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KRAWTCHOUK POLYNOMIALS, A UNIFICATION OF TWO DIFFERENT GROUP THEORETIC INTERPRETATIONS

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Krawtchouk polynomials, a unification of two different group theoretic interpretations*)

by

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ABSTRACT

The canonical matrix elements of irreducible unitary representations of SU(2) are written as Krawtchouk polynomials, with the orthogonality being the row orthogonality for the unitary representation matrix. Dunkl's interpretation of Krawtchouk polynomials as spherical functions on wreath products of symmetric groups is generalized to the case of intertwining functions. A conceptual unification is given of these two group theoretic interpretations of Krawtchouk polynomials.

KEY WORDS & PHRASES: *Krawtchouk polynomials; canonical matrix elements of irreducible representations of SU(2); spherical and intertwining functions on wreath products of symmetric groups; metaplectic representation*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let G be a compact group. Then matrix elements belonging to inequivalent irreducible unitary representations of G are orthogonal to each other. This phenomenon is lying at the background of many instances of group theoretic interpretations of orthogonal polynomials. However, if $\pi \in \hat{G}$ and $\pi_{m,n}(g)$ denotes the matrix elements of π with respect to an orthonormal basis then there is also a discrete orthogonality relation for $\pi_{m,n}(g)$ ($g \in G$ fixed) which is just the column or row orthogonality for the unitary matrix $(\pi_{m,n}(g))$. By looking at $\pi_{m,n}(g)$ in this way we may identify it with a quite different system of orthogonal polynomials. For instance, in the case G = SU(2) or U(2) the first kind of orthogonality is a group theoretic form of the orthogonality relations for Jacobi polynomials and the second kind of orthogonality is similarly related to Krawtchouk polynomials. Surprisingly enough, although the first fact is well-known, the second fact seems to have been unobserved in literature until now. Section 2 deals with this result.

Krawtchouk polynomials also have a group theoretic interpretation as spherical functions on wreath products of symmetric groups. It is no accident that this class of special functions has two group theoretic interpretations of so different nature. In Section 3 we give a conceptual proof that, for one special $g \in U(2)$, the corresponding canonical matrix elements can be expressed in terms of spherical functions on the wreath product of S_2 and S_N. A similar explanation can be given for the occurrence of Bessel functions both as generalized matrix elements for discrete series representations of $SL(2,\mathbb{R})$ and as spherical functions for the group of Euclidean motions. Weil's metaplectic representation here plays an important role. These things are shortly discussed in Section 4.

Not just spherical functions but also interwining functions on wreath products of symmetric groups can be written as Krawtchouk polynomials. This result, which seems to be new, is proved in Section 5. Finally, in Section 6 we describe a conceptual way to identify these intertwining functions with matrix elements for U(2), thus generalizing the results of Section 3.

The interpretation of Krawtchouk polynomials as matrix elements for representations of SU(2) is a suitable point of departure for several different lines of research. Here the author already announces some results, which he intends to publish in subsequent papers. First, the row orthogonality for unitary matrices yields group theoretic interpretations for several other classical orthogonal polynomials, by choosing suitable groups and bases (or double bases) for the representation spaces. We mention Meixner, Laguerre and Pollaczek polynomials for discrete series representations of $SL(2,\mathbb{R})$, Charlier polynomials for the Heisenberg group, Hahn polynomials for $SU(2) \times SU(2)$ (Clebsch-Gordan coefficients), Racah polynomials for $SU(2) \times SU(2) \times SU(2)$ (Racah coefficients). Next, a unification of two different group theoretic interpretations can also be given in the Hahn polynomial case (Clebsch-Gordan coefficients for SU(2) and spherical functions for the symmetric group, respectively). Finally, the group theoretic interpretations of classical orthogonal polynomials mentioned above lead to group theoretic proofs of certain formulas for these polynomials, for instance for the Poisson kernel.

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2. THE CANONICAL MATRIX ELEMENTS OF THE IRREDUCIBLE UNITARY REPRESENTATIONS OF SU(2)

Consider the natural representation T of $GL(2,\mathbb{C})$ on $\mathbb{C}^2$. The restriction of T to U(2) or SU(2) is a unitary representation, where we consider $\mathbb{C}^2$ as a Hilbert space with respect to the orthonormal basis $e_0 := (1,0)$, $e_1 := (0,1)$.

The N-fold tensor product $\otimes^N T$ of T is a representation of $GL(2,\mathbb{C})$ on $\otimes^N \mathbb{C}^2$. The space V^N of symmetric tensors in $\otimes^N \mathbb{C}^2$ is an invariant subspace of $\otimes^N \mathbb{C}^2$. Let T^N be the corresponding subrepresentation of $\otimes^N T$. A model for V^N is given by the space of all homogeneous polynomials of degree N in two complex variables with $GL(2,\mathbb{C})$ acting on V^N by

(2.1) $$\left(T^N\begin{pmatrix} a & b \\ c & d \end{pmatrix}F\right)(x,y) := F(ax+cy, bx+dy),$$

$$F \in V^N, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}), \quad x,y \in \mathbb{C}.$$

The space V^N has dimension $N+1$. A natural basis for V^N is given by the tensors f_n^N $(n = 0,1,\ldots,N)$:

(2.2) $$f_n^N := \frac{1}{N!} \sum_{\sigma \in S_N} e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(N)}},$$

$$i_1 = \ldots = i_{N-n} = 0, \quad i_{N-n+1} = \ldots = i_N = 1.$$

We have

(2.3) $$f_n^N(x,y) = x^{N-n} y^n, \quad x,y \in \mathbb{C}.$$

It follows from (2.2) that the Hilbert space norm of f_n^N is given by

$$\| f_n^N \|^2 = \frac{1}{(N!)^2}((N-n)!\,n!)^2 \binom{N}{n} = 1/\binom{N}{n},$$

so we have an orthonormal basis

(2.4) $$e_n^N(x,y) := \binom{N}{n}^{\frac{1}{2}} x^{N-n} y^n, \quad n = 0,1,\ldots,N,$$

for V^N. By construction, the restriction of T^N to $U(2)$ or $SU(2)$ is a unitary representation with respect to this orthonormal basis. It is well-known (cf. for instance HEWITT & ROSS [6, Theorems (29.20) and (29.27)]) that the representations T^N restricted to $SU(2)$ are irreducible and that each unitary irreducible representation of $SU(2)$ is equivalent to some T^N $(N = 0,1,2,\ldots)$.

Consider the subgroup

(2.5) $$K := \left\{ u_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \right\}$$

of $SU(2)$. We have

(2.6) $$T^N(u_\theta)e_n^N = e^{i(2n-N)\theta}e_n^N, \quad u_\theta \in K,$$

so T^N restricted to K splits as a direct sum of inequivalent irreducible representations of K. We call $\{e_n^N\}$ a K-*basis* for V^N.

For $g \in GL(2,\mathbb{C})$ let

(2.7) $$T_{m,n}^N(g) := (T^N(g)e_n^N, e_m^N), \quad m,n = 0,1,\ldots,N,$$

where (.,.) is the inner product with respect to the orthonormal basis $\{e_n^N\}$. We call $T_{m,n}^N(g)$ the *canonical matrix elements* of T^N. These matrix elements can be calculated from the generating function

(2.8) $$\binom{N}{n}^{\frac{1}{2}}(ax+cy)^{N-n}(bx+dy)^n = \sum_{m=0}^{N} T_{m,n}^N\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{N}{m}^{\frac{1}{2}} x^{N-m}y^m.$$

First of all, we conclude from (2.8):

(2.9) $$T_{m,n}^N\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T_{N-m,N-n}^N\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Binomial expansion of the left-hand side of (2.8) yields

(2.10) $$T_{m,n}^N\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{n}^{\frac{1}{2}}\binom{N}{m}^{-\frac{1}{2}} a^{N-n-m}b^n c^m \cdot$$
$$\cdot \sum_{\ell=0\vee(m+n-N)}^{m\wedge n} \binom{N-n}{M-\ell}\binom{n}{\ell}\left(\frac{ad}{bc}\right)^{\ell}.$$

This expression goes back to WIGNER [13,(15.21)]. In view of (2.9) we can suppose $m+n \le N$ without loss of generality. Thus it is possible to rewrite (2.10) in terms of the hypergeometric function

(2.11) $${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k,$$

where $(a)_k := a(a+1)\ldots(a+k-1)$. In general, the right-hand side of (2.11) is only defined if $|z| < 1$ and $c \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$. However, for $a = -n$, n nonnegative integer, the infinite series in (2.11) terminates:

$$(2.12)\qquad {}_2F_1(-n,b;c;z) = \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k\, k!} z^k$$

and the right-hand side of (2.12) remains meaningful for all complex z and for all $c \in \mathbb{C}\backslash\{0,-1,\ldots,-n+1\}$.

We obtain from (2.10) and (2.12):

$$(2.13)\qquad T^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{m!\,(N-m)!}{n!\,(N-n)!}\right)^{\frac{1}{2}} \binom{N-n}{m} \cdot$$

$$\cdot\, a^{N-n-m} b^n c^m\, {}_2F_1(-m,-n;N-n-m+1;ad/bc), \qquad m+n \le N.$$

Usually, this expression is rewritten in terms of Jacobi polynomials

$$(2.14)\qquad P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1(-n,n+\alpha+\beta+1;a+1;\tfrac{1-x}{2}),$$

by the use of the transformation

$$(2.15)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a,c-b;c;\tfrac{z}{z-1}),$$

cf. [3,2.1(22)]. Thus (2.13) takes the form

$$(2.16)\qquad T^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^m \left(\frac{m!\,(N-m)!}{n!\,(N-n)!}\right)^{\frac{1}{2}} a^{N-n-m} b^{n-m} (ad-bc)^m \cdot$$

$$\cdot\, P_m^{(N-n-m,n-m)}\left(1 - 2\frac{ad}{ad-bc}\right), \qquad m+n \le N.$$

For $\alpha,\beta > -1$, Jacobi polynomials are orthogonal polynomials on the interval $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. For integer α,β this orthogonality property can be derived from (2.16) combined with Schur's orthogonality relations on SU(2) and an expression for the Haar measure on SU(2) in terms of suitable coordinates. The observation that the $T^N_{m,s}$'s can be written in terms of Jacobi polynomials, goes probably back to GELFAND & ŠAPIRO [4,p.280].

However, we may also transform (2.13) by means of the formula

$$(2.17)\qquad {}_2F_1(-n,b;c;z) = \frac{(c-b)_n}{(c)_n}\, {}_2F_1(-n,b;b-c-n+1;1-z),$$

$$n = 0,1,2,\ldots;\ c-b,c \ne 0,-1,\ldots,-n+1,$$

cf. [3,10.8(13)] together with (2.14). Then we obtain

$$(2.18)\qquad T^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{m}^{\frac12}\binom{N}{n}^{\frac12} a^{N-n-m} b^n c^m \,{}_2F_1(-m,-n;-N;\tfrac{bc-ad}{bc}) =$$

$$= \binom{N}{m}^{\frac12}\binom{N}{n}^{\frac12} a^{m+n-N} b^{N-m} c^{N-n} \,{}_2F_1(-(N-m),-(N-n);-N;\tfrac{bc-ad}{bc}),$$

where the second identity follows from

$$(2.19)\qquad {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z),$$

cf. [3,2.1(23)]. Thus we have proved (2.18) for $m+n \le N$, but, in view of (2.9), the formula remains valid without this restriction.

For $N = 0,1,\ldots$, $n = 0,1,\ldots,N$ and $p \in \mathbb{C}\backslash\{0\}$ the *Krawtchouk polynomial* $K_n(x;p,N)$ is defined by

$$(2.20)\qquad K_n(x;p,N) := {}_2F_1(-n,-x;-N;p^{-1}).$$

By (2.12) this is a polynomial in x of degree n. For $0 < p < 1$ Krawtchouk polynomials are orthogonal polynomials on the set $\{0,1,\ldots,N\}$ with respect to the binomial distribution:

$$(2.21)\qquad \sum_{x=0}^{N} K_m(x;p,N)K_n(x;p,N)\binom{N}{x}p^x(1-p)^{N-x} = \left(\binom{N}{n}\left(\frac{p}{1-p}\right)^n\right)^{-1}\delta_{m,n}$$

(cf. SZEGÖ [10,§2.82]; we follow the modern notation as used in ASKEY [1,(2.41)]).

It follows from (2.18) and (2.20) that

$$(2.22)\qquad T^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{m}^{\frac12}\binom{N}{n}^{\frac12} a^{N-n-m} b^n c^m\, K_m(n;\tfrac{bc}{bc-ad},N).$$

In particular, put

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}, \qquad 0 < \psi < \pi,$$

which is in U(2). Then

(2.23) $$T^N_{m,n}\begin{pmatrix}\cos\psi & \sin\psi\\ \sin\psi & -\cos\psi\end{pmatrix} =$$

$$= \binom{N}{m}^{\frac{1}{2}}\binom{N}{n}^{\frac{1}{2}}(\cos\psi)^{N-m-n}(\sin\psi)^{m+n}K_m(n;\sin^2\psi,N).$$

Thus, for each value of the parameter $p \in (0,1)$ and for each N we can realize the Krawtchouk polynomials $K_n(x;p,N)$ in terms of the canonical matrix elements of the representation T^N of U(2). Furthermore, the left-hand side of (2.23) being a unitary matrix, the row orthogonality

(2.24) $$\sum_{n=0}^{N} T^N_{m,n}\begin{pmatrix}\cos\psi & \sin\psi\\ \sin\psi & -\cos\psi\end{pmatrix}\overline{T^N_{m',n}\begin{pmatrix}\cos\psi & \sin\psi\\ \sin\psi & -\cos\psi\end{pmatrix}} = \delta_{m,m'}$$

just yields the orthogonality relations (2.21) for Krawtchouk polynomials. This also holds for the column orthogonality, since

$$K_m(n;\sin^2\psi,N) = K_n(m;\sin^2\psi,N)$$

(cf. (2.20)).

3. IDENTIFICATION OF SPHERICAL FUNCTIONS ON A HAMMING SCHEME OVER AN ALPHABET OF TWO LETTERS WITH CANONICAL MATRIX ELEMENTS FOR SU(2)

Consider the abelian group of two elements $F := \{0,1\} = \mathbb{Z}(\text{mod } 2)$ and its N-fold direct product F^N (N = 1,2,...). Write elements of F^N as $x = (x_1,\ldots,x_N)$, $x_i \in F$. The space of all complex-valued functions on F^N becomes a Hilbert space $L^2(F^N)$, where the inner product is taken with respect to the normalized Haar measure on F^N:

(3.1) $$(f,g) := 2^{-N}\sum_{x\in F^N} f(x)\overline{g(x)}, \quad f,g \in L^2(F^N).$$

Note that $L^2(F^N)$ can be identified with the tensor product $\otimes^N L^2(F)$.

The characters on F are χ_0 and χ_1, defined by

(3.2) $$\chi_0(x) := 1, \quad \chi_1(x) := (-1)^x, \qquad x \in F,$$

and they form an orthonormal basis of $L^2(F)$. The characters on F^N are

$$\chi_y := \chi_{y_1} \otimes \chi_{y_2} \otimes \ldots \otimes \chi_{y_N}, \quad y = (y_1,\ldots,y_N) \in F^N, \tag{3.3}$$

i.e.

$$\chi_y(x) = \chi_{y_1}(x_1)\chi_{y_2}(x_2) \ldots \chi_{y_N}(x_N) = $$
$$= (-1)^{x_1y_1+x_2y_2+\ldots+x_Ny_N}, \quad x,y \in F^N,$$

and they form an orthonormal basis of $L^2(F^N)$. Since $\chi_y\chi_{y'} = \chi_{y+y'}$, $y,y' \in F^N$, the dual group of F^N can be identified with F^N. The *Fourier transform* $\mathcal{F}$ on $L^2(F^N)$ is given by

$$(\mathcal{F}f)(y) := 2^{-\frac{1}{2}N} \sum_{x\in F^N} f(x)\chi_y(x), \tag{3.4}$$

where we chose the constant $2^{-\frac{1}{2}N}$ such that $\mathcal{F}$ is a unitary transformation from $L^2(F^N)$ onto itself.

The symmetric group S_N acts as a group of automorphisms on F^N by

$$\sigma(x_1,\ldots,x_N) := (x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(N)}), \tag{3.5}$$
$$(x_1,\ldots,x_N) \in F^N, \quad \sigma \in S_N.$$

Let G be the semidirect product $F^N \circ S_N$ corresponding to this action. Then F^N can be identified with the homogeneous space G/S_N. This homogeneous space is called a *Hamming scheme* over the alphabet F of two letters. The terminology stems from coding theory, cf. for instance MAC WILLIAMS & SLOANE [7, Ch.21,§3]. Let λ be the regular representation of G on $L^2(F^N)$, i.e.

$$\begin{cases} (\lambda(0,\sigma)f)(x) = f(\sigma^{-1}x), & \sigma \in S_N, \\ (\lambda(y,id)f)(x) = f(x-y), & y \in F^N, \end{cases} \tag{3.6}$$

where $f \in L^2(F^N)$, $x \in F^N$. Then

$$\lambda(0,\sigma)\mathcal{F} = \mathcal{F}\lambda(0,\sigma), \quad \sigma \in S^N. \tag{3.7}$$

Hence, if $f \in L^2(F^N)$ is symmetric in $x_1,\ldots,x_N$ then $\mathcal{F}f$ is symmetric and

$$(3.8)\qquad (\mathcal{F}f)(y) = 2^{-\frac{1}{2}N} \sum_{x\in F^N} f(x)\chi_y(x) =$$

$$= 2^{-\frac{1}{2}N} \sum_{x\in F^N} f(x)\Big(\frac{1}{N!} \sum_{\sigma\in S^N} \chi_y(\sigma x)\Big).$$

The *Hamming distance* on F^N is defined by

$$(3.9)\qquad d(x,y) := |\{i \mid x_i \neq y_i\}|, \qquad x,y \in F^N.$$

It is translation invariant. The symmetric functions in $x \in F^N$ are just the functions which only depend on $d(x,0)$. The expression

$$\frac{1}{N!} \sum_{\sigma\in S^N} \chi_y(\sigma x)$$

occurring in (3.8) is symmetric both in x and y. Hence, for $n = 0,1,\dots,N$, we can define functions ϕ_n^N on F^N and $\tilde{\phi}_n^N$ on $\{0,1,\dots,N\}$ such that

$$(3.10)\qquad \tilde{\phi}^N_{d(y,0)}(d(x,0)) = \phi^N_{d(y,0)}(x) := \frac{1}{N!} \sum_{\sigma\in S_N} \chi_y(\sigma x), \qquad x,y \in F^N.$$

Note the similarity between these functions on the one hand and the Bessel functions in connection with the Fourier transform of rotation invariant functions on $\mathbb{R}^n$ on the other hand. In fact, the functions ϕ_n^N are the *spherical functions* on F^N with respect to the group G, i.e.:

PROPOSITION 3.1. $L^2(F^N)$ *is an orthogonal direct sum of* G*-invariant subspaces* H_n^N, $n = 0,1,\dots,N$, *where* $H_n^N := \text{span}\{\chi_y \mid d(y,0) = n\}$. *In each subspace* H_n^N *there is a unique* S_N*-invariant function which takes the value* 1 *in* 0, *namely the function* ϕ_n^N. *The subspaces* H_n^N *are irreducible under the action of* G.

The proof is immediate, by the use of (3.10). It follows from (3.10) that

$$\tilde{\phi}_n^N(m) = \frac{n!\,(N-n)!}{N!} \sum_{k=0}^{n\wedge m} (-1)^k \binom{m}{k}\binom{N-m}{n-k} =$$

$$= \frac{(N-m)!\,(N-n)!}{N!\,(N-m-n)!}\; {}_2F_1(-n,-m;N-m-n+1;-1).$$

Hence, by (2.17) and (2.20):

(3.11) $$\tilde{\phi}_n^N(m) = K_n(m;\tfrac{1}{2},N),$$

i.e., the spherical functions on F^N are Krawtchouk polynomials of order $p = \frac{1}{2}$. This result goes back to VERE-JONES [10].

On comparing (2.23) and (3.11) we find that

(3.12) $$T^N_{m,n}\begin{pmatrix} 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} \end{pmatrix} = 2^{-\frac{1}{2}N}\binom{N}{m}^{\frac{1}{2}}\binom{N}{n}^{\frac{1}{2}}\tilde{\phi}^N_m(n).$$

We will give now an other, more intrinsic proof of this relation, only using the group theoretic characterization of $T^N_{m,n}$ and $\tilde{\phi}^N_n$ and not any a priori knowledge that they can be expressed in terms of Krawtchouk polynomials.

Consider the natural action of U(2) on $L^2(F)$ with respect to the basis χ_0,χ_1 of $L^2(F)$, i.e. if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$ then

$$T\chi_0 = a\chi_0 + c\chi_1, \quad T\chi_1 = b\chi_0 + d\chi_1.$$

This yields a unitary action of U(2) on $L^2(F^N) = \otimes^N L^2(F)$, which commutes with the action of S_N on $L^2(F^N)$. Hence the space $L^2(S_N\backslash F^N)$ of symmetric functions on F^N is invariant under the action of U(2). We can make the following identification between the concepts from Sections 2 and 3, respectively:

(3.13) $$\begin{cases} \text{U(2)-module } \mathbb{C}^2 & \leftrightarrow \quad \text{U(2)-module } L^2(F), \\ \{e_0,e_1\} & \leftrightarrow \quad \{\chi_0,\chi_1\}, \\ \text{U(2)-module } \otimes^N \mathbb{C}^2 & \leftrightarrow \quad \text{U(2)-module } L^2(F^N), \\ \text{U(2)-module } V^N & \leftrightarrow \quad \text{U(2)-module } L^2(S_N\backslash F^N), \\ f^N_n & \leftrightarrow \quad \phi^N_n, \\ e^N_n & \leftrightarrow \quad \binom{N}{n}^{\frac{1}{2}}\phi^N_n. \end{cases}$$

Now the crucial point is to identify the Fourier transform F with the action of a certain element in U(2). Consider first the Fourier transform acting on $f \in L^2(F)$:

$$(Ff)(x) = 2^{-\frac{1}{2}}(f(0)\chi_x(0) + f(1)\chi_x(1)) =$$

$$= 2^{-\frac{1}{2}}(f(0)\chi_0(x) + f(1)\chi_1(x)).$$

Hence

$$F\chi_0 = 2^{-\frac{1}{2}}(\chi_0+\chi_1), \quad F\chi_1 = 2^{-\frac{1}{2}}(\chi_0-\chi_1),$$

i.e., F corresponds with the unitary matrix

$$(3.14) \qquad s_0 := \begin{pmatrix} 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} \end{pmatrix}.$$

Since F acting on $L^2(F^N)$ is the N-fold tensor product of F acting on $L^2(F)$, this correspondence is also valid on $L^2(F^N)$:

$$(3.15) \qquad T^N(s_0) \leftrightarrow F \text{ acting on } L^2(S_N\backslash F^N).$$

It follows from (3.13), (3.15) and (2.7) that

$$(3.16) \qquad \binom{N}{n}^{\frac{1}{2}}(F\phi_n^N)(x) = \sum_{m=0}^{N} T^N_{m,n}(s_0)\binom{N}{m}^{\frac{1}{2}}\phi_m^N(x).$$

The left-hand side of (3.16) can be evaluated by means of (3.10) and (3.4):

$$(F\phi_n^N)(x) = \frac{n!(N-n)!}{N!} \sum_{d(y,0)=n} (F\chi_y)(x) =$$

$$= \frac{n!(N-n)!}{N!} \sum_{d(y,0)=n} 2^{-\frac{1}{2}N} \sum_{z\in F^N} \chi_y(z)\chi_x(z) =$$

$$= \frac{2^{\frac{1}{2}N}n!(N-n)!}{N!}\, \delta_{d(x,0)n}.$$

Hence

$$(3.17)\qquad \sum_{m=0}^{N} T^N_{m,n}(s_0)\binom{N}{m}^{\frac{1}{2}}\tilde{\phi}^N_m(\ell) = 2^{\frac{1}{2}N}\binom{N}{n}^{-\frac{1}{2}}\delta_{\ell,n}.$$

Now multiply both sides of (3.17) with $T^N_{p,n}(s_0)$, sum over p and use that $T^N_{m,n}(s_0)$ is a unitary matrix with real entries (by (2.8)). It follows that

$$\binom{N}{p}^{\frac{1}{2}}\tilde{\phi}^N_p(\ell) = 2^{\frac{1}{2}N}\binom{N}{\ell}^{-\frac{1}{2}}T^N_{p,\ell}(s_0).$$

This settles (3.12).

4. CONNECTION WITH THE METAPLECTIC REPRESENTATION OF SL(2,$\mathbb{R}$)

Let us put the results of Section 3 in a more general framework. Let F be a locally compact abelian group and let F be isomorphic to the dual group F^* via the isomorphism $y \to \chi_y$. Let G be a locally compact group and let π be a unitary representation of G on $L^2(F)$ with the following properties:

(i) For some $s_0 \in G$, $\pi(s_0)$ is the Fourier transform $\mathcal{F}$ on $L^2(F)$.

(ii) For some closed subgroup H of G there is a function c on H×F such that

$$(\pi(h)f)(x) = c(h,x)f(x), \quad f \in L^2(F), \quad x \in F, \quad h \in H,$$

and

$$c(h,x) = c(h,y) \quad \text{for all } h \in H \Rightarrow x = y.$$

Then the Dirac measures on F form a (generalized) H-basis for $L^2(F)$ and $\pi(s_0)$ has (generalized) canonical matrix elements $(x,y) \to \chi_y(x)$ with respect to this basis.

Next suppose that for each natural number N there is a compact group K_N of automorphisms of F^N such that:

(i) K_N acting on $L^2(F^N)$ commutes with $\otimes^N \pi(G)$.

(ii) If $c(h,x_1)\ \dots\ c(h,x_N) = c(h,y_1)\ \dots\ c(h,y_N)$ for all $h \in H$ then $(x_1,\dots,x_N)$ and $(y_1,\dots,y_N)$ are in the same K_N-orbit.

Let $\mathcal{H}_N$ be the subspace of $L^2(F^N)$ consisting of K_N-invariant functions. Let π_N be the corresponding subrepresentation of $\otimes^N\pi$. Write $\tilde{x}$ for the K_N-orbit through $x \in F^N$ and put

$$\tilde{\phi}_{\tilde{y}}(\tilde{x}) = \phi_{\tilde{y}}(x) := \int_{K_N} \chi_{y_1}(k\cdot x_1)\ \dots\ \chi_{y_N}(k\cdot x_N)\,dk, \qquad x,y \in F^N.$$

Then $\phi_{\tilde{y}}$ is a spherical function on $F^N \circ K_N / K_N$. Now the Dirac measures on $K_N \backslash F^N$ form a (generalized) H-basis for $\mathcal{H}_N$ and $\pi_N(s_0)$ has (generalized) canonical matrix elements $(\tilde{x},\tilde{y}) \to \tilde{\phi}_{\tilde{y}}(\tilde{x})$.

In Section 3 we had $F = \{0,1\}$, $G = U(2)$, H is the subgroup of diagonal elements, π is the natural representation of $U(2)$ on $L^2(\{0,1\})$, K_N is the symmetric group and the spherical functions $\tilde{\phi}_{\tilde{y}}(\tilde{x})$ were Krawtchouk polynomials.

For another example let $F = \mathbb{R}$, G a two-fold covering group of $SL(2,\mathbb{R})$ and π the metaplectic representation of G on $L^2(\mathbb{R})$ (cf. WEIL [12] for the definition). Let K_N be the rotation group $SO(N)$. Then the functions $\tilde{\phi}_{\tilde{y}}(\tilde{x})$ can be expressed in terms of Bessel functions and the representations π_N are irreducible and belong to the discrete series. Thus we have a conceptual interpretation that discrete series representations of $SL(2,\mathbb{R})$ or its covering groups are related to the Hankel transform. See SALLY [9], GROSS & KUNZE [5] and RALLIS & SCHIFFMANN [8] for further information.

The analogy between the cases $F = \{0,1\}$ and $F = \mathbb{R}$ is not perfect, since $U(2)$ is not contained in the metaplectic group related to $\{0,1\}$, so π is not a metaplectic representation in this case.

5. KRAWTCHOUK POLYNOMIALS AS INTERTWINING FUNCTIONS ON HAMMING SCHEMES

Let G be the wreath product $(S_{k+1})^N \circ S_N$ $(k,N \in \mathbb{N})$, i.e. the semidirect product of $(S_{k+1})^N$ and S_N with S^N acting on $(S_{k+1})^N$ by

$$\tau\cdot(\sigma_1,\ldots,\sigma_N) := (\sigma_{\tau^{-1}(1)},\ldots,\sigma_{\tau^{-1}(N)}), \qquad \tau \in S_N,\ \sigma_1,\ldots,\sigma_N \in S_{k+1}.$$

Let $X := \{0,1,\ldots,k\}$. G acts transitively on X^N by

$$(\sigma_1,\ldots,\sigma_N)(x_1,\ldots,x_N) := (\sigma_1 x_1,\ldots,\sigma_N x_N), \qquad \sigma_i \in S^{k+1},\ x_i \in X,$$

and

$$\tau(x_1,\ldots,x_N) := (x_{\tau^{-1}(1)},\ldots,x_{\tau^{-1}(N)}), \quad \tau \in S^N,\ x_i \in X.$$

Let S_k denote the stabilizer of $0 \in X$ in S_{k+1}. Put $0 := (0,0,\ldots,0) \in X^N$. Then the stabilizer K of $0 \in X^N$ in G equals $(S_k)^N \circ S_N$. The homogeneous space

$X^N = G/K$ is called a *Hamming scheme* over the alphabet X of k+1 letters.

Fix an integer such that $0 \le q \le k-1$. Let $S_{q+1} \times S_{k-q}$ denote the stabilizer of the subset $\{0,1,\ldots,q\}$ of X in S_k. Let $L^2(X)$ be the space of all complex-valued functions on X provided with the inner product

$$(f,g) := (k+1)^{-1} \sum_{x \in F} f(x)\overline{g(x)}, \quad f,g \in L^2(X).$$

Let $\chi_0, \chi_1, \ldots, \chi_k$ be an orthonormal basis of $L^2(X)$ such that

$$\text{(5.1)} \qquad \chi_0(x) := 1, \quad x \in X,$$

$$\text{(5.2)} \qquad \chi_1(x) := \begin{cases} \left(\frac{k-q}{q+1}\right)^{\frac{1}{2}}, & x = 0,\ldots,q, \\ -\left(\frac{q+1}{k-q}\right)^{\frac{1}{2}}, & x = q+1,\ldots,k. \end{cases}$$

Note that χ_1 is $S_{q+1} \times S_{k-q}$-invariant.

The Hilbert space $L^2(X^N)$, provided with the inner product

$$(f,g) := (k+1)^{-N} \sum_{x \in X^N} f(x)\overline{g(x)}, \quad f,g \in L^2(X^N),$$

can be identified with the tensor product $\otimes^N L^2(X)$. Put

$$\text{(5.3)} \qquad \chi_y(x) := \chi_{y_1}(x_1) \ldots \chi_{y_N}(x_N), \quad x = (x_1,\ldots,x_N) \in X^N, \quad y = (y_1,\ldots,y_N) \in X^N.$$

Then the functions χ_y $(y \in X^N)$ form an orthonormal basis of $L^2(X^N)$. The Hamming distance on X^N is defined by

$$\text{(5.4)} \qquad d(x,y) := |\{i \mid x_i \neq y_i\}|, \quad x,y \in X^N.$$

PROPOSITION 5.1.

(a) $L^2(X^N)$ *is an orthogonal direct sum of* G*-invariant subspaces* H_n^N, $n = 0,1,\ldots,N$, *where*

$$\text{(5.5)} \qquad H_n^N := \text{span}\{\chi_y \mid d(y,0) = n\}.$$

(b) *Each space* H_n^N *contains a unique function* $\phi_n^{N,q}$ *which* (i) *is invariant under the subgroup* $H := (S_{q+1} \times S_{k-q})^N \circ S_N$ *of* G *and* (ii) *takes the value* 1 *in* $0 \in X^N$.

(c) *The spaces* H_n^N *are irreducible under* G.

PROOF. Part (a) is evident. For the proof of (b) let $f \in H_n^N$ satisfy (i). Then f is a linear combination of functions of the form

$$x \to \chi_1(x_{i_1})\chi_1(x_{i_2}) \ldots \chi_1(x_{i_n}), \qquad 1 \le i_1 < i_2 < \ldots < i_n \le N,$$

because of the $(S_{q+1} \times S_{k-q})^N$-invariance. By S_N-invariance we get

$$f(x) = \frac{C}{N!} \sum_{\tau \in S_N} \chi_1(x_{\tau(1)}) \ldots \chi_1(x_{\tau(n)})$$

for some constant C. If f also satisfies (ii) then

$$1 = C\left(\frac{k-q}{q+1}\right)^{\frac{1}{2}n}.$$

Hence (b) holds with

$$\text{(5.6)} \qquad \phi_n^{N,q}(x) = \left(\frac{q+1}{k-q}\right)^{\frac{1}{2}n} \cdot \frac{1}{N!} \sum_{\tau \in S_N} \chi_1(x_{\tau(1)}) \ldots \chi_1(x_{\tau(n)}).$$

Finally (c) follows from the case q = 0 (i.e. K = H) of (b). □

The functions $\phi_n^{N,q}$ are called *intertwining functions* because the G-intertwining operators from $L^2(G/K)$ into $L^2(G/H)$ can be written in terms of these functions. The functions $\phi_n^{N,0}$ are called the *spherical functions* on the homogeneous space G/K.

By using (5.6) we can evaluate the intertwining functions in terms of special functions. First note that an H-invariant function on X^N only depends on

$$\text{(5.7)} \qquad \tilde{x} := |\{i \mid q+1 \le x_i \le k\}|, \qquad x \in X^N.$$

Write

(5.8) $$\tilde{\phi}_n^{N,q}(\tilde{x}) := \phi_n^{N,q}(x).$$

It follows from (5.6) and (5.2) that

$$\tilde{\phi}_n^{N,q}(m) = \frac{n!\,(N-n)!}{N!}\left(\frac{q+1}{k-q}\right)^{\frac{1}{2}n}$$

$$\cdot \sum_{k=0}^{n\wedge m} \binom{m}{k}\binom{N-m}{n-k}\left(\frac{k-q}{q+1}\right)^{\frac{1}{2}(n-k)}\left(\frac{q+1}{k-q}\right)^{\frac{1}{2}k}(-1)^k =$$

$$= \frac{(N-m)!\,(N-n)!}{N!\,(N-m-n)!}\; {}_2F_1\left(-n,-m;N-m-n+1;-\frac{q+1}{k-q}\right).$$

Hence, by (2.17) and (2.20):

(5.9) $$\tilde{\phi}_n^{N,q}(m) = K_n\left(m;\frac{k-q}{k+1},N\right).$$

The spherical function case $q = 0$ of (5.9) is due to DUNKL [2]. The general case is probably new. Note that the set $\{(k-q)/(q+1) \mid 0\le q\le k-1,\ k=1,2,\ldots\}$ is just the set of rational numbers between 0 and 1. R. Askey suggested me that Krawtchouk polynomials of rational order might have a group theoretic interpretation as intertwining functions.

6. THE CONNECTION BETWEEN TWO DIFFERENT GROUP THEORETIC INTERPRETATIONS OF KRAWTCHOUK POLYNOMIALS OF GENERAL ORDER

Let F be the set $\{0,1\}$. Fix $0 < p < 1$ and let w be the weight function on F given by

(6.1) $$w(0) := 1-p, \quad w(1) := p.$$

Let $L^2(F;w)$ be the space of complex-valued functions on F with inner product

$$(f,g) := \sum_{x\in F} f(x)\overline{g(x)}w(x), \qquad f,g \in L^2(F;w).$$

We will now extend the results of Section 3 to the case of this weighted L^2-space. Let N be a natural number. Let

$$(6.2)\qquad W(x) := w(x_1)w(x_2)\ \dots\ w(x_N), \quad x \in F^N.$$

Then $L^2(F^N;W) = \otimes^N L^2(F;w)$. Let

$$(6.3)\qquad \chi_0(x) := 1, \quad x \in F,$$

$$(6.4)\qquad \chi_1(x) := \begin{cases} \left(\frac{p}{1-p}\right)^{\frac{1}{2}}, & x = 0, \\ -\left(\frac{1-p}{p}\right)^{\frac{1}{2}}, & x = 1. \end{cases}$$

Then $\{\chi_0, \chi_1\}$ is an orthonormal basis for $L^2(F;w)$ and the functions χ_y $(y \in F^N)$, given by

$$(6.5)\qquad \chi_y(x) = \chi_{y_1}(x_1)\ \dots\ \chi_{y_N}(x_N), \quad x \in F^N,$$

form an orthonormal basis of $L^2(F^N,W)$. By symmetrization of the basis functions (6.5) we obtain a basis for the symmetric functions in $L^2(F^N,W)$:

$$(6.6)\qquad \tilde{\psi}^{N,p}_{d(y,0)}(d(x,0)) = \psi^{N,p}_{d(y,0)}(x) := \frac{1}{N!} \sum_{\sigma \in S_N} \frac{\chi_y(\sigma x)}{\chi_y(0)}, \quad x \in F^N,$$

where the Hamming distance d on F^N is defined by (3.9). It follows from Section 5 that the intertwining functions $\phi^{N,q}_n$ are special cases of (6.6):

$$(6.7)\qquad \tilde{\phi}^{N,q}_n = \tilde{\psi}^{N,\frac{k-q}{k+1}}_n .$$

There is a natural unitary action of U(2) on $L^2(F;w)$ with respect to the basis χ_0, χ_1, just as in Section 3. Via the tensor product this yields a unitary action of U(2) on $L^2(F^N,W)$, which commutes with the action of S_N on $L^2(F^N,W)$. Thus, similarly to (3.13), we can make an identification between concepts from Sections 2 and 6, respectively:

$$(6.8)\quad \begin{cases} \text{U(2)-module } \mathbb{C}^2 & \leftrightarrow \quad \text{U(2)-module } L^2(F;w), \\ \{e_0, e_1\} & \leftrightarrow \quad \{\chi_0, \chi_1\}, \\ \text{U(2)-module } \otimes^N \mathbb{C}^2 & \leftrightarrow \quad \text{U(2)-module } L^2(F^N;W), \\ \text{U(2)-module } V^N & \leftrightarrow \quad \text{U(2)-module } L^2(S_N\backslash F^N;W), \\ f_n^N & \leftrightarrow \quad \left(\frac{p}{1-p}\right)^{\frac{1}{2}n} \psi_n^{N,p}, \\ e_n^N & \leftrightarrow \quad \binom{N}{n}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}n} \psi_n^{N,p} \end{cases}$$

The "Fourier" transform $\mathcal{F}$ on $L^2(F;w)$, defined by

$$(6.9)\qquad (\mathcal{F}f)(y) = \frac{1}{w(y)^{\frac{1}{2}}} \sum_{x\in F} f(x)\chi_y(x)w(x),$$

is clearly a unitary transformation from $L^2(F;w)$ onto itself. A calculation using (6.1), (6.3), (6.4) shows that this unitary transformation is given by the matrix

$$(6.10)\qquad s_p := \begin{pmatrix} (1-p)^{\frac{1}{2}} & p^{\frac{1}{2}} \\ p^{\frac{1}{2}} & -(1-p)^{\frac{1}{2}} \end{pmatrix}.$$

Let $\mathcal{F}$ acting on $L^2(F^N;W)$ be defined as the N-fold tensor product of $\mathcal{F}$ acting on $L^2(F;w)$. Then

$$(6.11)\qquad (\mathcal{F}f)(y) = \frac{1}{W(y)^{\frac{1}{2}}} \sum_{x\in F^N} f(x)\chi_y(x)W(x).$$

Just as in (3.15) we have the correspondence

$$(6.12)\qquad T^N(s_p) \leftrightarrow \mathcal{F} \text{ acting on } L^2(S_N\backslash F^N;W).$$

It follows from (6.8), (6.12) and (2.7) that

$$(6.13)\qquad \binom{N}{n}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}n} (\mathcal{F}\psi_n^{N,p})(x) =$$

$$= \sum_{m=0}^{N} T^N_{m,n}(s_p)\binom{N}{m}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}m} \psi_m^{N,p}(x).$$

The left-hand side of (6.13) can be evaluated by means of (6.9) and (6.6):

$$(\mathcal{F}\psi_n^{N,p})(x) = \binom{N}{n}^{-1}\left(\frac{1-p}{p}\right)^{\frac{1}{2}n} \frac{\delta_{d(x,0),n}}{p^{\frac{1}{2}d(x,0)}(1-p)^{\frac{1}{2}(N-d(x,0))}}.$$

Hence

$$\sum_{m=0}^{N} T^N_{m,n}(s_p)\binom{N}{m}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}m}\tilde{\psi}_m^{N,p}(\ell) = \binom{N}{n}^{-\frac{1}{2}} p^{-\frac{1}{2}\ell}(1-p)^{\frac{1}{2}(\ell-N)}\delta_{\ell,n}.$$

Now we use that $(T^N_{m,n}(s_p))$ is a real orthogonal matrix (cf. (2.8)). Finally

$$\text{(6.14)} \qquad T^N_{r,\ell}(s_p) = \binom{N}{r}^{\frac{1}{2}}\binom{N}{\ell}^{\frac{1}{2}} p^{\frac{1}{2}(\ell+r)}(1-p)^{\frac{1}{2}(N-\ell-r)}\ \tilde{\psi}_r^{N,p}(\ell).$$

In particular, by combination of (6.14) with (6.7), we have given a conceptual explanation that both the canonical matrix elements of SU(2) and the intertwining functions on Hamming schemes can be expressed in terms of the same special functions.

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