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MATRIX ELEMENTS OF IRREDUCIBLE REPRESENTATIONS
OF SU(2) X SU(2) AND VECTOR-VALUED ORTHOGONAL POLYNOMIALS

Preprint

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Matrix elements of irreducible representations of SU(2) × SU(2) and vector-valued orthogonal polynomials *)

by

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ABSTRACT

The matrix elements of irreducible representations of SU(2) × SU(2) in a diag(SU(2)×SU(2))-basis are expressed in terms of vector-valued orthogonal polynomials, which generalize the Jacobi polynomials.

KEY WORDS & PHRASES: *matrix elements of irreducible representations of* SU(2) × SU(2); *vector-valued orthogonal polynomials; generalized Jacobi polynomials; matrix elements of principal series representations of* SL(2,ℂ)

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*) This report will be submitted for publication elsewhere.

## 0. INTRODUCTION

It is well-known (cf. VILENKIN [11, Ch. 3]) that the matrix elements of the irreducible representations (irr. reps) of SU(2) in S(U(1)×U(1))-basis can be expressed in terms of Jacobi polynomials, such that the orthogonality relations for these polynomials are equivalent to Schur's orthogonality relations for the matrix elements. More generally, let G be a compact Lie group with closed subgroup K such that each irr. rep. of G, restricted to K, is multiplicity free. Consider the matrix elements of the irr. reps of G in a K-basis. Is it possible to express them in terms of some kind of orthogonal polynomials? For the case G = SU(2) × SU(2), K = diagonal in G, this paper will give a positive answer. (Note that this case is a covering of the pair (G,K) = (SO(4),SO(3)). The resulting polynomials are vector-valued and orthogonal on [-1,1] with respect to a positive definite matrix-valued weight function. It would be of interest to generalize these results to the cases (G,K) = (SO(n),SO(n-1)) or (U(n), U(n-1)).

The topic of this paper originated from work on the global approach to the representation theory of a noncompact semisimple Lie group G (cf. [7] for SL(2,ℝ), KOSTERS [8] for SL(2,ℂ)). In this approach one needs some knowledge of the matrix elements of the principal series reps of G in a K-basis (K maximal compact subgroup of G). These matrix elements have integral representations in terms of the matrix elements of irr. reps of K (cf. (4.1) in the case G=SL(2,ℂ)). Manipulation of these integral representations will be simplified if one can express the matrix elements for K in terms of orthogonal polynomials. Thus the results of the present paper will be useful for the analysis on $SO_0(4,1)$.

It is the author's feeling that the highly nontrivial example of vector-valued orthogonal polynomials presented here is interesting for its own sake. Hopefully this paper will also be useful for phycisists, who have already studied the matrix elements for SO(4) for a long time (cf. for instance FREEDMAN & WANG [3], SMORODINSKIĬ & SHEPELEV [10], BASU & SRINVASAN [1]). Many authors start with the matrix elements of the principal series reps of $SO_0(3,1)$ (cf. [1],[10]) and then obtain the matrix elements for the compact case by analytic continuation. In the present paper, with its emphasis on orthogonal polynomials, it seemed more natural to start with the compact case, but in the final section 4 the noncompact analogue is briefly discussed.

The other sections have the following contents. In section 1 matrix elements for SU(2) are reviewed, both as a tool needed later and as a motivating example. In section 2 Schur's orthogonality relations for matrix elements for SU(2) × SU(2) are expressed as an orthogonality for vector-valued functions on $[0,\pi]$ and good candidates are selected for the expected vector-valued orthogonal polynomials. In section 3 these polynomials are really obtained together with an integral representation and a power series expansion. There are two further matters of particular interest in section 3: First, a trick to deform the integral of an analytic function over SU(2) into the complexification SL(2,$\mathbb{C}$) by multiplication on the right of the integration variable with a particular element of SL(2,$\mathbb{C}$) (cf. the transition (3.3)→(3.6)) and, second, an unexpected symmetry (3.11) for the vector-valued polynomials.

## 1. THE MATRIX ELEMENTS FOR SU(2)

Let $\ell \in \frac{1}{2}\mathbb{Z}_+ := \{0,\frac{1}{2},1,\frac{3}{2},\ldots\}$. Let $\mathcal{H}_\ell$ be the space of homogeneous polynomials of degree $2\ell$ in two complex variables, made into a Hilbert space by the choice of orthonormal basis $\{\psi_n^\ell \mid n = -\ell, -\ell+1, \ldots, \ell\}$:

$$(1.1)\qquad \psi_n^\ell(x,y) := \binom{2\ell}{\ell-n}^{\frac{1}{2}} x^{\ell-n}\, y^{\ell+n}.$$

Define a rep $T^\ell$ of GL(2,$\mathbb{C}$) on $\mathcal{H}_\ell$ by

$$(1.2)\qquad \left(T^\ell\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}f\right)(x,y) := f(\alpha x+\gamma y, \beta x+\delta y).$$

The $T^\ell$'s form a complete system of representatives for $(SU(2))^\wedge$(cf. VILENKIN [11, Ch. 3]).

Write $T^\ell(g)\,(g\in GL(2,\mathbb{C}))$ as a matrix $(t_{mn}^\ell(g))$ with respect to the basis functions $\psi_n^\ell$:

$$(1.3)\qquad T^{\ell}(g)\psi_n^{\ell} = \sum_{m=-\ell}^{\ell} t_{mn}^{\ell}(g)\psi_m^{\ell}, \qquad g \in GL(2,\mathbb{C}).$$

If g is a diagonal matrix then so is $(t_{mn}^{\ell}(g))$. It follows from (1.1), (1.2), (1.3) that

$$(1.4)\qquad \binom{2\ell}{\ell-n}^{\frac{1}{2}} (\alpha x+\gamma y)^{\ell-n}(\beta x+\delta y)^{\ell+n} = \sum_{m=-\ell}^{\ell} t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \binom{2\ell}{\ell-m}^{\frac{1}{2}} x^{\ell-m} y^{\ell+m}.$$

Expansion of the left hand side of (1.4) yields

$$(1.5)\qquad t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = ((\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!)^{\frac{1}{2}}.$$

$$\cdot \sum_{r=0\vee(-n-m)}^{(\ell-n)\wedge(\ell-m)} \frac{\alpha^{\ell}\beta^{\ell-m-r}\gamma^{\ell-n-r}\delta^{m+n+r}}{r!(\ell-m-r)!(\ell-n-r)!(m+n+r)!}.$$

This implies the symmetries

$$(1.6)\qquad \beta^{m}\gamma^{n}t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \beta^{n}\gamma^{m}t_{nm}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

$$(1.7)\qquad t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t_{nm}^{\ell}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}.$$

From (1.4) and (1.7) we obtain the integral representation

$$(1.8)\qquad t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \left(\frac{(\ell-n)!(\ell+n)!}{(\ell-m)!(\ell+m)!}\right)^{\frac{1}{2}}.$$

$$\cdot \frac{1}{2\pi}\int_0^{2\pi} (\alpha e^{i\phi}+\beta e^{-i\phi})^{\ell-m}(\gamma e^{i\phi}+\delta e^{-i\phi})^{\ell+m} e^{2in\phi}d\phi.$$

The following symmetry is apparent from (1.8):

$$(1.9)\qquad t_{mn}^{\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t_{-m,-n}^{\ell}\begin{pmatrix}\delta & \gamma\\ \beta & \alpha\end{pmatrix}.$$

Now specialize to SU(2). We will use the notation

$$(1.10)\qquad k(\alpha,\beta) := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1,$$

$$(1.11)\qquad b_\theta := k(\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta),$$

$$(1.12)\qquad m_\phi := k(e^{\frac{1}{2}i\phi}, 0).$$

Note that

$$(1.13)\qquad t^{\ell}_{mn}(m_\phi) = e^{-in\phi}\delta_{mn}.$$

By the Cartan decomposition each element of SU(2) can be written as $m_\phi\, b_\theta\, m_\psi$ and the corresponding integration formula reads

$$(1.14)\qquad \int_{SU(2)} f(g)dg = \tfrac{1}{2}\int_0^\pi \int_0^{4\pi} \int_0^{4\pi} f(m_\phi\, b_\theta m_\psi)\, \sin\theta\, d\theta \frac{d\phi}{4\pi}\, \frac{d\psi}{4\pi}, \quad f \in C(SU(2)).$$

By Schur's orthogonality relations, (1.13) and (1.14) we obtain

$$\int_0^\pi t^{\ell}_{mn}(b_\theta)\overline{t^{\ell'}_{m,n}(b_\theta)}\sin\theta\; d\theta = 0, \qquad \ell \neq \ell'.$$

Suppose that $m + n \geq 0$, $m - n \geq 0$. Then the "lowest" element of the orthogonal system $\{t^{\ell}_{mn} \mid \ell = m, m+1, \ldots\}$ is $t^{m}_{mn}$. From (1.5) we obtain:

$$(1.15)\qquad t^{m}_{mn}(b_\theta) = (-1)^{m-n}\binom{2m}{m-n}^{\frac{1}{2}}(\sin\tfrac{1}{2}\theta)^{m-n}(\cos\tfrac{1}{2}\theta)^{m+n}.$$

Hence, if $\ell \neq \ell'$:

$$\int_0^\pi \frac{t^{\ell}_{mn}(b_\theta)}{t^{m}_{mn}(b_\theta)}\; \frac{t^{\ell'}_{mn}(b_\theta)}{t^{m}_{mn}(b_\theta)}\; (\sin\tfrac{1}{2}\theta)^{2m-2n+1}(\cos\tfrac{1}{2}\theta)^{2m+2n+1}d\theta = 0.$$

By (1.5) $t^{\ell}_{mn}(b_\theta)/t^{m}_{mn}(b_\theta)$ is a polynomial in $\cos\theta$ of degree $\le \ell - m$. It follows that

$$t^{\ell}_{mn}(b_\theta)/t^{m}_{mn}(b_\theta) = \text{const. } P^{(m-n,m+n)}_{\ell-m}(\cos\theta),$$

where the *Jacobi polynomial* $P^{(m-n,m+n)}_{\ell-m}$ is an orthogonal polynomial of degree $\ell - m$ with respect to the weight function $(1-x)^{m-n}(1+x)^{m+n}$ on the interval $(-1,1)$. Of course, this result has been derived in many other ways (cf. VILENKIN [11, Ch. 3]).

## 2. THE MATRIX ELEMENTS FOR SU(2) × SU(2)

Let $K := SU(2)$, $G := K \times K$, $K^* := \operatorname{diag}(K\times K)$, $A := \{a_\theta := (m_\theta, m_{-\theta})\}$ ($m_\theta$ is defined by (1.12)). Then $G = K^*AK^*$ is a Cartan decomposition. The corresponding integral formula is

$$(2.1) \qquad \int_G f(g)dg = \frac{1}{2\pi}\int_0^\pi \int_{K^*}\int_{K^*} f(k_1 a_\theta k_2)\sin^2\theta d\theta\, dk_1\, dk_2, \quad f \in C(G),$$

which is a special case of HELGASON [5, Prop. X.1.19].

A complete system of representatives for $\hat{G}$ is given by the reps $T^{\ell_1,\ell_2}$ $(\ell_1,\ell_2 \in \frac{1}{2}\mathbb{Z}_+)$:

$$(2.2) \qquad T^{\ell_1,\ell_2}(k_1,k_2) := T^{\ell_1}(k_1)\otimes T^{\ell_2}(k_2), \quad k_1, k_2 \in K.$$

The representation space $H_{\ell_1} \otimes H_{\ell_2}$ of $T^{\ell_1,\ell_2}$ can be identified with the space of polynomials in four complex variables x, y, u, v, homogeneous of degree $2\ell_1$ in x, y and homogeneous of degree $2\ell_2$ in u, v. An orthonormal basis of $H_{\ell_1} \otimes H_{\ell_2}$ is given by the polynomials

$$(x,y,u,v) \mapsto \psi^{\ell_1}_{j_1}(x,y)\psi^{\ell_2}_{j_2}(u,v).$$

<u>PROPOSITION 2.1</u>. (cf. [6, Theorems 3.1, 3.2]). *The functions* $\phi^{\ell_1,\ell_2}_{\ell,j}$ $(|\ell_1 + -\ell_2| \le \ell \le \ell_1+\ell_2, |j| \le \ell)$ *defined by*

$$(2.3)\qquad \phi^{\ell_1,\ell_2}_{\ell,j}(x,y,u,v) := (-1)^{\ell_1+\ell_2-\ell}\left(\frac{(2\ell+1)(2\ell_1)!(2\ell_2)!}{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right)^{\frac{1}{2}}\cdot$$

$$\cdot\ (xv-yu)^{\ell_1+\ell_2-\ell}\, t^{\ell}_{\ell_2-\ell_1,j}\begin{pmatrix} x & y \\ u & v\end{pmatrix}$$

*form an orthonormal basis of* $H_{\ell_1}\otimes H_{\ell_2}$ *such that*

$$(2.4)\qquad T^{\ell_1,\ell_2}(k,k)\phi^{\ell_1,\ell_2}_{\ell,j'} = \sum_{j=-\ell}^{\ell} t^{\ell}_{j,j'}(k)\phi^{\ell_1,\ell_2}_{\ell,j}, \qquad k\in K.$$

Define the matrix elements of $T^{\ell_1,\ell_2}$ with respect to this $K^*$-basis $\{\phi^{\ell_1,\ell_2}_{\ell,j}\}$ by

$$(2.5)\qquad T^{\ell_1,\ell_2}(g)\phi^{\ell_1,\ell_2}_{\ell',j'} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\ \sum_{j=-\ell}^{\ell} t^{\ell_1,\ell_2}_{\ell,j,\ell',j'}(g)\phi^{\ell_1,\ell_2}_{\ell,j}, \qquad g\in G.$$

Since the elements of A commute with the elements $(m_\theta,m_\theta)$ in $K^*$ and since

$$T^{\ell_1,\ell_2}(m_\theta,m_\theta)\phi^{\ell_1,\ell_2}_{\ell,j} = e^{-ij\theta}\phi^{\ell_1,\ell_2}_{\ell,j}$$

by (2.4) and (1.12), we conclude that

$$(2.6)\qquad t^{\ell_1,\ell_2}_{\ell,j;\ell',j'}(a_\theta) = 0 \qquad \text{if } j\neq j'.$$

By (2.4), (2.6) and the decomposition $G = K^*AK^*$ the matrix elements $t^{\ell_1,\ell_2}_{\ell,j;\ell',j'}$ will be known if we know the functions $t^{\ell_1,\ell_2}_{\ell,j;\ell',j}\Big|_A$.

PROPOSITION 2.2. *There are the orthogonality relations*

$$(2.7)\qquad \frac{1}{2\pi}\sum_{j=-(\ell\wedge m)}^{\ell\wedge m}\int_0^{\pi} t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)t^{\ell_1',\ell_2'}_{\ell,j;m,j}(d_\theta)\sin^2\theta\, d\theta =$$

$$= \frac{(2\ell+1)(2m+1)}{(2\ell_1+1)(2\ell_2+1)}\,\delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}.$$

PROOF. It follows from Schur's orthogonality relations, (2.1), (2.4) and (2.6) that

$$\frac{\delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}}{(2\ell_1+1)(2\ell_2+1)} = \frac{1}{2\pi}\int_0^\pi\int_{K^*}\int_{K^*} t^{\ell_1,\ell_2}_{\ell,p;m,p}(k_1a_\theta k_2)\overline{t^{\ell_1',\ell_2'}_{\ell,p;m,p}(k_1a_\theta k_2)}\sin^2\theta\, d\theta\, dk_1\, dk_2$$

$$= \sum_{j=-(\ell\wedge m)}^{\ell\wedge m}\ \sum_{j'=-(\ell\wedge m)}^{\ell\wedge m}\frac{1}{2\pi}\int_0^\pi\int_K\int_K t^{\ell}_{p,j}(k_1)\,t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)\,t^{m}_{j,p}(k_2)\,\cdot$$

$$\cdot\ \overline{t^{\ell}_{p,j'}(k_1)}\,\overline{t^{\ell_1',\ell_2'}_{\ell,j';m,j'}(a_\theta)}\,\overline{t^{m}_{j',p}(k_2)}\sin^2\theta\, d\theta\, dk_1\, dk_2 =$$

$$= \frac{1}{(2\ell+1)(2m+1)}\sum_{j=-(\ell\wedge m)}^{\ell\wedge m}\frac{1}{2\pi}\int_0^\pi t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)\,\overline{t^{\ell_1',\ell_2'}_{\ell,j;m,j}(a_\theta)}\sin^2\theta\, d\theta.$$

It follows from (2.5) and (2.3) that $t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)$ is real. □

From now on fix $\ell$ and m ($\ell, m \in \frac{1}{2}\mathbb{Z}_+$, $\ell - m \in \mathbb{Z}$) such that $\ell \le m$. (Because of unitariness of $T^{\ell_1,\ell_2}$ this last condition is not an essential restriction). Then the indices $\ell_1$, $\ell_2$ in $t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)$ can assume all values in $\frac{1}{2}\mathbb{Z}_+$ such that

(2.8) $\qquad \ell_1 + \ell_2 \ge m, \quad |\ell_1 - \ell_2| \le \ell, \quad \ell_1 + \ell_2 - \ell \in \mathbb{Z}$

(cf. Figure 1)

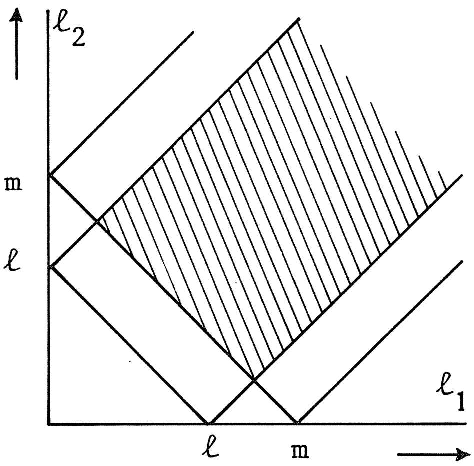


Figure 1.

and $j \in \{-\ell, -\ell+1, \ldots, \ell\}$. Thus, (2.7) can be viewed as the orthogonality relations for the vector-valued functions

$$(2.9)\qquad \theta \mapsto (t^{\ell_1,\ell_2}_{\ell,-\ell;m,-\ell}(a_\theta), t^{\ell_1,\ell_2}_{\ell,-\ell+1;m,-\ell+1}(a_\theta), \ldots, t^{\ell_1,\ell_2}_{\ell,\ell;m,\ell}(a_\theta)),$$

where $(\ell_1,\ell_2)$ run through all values satisfying (2.8). Like at the end of section 1 we pick the "lowest" elements of this orthogonal family. Candidates for these elements are all functions of the form (2.9) with $\ell_1 + \ell_2 = m$. Suppose that we can prove that for all $\theta$ in $(0,\pi)$ the matrix

$$(2.10)\qquad (t^{\frac{1}{2}(m+p),\frac{1}{2}(m-p)}_{\ell,j;m,j}(a_\theta))_{j,p\,=\,-\ell,\,-\ell+1,\,\ldots,\,\ell}$$

is nonsingular. Then, for $n = 0, 1, 2, \ldots$ and $k = -\ell, -\ell+1, \ldots, \ell$ we can define the real vector-valued functions

$$(2.11)\qquad x \mapsto P^{\ell,m}_{n,k}(x) = (P^{\ell,m}_{n,k,-\ell}(x), P^{\ell,m}_{n,k,-\ell+1}(x), \ldots, P^{\ell,m}_{n,k,\ell}(x))$$

on $(-1,1)$ by

$$(2.12)\qquad t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta) = \sum_{p=-\ell}^{\ell} t^{\frac{1}{2}(m+p),\frac{1}{2}(m-p)}_{\ell,j;m,j}(a_\theta) P^{\ell,m}_{\ell_1+\ell_2-m,\ell_2-\ell_1,p}(\cos\theta).$$

Also define

$$(2.13)\qquad W^{\ell,m}_{p,q}(\cos\theta) := \sin\theta \sum_{j=-\ell}^{\ell} t^{\frac{1}{2}(m+p),\frac{1}{2}(m-p)}_{\ell,j;m,j}(a_\theta)\, t^{\frac{1}{2}(m+q),\frac{1}{2}(m-q)}_{\ell,j;m,j}(a_\theta).$$

Then

$$(2.14)\qquad W^{\ell,m}(\cos\theta) := (W^{\ell,m}_{p,q}(\cos\theta))_{p,q\,=\,-\ell,\,\ldots,\,\ell}$$

is a positive definite real symmetric matrix for all $\theta$ in $(0,\pi)$ and it follows from (2.7), (2.12), (2.13) that the vector-valued functions $P^{\ell,m}_{n,k}$ satisfy the orthogonality relations

$$(2.15)\qquad \frac{1}{2\pi}\sum_{p,q=-\ell}^{\ell}\int_{-1}^{1} P^{\ell,m}_{n,k,p}(x)P^{\ell,m}_{n',k',q}(x)W^{\ell,m}_{p,q}(x)\,dx =$$

$$= \frac{(2\ell+1)(2m+1)}{(n+m+1)^2-k^2}\,\delta_{n,n'}\delta_{k,k'}.$$

In this paper we will show that the matrix (2.10) is indeed nonsingular for $\theta$ in $(0,\pi)$ and that $P^{\ell,m}_{n,k,p}$ is a polynomial of degree $n - |p+k|$. Hence the orthogonality relations (2.15) will characterize the vector-valued functions $P^{\ell,m}_{n,k}$ up to constant factors.

## 3. THE VECTOR-VALUED ORTHOGONAL POLYNOMIALS

First we derive an integral representation for the canonical matrix elements. Consider (2.5) with $g = a_\theta$ and evaluate both sides for $(x,y,u,v) = (\alpha,\beta,-\bar\beta,\bar\alpha)$, where $|\alpha|^2 + |\beta|^2 = 1$. In view of (2.3) and (2.6) we obtain

$$(-1)^{\ell_1+\ell_2-m}\left(\frac{(2m+1)(2\ell_1)!(2\ell_2)!}{(\ell_1+\ell_2-m)!(\ell_1+\ell_2+m+1)!}\right)^{\frac12}.$$

$$.\ t^{m}_{\ell_2-\ell_1,j}\begin{pmatrix} e^{\frac12 i\theta}\alpha & e^{-\frac12 i\theta}\beta \\ -e^{-\frac12 i\theta}\bar\beta & e^{\frac12 i\theta}\bar\alpha\end{pmatrix}(e^{i\theta}|\alpha|^2+e^{-i\theta}|\beta|^2)^{\ell_1+\ell_2-m} =$$

$$= \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (-1)^{\ell_1+\ell_2-\ell}\left(\frac{(2\ell+1)(2\ell_1)!(2\ell_2)!}{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right)^{\frac12}.$$

$$.\ t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta)\,t^{\ell}_{\ell_2-\ell_1,j}(k(\alpha,\beta)).$$

Hence, by Schur's orthogonality relations:

$$(3.1)\qquad t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta) = (-1)^{\ell-m}\left(\frac{(2\ell+1)(2m+1)(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}{(\ell_1+\ell_2-m)!(\ell_1+\ell_2+m+1)!}\right)^{\frac12}.$$

$$\cdot \int\limits_K (e^{i\theta}|\alpha|^2+e^{-i\theta}|\beta|^2)^{\ell_1+\ell_2-m}\, t^m_{\ell_2-\ell_1,j}\begin{pmatrix} e^{\frac{1}{2}i\theta}\alpha & e^{-\frac{1}{2}i\theta}\beta \\ -e^{-\frac{1}{2}i\theta}\overline{\beta} & e^{\frac{1}{2}i\theta}\overline{\alpha} \end{pmatrix}.$$

$$\cdot\ t^{\ell}_{\ell_2-\ell_1,j}(k(\overline{\alpha},\overline{\beta}))dk(\alpha,\beta).$$

Next, by some manipulations we will modify this integral representation into a form which is more suitable for our purpose. Substitution of (1.7) into (3.1) yields

$$t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta) = c^{\ell_1,\ell_2}_{\ell,j;m,j}\ \frac{1}{2\pi}\int\limits_K \int\limits_0^{2\pi} (e^{i\theta}|\alpha|^2+e^{-i\theta}|\beta|^2)^{\ell_1+\ell_2-m}$$

$$\cdot\ (\alpha e^{i(\phi+\frac{1}{2}\theta)}+\beta e^{-i(\phi+\frac{1}{2}\theta)})^{m+\ell_1-\ell_2}(-\overline{\beta}e^{i(\phi-\frac{1}{2}\theta)}+\overline{\alpha}\,e^{i(-\phi+\frac{1}{2}\theta)})^{m-\ell_1+\ell_2}$$

$$\cdot\ e^{2ij\phi}t^{\ell}_{\ell_2-\ell_1,j}(k(\overline{\alpha},\overline{\beta}))dk(\alpha,\beta)d\phi,$$

where

$$(3.2)\qquad c^{\ell_1,\ell_2}_{\ell,j;m,j} = (-1)^{\ell-m}\left(\frac{(2\ell+1)(2m+1)(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!(m-j)!(m+j)!}{(\ell_1+\ell_2-m)!(\ell_1+\ell_2+m+1)!(m+\ell_1-\ell_2)!(m-\ell_1+\ell_2)!}\right)^{\frac{1}{2}}.$$

In this last integral representation consider the K-integral as the inner integral and make the transformation of integration variable $k(\alpha,\beta)\mapsto k(\overline{\alpha},\overline{\beta})m_{-2\phi}$. Then the integrand no longer depends on $\phi$ and we obtain

$$(3.3)\qquad t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta) = c^{\ell_1,\ell_2}_{\ell,j;m,j}\int\limits_K (e^{i\theta}|\alpha|^2+e^{-i\theta}|\beta|^2)^{\ell_1+\ell_2-m}.$$

$$\cdot\ (\alpha e^{\frac{1}{2}i\theta}-\beta e^{-\frac{1}{2}i\theta})^{m-\ell_1+\ell_2}(\overline{\alpha}e^{\frac{1}{2}i\theta}+\overline{\beta}e^{-\frac{1}{2}i\theta})^{m+\ell_1-\ell_2}.$$

$$\cdot\ t^{\ell}_{\ell_2-\ell_1,j}(k(\alpha,\beta))dk(\alpha,\beta).$$

LEMMA 3.1. *Let* K *be a connected compact Lie group which has a complexification* $K_c$. *Let* f *be a complex analytic function on an open connected left*-K-*invariant subset* V *of* $K_c$ *containing* K. *Then*

$$(3.4)\qquad \int_K f(k)dk = \int_K f(kk')dk, \quad k' \in V.$$

PROOF. The right hand side is a complex analytic function of k' on V which is constant on K. □

Now observe that the integrand in (3.3) is the restriction to SU(2) of the complex analytic function

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto (e^{i\theta}\alpha\delta - e^{-i\theta}\beta\gamma)^{\ell_1+\ell_2-m} .$$

$$. \ (\alpha e^{\frac{1}{2}i\theta} - \beta e^{-\frac{1}{2}i\theta})^{m-\ell_1+\ell_2}(-\gamma e^{-\frac{1}{2}i\theta} + \delta e^{\frac{1}{2}i\theta})^{m+\ell_1-\ell_2} .$$

$$. \ t^{\ell}_{\ell_1-\ell_2,j}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \quad \text{on } SL(2,\mathbb{C}).$$

For $0 < \theta < \pi$ apply Lemma 3.1 to this function with K' chosen as

$$(3.5)\qquad g_\theta := e^{i\pi/4}(2\sin\theta)^{-\frac{1}{2}}\begin{pmatrix}e^{-\frac{1}{2}i\theta} & e^{\frac{1}{2}i\theta}\\ e^{\frac{1}{2}i\theta} & e^{-\frac{1}{2}i\theta}\end{pmatrix}.$$

We obtain:

$$(3.6)\qquad t^{\ell_1,\ell_2}_{\ell,j;m,j}(a_\theta) = c^{\ell_1,\ell_2}_{\ell,j;m,j}\ e^{3\pi im/2}(2\sin\theta)^m .$$

$$. \sum_{p=-\ell}^{\ell} t^{\ell}_{pj}(g_\theta) \int_K (2|\beta|^2\cos\theta + \alpha\bar{\beta} - \bar{\alpha}\beta)^{\ell_1+\ell_2-m} .$$

$$. \ \beta^{m-\ell_1+\ell_2}(-\bar{\beta})^{m+\ell_1-\ell_2}\, t^{\ell}_{\ell_2-\ell_1,p}(k(\alpha,\beta))\, dk(\alpha,\beta)$$

PROPOSITION 3.2. *We have*

$$(3.7)\qquad t^{\frac12(m+p),\frac12(m-p)}_{\ell,j;m,j}(a_\theta) = \left(\frac{(2\ell+1)(m-j)!(m+j)!(m-p)!(m+p)!}{(2m)!(m-\ell)!(m+\ell+1)!}\right)^{\frac12}.$$

$$.\ (-1)^{\ell+m}\, e^{3\pi i m/2}(2\sin\theta)^m t^{\ell}_{pj}(g_\theta).$$

*For* $0 < \theta < \pi$ *the matrix* $(t^{\frac12(m+p),\frac12(m-p)}_{\ell,j;m,j}(a_\theta))_{j,p=-\ell,\ldots,\ell}$ *is non-singular.*

PROOF. Formula (3.6), together with (1.13) and the invariance of the integral in (3.6) under right multiplication by $m_\phi$ yields

$$t^{\frac12(m+p),\frac12(m-p)}_{\ell,j;m,j}(a_\theta) = c^{\frac12(m+p),\frac12(m-p)}_{\ell,j;m,j}\, e^{3\pi i m/2}(2\sin\theta)^m.$$

$$.\ t^{\ell}_{pj}(g_\theta)\int_K \beta^{m-p}(-\bar\beta)^{m+p} t^{\ell}_{-p,p}(k(\alpha,\beta))\,dk(\alpha,\beta).$$

The integral can be evaluated by using (1.5), (1.14), the beta integral and the *Chu-Vandermonde sum*

$$(3.8)\qquad {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},\quad n = 0,1,\ldots;\ c-b,\ c \neq 0,-1,\ldots,\ -n+1.$$

Finally use (3.2). $\square$

THEOREM 3.3. *Formula (2.12) holds with*

$$(3.9)\qquad P^{\ell,m}_{n,k,p}(x) = A^{\ell,m}_{n,k,p}\int_K (2|\beta|^2x+\alpha\bar\beta-\bar\alpha\beta)^n\beta^{m+k}(-\bar\beta)^{m-k} t^{\ell}_{k,p}(k(\alpha,\beta))\,dk(\alpha,\beta),$$

*where*

$$(3.10)\qquad A^{\ell,m}_{n,k,p} := (-1)^{2\ell}\left(\frac{(2m+1)!(n+m-\ell)!(n+m+\ell+1)!(m-\ell)!(m+\ell+1)!}{n!(n+2m+1)!(m-k)!(m+k)!(m-p)!(m+p)!}\right)^{\frac12}.$$

*There are the symmetries*

$$(3.11) \qquad P^{\ell,m}_{n,k,p} = P^{\ell,m}_{n,p,k} = P^{\ell,m}_{n,-k,-p} = P^{\ell,m}_{n,-p,-k},$$

$$(3.12) \qquad P^{\ell,m}_{n,k,p}(-x) = (-1)^{n+k+p} P^{\ell,m}_{n,k,p}(x).$$

PROOF. Formula (3.9) follows from (3.7), (3.6) and (3.2). The symmetries are derived from (3.9) by the use of (1.6) and (1.9) in the case of (3.11) and by (1.13) in the case of (3.12). □

Of course, by the use of (2.12) and (3.7), the symmetries (3.11) imply certain symmetries for the matrix elements $t^{\ell_1,\ell_2}_{\ell,j;m,j}\big|_A$. It would be interesting to get a deeper understanding of the first of these symmetries.

Now expand the integrand in (3.9) with respect to x and use the invariance of the integral under right multiplication with $m_\phi$ and (1.13). We obtain

$$(3.13) \qquad P^{\ell,m}_{n,k,p}(x) = A^{\ell,m}_{n,k,p} \sum_{\substack{q=|p+k| \\ q+p+k \text{ even}}}^{n} d^{\ell,m}_{n,k,p,q}\, x^{n-q},$$

where

$$d^{\ell,m}_{n,k,p,q} = \frac{(-1)^{m-k+\frac{1}{2}(q-k-p)}\, 2^{n-q}\, n!}{(\frac{1}{2}(q-k-p))!\,(\frac{1}{2}(q+k+p))!\,(n-q)!}\,\cdot$$

$$\cdot \int_K \alpha^{\frac{1}{2}(q+k+p)}\, \overline{\alpha}^{\frac{1}{2}(q-k-p)}\, \beta^{m+n+\frac{1}{2}(k-p-q)}\, \overline{\beta}^{m+n+\frac{1}{2}(-k+p-q)}\,\cdot$$

$$\cdot\; t^{\ell}_{kp}(k(\alpha,\beta))\, dk(\alpha,\beta).$$

By using (1.5), (1.14) and the beta integral we obtain, for $k+p \geq 0$:

$$(3.15) \qquad d^{\ell,m}_{n,k,p,q} = d^{\ell,m}_{n,-k,-p,q} = \frac{(-1)^{\ell+m+\frac{1}{2}(q+k+p)}\, 2^{n-q}\, n!\,(\ell+m+n-\frac{1}{2}(q+k+p))!}{(\frac{1}{2}(q-k-p))!\,(n-q)!\,(k+p)!\,(\ell+m+n+1)!}.$$

$$\cdot \sqrt{\frac{(\ell+k)!\,(\ell+p)}{(\ell-k)!\,(\ell-p)}}\;\; {}_3F_2\left(\begin{matrix}-\ell+k,-\ell+p,\frac{1}{2}(q+k+p)+1\\ k+p+1,-\ell-m-n+\frac{1}{2}(q+k+p)\end{matrix}\;\middle|\;1\right).$$

For $q = p + k$ use (3.8). Then, for $k + p \geq 0$:

$$d^{\ell,m}_{n,k,p,k+p} = d^{\ell,m}_{n,-k,-p,k+p} = \frac{(-1)^{\ell+m+p+k}2^{n-p-k}n!\,(m+n-k)!\,(m+n-p)!}{(m-\ell+n)!\,(m+\ell+n+1)!\,(p+k)!\,(n-p-k)!}\,. \tag{3.16}$$

$$\cdot \left(\frac{(\ell+k)!\,(\ell+p)!}{(\ell-k)!\,(\ell-p)!}\right)^{\frac{1}{2}} \neq 0.$$

Hence $P^{\ell,m}_{n,k,p}$ is a polynomial of degree $n - |\,p+k\,|$.

<u>THEOREM 3.4</u>. *The vector-valued polynomial* $P^{\ell,m}_{n,k}$ *satisfies the conditions*

$$P^{\ell,m}_{n,k,p}(x) = \frac{(-1)^{\ell-m}2^n(m-k+1)_n(m+k+1)_n\delta_{k,-p}x^n}{(n!\,(2m+2)_n(m-\ell+1)_n(m+\ell+2)_n)^{\frac{1}{2}}} \tag{3.17}$$

$$+ \text{ polynomial of degree less than } n,$$

$$\sum_{p=-\ell}^{\ell} \int_{-1}^{1} P^{\ell,m}_{n,k,p}(x)x^{n'}W^{\ell,m}_{p,q}(x)\,dx = 0 \tag{3.18}$$

*for all* $q$ *in* $\{-\ell,\ldots,\ell\}$ *and all* $n'$ *in* $\{0,\ldots,n-1\}$.

<u>PROOF</u>. Use (3.13), (3.16) and (3.10) for (3.17), and (2.15) together with (3.17) for (3.18). □

Note that (3.17) and (3.18) completely determine $P^{\ell,m}_{n,k}$. They also imply (2.15) for $n \neq n'$. However, from the point of view of Theorem 3.4, the orthogonality relations (2.15) for $n = n'$, $k \neq k'$ are rather unexpected.

<u>REMARK 3.5</u>. Lemma 3.1 can also be applied in order to extract the factor $t^m_{mn}(b_\theta)$ from the integral representation (1.8) for $t^\ell_{mn}(b_\theta)$. Substitute $\alpha := \cos\frac{1}{2}\theta$, $\beta := \sin\frac{1}{2}\theta$ in (1.8) and make the successive transformations of

integration variable $\phi \mapsto z \mapsto \psi \mapsto \chi$, where $e^{2i\phi} = z = e^{i\psi}\operatorname{cotg}\frac{1}{2}\theta$, $\chi = 2\psi$:

$$\left(\frac{(\ell-m)!\,(\ell+m)!}{(\ell-n)!\,(\ell+n)!}\right)^{\frac{1}{2}} t^{\ell}_{mn}(b_\theta) =$$

$$= \frac{1}{2\pi i} \oint_{(0)} (z\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta)^{\ell-m}(-z\sin\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta)^{\ell+m} z^{n-\ell-1}\,dz =$$

$$= (\sin\tfrac{1}{2}\theta)^{m-n}(\cos\tfrac{1}{2}\theta)^{m+n}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} (e^{i\psi}\cos^2\tfrac{1}{2}\theta+\sin^2\tfrac{1}{2}\theta)^{\ell-m}\, e^{i\psi(n-\ell)}\,(1-e^{i\psi})^{\ell+m} d\psi =$$

$$= (-2i)^{\ell+m}(\sin\tfrac{1}{2}\theta)^{m-n}(\cos\tfrac{1}{2}\theta)^{m+n}.$$

$$\cdot\, \frac{1}{\pi}\int_0^{\pi} (\cos\chi+i\sin\chi\cos\theta)^{\ell-m}\, e^{2ni\chi}(\sin\chi)^{\ell+m}\,d\chi.$$

Now assume $m \geq n$ and use [2, 1.5 (29)]. Then

$$t^{\ell}_{mn}(b_\theta)/t^{m}_{mn}(b_\theta) = \tag{3.19}$$
$$= \text{const.}\int_0^{\pi} (\cos\chi+i\sin\chi\cos\theta)^{\ell-m}\, e^{2ni\chi}(\sin\chi)^{\ell+m} d\chi$$

with nonzero constant. Again by [2, 1.5 (29)], the right hand side of (3.19) is a polynomial of degree $\ell - m$ in $\cos\theta$ which takes a nonzero value if $\cos\theta = 1$. In GREINER & KOORNWINDER [4, § 1.3] the integral representation for Jacobi polynomials resulting from (3.19) is obtained in a quite different context.

## 4. THE NONCOMPACT ANALOGUE

Let now $G := SL(2,C)$ with Iwasawa decomposition $G = KAN$ such that $K = SU(2)$, $A = \{a_t := \begin{pmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t}\end{pmatrix} \mid t \in \mathbb{R}\}$, $N := \{\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \mid z \in \mathbb{C}\}$.

Let $k(\alpha,\beta)$ in K be defined by (1.10) and $m_\phi$ by (1.12). $M := \{m_\phi \mid 0 \le \phi < 4\pi\}$ is the centralizer of A in K.

Let $\pi^{\lambda,k}$ $(\lambda\in\mathbb{C}, k\in\frac{1}{2}\mathbb{Z})$ be the rep of G which is induced by the rep $m_\phi a_t n \mapsto e^{-ik\phi}e^{\lambda t}$ of MAN: a principal series rep. Then $\pi^{\lambda,k}|_K$ is unitary and decomposes as $\bigoplus_{\ell=k,k+1,\ldots} T^\ell$. Choose a K-basis for which $\pi^{\lambda,k}$ has matrix elements $\pi^{\lambda,k}_{\ell,p;m,q}$ $(\ell,m=k,k+1,\ldots;p=-\ell,\ldots,\ell;q=-m,\ldots,m)$ such that

$$\pi^{\lambda,k}_{\ell,p;m,q}(k) = \delta_{\ell,m}\, t^\ell_{p,q}(k), \quad k \in K. \text{ Then}$$

$$(4.1) \qquad \pi^{\lambda,k}_{\ell,j;m,j}(a_t) = (2\ell+1)^{\frac{1}{2}} \int_K (e^{-t}|\alpha|^2+e^t|\beta|^2)^{-\lambda-m-1}\,.$$
$$. \; t^m_{kj}\begin{pmatrix} e^{-\frac{1}{2}t}\alpha & e^{\frac{1}{2}t}\beta \\ -e^{\frac{1}{2}t}\overline{\beta} & e^{-\frac{1}{2}t}\overline{\alpha}\end{pmatrix} t^\ell_{kj}(k(\overline{\alpha},\overline{\beta}))\,dk(\alpha,\beta),$$

cf. RÜHL [9, § 3-5], KOSTERS [8, § 3.1].

Similary to (3.3) we derive from (4.1) that:

$$(4.2) \qquad \pi^{\lambda,k}_{\ell,j;m,j}(a_t) = c_{k,\ell,m,j} \int_K (e^{-t}|\alpha|^2+e^t|\beta|^2)^{-\lambda-m-1}\,.$$
$$. \; (e^{-\frac{1}{2}t}\alpha-e^{\frac{1}{2}t}\beta)^{m+k}(e^{-\frac{1}{2}t}\overline{\alpha}+e^{\frac{1}{2}t}\overline{\beta})^{m-k}\; t^\ell_{kj}(k(\alpha,\beta))\,dk(\alpha,\beta),$$

where

$$(4.3) \qquad c_{k,\ell,m,j} := \left(\frac{(2\ell+1)(2m+1)(m-j)!\,(m+j)!}{(m-k)!\,(m+k)!}\right)^{\frac{1}{2}}.$$

For $s > 0$ let

$$(4.4) \qquad h_s := (2\,\mathrm{sh}\, s)^{-\frac{1}{2}}\begin{pmatrix} e^{\frac{1}{2}s} & e^{-\frac{1}{2}s} \\ e^{-\frac{1}{2}s} & e^{\frac{1}{2}s}\end{pmatrix}.$$

Then we can apply Lemma 3.1 to (4.2) with $k' := h_s$ for $0 < t < s$. We obtain:

$$(4.5)\qquad \pi^{\lambda,k}_{\ell,j;m,j}(a_t) = c_{k,\ell,m,j}\, 2^m (\operatorname{sh} s)^{-m} \sum_{p=-\ell}^{\ell} t^{\ell}_{pj}(h_s)\,.$$

$$\cdot \int_K (\operatorname{ch} t - \operatorname{coth} s \operatorname{sh} t(|\alpha|^2-|\beta|^2)+(\alpha\bar\beta-\beta\bar\alpha)\frac{\operatorname{sh} t}{\operatorname{sh} s})^{-\lambda-m-1}\,.$$

$$\cdot\ (\alpha \operatorname{sh}\tfrac12(s-t)-\beta\operatorname{sh}\tfrac12(s+t))^{m+k}(\bar\alpha\operatorname{sh}\tfrac12(s-t)+\bar\beta\operatorname{sh}\tfrac12(s+t))^{m-k}\,.$$

$$\cdot\ t^{\ell}_{kp}(k(\alpha,\beta))\,dk(\alpha,\beta),\qquad 0 < t < s.$$

If $\operatorname{Re}\lambda \le m-1$ then the limit passage $s \downarrow t$ is certainly allowed in (4.5):

$$(4.6)\qquad \pi^{\lambda,k}_{\ell,j;m,j}(a_t) = c_{k,\ell,m,j}(-1)^{2m}(2\operatorname{sh} t)^m \sum_{p=-\ell}^{\ell} t^{\ell}_{pj}(h_t)\,.$$

$$\cdot \int_K (2|\beta|^2 \operatorname{ch} t+\alpha\bar\beta-\beta\bar\alpha)^{-\lambda-m-1}\beta^{m+k}(-\bar\beta)^{m-k}t^{\ell}_{kp}(k(\alpha,\beta))\,dk(\alpha,\beta).$$

Closer examination of the integral, using (1.14), shows that (4.6) holds with convergent integral if $\operatorname{Re}\lambda < 0$. Thus it is meaningful to study the vector-valued function $x \mapsto (P^{\ell,m}_{n,k,p}(x))_{p=-\ell,\ldots,\ell}$, defined by (3.9), for complex $n$, $\operatorname{Re} n > 0$, and for $x > 1$. In particular, this function has a nice asymptotics as $x \to \infty$.

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