $\|h\| \leq b$ and also Ω is contained in

$$(9.1) \qquad \{\zeta \mid -h\cdot\xi \leq \log 2, \ \forall h \in C \text{ with } \|h\| \leq b\}.$$

Hence Ω determines the convex cone $\mathbb{R}^n+i\,C$ in $\mathbb{C}^n$ and the convex homogeneous function H_Ω on $\mathbb{R}^n+i\,C$ by

$$(9.2) \qquad H_\Omega(z) = \sup_{\zeta\in\Omega} - \operatorname{Im} z\cdot\zeta.$$

$H_\Omega(z)$ is continuous up to $y = 0$, that is $H_\Omega(x)$ exists for $x \in \mathbb{R}^n$ and it follows from (9.1) that $H_\Omega(\tilde{z})$ is bounded by $(\log 2)/b+B$, where B is a bound for $\|\eta\|$, thus (8.5) may be applied. Also we have, see (4.2), (4.3), (4.4)

$$\Omega = \operatorname{int}\{\zeta \mid -\operatorname{Im} z\cdot\zeta \leq H_\Omega(z), \ z \in \mathbb{R}^n+i\,C\}.$$

Let Ω_m be an increasing sequence of convex closed subsets of Ω such that some ε-neighborhood $\Omega_{m,\varepsilon}$ of Ω_m is contained in Ω and $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$. Let H_m be the functions H_{Ω_m}, $m = 1,2,\ldots$. For $y \in C$, $h \in C$ let $s \in \mathbb{C}$ be such that

$$z+ish \in \mathbb{R}^n+i\,C,$$

so that $\mathrm{Re}\, s \geq -\alpha$ for some non-negative number α depending on y and h.

<u>LEMMA 9.1</u>. *Let* $z \in \mathbb{R}^n+i\,C$, $h \in C$ *and* $s \in \mathbb{C}$ *as above. Then the sequence*

$$\phi_{N,z}(\zeta) = e^{iz\cdot\zeta} \sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\zeta}-1)^k$$

converges for $N \to \infty$ *to* $\exp i(z+ish)\cdot\zeta$ *in all the spaces* $A(H_m+0;C)$, $m = 1,2,\ldots$.

<u>PROOF</u>. The space $A(H_m+0;C)$ is defined by (8.2) so that according to (5.9) $\exp iz\cdot\zeta \in A(H_m+0;C)$ when $z \in \mathbb{R}^n+i\,C$, hence $\exp i(z+ish)\cdot\zeta$ and $\exp(iz\cdot\zeta-kh\cdot\zeta)$, $k = 0,1,2,\ldots$, belong to $A(H_m+0;C)$ for all m. Since $A(H_m+0;C)$ is a Fréchet space, we have to show that for some $\varepsilon > 0$ small enough

$$(9.3) \qquad \sup_{\zeta\in\Omega_{m,\varepsilon}} |\phi_{N,z}(\zeta)| \exp \varepsilon\|\zeta\| \leq K ,$$

where K is independent of N.

In section 5 of [14] we defined subsets $\Omega(\varepsilon) \stackrel{\text{not}}{=\!=} \Omega_h(\varepsilon)$ of

$$\Omega_h = \{\zeta \mid |e^{-h\cdot\zeta}-1| < 1\}$$

by

$$\Omega_h(\varepsilon) \stackrel{\text{def}}{=\!=} \{\zeta \mid -h\cdot\xi < \log(2 \cos h\cdot\eta-\varepsilon)\}$$

and we showed that for all $\varepsilon > 0$ there is a ε_1 (here $\varepsilon_1 = \varepsilon/(6b)$) such that the ε_1-neighborhood of $\Omega_h(\varepsilon)$ is contained in $\Omega_h(\frac{1}{2}\varepsilon) \subset \Omega_h$. On the other hand, we will show that for all $\varepsilon > 0$ there is a ε_2 such that the boundary of the ε-neighborhood of

$$\text{(9.4)}\qquad \Omega(\varepsilon_2) \overset{\text{def}}{=\!=} \bigcap_{\substack{h\in C\\ \|h\|\le b}} \Omega_h(\varepsilon_2)$$

is contained in Ω^c.

First let us remark that $\Omega_h \subset \Omega_{\beta h}$ when $\beta \le 1$, since

$$\beta \log(2\cos x) \le \log(2\cos \beta x)\ , \quad |x| < \tfrac{1}{2}\pi\ .$$

Then $\zeta \in \partial\Omega(\varepsilon_2)$ means, that there is an $h \in C$, depending on ζ and δ, with $\|h\| = b$ and with

$$-h\cdot\xi \ge \log(2\cos h\cdot\eta - \varepsilon_2) - \delta.$$

Now we choose $\varepsilon_2 = \min(b^2\varepsilon^2/16,\ 1/171)$, $\delta = \tfrac{1}{4}\varepsilon_2$ and

$$\zeta_0 = \zeta + i\ \mathrm{sign}(\sin h\cdot\eta)\ \frac{4}{b}\sqrt{\varepsilon_2}\ \tilde{h},$$

where sign(0) = 1. Then $|\zeta-\zeta_0| \le \varepsilon$ and for some integer k

$$\tfrac{1}{2}\pi+4\sqrt{\varepsilon_2} \ > |\mathrm{Im}\ h\cdot\zeta_0+2k\pi| \ge 4\sqrt{\varepsilon_2}\ ,$$

so that when $|\mathrm{Im}\ h\cdot\zeta_0+2k\pi| \overset{\text{not}}{=\!=} |x| < \tfrac{1}{2}\pi$

$$-\mathrm{Re}\ h\cdot\zeta_0 \ge \log(2\cos h\cdot\eta-\varepsilon_2)-\tfrac{1}{4}\varepsilon_2 =$$

$$= \log\{2\cos(|x|-4\sqrt{\varepsilon_2})-\varepsilon_2\}-\tfrac{1}{4}\varepsilon_2 \ge \log(2\cos \mathrm{Im}\ h\cdot\zeta_0),$$

for, $\varepsilon_2 \le 1/171$ implies $\sin 4\sqrt{\varepsilon_2} \le 63/16\ \sqrt{\varepsilon_2}$, so that the following estimates with $|x| < \tfrac{1}{2}\pi$ hold

$$2\cos(|x|-4\sqrt{\varepsilon_2})-\varepsilon_2 \ge 2\cos x-16\varepsilon_2\cos x + 2|\sin x|\sin 4\sqrt{\varepsilon_2} -\varepsilon_2 \ge$$

$$\ge 2\cos x - 16\varepsilon_2(1-\tfrac{1}{3}x^2) + 2\tfrac{4}{7}x\ \tfrac{63}{16}\sqrt{\varepsilon_2} - \varepsilon_2$$

and the right hand side is larger than

$$2 \cos x + \varepsilon_2 \geq 2 \cos x \exp \tfrac{1}{4}\varepsilon_2$$

if

$$\frac{4}{9}\,\frac{16}{3}\sqrt{\varepsilon_2}\, x^2 + 2x - 8\sqrt{\varepsilon_2} \geq 0,$$

which is true when

$$x \geq 4\sqrt{\varepsilon_2} \geq [-1+(1+\frac{64}{27}\,8\varepsilon_2)^{\frac{1}{2}}]\Big/(\frac{64}{27}\sqrt{\varepsilon_2}) .$$

Hence $\zeta_0 \notin \Omega$ and also when $\frac{1}{2}\pi \leq |x| < \frac{1}{2}\pi+4\sqrt{\varepsilon_2}$, $\zeta_0 \notin \Omega$. Thus the sets $\Omega(1/m)$ defined by (9.4) with ε_2 replaced by $1/m$ may serve as the sets Ω_m.

From the formula above (5.3) in [14] we get for $\zeta \in \Omega_{m,\varepsilon}$, ε sufficiently small such that $\Omega_{m,\varepsilon} \subset \Omega(\varepsilon_1)$ for some ε_1,

$$|\phi_{N,z}(\zeta)| \leq C_1(\varepsilon_1) \exp \alpha\, h\cdot\zeta \exp -\mathrm{Im}\, z\cdot\zeta, \qquad \alpha > 0$$

$$|\phi_{N,z}(\zeta)| \leq C_2(\varepsilon_1)(1+\|\zeta\|) \exp -\mathrm{Im}\, z\cdot\zeta, \qquad \alpha = 0 .$$

For $\zeta \in \Omega$ outside a compact set and ε again sufficiently small

$$(y-\alpha h)\cdot\xi \geq \varepsilon\|\xi\| ,$$

hence there is a constant K such that for $\zeta \in \Omega$

$$-\mathrm{Im}\, z\cdot\zeta + \alpha h\cdot\xi \leq K - \varepsilon\|\zeta\|,$$

since $\|\eta\|$ is bounded in Ω. Now (9.3) follows when $\alpha > 0$ and for $\alpha = 0$ it follows by replacing ε by $\frac{1}{2}\varepsilon$. □

With the aid of lemma 9.1 and formula (8.3) the Newton series is derived for functions f belonging to $\widetilde{\mathrm{Exp}}(H_\Omega,T^C)$ given in (8.6), where H_Ω is defined by (9.2):

$$f(z+i(s+\alpha)h) = \langle\mu_\zeta, e^{iz\cdot\zeta-\alpha h\cdot\zeta-sh\cdot\zeta}\rangle =$$

$$= \langle e^{-\alpha h\cdot\zeta}\mu_\zeta,\ \lim_{N\to\infty} e^{iz\cdot\zeta}\sum_{k=0}^{N}\binom{s}{k}(e^{-h\cdot\zeta}-1)^k\rangle =$$

$$(9.5)\qquad = \sum_{k=0}^{\infty}\binom{s}{k}\langle e^{-\alpha h\cdot\zeta}\mu_\zeta,\ e^{iz\cdot\zeta}(e^{-h\cdot\zeta}-1)^k\rangle =$$

$$= \sum_{k=0}^{\infty}\binom{s}{k}\langle e^{-\alpha h\cdot\zeta}(e^{-h\cdot\zeta}-1)^k\mu_\zeta, e^{iz\cdot\zeta}\rangle = \sum_{k=0}^{\infty}\binom{s}{k}\Delta_{ih}^k f(z+i\alpha h)$$

valid for $z \in \mathbb{R}^n+i\,C$, $h \in C$, $\|h\| \le b$, $\mathrm{Re}\, s+\alpha \ge 0$, $\alpha \ge 0$ arbitrary. The sequence

$$\sum_{k=0}^{N}\binom{s}{k} e^{-\alpha h\cdot\zeta}(e^{-h\cdot\zeta}-1)^k\ \mu_\zeta$$

converges weakly in $A(H_m+0;C)'$ for some m depending on μ, which depends on f, and since $A(H_m+0;C)$ is a Montel space (see (8.2)), this sequence converges strongly in $A(H_m+0;C)'$, hence according to (8.7) the series (9.5) converges in the topology of $\widetilde{\mathrm{Exp}}(H_\Omega,T^C)$. Thus, reminding (8.5) we get, when f satisfies

$$(9.6)\qquad \forall k, \forall y \in C_k \text{ with } \|y\| > 1/k:\quad |f(z)| \le M_k \exp H_m(z),$$

for $\mathrm{Re}\, s \ge -\alpha$ with $\alpha \ge 0$, $h \in C$ with $\|h\| \le b$:

$$(9.7)\qquad \begin{aligned}&\forall\varepsilon > 0,\ \forall\ell > m,\ \forall p,\ \exists N_0(\varepsilon,\ell,p) \ge N_1(s),\ \forall z \in \mathbb{R}^n+i(1/p\,w+C),\ \forall N \ge N_0\\ &|f(z+i(s+\alpha)h) - \sum_{k=0}^{N}\binom{s}{k}\Delta_{ih}^k f(z+i\alpha h)| < \varepsilon\, A(s) \exp H_\ell(z)\ ,\end{aligned}$$

where $N_1(s)$ is determined by (5.1) [14] and $A(s)$ by (5.4) [14].

Replacing $z+i\alpha h$ by z in (9.5) we see that the Newton series

$$(9.8)\qquad f(z+ish) = \sum_{k=0}^{\infty}\binom{s}{k}\Delta_{ih}^k f(z)$$

valid for $y \in \eta w+C$, $h \in C$, $\|h\| \le b$, when $\mathrm{Re}\, s \ge -\alpha$, $\alpha > 0$ depending on $\eta > 0$ and h, such that $y-\alpha h \in \delta w+C$ for some $\delta > 0$, converges according to

$$(9.9)\qquad \begin{aligned}&\forall\varepsilon > 0,\ \forall\ell > m,\ \exists N_0(\varepsilon,\ell) \ge N_1(s),\ \forall z \in \mathbb{R}^n+i(\eta w+C),\ \forall N \ge N_0\\ &|f(z+ish) - \sum_{k=0}^{N}\binom{s}{k}\Delta_{ih}^k f(z)| < \varepsilon\, A(s) \exp H_\ell(z-i\alpha h)\ .\end{aligned}$$

We restate the results in

<u>THEOREM 9.1</u>. *Let* $h \in C$ *with* $\|h\| \le b$ *and let* f *be an element of* $\widetilde{\mathrm{Exp}}(H_\Omega, T^C)$ *where* H_Ω *is given in* (9.2). *If* $\alpha > 0$ *is such that* $y-\alpha h \in \delta w+C$, $\delta > 0$, *when* $y \in \eta w+C$ *for some* $\eta > \delta$, *then the Newton series* (9.8) *is valid for this* y *and* h, *when* $\mathrm{Re}\, s \ge -\alpha$. *The series* (9.8) *converges absolutely in one of the norms of* $\widetilde{\mathrm{Exp}}(H_\Omega, T^C)$ *or, more precisely, it converges according to* (9.9). *When* $\mathrm{Re}\, s \ge -\alpha$ *with* $\alpha \ge 0$ *arbitrary, the Newton series* (9.5) *holds for all* $y \in C$, $h \in C$, $\|h\| \le b$; *then the series* (9.5) *converges absolutely in the topology of* $\widetilde{\mathrm{Exp}}(H_\Omega, T^C)$ *or, more precisely, it converges according to* (9.7) *when* f *satisfies* (9.6). *In both cases* (9.5) *and* (9.8) *converge uniformly in* s *on compact subsets of* $\{s \mid s \in C, \mathrm{Re}\, s \ge -\alpha\}$.

Using (8.8) and (8.9) as in [14] section 7 we can derive the Newton series (9.5) for functions f satisfying

$$\forall k, \forall y \in C_k: \quad |f(z)| \le M_k(1+\|y\|^{-m}) \exp H_m(z).$$

This series holds for $z \in \mathbb{R}^n + iC$, $h \in C$ and $\mathrm{Re}\, s+\alpha \ge 0$, $\alpha \ge 0$ arbitrary and it converges in the topology of $\mathcal{D}'_F(a;C)$, namely according to

$$\forall \varepsilon > 0,\ \forall \ell > m,\ \forall p,\ \exists N_0(\varepsilon,\ell,p) \ge N_1(s),\ \forall z \in \mathbb{R}^n + i\{C_p \cup \{1/pw+C\}\},\ \forall N \ge N_0$$

$$\left|f(z+i(s+\alpha)h) - \sum_{k=0}^{N} \binom{s}{k} f(z+i\alpha h)\right| < \varepsilon\, A(s)(1+\|y\|^{-t}) \exp H_\ell(z),$$

where $t = m+n+2$ if $\alpha > 0$ or $t = m+n+3$ if $\alpha = 0$. This yields the convergence of the series (9.8) similarly to section 7 [14].

Actually, theorem 9.1 gives the condition f should satisfy in order that the Newton series holds when h ranges in a given domain. However, the function $H_\Omega(z)$ (formula (9.2)) arising in condition (9.6) is not given explicitely. This would be quite complicated (see [10] for entire functions and h complex). Therefore, we now start with a given class of functions and determine the domain of h the Newton series is valid in. For simplicity we will not give the largest possible domain, but still we get a considerable generalization of theorems 7.1 and 10.1 of [14].

The domain of convergence $\Omega_h = \{\zeta \mid |e^{-h\cdot\zeta}-1| < 1\}$ is determined by

$$-h\cdot\xi < \log(2\cos h\cdot\eta)$$

for $h \in \mathbb{R}^n$, see 4.1 and figure 4.1 of [14], here figure 9.1 when k = 0.

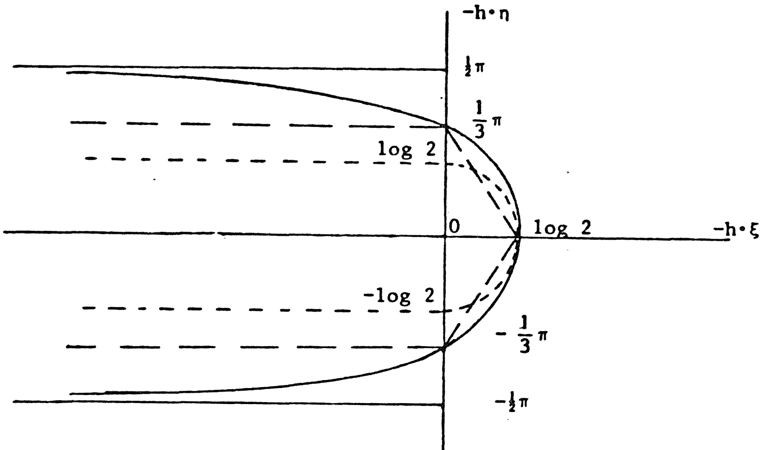


figure 9.1

Figure 9.1 gives the component of Ω_h that contains the origin. We approximate this domain from the inside by

(9.10) $$\{\zeta \mid -\tfrac{1}{3}\pi\ h\cdot\xi \pm \log 2\ h\cdot\eta < \tfrac{1}{3}\pi \log 2 \text{ when } -h\cdot\xi > 0 \text{ and } |h\cdot\eta| < \tfrac{1}{3}\pi \text{ when } -h\cdot\xi \le 0\} \subset \Omega_h.$$

Now let a convex homogeneous function $\tilde{a}(z)$ be given on $\mathbb{R}^n + iC$ with C a convex open cone in $\mathbb{R}^n$, such that $a(0,x)$ exists for $x \in \mathrm{pr}\ \mathbb{R}^n$. This function determines an open set Ω by

(9.11) $$\Omega = \mathrm{int}\{\zeta \mid -\mathrm{Im}\ z\cdot\zeta \le \tilde{a}(z),\ z \in \mathbb{R}^n + iC\}.$$

Let $\{\tilde{a}_m(z)\}_{m=1}^{\infty}$ be an increasing sequence convex homogeneous function with limit $\tilde{a}(z)$ and with $a_m(z)+\varepsilon_m \le a(z)$, $z \in \mathrm{pr}\ T^C$, for some $\varepsilon_m > 0$. Let Ω_m be the domain determined by the function a_m. Then from (9.10) and (9.11) it follows that $\bar{\Omega} \subset \bar{\Omega}_h$ when $h \in C$ satisfies

(9.12) $$\|h\| \le \min\left\{\frac{\frac{1}{3}\pi \log 2}{\tilde{a}(\frac{1}{3}\pi\tilde{h}, \pm\log 2\ \tilde{h})}, \frac{\frac{1}{3}\pi}{a(0,\pm\tilde{h})}\right\}.$$

Hence in that case $\Omega_m \subset \Omega_h$ for all m and we obtain

COROLLARY 9.1. *For functions* $f \in \widetilde{\mathrm{Exp}}(a;T^C)$ *the Newton series is valid, when* $h \in C$ *satisfies* (9.12).

However, when a is a rather constant function, a better condition for $\|h\|$ than (9.12) is obtained by approximating Ω_h from the inside by

$$\{\zeta \mid (h\cdot\xi)^2+(h\cdot\eta)^2 < \log^2 2 \text{ when } -h\cdot\xi > 0 \text{ and } |h\cdot\eta| < \log 2 \text{ when } -h\cdot\xi \le 0\} \subset \Omega_h .$$

This inclusion follows from $\log^2 2 \le \log^2(2\cos v)+v^2$, $|v| < \frac{1}{2}\pi$, which is true because

$$\log^2 2-\log^2(2\cos v) = (\log 2 - \log 2\cos v)(\log 2 + \log 2\cos v) \le$$

$$\le \log 2\cdot(2-2\cos v)\cdot 2\log 2 \le v^2 2\log^2 2 \le 0.98\, v^2 \le v^2 .$$

For $\zeta \in \Omega_m$ and $h \in C$ such that $-h\cdot\xi > 0$, we get

$$(h\cdot\xi)^2+(h\cdot\eta)^2 = -(-h\cdot\xi)h\cdot\xi-(-h\cdot\eta)h\cdot\eta \le \tilde{a}_m((-h\cdot\xi)h,(-h\cdot\eta)h) =$$

$$= \|h\|\{(h\cdot\xi)^2+(h\cdot\eta)^2\}^{\frac{1}{2}}\; a_m(\widetilde{(-h\cdot\xi)h,(-h\cdot\eta)h}),$$

hence

$$\{(h\cdot\xi)^2+(h.\eta)^2\}^{\frac{1}{2}} \le \|h\|\; a_m(\widetilde{\alpha h,\beta h})$$

for some $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. This is smaller than log2 when we require that

$$(9.13) \qquad \|h\| \le \min_{\substack{(\alpha,\beta)\in(\mathbb{R}^+,\mathbb{R})\\ \alpha^2+\beta^2=1}} \frac{\log 2}{a(\alpha\tilde{h},\beta\tilde{h})} .$$

In case $\zeta \in \Omega_m$ and $-h\cdot\xi \le 0$, $|h\cdot\eta| \le \|h\|\; a_m(0,\pm\tilde{h})$, so that $|h\cdot\eta| < \log 2$ if h satisfies (9.13). Thus for $h \in C$ with (9.13) satisfied, the domain Ω (9.11) is contained in Ω_h.

<u>COROLLARY 9.2</u>. *For functions* $f \in \widetilde{\mathrm{Exp}}(a;T^C)$ *the Newton series is valid, when* $h \in C$ *satisfies* (9.13).

For example, when $a(z)$ is a constant a on $\mathrm{pr}(\mathbb{R}^n+iC)$, the Newton series holds for $h \in C$ with $\|h\| \leq \log 2/a$, if the function f satisfies

$$\forall k: \; |f(z)| \leq M_k \exp a_m \|z\|, \qquad y \in C_k, \; \|y\| > 1/k, \; a_m < a.$$

This is a better bound than condition (9.12) since $\frac{1}{3}\pi \log 2\{(\frac{1}{3}\pi)^2+\log^2 2\}^{-\frac{1}{2}} <$ $< \log 2$. This condition for $\|h\|$ generalizes the one dimensional case of NÖRLUND [13] p.237.

In sections 7 and 10 of [14] we have seen that the bounds for $\|h\|$ were determined by the value of the convex homogeneous function a on C at the point $\tilde{h}$, namely $\|h\| \leq \log 2/a(\tilde{h})$ when $a(\tilde{h}) > 0$ or $\|h\|$ arbitrarily large when $a(\tilde{h}) \leq 0$, where the function f was of polynomial growth for $\|x\|$ large. Here the function f is of exponential growth also for $\|x\|$ large and the bounds for $\|h\|$ are determined by the values of a on

$$\{\beta\tilde{h}+i\alpha\tilde{h} \mid \alpha \geq 0, \; \beta \in \mathbb{R}, \; \alpha^2+\beta^2=1\},$$

see conditions (9.12) and (9.13). This bound is always positive and finite, except in one case, where the Newton series is valid for $h \in C$ with $\|h\|$ arbitrarily large, namely for functions f of exponential type, holomorphic in $\mathbb{R}^n+iC$, satisfying

$$\forall \varepsilon > 0: \; |f(\beta\tilde{h}+i\alpha\tilde{h})| \leq M_\varepsilon \exp \varepsilon(\alpha^2+\beta^2)^{\frac{1}{2}}, \qquad \alpha > 0, \; \beta \in \mathbb{R}.$$

This generalizes the case that $a(\tilde{h}) \leq 0$ in sections 7 and 10 of [14].

Finally we consider the case $\alpha < 0$ more carefully and we will find that in that case too the Newton series (9.8) is valid for all y such that $y-\alpha h \in C$, even if y does not belong to C. But first we have to modify the meaning of all the terms occurring in the series. We assume in the remaining of this section that for $\alpha < 0$

$$\mathrm{Re}\; s \geq -\alpha > m_0 \, ,$$

where m_0 is a non-negative integer.

Firstly, we consider the series

$$(9.14)\qquad \sum_{k=0}^{\infty} \binom{s}{k}(e^{-h\cdot\zeta}-1)^k = \sum_{k=0}^{\infty} \binom{s}{k} \left\{\sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} e^{-mh\cdot\zeta}\right\} = e^{-sh\cdot\zeta}$$

for $\zeta \in \Omega_h$. We will show that we can rearrange some of the terms in this series. Therefore, we remark that the series

$$\sum_{m=0}^{m_0} \sum_{k=m}^{\infty} \binom{s}{k}\binom{k}{m}(-1)^{k-m} \lambda_m$$

is absolutely convergent for arbitrary numbers λ_m, $m = 0,1,\ldots,m_0$, since by (5.1) [14]

$$\sum_{m=0}^{m_0} \sum_{k=N_1(s)}^{\infty} \left|\binom{s}{k}\right|\cdot\binom{k}{m}\cdot|\lambda_m| = \sum_{m=0}^{m_0} |\lambda_m|\cdot\left|\binom{s}{m}\right| \sum_{k=N_1(s)-m}^{\infty} \left|\binom{s-m}{k}\right| \le$$

$$\le \sum_{m=0}^{m_0} \frac{2}{|\Gamma(-s+m)|} |\lambda_m|\cdot\left|\binom{s}{m}\right| \sum_{k=1}^{\infty} k^{\alpha+m-1} < \infty,$$

because $\alpha+m < 0$. Hence

$$(9.15)\qquad \sum_{m=0}^{m_0} \sum_{k=m}^{\infty} \binom{s}{k}\binom{k}{m}(-1)^{k-m} \lambda_m = \sum_{k=m_0+1}^{\infty} \binom{s}{k} \sum_{m=0}^{m_0} \binom{k}{m}(-1)^{k-m} \lambda_m +$$

$$+ \sum_{k=0}^{m_0} \binom{s}{k} \sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} \lambda_m \overset{\text{not}}{=\!=} \Phi(s;\lambda_1,\ldots,\lambda_{m_0})$$

exists. Now we write (9.14) in the following way

$$e^{-sh\cdot\zeta}-\Phi(s;\lambda_1,\ldots,\lambda_{m_0}) = \lim_{N\to\infty}\left\{\sum_{k=0}^{k} \binom{s}{k}\Big[\sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} e^{-mh\cdot\zeta}\Big] -\right.$$

$$\left.- \sum_{k=m_0+1}^{N} \binom{s}{k}\Big[\sum_{m=0}^{m_0} \binom{k}{m}(-1)^{k-m} \lambda_m\Big] - \sum_{k=0}^{m_0} \binom{s}{k}\Big[\sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} \lambda_m\Big]\right\} =$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} \binom{s}{k}\Big[\sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m} \mu_m\Big],$$

where $\mu_m = \exp -mh\cdot\zeta$ for $m > m_0$ and $\mu_m = \exp(-mh\cdot\zeta)-\lambda_m$ for $m \le m_0$. In order to compute $\Phi(s;\lambda_1,\ldots,\lambda_{m_0})$ we derive from (9.15) and from Re $s > m$

$$\Phi(s;\lambda_1,\ldots,\lambda_{m_0}) = \sum_{m=0}^{m_0} \lambda_m\binom{s}{m} \sum_{k=0}^{\infty} \binom{s-m}{k}(-1)^k = \sum_{m=0}^{m_0} \lambda_m\binom{s}{m}(1-1)^{s-m} = 0$$

for any numbers λ_m. Choosing $\lambda_m = \exp -mh\cdot\zeta$ we obtain

$$(9.16)\qquad e^{-sh\cdot\zeta} = \sum_{k=m_0+1}^{\infty} \binom{s}{k} \sum_{m=m_0+1}^{k} \binom{k}{m}(-1)^{k-m}\, e^{-mh\cdot\zeta} =$$

$$= \sum_{k=0}^{\infty} \binom{s}{k} \sum_{m=0}^{k} \binom{k}{m}(-1)^{k-m}\, \mu_m,$$

where $\mu_m = \exp -mh\cdot\zeta$ for $m > m_0$ and μ_m arbitrary for $m \le m_0$. In fact, we have rearranged the terms in (9.14) so, that first the summation is performed over all the terms with $\exp -mh\cdot\zeta$ for $m \le m_0$ and it turns out that the series is independent of these terms.

Secondly, we give bounds to the functions

$$\phi_N(\zeta) = \sum_{k=0}^{N} \binom{s}{k}(e^{-h\cdot\zeta}-1)^k ,$$

when Re $s \ge -\alpha > m_0$. From p.27 [14] we get for $\zeta \in \Omega_h(\varepsilon)$

$$|\phi_N(\zeta)| \le 1+B_s\{1+(-\log\rho)^{-\alpha}\}$$

with $\rho = 1-\frac{1}{2}\varepsilon \exp -\mathrm{Re}\, h\cdot\zeta$, whence

$$|\phi_N(\zeta)| \le 1+B_s+C(s,\alpha)\varepsilon^{-\alpha}\, e^{\alpha\,\mathrm{Re}\, h\cdot\zeta}.$$

Therefore, we may conclude as in lemma 9.1 that the series $e^{-\alpha h\cdot\zeta}\,\phi_{N,z}(\zeta)$ converges in every space $A(H_m+0;C)$, when y is such that $y-\alpha h \in C$, $h \in C$ and that for $\mu_\zeta \in A(H_m+0;C)'$ the series

$$\sum_{k=0}^{N} \binom{s}{k} e^{-\alpha h\cdot\zeta} (e^{-h\cdot\zeta}-1)^k \mu_\zeta$$

converges strongly in $A(H_m+0;C)'$.

Now using (9.16) we derive that for $f \in \widetilde{\mathrm{Exp}}(H_\Omega,T^C)$ the following Newton series converges in the topology of $\widetilde{\mathrm{Exp}}\ A(_\Omega,T^C)$

$$f(z+i(s+\alpha)h) = \sum_{k=m_0+1}^{\infty} \binom{s}{k} < \sum_{m=m_0+1}^{k} \binom{k}{m}(-1)^{k-m} e^{-mh\cdot\zeta} e^{-\alpha h\cdot\zeta} \mu_\zeta, e^{iz\cdot\zeta}> =$$

$$= \sum_{k=m_0+1}^{\infty} \binom{s}{k} \sum_{m=m_0+1}^{k} \binom{k}{m}(-1)^{k-m} f(z+i(m+\alpha)h).$$

Replacing $z+i\alpha h$ by z and using the second part of (9.16) we find that the Newton series

$$(9.17) \qquad f(z+ish) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta_{ih}^{k*} f(z),$$

where the asterix means that in the points $\{z+imh \mid m = 0,1,...,m_0\}$ where f is singular or undefined we may take zero instead of $f(z+imh)$, is valid for all $h \in C$, $\|h\| \le b$, $\mathrm{Re}\ s \ge -\alpha > m_0 \ge 0$ and all y such that $y-\alpha h \in C$ and that it converges according to (9.9).

It may happen that $f \in \widetilde{\mathrm{Exp}}(H_\Omega,T^C)$ can be continued analytically outside the domain $\mathbb{R}^n+i\,C$, so that $f(z+imh)$ is defined for all m. But in fact, this is not essential and the series (9.17) has a meaning even if f is singular or undefined in some points $z+imh$, $m \le m_0$, as long as $\mathrm{Re}\ s > m_0$. Obviously, this is the generalization to several variables of the one dimensional case given in NÖRLUND [13] p.237 in the first example 123.

We conclude with

<u>THEOREM 9.1*.</u> *When* $\mathrm{Re}\ s \ge -\alpha > m_0 \ge 0$, *theorem* 9.1 *also holds for all* y *such that* $y-\alpha h \in C$; *then the modified Newton series* (9.17) *converges according to* (9.9).

APPENDIX

PASSAGE FROM LOCAL TO GLOBAL RELATIONS

In this appendix we discuss some well-known properties in the theory of functions of several complex variables. Except section I all sections are devoted to the problem how to extend local relations between holomorphic functions to global relations. As some readers may not be familiar with the topics used to solve this problem, we will go more into detail than merely copying definitions and theorems from litterature. We give those proofs that show how to use the various concepts (as sheaves and cohomology) in deriving the main result. In fact, since we want a quantitative result in section 7, we perform the same steps there as in section IV of this appendix, then taking care of estimates. Therefore, we also give the quantitative theorems these steps start from. Almost the same method HÖRMANDER [7] uses in his book is applied here and we repeatedly refer to this book.

I. DOMAINS OF HOLOMORPHY

In this section we give some definitions and theorems which are used in section 2, the case of holomorphic functions on compact sets.

Let Ω be an open set in $\mathbb{C}^n$. We denote by $A(\Omega)$ the space of all holomorphic functions in Ω with the topology of uniform convergence on compact subsets K of Ω. All functions holomorphic in a certain domain Ω in $\mathbb{C}^n$, $n \geq 2$, might be continued analytically into a larger domain. Domains for which this is not possible are called *domains of holomorphy*. Thus Ω is a domain of holomorphy if and only if there exists a function $f \in A(\Omega)$ which cannot be continued analytically beyond Ω, that is, it is not possible to find Ω_1 and Ω_2, with $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$ and with Ω_2 connected and not contained in Ω, and $f_1 \in A(\Omega_2)$ so that $f = f_1$ in Ω_1. One can decide whether a domain Ω is a domain of holomorphy by other means too. We will discuss some of these means which are most useful in applications.

For a compact set K of an open set Ω we define the $A(\Omega)$-hull $\hat{K}_\Omega$ of K by

(A1) $$\hat{K}_\Omega = \{z \mid z \in \Omega,\ |f(z)| \leq \sup_{z \in K} |f(z)| \text{ for all } f \in A(\Omega)\}.$$

If we choose $f(z) = \exp z\cdot\zeta$ we find that $\hat{K}_\Omega$ is contained in the convex hull ch(K) of K. Domains of holomorphy can be characterized by the following theorem, th.2.5.5(ii) of [7]:

THEOREM A1. *Ω is a domain of holomorphy if and only if from $K \subset\subset \Omega$ it follows that $\hat{K}_\Omega \subset\subset \Omega$.*

Hence convex open sets in $\mathbb{C}^n$ are domains of holomorphy. Conversely Bochner's theorem (th.2.5.12 of [7] or 17.5 of [16]) yields:

THEOREM A2. *A tube domain $\mathbb{R}^n+i\,O$, where O is a domain in $\mathbb{R}^n$, is a domain of holomorphy if and only if O is convex.*

A more geometrical characterization of domains of holomorphy is obtained by regarding them as pseudoconvex sets. These sets can be defined with the aid of plurisubharmonic functions. Rather than giving a precise definition (2.6.1 of [7]) we state some results. As in (A1) one can define a $P(\Omega)$-hull $\hat{K}^P_\Omega$ of K by requiring that f is plurisubharmonic instead of $f \in A(\Omega)$. Then like theorem A1 an open set Ω is pseudoconvex if and only if from $K \subset\subset \Omega$ it follows that $\hat{K}^P_\Omega \subset\subset \Omega$. Since the function $|f(z)|$ is plurisubharmonic if f is holomorphic, domains of holomorphy are pseudoconvex. The converse is also true (th.4.2.8 of [7]):

THEOREM A3. *An open pseudoconvex set is a domain of holomorphy.*

Actually, if K is a compact set of an open pseudoconvex set Ω, then $\hat{K}_\Omega$ equals $\hat{K}^P_\Omega$ (th.3.4.3 of [7]). Therefore, we will not distinguish between the concepts of pseudoconvex open set and of domains of holomorphy and we assume Ω to be one or the other where necessary.

THEOREM A5. *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and K a compact subset of Ω, such that $\hat{K}_\Omega = K$. Every function analytic in a neighborhood of K can then be approximated uniformly on K by functions in $A(\Omega)$.*

This is theorem 4.3.2 of [7].

DEFINITION A5. A domain of holomorphy $\Omega \subset \mathbb{C}^n$ is called a Runge domain if polynomials are dense in $A(\Omega)$, that is if every $f \in A(\Omega)$ can be uniformly approximated on an arbitrary compact set in Ω by analytic polynomials.

Since polynomials are dense in $A(\mathbb{C}^n)$ we might as well have considered arbitrary entire functions instead of polynomials in definition A5. For a compact set K we define

$$\tilde{K} = \{z \mid z \in \mathbb{C}^n,\ |P(z)| \le \sup_{z \in K} |P(z)| \text{ for all polynomials } P\}.$$

Then $\tilde{K} = \hat{K}_{\mathbb{C}^n}$ and compact sets K with $K = \tilde{K}$ are called *polynomially convex*. However, we even have (th.2.7.3 of [7]):

<u>THEOREM A6</u>. *Ω is a Runge domain if and only if for every compact set K in Ω $\tilde{K} = \hat{K}_\Omega$.*

This theorem is a special case (namely when $\Omega_1 = \Omega$ and $\Omega_2 = \mathbb{C}^n$) of the following theorem (th.4.3.3 of [7]):

<u>THEOREM A7</u>. *Let $\Omega_1 \subset \Omega_2$ be domains of holomorphy. Then every function in $A(\Omega_1)$ can be approximated by functions in $A(\Omega_2)$ uniformly on every compact subset of Ω_1 if and only if for every compact subset K of Ω_1 we have $\hat{K}_{\Omega_2} = \hat{K}_{\Omega_1}$.*

II. THE $\bar{\partial}$-OPERATOR

In this section we define the $\bar{\partial}$-operator and give some existence theorems.

Let u be a complex valued differentiable function in $\Omega \subset \mathbb{C}^n$. We denote $z = x+iy \in \Omega$ also as $z = (y,x)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, where now Ω is regarded as an open set in $\mathbb{R}^{2n}$ (the reason for not writing $z = (x,y)$ is, that, when we do so for $\zeta = \xi+i\eta \in \mathbb{C}^n$, $\zeta = (\xi,\eta)$, then $-\mathrm{Re}(iz\cdot\zeta)$ can be written as inproduct between the vectors (y,x) and (ξ,η) in $\mathbb{R}^{2n}$). The components of z are denoted by $z_j = x_j+iy_j$ and $\bar{z}_j = x_j-iy_j$. When differentiation takes place, we rather use z_j and $\bar{z}_j$, $j = 1,\dots,n$, as coordinates than (y,x), so that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) \quad\text{and}\quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right).$$

Then we get

$$du = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j$$

and with

$$\partial u = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial} u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j$$

we may also write

$$du = \partial u + \bar{\partial} u \ .$$

When we write $\bar{\partial} u = 0$ in Ω, we mean that every component $\bar{\partial}_j u = \partial u / \partial \bar{z}_j$ must vanish in Ω. These are exactly the Cauchy Riemann equations, so that we get

<u>THEOREM A8</u>. *A function* u *in* $C^1(\Omega)$ *is holomorphic in the open set* Ω *if and only if* $\bar{\partial} u = 0$ *in* Ω.

In the above $\bar{\partial} u$ is a (0,1)-form. We call g a (0,1)-form in Ω if it can be written as

$$g(z) = \sum_{k=1}^{n} g_k(z) \, d\bar{z}_k, \qquad z \in \Omega,$$

where g_k, $k = 1,\dots,n$, are functions in Ω. We will give a condition when a (0,1)-form g can be written as $\bar{\partial} u$ for some function u. A necessary condition on g is $\bar{\partial} g = 0$, where we define

$$\bar{\partial} g = \sum_{m=1}^{n} \sum_{k=1}^{n} \frac{\partial g_k}{\partial \bar{z}_m} d\bar{z}_m \wedge d\bar{z}_k$$

when the functions g_k are differentiable. Here we may use the rule

$$\text{(A2)} \qquad d\bar{z}_m \wedge d\bar{z}_k = -d\bar{z}_k \wedge d\bar{z}_m, \qquad k,m = 1,\dots,n$$

and $\bar{\partial} g = 0$ if the coefficients of all the $d\bar{z}_m \wedge d\bar{z}_k$ $(m < k)$ vanish. It is easy to see that for any $u \in C^2(\Omega)$ $\bar{\partial}\,\bar{\partial}\, u = 0$, so that indeed $\bar{\partial} g = 0$ is a necessary condition. When Ω is pseudoconvex, it is also a sufficient condition. This we state in a theorem, which we give in a more general form, namely for $(0,q)$-forms. We say that g is a $(0,q)$-form in Ω $(q = 0,1,\ldots,n)$ if it can be written in the form

$$g = \sum_{|I|=q} g_I(z)\, d\bar{z}^I, \qquad z \in \Omega$$

where $I = (k_1,\ldots,k_q)$ is a multiindex and $d\bar{z}^I = d\bar{z}_{k_1}\wedge\ldots\wedge d\bar{z}_{k_q}$ and where the summatition is performed over all multiindices I with $k_1 < k_2 < \ldots < k_q$ (for again we may use the rule (A2)). Thus g has $\binom{n}{q}$ coefficients g_I. We define

$$\bar{\partial} g = \sum_{k=1}^{n} \sum_{|I|=q} \frac{\partial g_I}{\partial \bar{z}_k}\, d\bar{z}_k \wedge d\bar{z}^I$$

where (A2) should be used. It is clear that $\bar{\partial}\,\bar{\partial}\, g = 0$. Now the following existence theorem for the $\bar{\partial}$-operator holds (cor.4.2.6 of [7]):

<u>THEOREM A9</u>. *Let the coefficients* g_I *of the* $(0,q+1)$*-form* g *in the pseudoconvex open set* Ω *be* C^∞*-functions and let* $\bar{\partial} g = 0$. *Then there exists a* $(0,q)$*-form* u *with* C^∞*-coefficients in* Ω *such that* $\bar{\partial} u = g$.

Next we state a similar theorem, where besides the existence of u also estimates of u in terms of estimates for g are given. We use the measure $e^{-\phi} d\lambda$, where $d\lambda$ is the Lebesgue measure in $\mathbb{C}^n$, and $\phi(y,x)$ is a plurisubharmonic function. We do not give the definition of a plurisubharmonic function (see 2.6.1 of [7]), but we merely state that a convex function $\tilde{a}(y,x)$ is plurisubharmonic, that $\log(1+\|z\|^2)$ is plurisubharmonic and that $\alpha\phi+\beta\psi$ is plurisubharmonic for $\alpha \ge 0$, $\beta \ge 0$ whenever ϕ and ψ are plurisubharmonic. These facts will be sufficient for the applications we make. For a $(0,q)$-form f in Ω $(q = 0,1,\ldots,n)$, where the coefficients f_I are locally square integrable functions, we write

$$|f(z)|^2 = \sum_{|I|=q} |f_I(z)|^2, \qquad z \in \Omega$$

and

$$\|f\|_\phi = \int_\Omega |f(z)|^2\, e^{-\phi(z)}\, d\lambda(z) \;.$$

We remark that for such an f we must take the weak derivative in $\bar{\partial} f$, thus derivatives in distributional sense. Then we state the following theorem (th.4.4.2 of [7]):

THEOREM A10. *Let* Ω *be a pseudoconvex open set in* $\mathbb{C}^n$ *and* ϕ *any plurisubharmonic function in* Ω. *For every* (0,q+1)*-form* g *with locally square integrable coefficients, with* $\|g\|_\phi$ *finite and with* $\bar{\partial} g = 0$, *there is a* (0,q)*-form* u *in* Ω *with locally square integrable coefficients, such that* $\bar{\partial} u = g$ *and*

$$\int_\Omega |u(z)|^2 e^{-\phi(z)} (1+\|z\|^2)^{-2} \, d\lambda(z) \le \|g\|_\phi^2 .$$

Here u *depends on* ϕ, *when the right hand side is finite for more than one function* ϕ.

III. ANALYTIC SHEAVES

In this section we discuss some properties of analytic sheaves and we formulate the main problem of this appendix. We do not give a general definition of a sheaf on an open set Ω in $\mathbb{C}^n$, but we just give the properties we need in this paper. A more complete description can be found in [6] or [7].

For $z \in \Omega$ we denote by A_z the set of equivalence classes of functions f which are analytic in a neighborhood of z, under the equivalence relation $f \sim g$ if f = g in a neighborhood of z in Ω. The residue class f_z of f in A_z is called the germ of f at z.[1)] It is clear that A_z is a ring. Let

$$A = \bigcup_{z \in \Omega} A_z$$

1) Since an analytic function is determined completely when it is given in an open set, the residue class of f is trivial: it consists of f only. But when we consider the restriction of f to a variety V in Ω, we get a sheaf on V (V is a simultaneous zero set of holomorphic functions in Ω) and the equivalence classes are no longer trivial, see [6] def.IV D.5, p.143 and see also section VI. Also, when we consider C^∞-functions instead of analytic functions, it has sense to define the germ of f at z as a residue class.

where the rings A_z are considered as disjoint sets. Furthermore, let the collection of subsets of A of the form

$$\{f_z \mid z \in \omega \subset \Omega, \text{ where } \omega \text{ is open and } f \in A(\omega)\} ,$$

where ω runs over the collection of open subsets of Ω and f runs over the elements of $A(\omega)$, be a basis for the topology of A. Then for every open subset ω of Ω and every $f \in A(\omega)$ the map ϕ from ω into A with $\phi(z) = f_z$ is open and continuous.

Let π be the map from A into Ω which maps A_z onto z. Then $\pi\phi = \text{identity}$. In general we call the image of a subset U of Ω under a continuous map ϕ: $U \to A$, with $\pi\phi = \text{identity}$, or the map ϕ itself, a *section of* A *over* U. The set of all sections of A over U is denoted by $\Gamma(U,A)$. In fact an element of $\Gamma(U,A)$ is the restriction to U of a holomorphic function in a neighborhood of U in Ω or if U itself is open, it is a holomorphic function in U.

The space A is an example of a *sheaf* on Ω. Since A_z is a ring for each $z \in \Omega$, we can consider a sheaf F such that F_z is an A_z-module for each $z \in \Omega$ and such that the product of a section in A and a section in F is a section in F. Such a sheaf is called an *analytic sheaf*. In particular we will consider ideals in A_z and modules in A_z^p. Since the ring A_z is a noetherian ring ([7] th.6.3.3 or [6] th.II.B.9) the ideals in A_z and the modules in A_z^p are finitely generated.

For example, let U be an open subset of Ω with $\emptyset \neq U \neq \Omega$ and let an analytic sheaf F be given by $F_z = A_z$ if $z \in U$ and $F_z = 0$ if $z \in \Omega\backslash U$. A section of this sheaf over a connected open set intersecting $\Omega\backslash U$ must be zero by the uniqueness of analytic continuation. In any point $z \in \Omega$, F_z is finitely generated, but in any neighborhood ω of a boundary point of U in Ω F is not finitely generated by the sections over ω.

Thus although F_z is finitely generated in any point $z \in \Omega$, we cannot always use the same generators for all z in a neighborhood of any point. However, we consider sheafs where this property indeed is satisfied. Namely, an analytic sheaf F is said to be *locally finitely generated* if for every given point in Ω there exists a neighborhood ω in Ω and a finite number of sections $f_1,\dots,f_q \in \Gamma(\omega,F)$ so that F_z is generated by $(f_1)_z,\dots,(f_q)_z$ as an A_z-module for every $z \in \omega$. In particular we will consider locally finitely generated subsheaves $\mathcal{F}$ of A^p, so that then in the above definition for $k = 1,\dots,q$ f_k is a p-tuple of analytic functions $f_k^j \in A(\omega)$ in ω, $j = 1,\dots,p$ with $(f_k)_z = (f_k^1(z),\dots,f_k^p(z))$.

Let $\mathcal{F}$ be a locally finitely generated analytic sheaf, let $f_1,\dots,f_q$ be sections over an open set U of Ω and let for any $z \in U$

$$\mathcal{R}_z(f_1,\dots,f_q) = \{(g^1,\dots,g^q) \in A_z^q \mid \sum_{k=1}^{q} g^k\ (f_k)_z = 0\}.$$

$\mathcal{R}_z$ is a submodule of A_z^q, called the module of relations between $f_1,\dots,f_q$ at z. Then

$$\mathcal{R}(f_1,\dots,f_q) = \bigcup_{z\in U} \mathcal{R}_z(f_1,\dots,f_q)$$

is a subsheaf of A^q on U, called the *sheaf of relations* between $f_1,\dots,f_q$.

A locally finitely generated analytic sheaf $\mathcal{F}$ is called a *coherent* analytic sheaf, if $\mathcal{R}(f_1,\dots,f_q)$ is locally finitely generated for all $U \subset \Omega$, all $f_k \in \Gamma(U,\mathcal{F})$, $k = 1,\dots,q$ and all q. When $\mathcal{F}$ is a locally finitely generated subsheaf of A^p, the last condition is always satisfied. For by Oka's theorem ([7] th.6.4.1 and th.7.1.5 or [6] th.IV.C.1 and IV.B.7 and 8) every locally finitely generated subsheaf $\mathcal{F}$ of A^p is coherent; that is for any point in $U \subset \Omega$ one can find a neighborhood $\omega \subset U$ and finitely many elements $G_1,\dots,G_r \in \Gamma(\omega,\mathcal{R}(f_1,\dots,f_q))$ (thus for $\ell = 1,\dots,r$ $G_\ell = (g_\ell^1,\dots,g_\ell^q) \in$ $\in A(\omega)^q$ and for $z \in \omega$

$$\text{(A3)}\qquad \sum_{k=1}^{q} g_\ell^k(z)\ f_k^j(z) = 0, \quad j = 1,\dots,p\ ,$$

$\ell = 1,\dots,r$), so that $\mathcal{R}_z$ for every $z \in \omega$ is equal to the A_z-module generated by $(G_1)_z,\dots,(G_r)_z$.

If two of the sheafs of the exact sequence (that is the image of one map is the kernel of the next map)

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

are coherent, then the third sheaf is coherent too, see [6] th.IV.B.13.

From now on we only consider coherent analytic sheaves and we do not state this all the time.

Let now in (A3) the functions $f_k^j \in A(U)$ be polynomials $P_{jk} = f_k^j$, $j = 1,\dots,p$, $k = 1,\dots,q$ and let $U = \Omega$. Then we consider the sheaf homomorphism

$$P: A^q \to A^p$$

defined by mapping $(g^1(z),\dots,g^q(z)) \in A_z^q$ to

$$\left(\sum_{k=1}^{q} P_{1k}(z)\, g^k(z),\dots,\sum_{k=1}^{q} P_{pk}(z)\, g^k(z)\right) \in A_z^p, \qquad z \in \Omega.$$

We have seen that the image and the kernel of this map are coherent analytic sheaves and in particular it follows from the proof of the Oka theorem (th.6.4.1 of [7]), that the functions g_p^k in (A3), $\ell = 1,\dots,r$, $k = 1,\dots,q$, can be chosen to be polynomials. Thus the kernel R_P of this map is generated by the germs of all q-tuples $Q = (Q_1,\dots,Q_q)$ with Q_k polynomials for $k = 1,\dots,q$, such that

$$\text{(A4)} \qquad \sum_{k=1}^{q} P_{jk}(z)\, Q_k(z) = 0, \qquad z \in \Omega.$$

Furthermore, since the polynomial ring (over $\mathbb{C}$) is noetherian, the module of all $Q = (Q_1,\dots,Q_q)$ with Q_q polynomials satisfying (A4) is finitely generated over the polynomial ring. Thus since all the generators, that is in a neighborhood of all the points of Ω, are polynomial q-tuples Q, R_P is generated by a finite number of such Q, say by $Q_\ell = (Q_{1\ell},\dots,Q_{1\ell})$, $\ell = 1,\dots,r$, where $Q_{k\ell}$ is a polynomial. Summarizing we get the exact sequences of sheaf homomorphisms

$$\text{(A5)} \qquad 0 \to R_P \to A^q \xrightarrow{P} F \to 0\ ,$$

where F is the image of P and

$$\text{(A6)} \qquad 0 \to R_Q \to A^r \xrightarrow{Q} R_P \to 0.$$

A section $f = (f_1,\dots,f_p)$ in F is a p-tuple holomorphic functions in Ω, thus $f_j \in A(\Omega)$, $j = 1,\dots,p$, satisfying locally, that is in a neighborhood ω_s of each point in Ω,

$$\text{(A7)} \qquad f_j(z) = \sum_{k=1}^{q} P_{jk}(z)\, g_k^s(z), \qquad z \in \omega_s,\ g_k^s \in A(\omega_s),\ j = 1,\dots,p.$$

In $\omega_s \cap \omega_t$ the functions $g_k^s \in A(\omega_s)$ are not necessarily equal to the functions $g_k^t \in A(\omega_t)$, for they may differ by a section h^{st} in R_p over $\omega_s \cap \omega_t$. We would like that $h^{st} = 0$, thus that (A7) holds globally, that is we would like to find $g_k \in A(\Omega)$ such that (A7) holds for all $z \in \Omega$. The main problem of this appendix is to prove that such functions g_k, $k = 1,\dots,q$, exist. We can formulate this as: the problem is to prove that the following sequence of sections is exact

$$\text{(A8)} \qquad 0 \longrightarrow \Gamma(\Omega,R_p) \longrightarrow \Gamma(\Omega,A^q) \xrightarrow{P} \Gamma(\Omega,F) \longrightarrow 0.$$

That the sequence is exact in the first two places is clear, but our attention is paid to the exactness in the last place, thus to prove that the map P is surjective. We will find that (A8) is indeed exact, when Ω is pseudoconvex. Then starting with (A6) we would at the same time have solved the problem:

<u>THEOREM A11</u>. *If the function* $f_k \in A(\Omega)$ *satisfy*

$$\sum_{k=1}^{q} P_{jk}(z)\, f_k(z) = 0, \qquad z \in \Omega,\ j = 1,\dots,p\,,$$

then there are functions $g_\ell \in A(\Omega)$, $\ell = 1,\dots,r$, *such that*

$$f_k(z) = \sum_{\ell=1}^{r} Q_{k\ell}(z)\, g_\ell(z), \qquad z \in \Omega,\ k = 1,\dots,q\,,$$

when Ω *is pseudoconvex.*

IV. COHOMOLOGY GROUPS WITH VALUES IN A SHEAF

In this section we define cohomology groups and show how they are used to solve the problem formulated in section III.

We consider the sheaf $\mathcal{F}$ as an additive commutative group. Let $\mathcal{U} = \{I_i\}_{i\in I}$ be an open covering of the open set Ω in $\mathbb{C}^n$. If p is a non-negative integer, we denote by $s = (s_o,\ldots,s_p)$ any element in I^{p+1} and we set $U_s = U_{s_0} \cap \ldots \cap U_{s_p}$. A map assigning to every $s \in I^{p+1}$ a section $c_s \in \Gamma(U_s,\mathcal{F})$ so that c_s is an alternating function of s (that is, c_s changes sign if two indices in s are permuted) is called a p-*cochain* of the covering $\mathcal{U}$ with values in $\mathcal{F}$. Here we define $\Gamma(\emptyset,\mathcal{F}) = 0$, the abelian group with one element. Then the set $C^p(\mathcal{U},\mathcal{F})$ of all p-cochains is an abelian group.

A map δ from $C^p(\mathcal{U},\mathcal{F})$ into $C^{p+1}(\mathcal{U},\mathcal{F})$, called the *coboundary operator*, is defined as follows: if $c \in C^p(\mathcal{U},\mathcal{F})$, then for $s \in I^{p+2}$

$$(\delta c)_s = \sum_{j=0}^{p+1} (-1)^j\, c_{s_0\ldots\hat{s}_j\ldots s_{p+1}}$$

where the notation $\hat{s}_j$ means that the index s_j should be removed. We introduce the group of p-*cocycles*

$$Z^p(\mathcal{U},\mathcal{F}) = \{c \mid c \in C^p(\mathcal{U},\mathcal{F}),\ \delta c = 0\}$$

and the group of p-*coboundaries*

$$B^p(\mathcal{U},\mathcal{F}) = \{\delta c \mid c \in C^{p-1}(\mathcal{U},\mathcal{F})\},$$

where $C^{-1} = 0$. Since for all $c \in C^p(\mathcal{U},\mathcal{F})$ $\delta\delta c = 0$, B^p is a subgroup of Z^p. We can, therefore, define the quotient group

$$H^p(\mathcal{U},\mathcal{F}) = Z^p(\mathcal{U},\mathcal{F}) \Big/ B^p(\mathcal{U},\mathcal{F}) ,$$

which is called the p^{th} *cohomology group* of $\mathcal{U}$ with values in $\mathcal{F}$.

For example, if c is a 0-cocycle, then $c_{s_0} - c_{s_1} = 0$ in $U_{s_0} \cap U_{s_1}$ for all

s_0 and s_1, which means that there is a section $f \in \Gamma(\Omega,F)$ with the restriction c_s to U_s for every s. Hence

(A9) $$H^0(\mathcal{U},F) \cong \Gamma(\Omega,F) .$$

Let $\mathcal{V} = \{V_j\}_{j\in J}$ be another covering of Ω, which is a refinement of $\mathcal{U}$. This means that there is a map ρ from J into I such that $V_j \subset U_{\rho(j)}$ for every $j \in J$. If $c \in C^p(\mathcal{U},F)$, we can then define a cochain $\rho c \in C^p(\mathcal{V},F)$ by setting $(\rho c)_s$ equal to the restriction of $c_{\rho(s)_p\ldots\rho(s_p)}$ to V_s. One easily sees that ρ commutes with the coboundary operators in $C^p(\mathcal{U},F)$ and $C^p(\mathcal{V},F)$ and, therefore, it induces a map ρ^* from $H^p(\mathcal{U},F)$ into $H^p(\mathcal{V},F)$. This map ρ^* is independent of the choice of ρ (see prop.7.3.1 of [7]).

Let E be the sheaf of germs of C^∞-functions on Ω (see the footnote on page 85).

THEOREM A12. *Let F be a sheaf of E-modules on Ω, then $H^p(\mathcal{U},F) = 0$ for $p \geq 1$ and every covering $\mathcal{U}$ of Ω.*

PROOF. Let ϕ_ν be a partition of unity subordinate to the covering $\mathcal{U}$, that is

i) ϕ_ν is a C^∞-function with compact support in U_{i_ν} for a certain index i_ν;

ii) all but a finite number of functions ϕ_ν vanish identically on any compact subset of Ω;

iii) $\sum_\nu \phi_\nu = 1$ on Ω.

For $c \in Z^p(\mathcal{U},F)$ we put, when $s \in I^p$,

$$g_s = \sum_\nu \phi_\nu \, c_{i_\nu s},$$

which defines a cochain g in $C^{p-1}(\mathcal{U},F)$. Since with $s \in I^{p+1}$

$$(\delta c)_{i_\nu s} = c_s + \sum_{j=0}^{p+1} (-1)^{j+1} \, c_{i_\nu s_0\ldots\hat{s}_j\ldots s_p} = 0$$

we get

$$(\delta g)_s = \sum_\nu \sum_{j=0}^{p+1} \phi_\nu (-1)^j \, c_{i_\nu s_0\ldots\hat{s}_j\ldots s_p} = \sum_\nu \phi_\nu \, c_s = c_s .$$

Thus c is a coboundary. □

Let F, G, H be three sheaves of abelian groups on Ω and let ϕ and ψ be sheaf homomorphisms such that the sequence

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\psi} H \longrightarrow 0$$

is exact (thus ϕ is injective, ψ is surjective, the kernel of ψ is the image of ϕ). This defines exact sequences between sections; thus we get the exact sequence

$$0 \longrightarrow C^p(U,F) \longrightarrow C^p(U,G) \longrightarrow C^p(U,H) \quad ,$$

but the last map is not necessarily surjective. We denote its image by $C^p_a(U,H)$ and call it the group of *liftable* cochains. We then have an exact sequence

$$0 \longrightarrow C^p(U,F) \longrightarrow C^p(U,G) \longrightarrow C^p_a(U,H) \longrightarrow 0 \ .$$

Since δ commutes with ψ, $C^p_a(U,H)$ is mapped by δ into $C^{p+1}_a(U,H)$ and we can define the cohomology groups

$$H^p_a(U,H) = Z^p_a \Big/ B^p_a \ ,$$

where $Z^p_a(B^p_a)$ is the group of all liftable p-cocycles (coboundaries of liftable (p-1)-cochains). Then we have the commutative diagram with exact columns:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C^{p-1}(U,F) & \xrightarrow{\delta} & C^p(U,F) & \xrightarrow{\delta} & C^{p+1}(U,F) \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
C^{p-1}(U,G) & \xrightarrow{\delta} & C^p(U,G) & \xrightarrow{\delta} & C^{p+1}(U,G) \\
\downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
C^{p-1}_a(U,H) & \xrightarrow{\delta} & C^p_a(U,H) & \xrightarrow{\delta} & C^{p+1}_a(U,H) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Now we construct a map δ^* from $H^p_a(\mathcal{U},\mathcal{H})$ into $H^{p+1}(\mathcal{U},\mathcal{F})$ as follows: If $f \in Z^p_a(\mathcal{U},\mathcal{H})$, then $f = \psi g$ for some $g \in C^p(\mathcal{U},\mathcal{G})$ and $\psi\delta g = \delta\psi g = \delta f = 0$, hence $\delta g = \phi c$ for some $c \in C^{p+1}(\mathcal{U},\mathcal{F})$. We put $\delta^* f = c$ and we have $\phi\delta c = \delta\phi c = \delta\delta g = 0$, hence $\delta c = 0$, that is $c \in Z^{p+1}(\mathcal{U},\mathcal{F})$, since ϕ is injective.

Another representative of f in $H^p_a(\mathcal{U},\mathcal{H})$ differs from f by a coboundary $f_1 \in B^p_a(\mathcal{U},\mathcal{H})$. Then $f_1 = \psi g_1$ for some $g_1 \in C^p(\mathcal{U},\mathcal{G})$ and also $f_1 = \delta f'$ for some $f' \in C^{p-1}_a(\mathcal{U},\mathcal{H})$. Furthermore there is a $g' \in C^{p-1}(\mathcal{U},\mathcal{G})$ with $\psi g' = f'$ and we have $\psi(g_1-\delta g') = \psi g_1-\delta\psi g' = f_1-\delta f' = 0$, thus $g_1-\delta g' = \phi c'$ for some $c' \in C^p(\mathcal{U},\mathcal{F})$. Let $c_1 = \delta c' \in B^{p+1}(\mathcal{U},\mathcal{F})$, then $\delta g_1 = \delta\phi c'+\delta\delta g' = \phi\delta c' = \phi c_1$, hence $\delta^* f_1 = c_1 \in B^{p+1}(\mathcal{U},\mathcal{F})$.

Thus indeed δ^* is a homomorphism between the cohomology groups

(A10) $$\delta^*: H^p_a(\mathcal{U},\mathcal{H}) \longrightarrow H^{p+1}(\mathcal{U},\mathcal{F}).$$

The kernel of δ^* consists of those $f \in Z^p_a(\mathcal{U},\mathcal{H})$ mapped by δ^* on coboundaries $c \in B^{p+1}(\mathcal{U},\mathcal{F})$. For such an f we have $\delta^* f = c = \delta c''$ with $c'' \in C^p(\mathcal{U},\mathcal{F})$; hence $\psi(g-\phi c'') = \psi g = f$ and $\delta(g-\phi c'') = \phi c-\phi\delta c'' = 0$, thus f is the image under ψ of a cocycle in $Z^p(\mathcal{U},\mathcal{G})$. Conversely, the image f under ψ of a cocycle g in $Z^p(\mathcal{U},\mathcal{G})$ is mapped by δ^* to 0, since $0 = \delta g = \phi\delta^* f$ and ϕ is injective.

The image of δ^* consists of those c in $Z^{p+1}(\mathcal{U},\mathcal{F})$ mapped by ϕ into coboundaries of $B^{p+1}(\mathcal{U},\mathcal{G})$, because it follows from the construction of δ^* that $\phi\delta^* f = \phi c = \delta g$. Conversely, if $c \in Z^{p+1}(\mathcal{U},\mathcal{F})$ is such that $\phi c = \delta g$ for some $g \in C^p(\mathcal{U},\mathcal{G})$, then $0 = \psi\phi c = \psi\delta g = \delta\psi g$, thus $\psi g = f$ is a cocycle in $Z^p_a(\mathcal{U},\mathcal{H})$ with $\delta^* f = c$.

Therefore, we have obtained an exact sequence

(A11) $$0 \longrightarrow H^0(\mathcal{U},\mathcal{F}) \xrightarrow{\phi^*} H^0(\mathcal{U},\mathcal{G}) \xrightarrow{\psi^*} H^0_a(\mathcal{U},\mathcal{H}) \xrightarrow{\delta^*} H^1(\mathcal{U},\mathcal{F}) \xrightarrow{\phi^*} H^1(\mathcal{U},\mathcal{G})$$
$$\xrightarrow{\psi^*} H^1_a(\mathcal{U},\mathcal{H}) \xrightarrow{\delta^*} H^2(\mathcal{U},\mathcal{F}) \longrightarrow \dots ,$$

where the maps ϕ^* and ψ^* are obtained from ϕ and ψ in the obvious way, using the fact that the maps of cochains defined by ϕ and ψ commute with the coboundary operators.

We shall now prove that existence theorems for the $\bar\partial$-operator are equivalent to statements involving $H^p(\mathcal{U},A)$.

<u>THEOREM A13</u>. *When* $\Omega \in \mathbb{C}^n$ *is covered by an open covering* $\mathcal{U} = \{U_i\}_{i\in I}$, *where each* U_i *is pseudoconvex, then for* $p \geq 1$ $H^p(\mathcal{U},A)$ *is isomorphic to the quotient space*

$$\{f \mid f \text{ is a } (0,p)\text{-form with } C^\infty\text{-coefficients in } \Omega \text{ and with } \bar\partial f = 0\} \Big/ \{\bar\partial g \mid g \text{ is a } (0,p-1)\text{-form with } C^\infty\text{-coefficients in } \Omega\} \quad .$$

<u>PROOF</u>. Denote by $\mathcal{E}_q$ the sheaf of germs of (0,q)-forms with C^∞-coefficients and by $\mathcal{Z}_q$ the sheaf of germs of (0,q)-forms f with C^∞-coefficients and with $\bar\partial f = 0$. Then it follows from theorem A9 that the sequence

$$0 \longrightarrow \mathcal{Z}_q \longrightarrow \mathcal{E}_q \xrightarrow{\bar\partial} \mathcal{Z}_{q+1} \longrightarrow 0$$

is exact and that this also holds for the sequence of sections

$$0 \longrightarrow C^p(\mathcal{U},\mathcal{Z}_q) \longrightarrow C^p(\mathcal{U},\mathcal{E}_q) \longrightarrow C^p(\mathcal{U},\mathcal{Z}_{q+1}) \longrightarrow 0 \ ,$$

since intersections of pseudoconvex sets are pseudoconvex. Thus $C^p_a(\mathcal{U},\mathcal{Z}_{q+1}) = C^p(\mathcal{U},\mathcal{Z}_{q+1})$ and using (A11) and (A9) we get the exact sequence

$$0 \longrightarrow \Gamma(\Omega,\mathcal{Z}_q) \longrightarrow \Gamma(\Omega,\mathcal{E}_q) \longrightarrow \Gamma(\Omega,\mathcal{Z}_{q+1}) \longrightarrow H^1(\mathcal{U},\mathcal{Z}_q) \longrightarrow H^1(\mathcal{U},\mathcal{E}_q) \longrightarrow$$
$$\longrightarrow H^1(\mathcal{U},\mathcal{Z}_{q+1}) \longrightarrow H^2(\mathcal{U},\mathcal{Z}_q) \longrightarrow H^2(\mathcal{U},\mathcal{E}_q) \longrightarrow \dots \ .$$

Theorem A12 yields $H^p(\mathcal{U},\mathcal{E}_q) = 0$ for $p \geq 1$ and, therefore, we get

$$H^p(\mathcal{U},\mathcal{Z}_{q+1}) \cong H^{p+1}(\mathcal{U},\mathcal{Z}_q), \qquad p \geq 1$$

and

$$H^1(\mathcal{U},\mathcal{Z}_q) \cong \Gamma(\Omega,\mathcal{Z}_{q+1}) \Big/ \Gamma(\Omega,\mathcal{E}_q) \ .$$

So using theorem A8 we get for $p \geq 1$

$$H^p(\mathcal{U},Z) = H^p(\mathcal{U},Z_0) \cong H^{p-1}(\mathcal{U},Z_1) \cong \dots \cong H^1(\mathcal{U},Z_{p-1}) \cong$$

$$\cong \Gamma(\Omega,Z_p) \Big/ \Gamma(\Omega,E_{p-1}) \; . \qquad \square$$

In particular, it follows from theorem A9, that if Ω itself is pseudoconvex

$$\text{(A12)} \qquad H^p(\mathcal{U},A) = 0, \qquad p \geq 1$$

for all open coverings $\mathcal{U}$ of Ω consisting of pseudoconvex sets.

This result holds more generally for all coherent analytic sheaves F, which is Cartan's theorem B (th.7.4.3 of [7] or th.VIII.A.14 of [6]). We will prove this only for subsheaves F of A^p that are finitely generated by polynomial vectors in $A(\Omega)^p$ [1)], which is all we need in this paper. For the general case we only indicate where the proof follows the same pattern, which will be sufficient to show why F should be coherent. Moreover, we assume that the covering $\mathcal{U}$ is such that more than M distinct sets $U_i \in \mathcal{U}$ have empty intersection, although this requirement is not necessary (see the footnote on page 98).

<u>THEOREM A14</u>. *Let Ω be an open pseudoconvex set in $\mathbb{C}^n$, let $\mathcal{U}$ be an open covering of Ω consisting of pseudoconvex sets such that the intersection of more than any* M *elements of $\mathcal{U}$ is empty and let F be a subsheaf of A^p on Ω finitely generated by polynomial vectors. Then*

$$H^p(\mathcal{U},F) = 0 \qquad \textit{for } p \geq 1.$$

1) The fact that a coherent analytic sheaf F is generated in each point by sections over Ω is Cartan's theorem A (th.7.2.8 of [7] or th.VIII.A.13 of [6]). Here we only assume that there is a finite number of sections generating F in *all* points of Ω and that these sections consist of polynomials.

PROOF. Let F be generated by $h_1,\dots,h_q \in \Gamma(\Omega,F)$, thus each $h_k = (h_k^1,\dots,h_k^r) \in A(\Omega)^r$, $k = 1,\dots,q$ and each $h_k^j \overset{\text{not}}{=\!=} P_{jk}$ is a polynomial. Let us suppose that the problem of section III is solved, that is the sequence (A8) is exact when Ω is pseudoconvex. This means that the cochains in $C^p(U,F)$ are liftable, hence from the exact sequence

$$0 \longrightarrow R_P \longrightarrow A^q \xrightarrow{P} F \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow C^p(U,R_P) \longrightarrow C^p(U,A^q) \longrightarrow C^p(U,F) \longrightarrow 0 .$$

As in (A11) we obtain the exact sequence

$$H^p(U,A^q) \longrightarrow H^p(U,F) \longrightarrow H^{p+1}(U,R_P) \longrightarrow H^{p+1}(U,A^q) .$$

From (A12) it follows that the right and left hand terms are zero for $p \geq 1$, thus

$$H^p(U,F) \cong H^{p+1}(U,R_P) .$$

From (A6) it follows that also R_P is a sheaf which is finitely generated by polynomial vectors. Thus if we have shown that $H^{t+1}(U,G) = 0$ for every sheaf G finitely generated by polynomial vectors, it follows that $H^t(U,H) = 0$ for every sheaf H finitely generated by polynomial vectors, $t \geq p$, in particular that $H^p(U,F) = 0$. But $H^M(U,G) = 0$, hence the theorem is proved. □

The above proof is based on the fact that when F is a sheaf which is finitely generated by sections $h_1,\dots,h_q$, then also $R(h_1,\dots,h_q)$ is a sheaf with this property. For that reason we had to require that the vector h_k consists of polynomials (see (A6)). In the general case, F is just a coherent analytic sheaf. Then it follows from Cartan's theorem A (see footnote on page 95) and the Heine-Borel theorem that F is finitely generated by sections $h_1,\dots,h_q \in \Gamma(\Omega,F)$ in the interior Ω' of any compact subset of Ω. Let U' be the covering of Ω' consisting of the sets $U_i' = U_i \cap \Omega$. Since F is coherent, also the sheaf $\mathcal{R}(h_1,\ldots,h_q)$ is finitely generated in Ω' by sections over Ω, and the above shows that $H^p(\mathcal{U}',\mathcal{F}) = 0$ for $p \geq 1$. For the passage from $\mathcal{U}'$ to $\mathcal{U}$ see parts (a) and (b) of the proof of theorem 7.4.3. in [7].

We still have to prove that (A8) is exact. Briefly we can say that by definition all cochains are locally liftable and that by theorem A14 locally liftable cochains are globally liftable in Ω, when Ω is pseudoconvex. Let us investigate this statement more precisely.

We assume that either Ω is an open pseudoconvex set whose closure is compact in the open pseudoconvex set Ω' and that $\mathcal{F}$ is a coherent analytic sheaf on Ω', or that Ω is an open pseudoconvex set and that $\mathcal{F}$ is a coherent analytic sheaf on Ω such that $\mathcal{F}$ is generated in any point of Ω by finitely many sections H^j, $j = 1,\ldots,q$, over Ω, such that $\mathcal{R}_H$ is generated in any point by finitely many sections $S_1^j = (S_1^{j1},\ldots,S_1^{jq})$, $j = 1,\ldots,r$, over $\Omega,\ldots,$ such that $\mathcal{R}_{S_{m-1}}$ is generated by finitely many sections $S_m^j = (S_m^{j1},\ldots,S_m^{jr_{m-1}})$, $j = 1,\ldots,r_m$, over Ω, $m = 2,3,\ldots$. For example, when $\mathcal{F}$ is generated by polynomial vectors, we deal with the last case. In both cases we can find for any $z_j \in \Omega$, any m and any $f \in \Gamma(\Omega,\mathcal{F})$, $(c^k)_{z_j} \in (\mathcal{R}_{S_k})_{z_j}$, $k = 0,1,\ldots,m-1$, $S_0 = H$, an open neighborhood ω_j^m of z_j in Ω, such that in the following sequence

$$\text{(A13)}\qquad A(\omega_j^m)^{r_m} \xrightarrow{S_m} A(\omega_j^m)^{r_{m-1}} \xrightarrow{S_{m-1}} \cdots \to A(\omega_j^m)^{r_1} \xrightarrow{S_1} A(\omega_j^m)^{q} \xrightarrow{H} \Gamma(\omega_j^m,\mathcal{F})$$

$f|_{\omega_j^m}$ belongs to the image of H and $c^k \in A(\omega_j^m)^{r_k}$ belongs to the image of S_{k+1} for $k = 0,1,\ldots,m-1$ ($r_0 = q$). ω_j^m depends moreover on f and c^k, $k = 0,\ldots,m-1$ and it is clear that the above property also holds with ω_j^m replaced by an open subset of ω_j^m, hence $\omega_j^{m+1} \subset \omega_j^m$. Now $\mathcal{U}^{(k)} = \{\omega_j^k \mid z_j \in \Omega\}$ is an open covering of Ω and $\mathcal{U}^{(\ell)}$ is an open refinement of $\mathcal{U}^{(k)}$ when $\ell > k$; we denote the restriction map from $C^p(\mathcal{U}^{(k)},\mathcal{G})$ into $C^p(\mathcal{U}^{(\ell)},\mathcal{G})$ induced by the map from $\mathcal{U}^{(\ell)}$ into $\mathcal{U}^{(k)}$ by $\rho_{k,\ell}$ ($\mathcal{G}$ is any sheaf on Ω).

Actually we will show that there is an open refinement $\mathcal{V}$ of $\mathcal{U}^{(0)}$ such that in the exact sequence (A11) f is liftable and that δ^* maps f onto a coboundary of $B^1(\mathcal{V},\mathcal{R}_H)$, that is $\delta^* f = 0$, so that H is surjective. The proof is in fact the same as that of theorem A14, only we develop the sequence

(A11) explicitely using (A10).

Let $f \in \Gamma(\Omega,\mathcal{F})$, then $f = Hg_j^0$ in ω_j^0 for some $g_j^0 \in A(\omega_j^0)^{r_0}$, and we regard f as a cocycle in $C^0(\mathcal{U}^{(0)},\mathcal{F})$. The set $g^0 = \{g_j^0 \mid z_j \in \Omega\}$ determines a cochain in $C^0(\mathcal{U}^{(0)},\mathcal{A}^{r_0})$. Let $c^0 = \delta g^0$, then $Hc^0 = \delta Hg^0 = \delta f = 0$, hence c^0 is a cocycle in $C^1(\mathcal{U}^{(0)},\mathcal{R}_H)$ (in fact $c^0 = \delta^* f$ by (A10) and (A5) with P = H). According to (A13) there is a $g^1 \in C^1(\mathcal{U}^{(1)},\mathcal{A}^{r_1})$ with $\rho_{0,1}c^0 = S_1 g^1$. Let $c^1 = \delta g^1$, then $S_1 c^1 = \delta S_1 g^1 = \rho_{0,1}\delta c^0 = 0$, hence $c^1 \in C^2(\mathcal{U}^{(1)},\mathcal{R}_{S_1})$ (in fact $c^1 = \delta^*\rho_{0,1}c^0$ by (A10) and (A6) with P = H and Q = S_1).

Generally we find cochains $g^k \in C^k(\mathcal{U}^{(k)},\mathcal{A}^{r_k})$ and cocycles

$$c^k = \delta g^k \in C^{k+1}(\mathcal{U}^{(k)},\mathcal{R}_{S_k}), \qquad k = 0,1,\dots,m ,$$

since $S_k c^k = \delta S_k g^k = \rho_{k-1,k}\delta c^{k-1} = 0$, so that

$$\rho_{k,k+1}c^k = S_{k+1}g^{k+1}, \qquad k = 0,1,\dots,m-1 .$$

In the next section we show that any open covering of Ω has a refinement consisting of pseudoconvex open sets such that the intersection of more than M of these sets is empty. Let m = M-1 and let $\mathcal{V}$ be such a refinement of $\mathcal{U}^{(m)}$; we denote the restriction map from $C^p(\mathcal{U}^{(k)},\mathcal{G})$ into $C^p(\mathcal{V},\mathcal{G})$ by ρ_k.

Now $c^{M-1} = 0$ [1], so certainly we may write $\rho_{M-1}c^{M-1} = \delta\tilde{c}^{M-1}$ with $\tilde{c}^{M-1} \in C^{M-1}(\mathcal{V},\mathcal{R}_{S_{M-1}})$. Assume that for $k \le M-1$

$$\rho_k c^k = \delta\tilde{c}^k, \qquad \tilde{c}^k \in C^k(\mathcal{V},\mathcal{R}_{S_k}) .$$

Let $\tilde{g}^k = \rho_k g^k - \tilde{c}^k$, then $\delta\tilde{g}^k = \rho_k c^k - \rho_k c^k = 0$. Since Ω is pseudoconvex, by (A12) there is a cochain $f^{k-1} \in C^{k-1}(\mathcal{V},\mathcal{A}^{r_k})$ with $\tilde{g}^k = \delta f^{k-1}$. Then we define $\tilde{c}^{k-1} = S_k f^{k-1}$, so that $\tilde{c}^{k-1} \in C^{k-1}(\mathcal{V},\mathcal{R}_{S_{k-1}})$ and

$$\delta\tilde{c}^{k-1} = S_k\delta f^{k-1} = S_k\tilde{g}^k = \rho_k S_k g^k - S_k\tilde{c}^k = \rho_k\rho_{k-1,k}c^{k-1} = \rho_{k-1}c^{k-1} .$$

---

1) The Hilbert syzygy theorem says that $\mathcal{R}_{S_n} = 0$, hence $c^n = 0$, see [6] IV.C.th.4. So, neither here nor in theorem A14 we have to require that more than M sets of the covering have empty intersection. However, the Hilbert syzygy theorem is not proved here.

Thus this holds for all k, in particular for k = 0:

$$\rho_0 c^0 = \delta \tilde{c}^0, \qquad \tilde{c}^0 \in C^0(\mathcal{V}, \mathcal{R}_H)$$

(that is $\rho_0 c^0$ is a coboundary, thus $\delta^* \rho_0 f = 0$). Hence we have

$$f = H(g_k^0 - \tilde{c}_j^0) \quad \text{in} \quad V_j \in \mathcal{V},\ k = \rho_0(j)$$

$$f = H(g_\ell^0 - \tilde{c}_i^0) \quad \text{in} \quad V_i \in \mathcal{V},\ \ell = \rho_0(i),$$

while

$$(\delta \tilde{c}^0)_{ji} = \tilde{c}_i^0 - \tilde{c}_j^0 = (\rho_0 c^0)_{ji} = (\delta g^0)_{k\ell} = g_\ell^0 - g_k^0$$

yields

$$g_k^0 - \tilde{c}_j^0 = g_\ell^0 - \tilde{c}_i^0 \quad \text{in} \quad V_j \cap V_i \ .$$

Thus there is a holomorphic vector function $g \in A(\Omega)^q$ with $f = Hg$ in Ω, namely for all j

$$g = g^0_{\rho_0(j)} - \tilde{c}_j^0 \quad \text{in} \quad V_j \in \mathcal{V} \ .$$

So we have solved the main problem of this appendix:

<u>THEOREM A15</u>. *When Ω is pseudoconvex, the following sequence is exact*

(A8) $$0 \longrightarrow \Gamma(\Omega, \mathcal{R}_P) \longrightarrow \Gamma(\Omega, \mathcal{A}^q) \xrightarrow{\ P\ } \Gamma(\Omega, \mathcal{F}) \longrightarrow 0.$$

We have proved theorem A11 too, so that the sequence (A13) is exact for any open pseudoconvex set ω_j^m. In the general case when $\mathcal{F}$ is not generated by polynomial vectors, theorem A of CARTAN (see footnote on page 95) and consequently theorem B, as we have shown, follow from the next result due to CARTAN ([7] th.7.2.1.(ii)):
Let Ω be pseudoconvex and K a compact subset of Ω with $K = \hat{K}_\Omega$ (see (A1)) and let $h_1, \ldots, h_q$ be sections over a neighborhood of K of a coherent analytic sheaf $\mathcal{F}$ on a neighborhood of K, which generate $\mathcal{F}$ there. If f is an arbitrary

section of F over a neighborhood of K, then there are $g_1,\ldots,g_q$ analytic in a neighborhood of K so that $f = \sum_{k=1}^{q} h_k\, g_k$ there.

In section 7 we give a quantitative version of theorems A14 and A15, when H is a polynomial matrix. For that purpose we need a quantitative version of the above semilocal result. This is the following modification of th.7.6.5 in [7], which is actually proved there (or th.III 3.4.(3) when $p = q = 1$ and the general case is contained in th.III 3.6 in [3]):

THEOREM A16. *For any polynomial matrix* P, *there is an integer* $t > 1$, *such that for any neighborhood* ω *of* 0 *and every* $u \in A(t\omega+z)^q$, $z \in \mathbb{C}^n$, *there is a* $v \in A(\omega+z)^q$ *with* $Pv = Pu$ *and*

$$\text{(A14)} \qquad \sup_{w\in\omega+z} |v(w)| \le C\,(1+\|z\|)^N \sup_{w\in t\omega+z} |P(w)\,u(w)|\,,$$

where C *is a constant depending on* P *and* ω *(the smaller* ω *the larger* C*) and where* N *only depends on* P. *Here* $t\omega+z$ *denotes* $\{w \mid w = t\zeta+z,\ \zeta\in\omega\}$ *and* $|v(w)|^2 = \sum_{k=1}^{q} |v_k(w)|^2$.

In section 7 we perform all the steps of the proofs in this section again, then taking care of the bounds. The sets $\omega+z_j$ and $t\omega+z_j$ in theorem A16 in fact will be just the sets ω_j^m and ω_j^{m-1}, respectively, in a quantitative semilocal version of (A13).

V. SPECIAL COVERINGS

In this section we show that any open covering of the open set Ω has a refinement that satisfies properties (A15)(i) and (ii) below. This is based on a theorem of dimension theory, th.3,§2,Ch.7, p.278 [4]. Moreover, we construct a special covering of Ω with refinements satisfying some additional properties needed in section 7. The essential idea for this construction has already been used by WHITNEY in [17], whcih can be found in [8] too.

Let Ω be an open set in $\mathbb{R}^n$ and let $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$ be an open covering of Ω. Each point in Ω has a bounded open neighborhood whose closure in $\mathbb{R}^n$ is contained in some open set O_α. Hence there exists an open refinement of $\mathcal{O}$ consisting of open sets whose closure is compact and contained in Ω.

Since Ω is paracompact, we can find a locally finite open refinement $\mathcal{V} = \{V_j\}_{j=1}^{\infty}$, where each $\bar{V}_j$ is compact and contained in Ω (such a refinement is necessarily countable, because Ω is separable). According to [4] 7.2.th.3 and 7.3.th.13 $\mathcal{V}$ has an open shrinking $\mathcal{W} = \{W_j\}_{j=1}^{\infty}$ (which, therefore, is also locally finite and $W_j \subset\subset \Omega$), such that more than n+1 distinct sets W_j have empty intersection (that is the order of $\mathcal{W}$ is at most n). Since Ω is normal, lemma 1 to th.3,§1,Ch.5 [4] yields an open shrinking $\mathcal{W}' = \{W_j'\}_{j=1}^{\infty}$ of $\mathcal{W}$ such that the closure with respect to Ω of each W_j' is contained in W_j, but since $\bar{W}_j \subset \Omega$, this yields $\bar{W}_j' \subset W_j$ for all j. Of course $\mathcal{W}'$ is a locally finite open covering of order at most n.

For each j $\bar{W}_j'$ is compact and contained in W_j and, therefore, we can find finitely many open convex sets $U_{j,k}$, $k = 1,\ldots,m_j$ with $U_{j,k} \subset W_j$ and $W_j' \subset \bigcup_{k=1}^{m_j} U_{j,k}$, such that more than M' distinct sets $U_{j,k}$ have empty intersection, where M' is a positive integer independent of j. For example, this can be done by covering W_j' by sufficiently small closed hypercubes in W_j (so, that the vertices form a rectangular lattice) and by taking sufficiently small convex open neighborhoods of these cubes. Then we get $M' = 2^n$, but it is also possible (by choosing sufficiently small convex open neighborhoods of some cubes and sufficiently large convex open sets contained in the other cubes) to obtain M' = n+1.

Since each point in Ω has a neighborhood that intersects a finite number of the sets W_j, this neighborhood also intersects a finite number of the sets $U_{j,k}$. Furthermore, each point in Ω is contained in at least one set W_j' and in at most n+1 sets W_j, thus in at least one and at most $M = (n+1)M'$ sets $U_{j,k}$. Therefore, the covering $\mathcal{U} = \{U_{j,k}\}_{j=1,\ k=1}^{\infty\ \ \ m_j}$ is a locally finite open refinement of $\mathcal{O}$ consisting of convex open sets, such that more than M distinct sets of $\mathcal{U}$ have empty intersection.

Since convex sets in $\mathbb{C}^n$ are pseudoconvex, we have obtained the

COROLLARY. *Let Ω be an open set in $\mathbb{C}^n$ and let $\mathcal{O}$ be an open covering of Ω. Then there exists a locally finite open refinement $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of $\mathcal{O}$ with the properties*

(A15)
- (i) for every i U_i is pseudoconvex and $U_i \subset\subset \Omega$,
- (ii) there is an integer M such that more than M distinct sets in $\mathcal{U}$ have empty intersection.

Now we construct coverings of Ω that satisfy some additional properties. Let Ω be an open set in $\mathbb{C}^n$ and let $\{\Omega_k\}_{k=1}^{\infty}$ be an increasing sequence of open subsets with union Ω and with

$$\forall k,\ \exists \varepsilon = \varepsilon(k):\ \Omega_k(\varepsilon) \subset \Omega_{k+1}$$

where $\Omega_k(\varepsilon)$ is the ε-neighborhood of Ω_k.

Choose positive integers m_k with $m_{k+1} > m_k$ such that a cube with side $1/m_k$ is contained in the ball with radius $\varepsilon(k)$, for $k = 1,2,\ldots$ and let $m_0 = 1$. Divide $\mathbb{C}^n$ into a collection $\mathcal{U}'$ of closed cubes with side 1 (such that the vertices form a rectangular lattice) and select those cubes contained in Ω_1. Call the collection of these cubes $\mathcal{U}'_0$. Divide the remaining cubes and parts of cubes of $\mathcal{U}'$ into a collection of cubes with side $1/m_1$ and let $\mathcal{U}'_1$ be the collection of those cubes that are contained in Ω_2. Generally when we have defined sets $\mathcal{U}'_0,\ldots,\mathcal{U}'_{k-1}$ of cubes, we define the set $\mathcal{U}'_k$ of cubes obtained by dividing the remaining cubes and parts of cubes of $\mathcal{U}'$ into a collection of cubes with side $1/m_k$ and by selecting those cubes that are contained in Ω_{k+1}.

Then the union of $\mathcal{U}'_0,\mathcal{U}'_1,\ldots$ covers Ω, since Ω_k is covered by the union of $\mathcal{U}'_0,\mathcal{U}'_1,\ldots,\mathcal{U}'_k$. For, a point $x \in \Omega_k$ either belongs to one cube of $\mathcal{U}'_0$ or ... or $\mathcal{U}'_{k-1}$, since these cubes are all contained in Ω_k, or it belongs to some cube of $\mathcal{U}'_k$, since any cube with side $1/m_k$ containing x is contained in Ω_{k+1}. Hence any cube in $\mathcal{U}'_k$ can intersect only cubes of $\mathcal{U}'_\ell$ for $\ell = k-1$, k or $k+1$. Furthermore, the intersection of more than 2^{2n} distinct cubes is empty.

Now we will define sufficiently small open neighborhoods of the cubes of $\mathcal{U}'_0,\mathcal{U}'_1,\ldots$, so that we get an open covering. Define the map α by mapping a cube $K' \in \mathcal{U}'_k$ to the enlargement of the interior of K' by a factor $1+m_k/m_{k+1}$, the center kept fixed. Then $\alpha K' = K$ is an open cube. Let for each k $\mathcal{U}_k^{(0)}$ be the set $\mathcal{U}_k^{(0)} = \{\alpha K' \mid K' \in \mathcal{U}'_k\}$ and let

$$\mathcal{U}^{(0)} = \bigcup_{k\geq 0} \mathcal{U}_k^{(0)}.$$

Then $\mathcal{U}^{(0)}$ is a covering of Ω that satisfies besides properties (A15)(i) and (ii) the following properties for $\lambda = 0$

(A15) (iii) all the sets in the covering $\mathcal{U}^{(\lambda)}$ intersecting Ω_k have a minimum size and are contained in $\Omega_{\ell(k)}$ with $\ell(k) = k+3$;

(iv) when a set in $\mathcal{U}^{(\lambda)}$ intersects Ω_k, it intersects not more than $N_k^{(\lambda)}$ elements of the covering $\mathcal{U}_k^{(\lambda)}$, where $N_k^{(\lambda)}$ is some number depending only on k.

The proof follows from the fact that two cubes K_j and K_i in $\mathcal{U}^{(0)}$ have a non-empty intersection if and only if $\alpha^{-1}K_j = K_j'$ intersects $\alpha^{-1}K_i = K_i'$. To prove this, assume that $K_j' \in \mathcal{U}_k'$, $K_i' \in \mathcal{U}_\ell'$, $\ell \geq k$, thus $\ell - k = m \geq 0$ and that $K_j' \cap K_i' = \emptyset$. Since cubes in $\mathcal{U}_p'$ can intersect cubes in $\mathcal{U}_q'$ only when $q = p-1$, p or p+1, the distance between K_j' and K_i' is at least

$$\frac{1}{m_{k+1}} + \frac{1}{m_{k+2}} + \ldots + \frac{1}{m_{k+m-1}} \geq \frac{1}{m_{k+1}}$$

when $m \geq 2$, or

$$\frac{1}{m_{k+1}}$$

when m equals zero or one. The distance from the boundary of K_j to K_j' is by definition of α

(A16) $$\tfrac{1}{2}\left[\frac{1}{m_k}\left(1 + \frac{m_k}{m_{k+1}}\right) - \frac{1}{m_k}\right] = \frac{1}{2m_{k+1}}$$

and the distance from the boundary of K_i to K_i' is $1/(2m_{\ell+1})$, so that the distance between K_j and K_i is at least

$$\frac{1}{m_{k+1}} - \frac{1}{2m_{k+1}} - \frac{1}{2m_{k+m+1}} > \frac{1}{m_{k+1}} - \frac{1}{2m_{k+1}} - \frac{1}{2m_{k+1}} = 0$$

when $m \geq 1$, or

$$\frac{1}{m_{k+1}} - \frac{1}{2m_{k+1}} - \frac{1}{2m_{k+1}} = 0$$

when $m = 0$. Only in this case the boundaries of K_j and K_i might touch each other, but since K_j and K_i are open, $K_j \cap K_i = \emptyset$.

Now property (A15)(ii) follows from the same property for the cubes of

$\mathcal{U}'_0, \mathcal{U}'_1, \ldots$. Let the cube K in $\mathcal{U}^{(0)}$ intersect Ω_k. If $\alpha^{-1}K$ does not intersect Ω_{k+1}, $\alpha^{-1}K$ does not intersect the elements of $\mathcal{U}'_0, \ldots, \mathcal{U}'_k$, hence K does not intersect the elements of $\mathcal{U}^{(0)}_0, \ldots \mathcal{U}^{(0)}_k$, the union of which contains Ω_k. Thus $\alpha^{-1}K$ intersects Ω_{k+1}, hence $\alpha^{-1}K$ is contained in Ω_{k+2}, so that K is contained in Ω_{k+3}. Thus K has a minimum size, namely the size of $\alpha^{-1}K$ is at least $1/m_{k+2}$. Property (A15)(iv) follows from property (iii) and the same property for $\mathcal{U}'_0, \mathcal{U}'_1, \ldots$.

Finally we construct open refinements $\mathcal{U}^{(\lambda)}$ of the covering $\mathcal{U}^{(0)}$ satisfying besides the properties (A15)(i),(ii),(iii) and (iv) the following properties

(A15)(v) for each λ $\mathcal{U}^{(\lambda+1)}$ is a refinement of $\mathcal{U}^{(\lambda)}$ and moreover each open cube $K^{(\lambda)}_j \in \mathcal{U}^{(\lambda)}$ enlarged $2^{\lambda-\mu}$ times with the center kept fixed is contained in some $K^{(\mu)}_{i_\mu} \in \mathcal{U}^{(\mu)}$ for every $\mu = 0,1,\ldots,\lambda-1$; we denote the map ρ between the index sets of $\mathcal{U}^{(\lambda)}$ and $\mathcal{U}^{(\mu)}$ with $\rho(i) = i_\mu$ by $\rho_{\mu,\lambda}$;

(vi) when $K^{(\lambda)}_j \in \mathcal{U}^{(\lambda)}$ intersects Ω_k, there are at most $M_{\lambda,\mu}(k)$ indices i_p with $\rho_{\lambda,\mu}(i_p) = j$, $p = 1,\ldots,M_{\lambda,\mu}(k)$ $(\mu > \lambda)$.

Eventually by taking larger integers m_k, we may assume that each $m_{k+1} = p_{k+1}\, m_k$ for some integer $p_{k+1} \geq 2$, $k = 0,1,\ldots$. Let $m^{(0)}_k = m_k$ and let $m^{(\lambda)}_k = 2^\lambda m_{k+\lambda}$ for $\lambda = 1,2,\ldots$, then $m^{(\lambda)}_k = 2p_{k+\lambda} m^{(\lambda-1)}_k$ and $m^{(\lambda)}_k = 2m^{(\lambda-1)}_{k+1}$; $(m^{(\lambda)}_k)^{-1}$, $k = 0,1,\ldots$, will be the length of the sides of the closed cubes the covering $\mathcal{U}^{(\lambda)}$ is constructed from similarly to the construction of $\mathcal{U}^{(0)}$. Namely, let K'_λ be a closed cube with side $(m^{(\lambda)}_k)^{-1}$, then the enlargement with a factor $(1+m^{(\lambda)}_k/m^{(\lambda)}_{k+1})$ of the interior of K'_λ will be a cube K_λ of $\mathcal{U}^{(\lambda)}$ as in the construction of $\mathcal{U}^{(0)}$. Then $\mathcal{U}^{(\lambda)}$ satisfies the same properties on $\mathcal{U}^{(0)}$. So, let us assume that the coverings $\mathcal{U}^{(0)},\ldots,\mathcal{U}^{(\lambda-1)}$ with the desired properties have been constructed in the same way as $\mathcal{U}^{(0)}$ have been constructed from closed cubes.

We divide the closed cubes $K'_{\lambda-1}$, with side $(m^{(\lambda-1)}_k)^{-1}$, the sets $K_{\lambda-1} \in \mathcal{U}^{(\lambda-1)}$ are constructed from into $(2p_{k+\lambda})^{2n}$ closed cubes K'_λ with side $(m^{(\lambda)}_k)^{-1}$ and the covering $\mathcal{U}^{(\lambda)}$ is defined as the set of open cubes K_λ being the enlargement of the interior of the cubes K'_λ by the above factor, $k = 0,1,\ldots$. Then the difference of two times half the side of K_λ and half the side of K'_λ satisfies

$$2\left[\tfrac{1}{2}\left(1+\frac{m_k^{(\lambda)}}{m_{k+1}^{(\lambda)}}\right)\frac{1}{m_k^{(\lambda)}}\right]-\tfrac{1}{2}\,\frac{1}{m_k^{(\lambda)}}\le\frac{1}{m_k^{(\lambda)}}=\frac{1}{2m_{k+1}^{(\lambda-1)}}\,,$$

where the right hand side equals the distance from the boundary of $K_{\lambda-1}$ to $K'_{\lambda-1}$ according to (A16). Hence two times K_λ, with the center kept fixed, is contained in $K_{\lambda-1}$, so that property (A15)(v) follows. Furthermore, $K'_{\lambda-1}$ contains $(2p_{k+\lambda})^{2n}$ cubes K'_λ, hence $\rho_{\lambda-1,\lambda}$ maps not more than $(2p_{k+\lambda})^{2n}$ sets K_λ onto the same $K_{\lambda-1}$. From this and from property (A15)(iii) the above property (A15)(vi) follows.

REMARK. Although we use property (A15)(iv) in section 7, this could be avoided. However, the coverings $\mathcal{U}^{(\lambda)}$ satisfy (A15)(iv) anyhow.

VI. NULLSTELLENSATZ AND FUNDAMENTAL PRINCIPLE

In this section we discuss Hilbert's Nullstellensatz, Ehrenpreis' generalization and fundamental principle.

Consider an ideal $\mathcal{I}'_z$ in $\mathcal{A}_z$ generated by the germs $(h_1)_z,\dots,(h_q)_z$ at z of functions $h_1,\dots,h_q$ holomorphic in some neighborhood U of z. We define the set

$$V = \{w \mid h_1(w)=0,\dots,\ h_q(w)=0,\ w\in U\}$$

and let V_z be the equivalence class of V under the equivalence relation $V\sim W$ if there is a neighborhood ω of z with $V\cap\omega = W\cap\omega$. V_z is called the germ at z of V. It is clear, that the ideal $\mathcal{I}'_z$ is not trivial only if $h_1(z) = \dots = h_q(z) = 0$. When $f_z \in \mathcal{I}'_z$ we denote by f a holomorphic function in a neighborhood of z such that f_z is the germ of f at z. Then for any $f_z \in \mathcal{I}'_z$, $z\in V$, there is a neighborhood ω of z with

$$f(w) = 0, \qquad w\in V\cap\omega.$$

Conversely, let us consider the ideal $\mathcal{I}_z$ in $\mathcal{A}_z$ of all the germs at z of holomorphic functions vanishing on V_z, that is

(A17) $$\mathcal{I}_z = \{f_z \mid \text{there is a neighborhood } \omega \text{ of } z \text{ and } f\in A(\omega) \text{ with } f(w)=0 \text{ for } w\in V\cap\omega\}.$$

It is clear that I_z is an ideal and that $I'_z \subset I_z$.

Hilbert's Nullstellensatz says that, if $f_z \in I_z$, there is a positive integer m with $f_z^m \in I'_z$ or

$$I_z = \text{rad } I'_z = \{f_z \mid f_z^m \in I'_z \text{ for some m depending on } f_z\},$$

see [6] II.E.th.20. Obviously, when I'_z is a prime ideal this yields ([6] III.A.7)

(A18) $$I'_z = I_z ,$$

that is, $f_z \in I_z$ can be written as $f(w) = \sum_{k=1}^{q} g_k(w)\, h_k(w)$ for w in some neighborhood ω of z and for some $g_k \in A(\omega)$, $k = 1,\dots,q$.

EHRENPREIS has generalized this result in the following way (see [3] chapter II): let the functions $h_1,\dots,h_q$ be polynomials, let for example

$$\frac{\partial h_1}{\partial z_1}(z) = 0$$

(of course, also $h_1(z) \dots = h_q(z) = 0$) and let V'_z be the germ at z of

$$V' = \{w \mid \frac{\partial h_1}{\partial z_1}(w) = 0,\ w \in U\}.$$

Then we require that $f_z \in I_z$ moreover satisfies in some neighborhood ω of z

$$\frac{\partial f}{\partial z_1}(w) = 0, \qquad w \in V' \cap \omega.$$

Now let W_z be defined as (V_z, V'_z), where this should be understood in the following way: a function f holomorphic in a neighborhood ω of z vanishes on W_z if f vanishes on $V \cap \omega$ and $\partial f/\partial z_1$ vanishes $V' \cap \omega$.

The same can be done for higher order derivatives and the other polynomials h_k. The characterization of W_z is not immediately clear from the polynomials $h_1,\dots,h_q$ (see example 4,II.2 in [3]). Anyhow, W_z can be defined in such a way that, if I_z is the ideal in A_z of germs of functions vanishing on W_z, we always have (A18), that is I_z is the ideal in A_z generated by $(h_1)_z,\dots,(h_q)_z$ (th.II 2.4 of [3]).

V_z in the Nullstellensatz is called the germ at z of a variety and W_z in Ehrenpreis' formulation is called the germ at z of a multiplicity variety. In case of modules in A_z^p instead of ideals, it is possible to define p germs $(W_1)_z,\dots,(W_p)_z$ of multiplicity varieties and so we get the germ $\vec{W}_z = ((W_1)_z,\dots,(W_p)_z)$ of a vector multiplicity variety. This can be done in such a way, that the analogue of (A18) holds, namely (th.II 2.6 of [3]):

<u>THEOREM A17</u>. *Let* P_{jk} *be polynomials,* $j = 1,\dots,p$, $k = 1,\dots,q$. *Then it is possible for each* z *to define the germ* $\vec{W}_z$ *at* z *of a vector multiplicity variety, such that each* p*-tuple of functions* f_j, $j = 1,\dots,p$, *holomorphic in a neighborhood of* z, *whose germ at* z *vanishes on* $\vec{W}_z$, *can be written as*

$$f_j(w) = \sum_{k=1}^{q} P_{jk}(w)\, g_k(w), \qquad j = 1,\dots,p$$

for w *in some neighborhood* ω *of* z *and for some functions* $g_k \in A(\omega)$, $k = 1,\dots,q$.

Next we consider a sheaf I' of ideals generated in each point of an open pseudoconvex set Ω by polynomials $h_1,\dots,h_q$, thus $p = 1$. Their simultaneous zero-set defines a variety $V = \bigcup_{z\in\Omega} V_z$ in Ω (at points z where some $h_k(z) \neq 0$ V_z is empty). Similarly we can define a multiplicity variety W in Ω (see [3]). We will consider sheafs of functions on V; the same can be done for a multiplicity variety W. Let I be the sheaf on Ω

$$I = \bigcup_{z\in\Omega} I_z ,$$

where I_z is defined by (A17); note that $I_z = A_z$ when $z \in \Omega\backslash V$. We can define a sheaf F on Ω by

$$F_z = A_z \big/ I_z , \qquad z \in \Omega,$$

that is the following sequence is exact

$$0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0.$$

For $z \in \Omega\backslash V$ $\mathcal{I}_z = \mathcal{A}_z$, thus $\mathcal{F}_z = 0$. Thus $\mathcal{F}$ is only non-trivial in points of V, so we consider the restriction $\mathcal{F}'$ to V

$$\mathcal{F}' = \bigcup_{z\in V} \mathcal{F}_z$$

which is a sheaf on V. In accordance with the footnote on page 85 we can regard $\mathcal{F}'$ as the sheaf of germs of analytic functions on V. A section f in $\Gamma(V,\mathcal{F}')$ is a holomorphic function in V; regarded as a section f_1 in $\Gamma(\Omega,\mathcal{F})$ we would have $f_1(z) = f(z)$ for $z \in V$ and $f_1(z) = 0$ for $z \in \Omega\backslash V$. So, we may just as well consider the sections in $\Gamma(\Omega,\mathcal{F})$ as the holomorphic functions in V. In case $\mathcal{I}'_z$ is a prime ideal for all $z \in \Omega$, (A18) holds and the sheaf $\mathcal{I}$ is finitely generated by polynomials. This also holds when we consider a sheaf of ideals on a multiplicity variety. Hence theorem A14 may be applied. Also, generally for any sheaf $\mathcal{I}$ of ideals Cartan's theorem B may be applied, since $\mathcal{I}$ is coherent ([6] IV.D.2). However, in the case occurring in this paper $\mathcal{I}'_z$ is prime for all $z \in \Omega$. Hence in the same way as theorem A15 was obtained from Cartan's theorem B we here get

$$0 \longrightarrow \Gamma(\Omega,\mathcal{I}) \longrightarrow \Gamma(\Omega,\mathcal{A}) \longrightarrow \Gamma(\Omega,\mathcal{F}) \longrightarrow 0,$$

so that

$$\text{(A19)} \qquad A(\Omega) \Big/ \Gamma(\Omega,\mathcal{I}) \;\cong\; \Gamma(\Omega,\mathcal{F}) \;\cong\; \Gamma(V,\mathcal{F}') \;.$$

Thus any function holomorphic in V is the restriction of a function in $A(\Omega)$ and in case (A18) holds any function f in $A(\Omega)$ that vanishes on V can be written as $f(z) = \sum_{k=1}^{q} h_k(z)\, g_k(z)$, $z \in \Omega$ for some $g_k \in A(\Omega)$. In this paper we will derive a quantitative version of (A19) for a special variety V.

When $\Omega = \mathbb{C}^n$, when $\mathcal{I}'$ is an ideal generated by polynomials and when W is its associated multiplicity variety (thus (A18) holds), the isomorphism (A19) with V replaced by W and with bounds (that is all the occurring functions satisfy moreover certain estimates at infinity) is Ehrenpreis' fundamental principle (theorem IV 4.1 in [3]; a survey of this theorem and its proof can be found in [1] IV). The fundamental principle holds for modules generated by polynomials too (th. IV 4.2.[3]), however, in that case the definition of global vector multiplicity varieties is a more delicate question (see page 100 [3]).

LITERATURE

[1] BERENSTEIN, A.C. & M.A. DOSTAL, *Analytically uniform spaces and their applications to convolution equations*, Lecture notes in Mathematics no. 256, Springer-Verlag, Berlin, 1972.

[2] EHRENPREIS, L., *The fundamental principle for linear constant coefficient partial differential equations*, Proc. Intern. Symp. Lin. Spaces, 161-174, Jerusalem, 1961.

[3] EHRENPREIS, L., *Fourier analysis in several complex variables*, Wiley, New York, 1970.

[4] ENGELKING, R., *Outline of general topology*, North-Holland, Amsterdam, 1968.

[5] GELFOND, A.O., *Differenzenrechnung*, Deutscher Verlag der Wissenschaften, 1958.

[6] GUNNI, R.C. & H. ROSSI, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, 1965.

[7] HÖRMANDER, L., *An introduction to complex analysis in several variables*, second edition, North-Holland, Amsterdam, 1973.

[8] HÖRMANDER, L., *On the division of distributions by polynomials*, Ark. Mat., 3 (1958) 555-568.

[9] KAWAI, T., *On the theory of Fourier hyperfunctions and its applications to partial differential with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect IA, 17 (1970) 467-517.

[10] KIOUSTELIDIS, J., *Eine einheitliche Methode zur Herleitung von Reihenentwicklungen für ganze Funktionen vom Exponentialtyp*, Compositio Math., 26 (1973) 201-232.

[11] MARTINEAU, A., *Les hyperfonctions de M. Sato*, Séminaire Bourbaki, 13 (1960-61), No. 214.

[12] MARTINEAU, A., *Sur les fonctionelles analytiques et la transformation de Fourier-Borel*, J. Analyse Math. 9 (1963) 1-164.

[13] NÖRLUND, N.E., *Vorlesungen über Differenzenrechnung*, Chelsea Publ. Co., N.Y., 1954.

[14] ROEVER, J.W. DE, *Fourier transforms of holomorphic functions and application to Newton interpolation series, I*, Math. Centrum, Amsterdam, Afd. Toegepaste Wiskunde, Rep. TW 142 (1974).

[15] TREVES, F., *Topological vector spaces, distributions and kernels*, Academic Press, N.Y., 1967.

[16] VLADIMIROV, V.S., *Methods of the theory of functions of many complex variables*, The M.I.T. Press, 1966.

[17] WHITNEY, H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36 (1934) 63-89.

[18] WLOKA, J., *Grundräume und verallgemeinerte Funktionen*, Lecture Notes in Mathematics no. 82, Springer-Verlag, Berlin, 1969.

[19] KOMATSU, H., *Projective and injective limits of weakly compact sequences of locally convex spaces*, J. Math. Soc. Japan, 19 (1967) 366-383.