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MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

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A computational method for the water hammer problem

by

R.P.van de Riet

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Introduction

We consider a network of conduits through which a fluid is transported from a pump P to reservoirs R_i. An example of such a network is shown in fig.1.

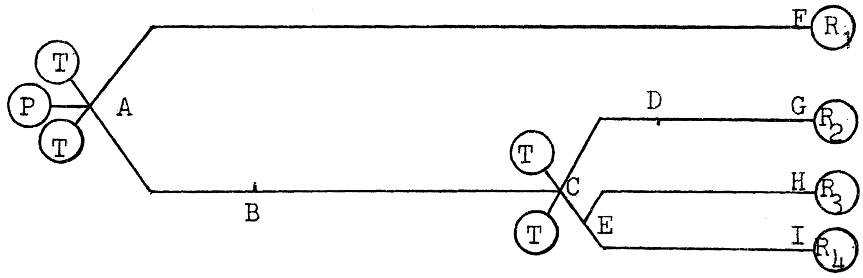


fig.1

In the steady case the pump maintains a constant velocity in the conduits.
When the pump conditions are suddenly changed, a pressure discontinuity is created, which travels through the conduits. This phenomenon is called water hammer. To prevent water hammer causing serious damage to the system, surge tanks or air vessels T are placed in the network.

In this report we give a computational method by which the propagation of this pressure discontinuity can be calculated on an electronic digital computer.*)

Let a point P of some conduit C be characterized by the space coordinate x and the time coordinate t, then the hydrostatic pressure h(x,t) and the mean velocity v(x,t) over the cross-section of C satisfy the system of partial differential equations

$$
(1) \qquad
\begin{aligned}
&\frac{\partial h}{\partial x} + \frac{1}{g}\frac{\partial v}{\partial t} = -\frac{\alpha}{2g} v.|v| - \frac{1}{g} v \frac{\partial v}{\partial x} \\
&\frac{\partial h}{\partial t} + \frac{a^2}{g}\frac{\partial v}{\partial x} = 0.
\end{aligned}
$$

The first equation is the equation of motion with the friction term $-\frac{\alpha}{2g} v.|v|$, where α is a constant depending on the level of turbulence

*) The investigations originated from computations, carried out in behalf of "Gemeentewaterleidingen - Amsterdam"

and the roughness of the conduit. In the cases considered here, α is of the order .01 m^{-1}.

g is the constant of gravity.

The second equation is the continuity equation. The constant a, which is called the wave celerity depends on the elastic properties of the conduit and the fluid. In fact the following relation holds

$$a^2 = \frac{g}{\bar{\omega}\left(\frac{1}{\varepsilon} + \frac{D}{E\,e}\right)} \tag{2}$$

where $\bar{\omega}$ is the specific weight of the fluid, ε the modulus of elasticity of the fluid, D the diameter of the conduit (which we assume to be cilyndrical), E the modulus of elasticity of the material of which the conduit is constructed and e is the thickness of the conduit wall. See Bergeron [1] p.289 and Jaeger [2] p.279.

In the cases treated here, the conduits are constructed of concrete or steel and the wave celerity for water has then the order of magnitude 1000 m/sec.

It follows from the second equation in (1), that

$$\frac{1}{g}\, v\, \frac{\partial v}{\partial x} = -\frac{1}{a^2}\, v\, \frac{\partial h}{\partial t}$$

Assuming $\frac{\partial h}{\partial t}$ not too large, we may neglect this term in comparison with $\frac{\alpha}{2g}\, v.|v|$, since $v \ll a$, and we obtain the system of partial differential equations

$$\begin{aligned} &\frac{\partial h}{\partial x} + \frac{1}{g}\frac{\partial v}{\partial t} = -\frac{\alpha}{2g}\, v.|v| \\ &\frac{\partial h}{\partial t} + \frac{a^2}{g}\frac{\partial v}{\partial x} = 0 \end{aligned} \tag{3}$$

In section 1 the system (3) is transformed into a characteristic system and the formulae, by which the pressure and the velocity in each conduit can be computed, are presented in sections 2 and 3. In order to obtain an efficient calculation scheme it is very important to have a rule defining the order in which all the conduits should be consecutively calculated; this rule is established in section 4. Section 5 may be useful for readers, who are familiar with ALGOL 60 (see [3]). Here we give an outline of the general ALGOL 60 program which has been used in an actual calculation on the X1 computer of the Mathematical Centre. It may be applied for the calculation of any other practical case.

1. Reduction of the system of differential equations

Considering the system (3), it is obvious, that the characteristics are given by straight lines with slopes $\pm$ a. If we introduce characteristic coordinates

$$\sigma = x - at \qquad (1.1)$$
$$\tau = x + at$$

the equations (3) become

$$\frac{\partial(h - \frac{a}{g} v)}{\partial\sigma} + \frac{\partial(h + \frac{a}{g} v)}{\partial\tau} = - \frac{\alpha}{2g} v \cdot |v|$$

$$\frac{\partial(h - \frac{a}{g} v)}{\partial\sigma} - \frac{\partial(h + \frac{a}{g} v)}{\partial\tau} = 0.$$

After addition and substraction we obtain

$$\frac{\partial(h - \frac{a}{g} v)}{\partial\sigma} = - \frac{\alpha}{4g} v.|v| \qquad (1.2)$$

$$\frac{\partial(h + \frac{a}{g} v)}{\partial\tau} = - \frac{\alpha}{4g} v.|v|.$$

Finally, integrating the first equation along a τ- characteristic from R to P and the second equation along a σ-characteristic from Q to P (see fig.2), we get the equations

$$
(1.3)\quad
\begin{aligned}
\{h(P) - \frac{a}{g}\, v(P)\}-\{h(R) - \frac{a}{g}\, v(R)\} &= \int_R^P \frac{\alpha a\ v.|v|}{2g\sqrt{(1+a^2)}}\, ds_1 \\
\{h(P) + \frac{a}{g}\, v(P)\}-\{h(Q) + \frac{a}{g}\, v(Q)\} &= -\int_Q^P \frac{\alpha a\ v.|v|}{2g\sqrt{(1+a^2)}}\, ds_2,
\end{aligned}
$$

where s_1 and s_2 are the arclengths of characteristic segments on RP respectively QP; see fig.2.

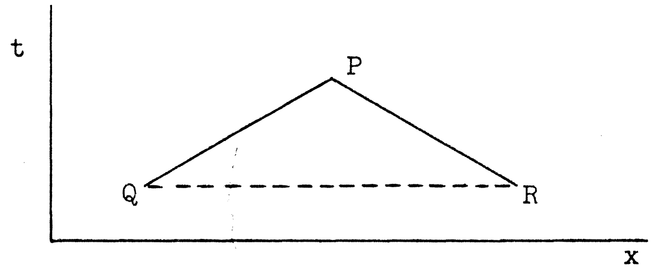


fig.2

2. Numerical approximation

In the (x,t) plane we draw a rectangular grid with grid widths Δx and Δt; Δx is chosen equal to $a.\Delta t$ and hence the diagonals of the grid are characteristic lines (see fig.3).

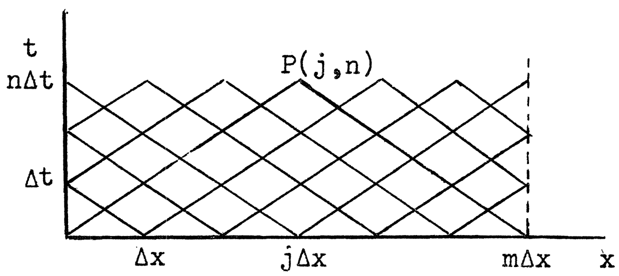


fig.3

Δx equals $\frac{L}{m}$ where L is the length of the conduit C and m is an integer. The choice of the integer m depends on the accuracy required.

The grid points P(x,t) will be denoted by P(j,n) where $x = j\,\Delta\, x$ and $t = n\,\Delta\, t$.

Let us consider a point P(j,n), which lies not on the boundaries of the conduit , so that $0 < j < m$. We shall derive formulae by which the values of h and v at the point P(j,n) are defined in terms of the values of h and v at the points P(j-1, n-1) and P(j+1, n-1).

Putting $h(P(j,n)) = h_{j,n}$ and $v(P(j,n)) = v_{j,n}$, we obtain from equation (1.3)

$$\{h_{j,n} - \frac{a}{g} v_{j,n}\} - \{h_{j+1,n-1} - \frac{a}{g} v_{j+1,n-1}\} = \int_{P(j+1,n-1)}^{P(j,n)} \frac{\alpha a}{2g} \frac{v \cdot |v|}{\sqrt{(1+a^2)}} ds_1$$

(2.1)

$$\{h_{j,n} + \frac{a}{g} v_{j,n}\} - \{h_{j-1,n-1} + \frac{a}{g} v_{j-1,n-1}\} = -\int_{P(j-1,n-1)}^{P(j,n)} \frac{\alpha a}{2g} \frac{v \cdot |v|}{\sqrt{(1+a^2)}} ds_2$$

Both integrals in (2.1) are unknown, but they can be approximated if we estimate v along the path of integration. Let this estimate be given by:

$$\text{(2.2)} \qquad \begin{aligned} v(s_1) &\approx (v_{j+1,n-1} + v_{j,n})/2 \\ v(s_2) &\approx (v_{j-1,n-1} + v_{j,n})/2 \,. \end{aligned}$$

Then we get from (2.1)

$$\text{(2.3)} \qquad \begin{aligned} &\{h_{j,n} - \frac{a}{g} v_{j,n}\} - \{h_{j+1,n-1} - \frac{a}{g} v_{j+1,n-1}\} = \\ &= \frac{\alpha}{8g} \Delta x (v_{j,n} + v_{j+1,n-1}) |v_{j,n} + v_{j+1,n-1}| \,, \\ &\{h_{j,n} + \frac{a}{g} v_{j,n}\} - \{h_{j-1,n-1} + \frac{a}{g} v_{j-1,n-1}\} = \\ &= -\frac{\alpha}{8g} \Delta x (v_{j,n} + v_{j-1,n-1}) |v_{j,n} + v_{j-1,n-1}| \,. \end{aligned}$$

Within the accuracy required, we may assume that

$$\text{(2.4)} \qquad \operatorname{sgn}(v_{j,n}) = \operatorname{sgn}(v_{j-1,n-1}) = \operatorname{sgn}(v_{j+1,n-1}) = \operatorname{sgn}(v_{j,n-1}).$$

When this is not true, v will be small and the error introduced will not lead to large deviations, at least when the grid is sufficiently fine.

If β is defined by

$$\text{(2.5)} \qquad \beta = \frac{\alpha}{8g} \Delta x \operatorname{sgn}(v_{j,n-1})$$

then we get from (2.4)

$$h_{j,n} = \frac{1}{2}\{h_{j-1,n-1} + h_{j+1,n-1} + \frac{a}{g}(v_{j-1,n-1} - v_{j+1,n-1}) + \tag{2.6}$$
$$+ \beta\{(v_{j,n} + v_{j+1,n-1})^2 - (v_{j,n} + v_{j-1,n-1})^2\}\},$$

$$v_{j,n} = \frac{g}{2a}\ \{h_{j-1,n-1} - h_{j+1,n-1} + \frac{a}{g}(v_{j-1,n-1} + v_{j+1,n-1}) + \tag{2.7}$$
$$- \beta\{(v_{j,n} + v_{j+1,n-1})^2 + (v_{j,n} + v_{j-1,n-1})^2\}\}.$$

Formula (2.7) is a quadratic equation in $v_{j,n}$.
If we define A, B, C and D by

$$A = v_{j+1,n-1} + v_{j-1,n-1}\ ,\ B = v_{j+1,n-1}^2 + v_{j-1,n-1}^2\ ,$$

$$C = \frac{g}{a}\{h_{j-1,n-1} - h_{j+1,n-1}\} + A \text{ and } D = \frac{\beta g}{a}\ ,$$

then (2.7) reads as:

$$2\,D\,v_{j,n}^2 + 2\,(A\,D + 1)\,v_{j,n} + B\,D - C = 0 \tag{2.8}$$

from which we obtain the two solutions:

$$v_{j,n} = \frac{-(A\,D + 1) \pm \{(A\,D + 1)^2 + 2\,D\,(C - B\,D)\}^{1/2}}{2\,D}\ . \tag{2.9}$$

We remark that the minus sign leads to a contradiction, therefore we take the solution with the positive sign.
Moreover we see that, due to the smallness of D, loss of precision may occur in actual calculations. We therefore use the numerically better formula

$$v_{j,n} = \frac{B\,D - C}{-(A\,D + 1) - \{(A\,D + 1)^2 + 2\,D(C - B\,D)\}^{1/2}} \tag{2.10}$$

From the definitions of A, B, C and D, it follows directly that the formula (2.10) defines $v_{j,n}$ in terms of the values of h and v at the time $t = (n-1)\Delta t$.

The formula (2.6) together with (2.10) defines finally also $h_{j,n}$ in terms of the values of h and v at the time $t = (n-1)\Delta t$.

3. Boundary conditions

In the preceding section we have found a set of formulae for calculating h and v in the points of the conduit, not lying on its boundaries. To complete these formulae, it is therefore necessary to study the boundary conditions of the conduit.

Each conduit has an a priori chosen fixed coordinate system (x,t), the point x = 0 is called henceforth the entrance and the point x = L the exit. (we emphasize that both names do not refer to the direction of the flow.)

We call a boundary point a junction point of the first kind when the conduit is only connected with other conduits at this point (see fig.1 points B, D and E).

Whenever, in contrast to this case, the conduit is also connected at the boundary point with some mechanism (e.g. a pump, a surge tank or a reservoir, see fig.1 points A, C, F, G, H and I), we call this point a junction point of the second kind. The velocity and pressure satisfy certain boundary conditions at these junction points. The nature of these conditions are different according as the junction point is of the first or the second kind.

3.1 Junction points of the first kind

We consider a junction point S of the first kind.

Let us assume that k conduits are connected with each other in the junction point S, that S is the exit of k_1 conduits, say $C_1,\ldots,C_{k_1}$, and the entrance of the $k-k_1$ conduits $C_{k_1+1},\ldots,C_k$;

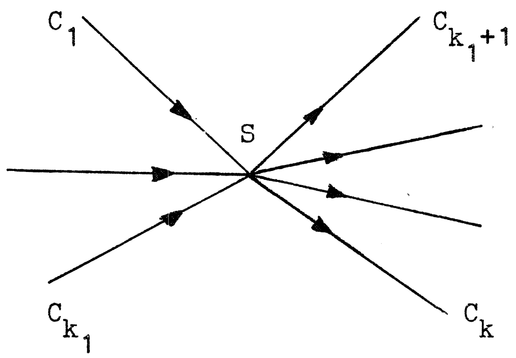


fig.5

this situation is illustrated by figure 5, where the arrows indicate the direction of increasing x. All the conduits C_i have their own grids with grid widths Δx_i and Δt_i.

x_i equals $\frac{L_i}{m_i}$ and $\Delta t_i = \frac{\Delta x_i}{a}$.

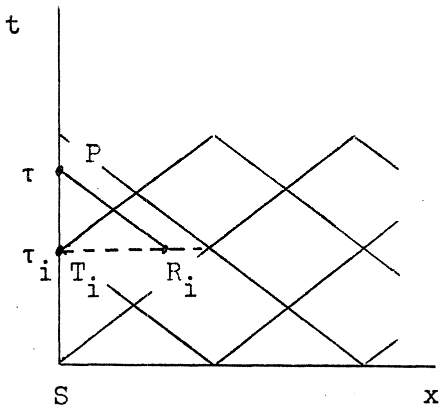
L_i is the length and m_i+1 the number of grid points of the conduit C_i.
Inpractice Δx_i is chosen large for long conduits and small for short conduits.

For $1 \leqq i \leqq k_1$ the junction point S at the time $n\Delta t_i$ corresponds to the point $P_i(m_i,n)$ of the grid of C_i, while this point corresponds for $k_1+1 \leqq i \leqq k$ to the point $P_i(0,n)$ of the grid of C_i.
We wish to calculate the conduit C_j at the point P with $P = P_j(0,\frac{\tau}{\Delta t_j})$or $P = P_j(m_j, \frac{\tau}{\Delta t_j})$ according as $j > k_1$ or $j \leqq k_1$, assuming that τ is an integral multiple of Δt_j.
Since the Δt_i are different for different conduits, the point P will in general not be a point of the grid of C_i $(i \neq j)$.

Let us assume that the neighbouring conduits C_i of C_j are already calculated up to the times $t = \tau_i$, with

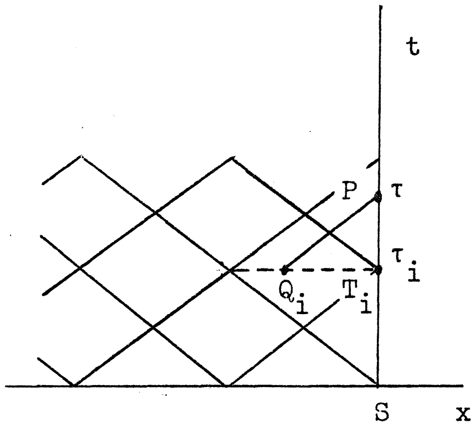
$$\tau_i \leqq \tau \leqq \tau_i + \Delta t_i,$$

while the conduit C_j is calculated up to the time $t = \tau_j = \tau - \Delta t_j$.
In the figures 6 and 7 we show the x,t planes of the conduits C_i for $i > k_1$ and for $i \leqq k_1$ respectively.



$k \geq i > k_1$

fig.6



$1 \leq i \leq k_1$

fig.7

From equation (1.5) we obtain for $i = k_1+1,\ldots,k$

$$(3.1)\qquad \{h_i(P) - \frac{a}{g}\, v_i(P)\} - \{h_i(R_i) - \frac{a}{g}\, v_i(R_i)\} = \int_{R_i}^{P} \frac{\alpha_i a}{2g} \frac{v_i|v_i|}{\sqrt{(1+a^2)}}\, ds_1$$

and for $i = 1,\ldots,k_1$

$$(3.2)\qquad \{h_i(P) + \frac{a}{g}\, v_i(P)\} - \{h_i(Q_i) + \frac{a}{g}\, v_i(Q_i)\} = -\int_{Q_i}^{P} \frac{\alpha_i a}{2g} \frac{v_i|v_i|}{\sqrt{(1+a^2)}}\, ds_2 .$$

For all i $(i \neq j)$ the values of v and h in the points R_i and Q_i are in general unknown, but they can be calculated by interpolation between other points where v and h are assumed to be known.
The two integrals at the right-hand sides of (3.1) and (3.2) may be approximated with the aid of the following estimation of v_i

$$(3.3)\qquad \begin{aligned} v_i(s_1) &\approx \frac{v_i(R_i) + v_i(T_i)}{2} \quad \text{and} \\ v_i(s_2) &\approx \frac{v_i(Q_i) + v_i(T_i)}{2} . \end{aligned}$$

The estimation (2.2), which is more accurate, will not be used here as this would lead to a system of quadratic equations, which are rather complicated to solve in this case.

The right-hand sides of (3.1) and (3.2) take on respectively the form:

$$\beta_i \; \{v_i(R_i) + v_i(T_i)\}^2 \text{ and } -\beta_i\{v_i(Q_i) + v_i(T_i)\}^2$$

with

$$(3.4) \quad \beta_i = \frac{\alpha_i}{8g} \, \Delta x_i \, \frac{\tau - \tau_i}{\Delta t_i} \circ \operatorname{sgn}\, (v_i(T_i)).$$

Putting

$$A_i = h_i(R_i) - \frac{a}{g} \, v_i(R_i) + \beta_i \; \{v_i(R_i) + v_i(T_i)\}^2 \text{ for } i > k_1$$

and

$$A_i = h_i(Q_i) + \frac{a}{g} \, v_i(Q_i) - \beta_i\{v_i(Q_i) + v_i(T_i)\}^2 \quad \text{for } i \leqq k_1$$

then (3.1) and (3.2) may be rewritten as

$$(3.5) \qquad h_i(P) - \frac{a}{g} \, v_i(P) = A_i \quad (i > k_1)$$

$$(3.6) \qquad h_i(P) + \frac{a}{g} \, v_i(P) = A_i \quad (i \leqq k_1)$$

The expressions A_i contain only values of v_i and h_i at the time τ_i, and hence, they are assumed to be known.

The equations (3.5) and (3.6) constitute k linear algebraic equations with 2 k unknowns, and thus there are still k additional conditions required. These are obtained from the condition of continuity at the junction point.

Let the area of the cross-section of the conduit C_i be denoted by F_i. Then the equation of continuity may be written as

$$(3.7) \qquad \sum_{i=1}^{k_1} F_i v_i(P) = \sum_{i=k_1+1}^{k} F_i v_i(P)$$

while we may assume for the pressures

(3.8) $$h_1(P) = h_2(P) = \ldots = h_k(P).$$

If we multiply the formulae (3.5) and (3.6) with F_i and add, we get

(3.9) $$\sum_{i=1}^{k} h_i(P)F_i - \frac{a}{g}\{\sum_{i=k_1+1}^{k} v_i(P)F_i - \sum_{i=1}^{k_1} v_i(P)F_i\} = \sum_{i=1}^{k} A_i F_i.$$

In view of (3.7) and (3.8) we obtain finally

(3.10) $$h_j(P) = \left(\sum_{i=1}^{k} A_i F_i\right)\left(\sum_{i=1}^{k} F_i\right)^{-1}.$$

Furthermore we have

(3.11) $$v_j(P) = \frac{g}{a}(h_j(P) - A_j) \quad \text{when } j > k_1$$

and

(3.12) $$v_j(P) = \frac{g}{a}(-h_j(P) + A_j) \quad \text{when } j \leq k_1.$$

These formulae enable us to calculate the pressure and the velocity in the boundary point P of the conduit C_j, provided that the neighbouring conduits are already calculated up to the times $t = \tau_i$, with

(3.13) $$\tau_i \leq \tau \leq \tau_i + \Delta t_i$$

and that the conduit C_j itself is calculated up to the time $t = \tau_j = \tau - \Delta t_j$.

3.2. Junction points of the second kind

At a junction point S of the second kind the system is directly connected with a mechanism M, which imposes a certain functional relation between the pressure h_M and the velocity v_M at the mechanism, viz.:

(3.14) $$h_M = H(v_M, t).$$

Let the area of the cross-section, of the short pipe which connects M and S, be F_M.

We adopt the same notation as in section 3.1. The junction point S is shown in fig.8.

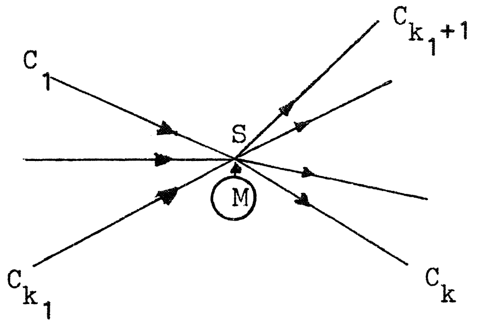


fig.8

The continuity equation reads now as

$$(3.15)\quad \sum_{i=k_1+1}^{k} F_i v_i(P) = \sum_{i=1}^{k_1} F_i v_i(P) + F_M v_M(P).$$

and for the pressures at S we have the equalities

$$(3.16)\qquad h_1(P) = h_2(P) = \ldots = h_k(P) = h_M(P).$$

Of course, the equation (3.9) is still correct in this case, so that we obtain in view of (3.15) and (3.16)

$$(3.17)\qquad h_j(P) = \{\frac{a}{g} F_M v_M(P) + \sum_{i=1}^{k} A_i F_i\} \left(\sum_{i=1}^{k} F_i\right)^{-1}.$$

Substraction of (3.17) from (3.14) yields

$$(3.18)\qquad g(v_M) = H(v_M,\tau) - \{\frac{a}{g} F_M v_M + \sum_{i=1}^{k} A_i F_i\} \left(\sum_{i=1}^{k} F_i\right)^{-1} = 0$$

A numerical value of v_M can now be obtained by solving this equation with the aid of e.g. the Regula Falsi.

Finally $h_j(P)$ can be calculated with the aid of (3.17) and $v_j(P)$ with the aid of (3.11) and (3.12).

3.3. The case of a reservoir

Let us now consider the end point S of a conduit C which is connected with a reservoir for which the pressure is constant (say h_o) see fig.9.

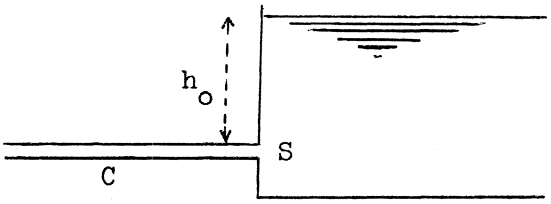


fig.9

This is of course a special case of the preceding investigations in section 3.2. ($k_1=1$, $k=1$, $h_M=H(v_M,t) \equiv h_o$) The reason, however, why we treat this case independently is, that we can use, due to the simple boundary conditions, the better estimation (2.2) instead of (3.3) which was applied above. Let S have the space coordinate $m\,\Delta x$. Assume that we have already calculated pressure and velocity in S at the time $t = (n-1)\Delta t$. From the equations (2.3) and (2.5) we get

$$(3.19) \quad h_{m,n} + \frac{a}{g} v_{m,n} = h_{m-1,n-1} + \frac{a}{g} v_{m-1,n-1} - \beta\,(v_{m,n} + v_{m-1,n-1})^2.$$

When we substitute h_o for $h_{m,n}$ we find in the same way as in the derivation of formula (2.10)

$$v_{m,n} = (-\beta v_{m-1,n-1}^2 + \frac{a}{g} v_{m-1,n-1} - h_o + h_{m-1,n-1}).$$

$$(3.20) \quad \left[\beta v_{m-1,n-1} + \frac{a}{2g} + \{(\beta v_{m-1,n-1} + \frac{a}{2g})^2 + \beta\,(\frac{a}{g} v_{m-1,n-1} + - \beta v_{m-1,n-1}^2 - h_o + h_{m-1,n-1})\}^{1/2}\right]^{-1},$$

which is the desired formula.

Recollecting the results, we may conclude that we have found the formuluae (2.6) and (2.10), by which we can calculate the conduits in inner points.
Furthermore we derived formulae for the boundary points. All these formulae are approximate formulae, since we have estimated the

friction term

$$\int \frac{\alpha a \; v . |v|}{2g\sqrt{(1+a^2)}} \, ds .$$

Although this term is not so small as to be neglected, it is however small compared with the other terms and therefore our approximation is fairly good.

By dimishing the grid widths, the error, due to the approximation, is reduced.
However, we observed during actual calculation that a rather rough grid gives already satisfactory results.

4. The computation scheme

This section is devoted to the arrangement in which the calculation may actually be performed.
Assume there are N conduits in the network.
We order these conduits according to increasing Δt, say

$$C_1, C_2, \ldots, C_N \quad \text{with } \Delta t_1 \leq \Delta t_2 \leq \ldots \leq \Delta t_N .$$

Moreover, let us denote all the neighbouring conduits of C_i, as well at its entrance as at its exit, by $C_{s_{i,1}}$, $C_{s_{i,2}}, \ldots, C_{s_{i,l_i}}$, which are ordered in the same way as above, i.e.

$$\Delta t_{s_{i,j}} \leq \Delta t_{s_{i,j+1}} \qquad \text{for } j = 1,2,\ldots,l_i-1$$

Let us assume that the velocity v and the pressure h are known in all the conduits at the time t = 0 (e.g. up to the time t = 0 the steady state is maintained).
Using the same notation as in section 3, all τ_i are zero at t = 0.
A consequence of ordering the sequence $C_1, C_2, \ldots, C_N$ according to increasing Δt_i is, that the conditions (3.13) are automatically satisfied for the conduit C_1. Hence C_1 can be calculated in all

its grid points at the time $t = \Delta t_1$.
The formulae, needed for the calculation of velocity and pressure at inner points have been derived in section 2, while those for the boundary points have been treated in section 3. Whether these calculations are performed from left to right or vice versa is irrelevant.

C_1 being calculated, we raise the value of τ_1 by an increment of magnitude Δt_1, whereas the other τ_i remain zero.
If the conditions (3.13) are still satisfied for C_1, C_1 can be calculated again and the value of τ_1 becomes $2\Delta t_1$. We repeat this procedure until the conditions (3.13) are violated, the conduit C_2 may now be calculated and the value of τ_2 becomes Δt_2. The conduit to be calculated next is one of the conduits C_1, C_2 and C_3 and we proceed in the same way as before. A flow-diagram illustrating the course of the calculation, is given below. The conditions (3.13) are expressed in the following form

$$(4.1) \qquad \tau_{s_{i,j}} \leqq \tau \leqq \tau_{s_{i,j}} + \Delta t_{s_{i,j}} \qquad \text{for } j = 1,2,\ldots, l_i$$

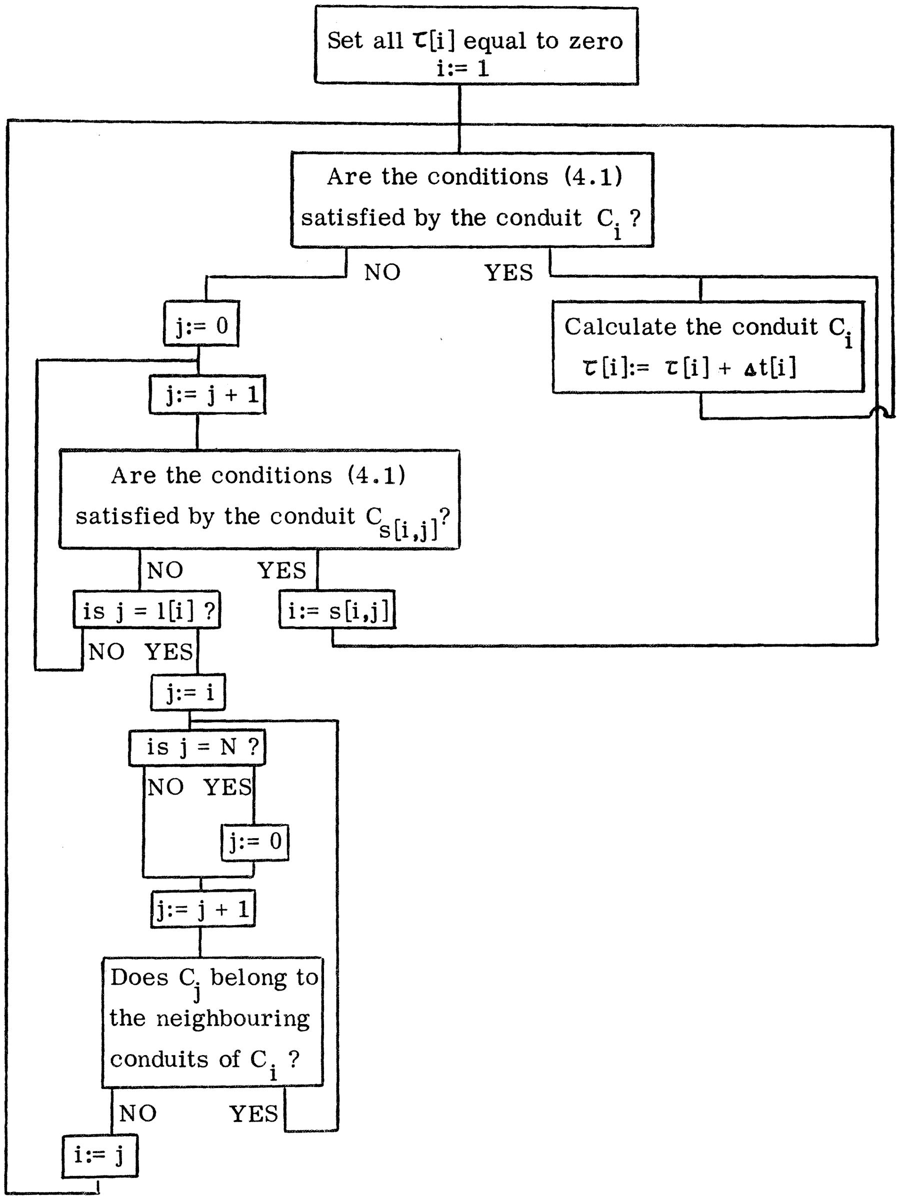
where $\tau = \tau_i + \Delta t_i$ is the time at which we wish to calculate the conduit C_i.
In the flow-diagram we abbreviate the phrase:

"Set the value of a equal to b" by " a := b".

Moreover we adopt a square-bracket notation for subscripted variables, e.g.

$$\tau[s[i,j]] \quad \text{means} \quad \tau_{s_{i,j}}.$$

The ALGOL procedure COMPUTATION SCHEME, corresponding to this flow diagram is given in the following section.



Flow diagram of the Computation scheme

5. The ALGOL 60 program

In this section we give an outline of the ALGOL 60 program used in an actual calculation. It can be used in any practical case if proper values are assigned to the variables listed at the beginning of the program and if the procedures hstar and CALC are properly defined.

```
begin real a, g, epsh; integer N, N1, N2, M; real array beta, tau, dt, he,
   F[1:N], v, h, V, H[1:N,0:M], FM, va, vb[1:N1]; integer array m, l[1:N],
   s[1:N,1:N], k[1:N2], J[1:N2,1:N]; Boolean fi;
   procedure NS1 (i, j); integer i, j;
   comment NS1 calculates the velocity and pressure in the point
   x = j × Δ x_i of the conduit C_i at the time t = tau[i] + dt[i], according
   to the formulae (2.6) and (2.10);
   begin real A, c, Beta, d, B;
      c:= v[i,j+1] × v[i,j+1]; d:= v[i,j-1] × v[i,j-1]; Beta:= beta[i] × sign
      (v[i,j]); A:= (h[i,j-1] - h[i,j+1])/a × g + v[i,j+1] + v[i,j-1]; B:= -
      Beta/a × g; V[i,j]:= (B × (c + d) + A)/(-B × (v[i,j+1] + v[i,j-1]) +
      1 + sqrt ((B × (v[i,j+1] + v[i,j-1]) - 1)↑2 - 2 × B × (B × (c + d)
      + A))); H[i,j]:= (h[i,j+1] + h[i,j-1] + a/g × (v[i,j-1] - v[i,j+1]) + Beta ×
      (V[i,j] × 2 × (v[i,j+1] - v[i,j-1]) + c - d))/2; v[i,j-1]:= V[i,j-1];
      h[i,j-1]:= H[i,j-1]
   end NS1;
   procedure NS2 (i, n, K); integer i, n, K;
   comment NS2 calculates the velocity and pressure in the conduit C_i
   at a junction point S, according to the formulae of section 3.2.
   There are N2 different junction points in the network, which should be
   ordered in some way. The junction point S considered here is the n^th
   one. There are k[n] conduits connected in S, denoted by C_abs(J[n,p])
   where p = 1,..., k[n]. S is the exit or entrance of a conduit according
   as sign(J[n,p]) is +1 or -1 (so i is one of the numbers abs(J[n,p])).
   If K > 0 then S is moreover connected with the mechanism M_K. NS2
```

```
uses the non-local procedure hstar.;
begin real h1, t, S1, S2; integer p, q, l, si; array R, x, vv, hh[1:N];
   integer array nj, sj, j[1:N];
   procedure IP (p, s, q, n); integer p, s, q, n;
   begin real a, b; if n = 2 then
      begin a:= x[p] × (x[p]-1)/2 × (v[p,s] - 2 × v[p,s+q] + v[p,s+2×q]);
         b:= x[p] × (x[p]-1)/2 × (h[p,s] - 2 × h[p,s+q] + h[p,s+2×q])
      end else a:= b:= 0; vv[p]:= v[p,s] + x[p] × (v[p,s+q] - v[p,s]) + a;
      hh[p]:= h[p,s] + x[p] × (h[p,s+q]-h[p,s]) + b
   end IP;
   for p:= 1 step 1 until k[n] do
   begin sj[p]:= sign (J[n,p]); nj[p]:= (sj[p] + 1)/2; j[p]:= abs (J[n,p])
   end; t:= tau[i] + dt[i]; x[i]:= 1; S1:= S2:= 0; q:= 0;
B: q:= q + 1; p:= j[q]; l:= nj[q] × (m[p] - 1); if p ≠ i then
   begin x[p]:= (t-tau[p])/dt[p]; IP (p, l + nj[q], -sj[q], if m[p] > 1
      then 2 else 1)
   end else
   begin si:= sj[q]; vv[i]:= v[i,l+nj[q]-si]; hh[i]:= h[i,l+nj[q]-si] end;
   R[p]:= hh[p] + sj[q] × a/g × vv[p] - sj[q] × beta[p] × x[p] × sign
   (v[p,l]) × (v[p,l] + v[p,l+1])↑2; S1:= S1 + F[p] × R[p]; S2:= S2 + F[p];
   if q < k[n] then goto B; h1:= S1/S2; if K > 0 then
   begin real v1, ha1, h2; v1:= 2 × va[K] - vb[K]; fi:= true;
   RF: h2:= hstar (K, v1, t, ha1) - h1 - FM[K] × a/g × v1/S2; ha1:=
      ha1 - a/g × FM[K]/S2; v1:= v1 - h2/ha1; fi:= false; if abs (h2)
      > epsh × abs (h1) then goto RF; h1:= h1 + FM[K] × a/g × v1/S2;
      vb[K]:= va[K]; va[K]:= v1
   end; if si > 0 then
   begin v[i,m[i]]:= (R[i] - h1)/a × g; h[i,m[i]]:= h1; v[i,m[i]-1]:=
      V[i,m[i]-1]; h[i,m[i]-1]:= H[i,m[i]-1]
   end else begin V[i,0]:= (h1 - R[i])/a × g; H[i,0]:= h1 end
end NS2;
procedure NS3 (i); integer i;
comment NS3 calculates, with the aid of formula (3.20), the velo-
city at an endpoint x = m[i] × Δx_i of the conduit C_i , the pressure
```

```
is equal to he[i] = h0 .;
begin real Beta; Beta:= beta[i] × sign (v[i,m[i]]); v[i,m[i]]:= (-Beta ×
    v[i,m[i] - 1]↑2 + a/g × v[i,m[i] -1] - he[i] + h[i,m[i] - 1])/(Beta ×
    v[i,m[i]-1] + a/g/2 + sqrt ((Beta × v[i,m[i]-1] + a/g/2)↑2 + Beta
    × (a/g × v[i,m[i]-1] - Beta × v[i,m[i]-1]↑2 - he[i] + h[i,m[i]-1])));
    h[i,m[i]]:= he[i]; v[i,m[i]-1]:= V[i,m[i]-1]; h[i,m[i]-1]:= H[i,m[i]-1]
end NS3;
real procedure hstar (K, v0, t, hst); real v0, t, hst; integer K;
comment hstar should calculate the value of the function h = H (v0, t),
belonging to the mechanism M_K (see formula (3.13)), and the deriva-
tive of this function with respect to v0. The results should be stored
in hstar and hst resp. Since hstar is called several times in NS2
during the iteration proces, it is worth while to know when hstar is
called for the first time. This is the case when fi = true , the other
times fi = false .;
procedure CALC (i); integer i;
comment CALC should define the calculation of the conduit C_i and the
output desired.;
procedure COMPUTATION SCHEME;
begin integer i, j; Boolean array B[1:N];
    Boolean procedure COND (i); value i; integer i;
    begin integer n; for n:= 1 step 1 until l[i] do
        begin if ¬ ((tau[s[i,n]] ≤ (tau[i] + dt[i])) ∧ ((tau[i] + dt[i]) ≤
            (tau[s[i,n]] + dt[s[i,n]]))) then begin COND:= false; goto A end
        end; COND:= true;
    A: end COND;
    i:= 1;
AA: if COND (i) then goto A2 else
    begin for j:= 1 step 1 until l[i] do
        begin if COND (s[i,j]) then begin i:= s[i,j]; goto A2 end end
    end; for j:= 1 step 1 until N do B[j]:= true; for j:= 1 step 1
    until l[i] do B[s[i,j]]:= false;
A1: if i = N then i:= 0; i:= i + 1; if B[i] then goto AA else goto A1;
A2: CALC (i); tau[i]:= tau[i] + dt[i]; goto AA
end COMPUTATION SCHEME;
```

INPUT: comment this part of the program should assign values to all the non-local variables except V, H and fi.

When a variable A refers to the corresponding variable B, occurring in, say, formula (n) or section n, then we denote this by: A ⇝ B: f(n) respectively by: A ⇝ B: s.n.

We have: a ⇝ a: f(1), g ⇝ g: f(1), N ⇝ N: s.4, tau[i] ⇝ τ_i: s.3.1, dt[i] ⇝ Δt_i: f(3.13), he[i] ⇝ h_0: f(3.20), (see the comment of NS3), F[i] ⇝ F : f(3.7), v[i,j] ⇝ v: f(1), h[i,j] ⇝ h: f(1), (v[i,j] and h[i,j] are the velocity and the pressure in the point $x = j \times \Delta x_i$ of the conduit C_i at the time $t = \tau_i$), FM[K] ⇝ F_{M_K} : f(3.15), m[i] ⇝ m: s.3.1.

epsh corresponds to the relative accuracy needed for calculating the pressure in NS2.

N1 corresponds to the number of different mechanisms.

N2 ,, ,, ,, ,, ,, junction points.

k[i] ,, ,, ,, ,, ,, conduits connected at the i^{th} junction point, J[i,j] defines the k[i] conduits connected at this point (See the comment of NS2).

M corresponds to the maximum of all m[i].

beta[i] corresponds to the absolute value of β belonging to the i^{th} conduit (see formula (2.5)). The value of the velocity in the mechanism M_K at the initial time should be assigned to va[K] and vb[K].

The definitions of s[i,j] and l[i] are given in section 4, the following program however, calculates from known arrays J and k the arrays s and l;

```
begin real TAU; integer t, K, I, p; array taul[1:N]; integer array
   ss[1:N,1:2×(N-1)];
   procedure MIN;
   begin real r, s; t:= j:= 1; if j ≤ l[i] then s:= taul[ss[i,j]]; goto C;
    B: j:= j + 1; r:= taul[ss[i,j]]; if r < s then begin s:= r; t:= j end;
    C: if j < l[i] then goto B
   end;
   for i:= 1 step 1 until N do l[i]:= 0; t:= 1; for i:= 1 step 1 until N2 do
   begin K:= 1; AA: I:= abs (J[i,K]); for j:= K + 1 step 1 until k[i], 1
      step 1 until K - 1 do begin ss[I,l[I]+t]:= abs (J[i,j]); t:= t + 1 end;
```

```
    l[I]:= l[I] + t - 1; t:= 1; K:= K + 1; if K ≤ k[i] then goto AA
  end; TAU:= 0; for i:= 1 step 1 until N do
  begin K:= 1; AA: for j:= K + 1 step 1 until l[i] do
    begin if ss[i,K] = ss[i,j] then
      begin for p:= j step 1 until l[i] - 1 do ss[i,p]:= ss[i,p+1];
          l[i]:= l[i] - 1
      end
    end; K:= K + 1; if K < l[i] then goto AA; tau1[i]:= tau[i];
    TAU:= TAU + tau[i]
  end; for i:= 1 step 1 until N do
  begin p:= 0; A: p:= p + 1; MIN; s[i,p]:= ss[i,t]; tau1[ss[i,t]]:= TAU;
    if p < l[i] then goto A; for j:= 1 step 1 until N do tau1[j]:= tau[j]
  end
end;
The actual calculation is started by: COMPUTATION SCHEME
end
```

References

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typ: JvS