Linear System Representations

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1. Introduction

The theory of system representations is concerned with the various ways in which a 'system' (a dynamical relation between several variables) can be described in mathematical terms. This paper will concentrate on the class of linear, time-invariant, deterministic, finite-dimensional systems, for which there exists indeed a variety of representations. The study of system representations is of interest for two reasons, which correspond to two different points of view. First of all, even when representation types (or 'model classes') are mathematically equivalent, the ease with which a particular problem is handled may be quite representation-dependent. Also, it may happen for instance that a problem is best understood theoretically in one representation, but that another representation is most useful for the numerical solution. Thus, one should be able to switch from one representation to another. The study of the corresponding transformations belongs to representation theory. The second reason for interest in system representations is connected with the modeling problem. Often, a model for a physical system is built up by writing down equations for the components and for the connection constraints. In this way, one obtains a system representation. It may be useful, though, to rewrite the equations; the derivation of the Euler-Lagrange equations of mechanics could be cited as an example. Again, we have here a problem of transformation between system representations.

Interest in the theory of system representations has been stimulated in recent years by a series of papers by J. C. Willems [64, 68-70, 72, 73]. In this work, the 'modeling' point of view has been emphasized. As noted by Willems, even such raw data as an observed time series can already be taken as a system representation, and the identification problem then becomes a problem of transformation of representations. In this paper, we shall concentrate on representations by equations rather than by measured data. A survey of system representations and transformations will be presented in the spirit of [71]. We shall use the notion of 'external equivalence', again following

Willems.

The next section contains a brief historical survey of system representations in connection with control theory, centered on the description of linear, finite-dimensional, deterministic systems. After that, we shall attempt to give an up-to-date account of the results concerning the representation of this class under external equivalence. Section 4 will be devoted to an application of the theory to the idea of a factor system, and the paper will be closed with conclusions and research perspectives.

2. System representations: a historical sketch

The birth of mathematical control theory is often dated 1868, the year of the publication of J.C. Maxwell's paper "On Governors" [42]. In this paper, Maxwell deals with a number of contrivances that in his time were in use to regulate the operation of steam engines. Maxwell uses second order equations to describe the motions of the engine itself and the regulators. He takes the coupling of the different parts into account and linearizes to obtain a coupled set of second-order linear differential equations. As an example, the following equations appear for a steam engine regulated by a combination of Thomson's governor with Jenkin's governor (in Maxwell's notation):

$$\begin{aligned} A\frac{d^2\theta}{dt^2} + X\frac{d\theta}{dt} + K\frac{d\phi}{dt} + T\phi + J\psi &= P - R \\ B\frac{d^2\phi}{dt^2} + Y\frac{d\phi}{dt} - K\frac{d\theta}{dt} &= 0 \\ C\frac{d^2\psi}{dt^2} + Z\frac{d\psi}{dt} - T\phi &= 0. \end{aligned}$$

Here, $P - R$ denotes the effective driving torque. The main variable is θ, which represents the deviation of the main shaft angle from its nominal value. The variables ϕ and ψ correspond to the two governors. Maxwell then writes the general solution for θ, which, by the standard theory of ordinary differential equations, involves a linear combination of exponential functions. These exponential functions are determined by the roots of a polynomial equation that can be derived readily from the given system. Maxwell writes n for the unknown, and obtains a fifth-degree equation by setting

$$\begin{vmatrix} An^2 + Xn & Kn + T & J \\ -K & Bn + Y & 0 \\ 0 & -T & Cn^2 + Zn \end{vmatrix} = 0$$

(a factor n has been cancelled right away in the second row). He is then confronted with the problem of determining conditions on the coefficients under which all solutions of this equation are located in the left half of the complex plane. This, of course, led to the work of Routh on conditions for the stability of polynomials of arbitrary degree. We see that Maxwell's fifth-order equation arises from the application of a fourth-order controller to a second-order system, and that the conditions for stability are given by him in terms of the zeros of a polynomial matrix that is obtained directly from a standard modeling procedure.

Maxwell used second-order differential equations, but it gradually became standard in the nineteenth century to write differential equations in first-order form. The fact that a higher-order differential equation in one variable may be replaced by a first-order equation in several variables

was actually already known in Cauchy's time. The Lagrangian equations of mechanics were later put into a suitable first-order form by Hamilton; towards the end of the century, Poincaré and Lyapunov used first-order vector representations systematically. Naturally, therefore, representations of this type (called *state* representations later on) have dominated control-theoretical work that was done in close connection with the theory of ordinary differential equations. This concerned mainly linear stability theory at first, but later, in the first decades of the twentieth century, attention shifted to nonlinear problems. This line of research was held up high especially in the USSR (see for instance the survey by Minorsky in [45]).

The work in connection with differential equations had a natural tendency to emphasize closed-loop systems, obtained by combining a given system with a given controller. Indeed, for such systems one may readily apply the powerful methods from the theory of ordinary differential equations and allied disciplines, such as the theory of differential-difference (delay) equations. The analysis by Maxwell, as briefly described above, is an example of this approach. The closed-loop point of view is quite satisfactory for many problems in mechanical engineering. To the communications engineer, however, it is more natural to use an open-loop point of view, in which a system is viewed as an operation that acts on an input signal and produces an output signal. This 'operational' point of view called for a representation which would express the output signal as the result of some operator acting on the input signal. Such a representation is provided, at least for linear systems, by the convolution integral. However, competing representations were soon to appear. Indeed, the use of complex quantities for the representation of complex signals, the Fourier and Laplace transforms, and Heaviside's Operational Calculus were all in principle available by the turn of the century. The value of these techniques was gradually recognized among electrical engineers, be it certainly not without resistance (see for instance [46]). From the mathematical point of view, the use of operational methods led to the introduction of techniques quite different from the ones usually found in the theory of differential equations. Applications of complex function theory were limited at first to partial fraction expansions and computation of integrals, but the appearance of the Nyquist criterion [47] made engineers realize that full-fledged function theory was a natural tool to use in the analysis of linear systems [10, p. 9]. Function-theoretic tools, in particular Cauchy's theorem, were used extensively by Bode in his book [9], which incorporated the celebrated Bode gain-phase relation and the minimum phase concept. The development of the root locus method by Evans in 1948 [18] firmly established the view of the transfer function as a function defined on the complex plane rather than just on the real frequency axis. For a more extensive discussion of the development of frequency-domain methods, we refer to [40]. We will not at all review the developments in the area of stochastic systems. In connection with what just has been said, however, it is interesting to quote Wiener on some of the differences between his own work and that of Kolmogorov:

> ... my work, unlike the explicitly published work of Kolmogoroff, concerns the instrumentation which is necessary to realize the theory of prediction in automatic apparatus for shooting ahead of an airplane. This engineering bias leads me to emphasize more than does Kolmogoroff the problem of prediction in terms of linear operators in the scale of frequency, rather than in similar operators on the scale of time. [63, p. 308]

While the communication engineers developed their own methods, work on the ODE-type approach to control systems was still continuing, in particular in the Soviet Union. During the Second World War, a research centre was formed in Kazan where work on applied problems was done by outstanding mathematicians such as L. S. Pontryagin, who had already acquired fame

because of his pre-war contributions to topological algebra. After the war, research efforts in control theory continued at various mathematical institutes in the USSR. One important research direction centered around 'Aizerman's conjecture' [1], a nonlinear generalization of the Nyquist criterion. This problem called for a representation of systems with an explicitly appearing input variable, unlike the setting that was mainly used before in the 'ODE' framework. Systems with one input were studied first, in line with the original work of Nyquist, but the extension to several inputs was a natural one. For instance, Letov [35] considered in 1953 the following system (in original notation):

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + n_{k1}\xi_1 + n_{k2}\xi_2 \qquad (k = 1, \cdots, n),$$

$$\dot{\xi}_1 = f_1(\sigma_1), \quad \sigma_1 = \sum_{\alpha=1}^{n} p_{1\alpha}\eta_\alpha - r_{11}\xi_1 - r_{12}\xi_2 ,$$

$$\dot{\xi}_2 = f_2(\sigma_2), \quad \sigma_2 = \sum_{\alpha=1}^{n} p_{2\alpha}\eta_\alpha - r_{21}\xi_1 - r_{22}\xi_2 .$$

We recognize the first equation (with hindsight, perhaps) as a linear state equation with two inputs.

The early fifties saw the rise of modern *optimal control theory*. One of the first problems to be studied was time-optimal control. In some applications, it is natural to consider control strategies in which one switches between full power in one direction and full power in the reverse direction. This motivated a study of differential equations with discontinuous forcing terms by D. W. Bushaw at Princeton University [11]. Bushaw noted that the switching instant could be optimized to obtain a transfer from one state to another in minimal time. Subsequently, J. P. LaSalle observed that 'bang-bang' policies would be optimal among all possible control policies which lead from a given state to another. LaSalle used a nonlinear formulation, but later on Bellman *et al.* considered linear systems [7]. In this paper, Bellman and his co-authors required invertibility of the input matrix (as we would now call it), so in particular they let the number of inputs be equal to the number of states. In independent work, Gamkrelidze [25] considered shortest time problems for linear systems with n states and r inputs. He writes the following state equation [25, p. 451]:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}_1 u^1 + \cdots + \mathbf{b}_r u^r$$

which is practically the formula $\dot{x} = Ax + Bu$ that has become ubiquitous in control theory.

By the end of the fifties, the time had come for an amplification of the notion of 'state' far beyond its meaning as the vector that appears when dynamic equations are written in a first-order form. This was due to the role that this concept had to play in Bellman's dynamic programming method, but also to developments in the theory of automata (finite state machines; Nerode equivalence).

In control theory, the announcement by Pontryagin of his Maximum Principle at the International Mathematical Congress in Edinburgh in 1958 had a tremendous impact on research in optimal control. Bang-bang control problems, in which one seeks to steer from one state to another, naturally led to the formulation of the concept of *controllability* by R. E. Kalman. This concept, and the dual notion of observability, turned out to play a crucial role in what Kalman called the *realization problem*:

> Given an (experimentally observed) impulse response matrix, how can we identify the linear dynamical system which generated it? [30, p. 153]

The word 'realization' is used here in a sense that is different from the traditional usage in electrical engineering. There, one would look for realization of a given driving-point impedance as an actual or idealized electrical circuit (cf. also the use of the term 'realize' in the quotation from Wiener given above). Although Kalman did advertize the state space realization as a 'blueprint' which could serve as a basis for implementation in an analog network [28], this connection was hardly emphasized in subsequent research.

In the newly founded SIAM Journal on Control, E. Gilbert argued that the transfer representation was misleading and could lead to erroneous results. His point was that unobservable and/or uncontrollable states could be created by system composition:

> Thus transfer-function matrices may satisfactorily represent all the dynamic modes of the subsystems but fail to represent all those of the composite system. Furthermore, the loss of hidden response modes is not easy to detect because of the complexity of the transfer-function matrices and matrix algebra. [27, p. 140]

To develop linear control theory from the state space point of view, it had to be shown that the familiar concepts from the frequency domain could be translated to state space terms. For this, the new realization theory was an indispensible tool. Gilbert [27] used partial fraction expansion (much in the tradition of Heaviside, one might say) to obtain a state space realization for a transfer matrix having only simple poles. This method can be extended to the general situation (not necessarily simple poles), but then becomes somewhat involved (see [50]). A more elegant realization algorithm was published by Kalman and B. L. Ho in 1966 [28]. The algorithm was based on a new parametrization of the transfer matrix — new at least to control theory: in 1894, A. A. Markov had already used essentially the same parametrization for a study of continued fractions [41]. The 'Markov parameters' are the first (matrix) coefficients in the power series development around infinity of a proper rational matrix.

For a while, 'realization theory' was, at least to the system theorist, practically equivalent to the determination of a state space representation from a transfer matrix given through its Markov parameters. The seventies, however, brought a renewed interest in polynomial representations. An important impetus for this development came from the appearence of Rosenbrock's book [51] on multivariable systems. In this work, Rosenbrock considered input/output systems given in the form

$$T(s)\xi = U(s)u$$

$$y = V(s)\xi + W(s)u$$

where all matrices are polynomial. Great emphasis was placed on the study of *equivalence* notions. Rosenbrock found a 'lifting' of Kalman's system equivalence concept to the more general representation displayed above, which he called *strict system equivalence*. It seems safe to say that the systematic development of the theory of system representations, system equivalence and system transformations starts with [51].

From Rosenbrock's system matrix, the transfer matrix is represented as $V(s)T^{-1}(s)U(s) + W(s)$, i.e., as a ratio of polynomial matrices. It is not difficult to see that, in fact, every rational matrix can be written in either of the two forms $V(s)T^{-1}(s)$ or $T^{-1}(s)U(s)$, where, moreover, a *coprimeness* condition may be imposed. These *coprime fractional representa-*

tions were used very successfully by Kučera [31, 32] and by Youla *et al.* [79, 80] to give a parametrization of all stabilizing controllers for a given plant. This is an example of a result that appears quite naturally in one representation but would be awkward to derive in some other representations. At the same time, fractional matrix representations were also used in work on infinite-dimensional realization problems done at Harvard University by R. W. Brockett, J. S. Baras, and P. A. Fuhrmann (see for instance [6]). In the infinite-dimensional context, the available mathematical tools strongly suggested to replace polynomials by functions analytic on the unit disk (in the discrete-time case — for continuous-time systems, the class to use would be the set of functions that are analytic on the right half plane). This idea was picked up by researchers in finite-dimensional system theory, who discovered that some difficulties with the Kučera-Youla parametrization could be ironed out by using the ring of rational functions that have no poles in the closed right half plane (including the point at infinity) rather than the ring of polynomials (see for instance [16]). The fractional representation over the ring of proper and stable rational functions was subsequently used extensively in the emerging H^∞-theory, which is in itself an example of an application of function-theoretic techniques to control problems in a way that would probably have been quite beyond the imagination of Nyquist and Bode. On the other hand, H^∞-theory has also relied heavily on state space representations, since the representation in terms of constant matrices makes it possible to use standard numerical software. The cooperation between the two representations was facilitated by the discovery (attributed to D. Aplevich in [62]) that there is an easy way to pass from a state space representation to a fractional representation over the ring of proper stable rational functions. (Fractional representations over the ring of polynomials cannot be obtained in a comparable way from a state space representation.)

Nevertheless, polynomial representations were emphasized again in the mid-seventies when Fuhrmann worked out an elegant procedure to go from a polynomial matrix fraction representation to a state space representation [23]. The discovery of this procedure, now known as *Fuhrmann's realization*, spurred considerable research on the relation between state space concepts, as developed in particular in the 'geometric approach' to linear systems [77], and polynomial or transfer matrix concepts. For an introduction to this, see for example Chapter 1 of [24].

Polynomial matrices, even when less suitable for a number of purposes than stable proper rational matrices, are important in system theory because they arise naturally in modeling. Indeed, a polynomial matrix representation can be written down immediately from a set of linear differential and algebraic equations describing a given system. Maxwell's equations for the controlled steam engine, as given above, may serve as an example. Of course, by the old trick of replacing higher-order derivatives by new variables, it is also possible to obtain a first-order representation. Instead of the Rosenbrock form discussed above, one then gets a representation in the form

$$E\dot{x} = Ax + Bu$$

$$y = Cx + Du,$$

where E, A, B, C, and D are constant matrices. The variable "x" which appears here was called the *descriptor variable* by Luenberger, who was first to make an extensive study of this representation in system theory [37, 38]. Contrary to the standard state space representation, the descriptor form is capable of representing systems having a non-proper transfer function (also called 'non-causal systems' or 'singular systems'). Through the years, the term 'descriptor system' has come to be used almost exclusively for such systems, although this was certainly not Luenberger's original

intention — he was trying to emphasize the modeling issue, rather than the question of causality. The descriptor form was used by Verghese [60] to define an equivalence concept which deals neatly with pole/zero cancellations at infinity. This cleared up a problem which had remained unsolved in Rosenbrock's work. Alternative solutions were given later by Anderson, Coppel and Cullen [2] and by Pugh, Hayton and Fretwell [48, 49]. The fact that the notions of equivalence defined by these authors are indeed the same was established by Ferreira [19]. Further comments on descriptor systems will be given in the next section.

In recent years, the study of system representations has been stimulated by the work of J. C. Willems. There are several important points where his approach is different from other approaches discussed above. First of all, Willems uses an intrinsic definition of system equivalence (i.e., one that does not depend on a specific representation). He does this by defining a 'system' simply as 'a family of trajectories of given variables' (such as the port voltages and currents of an electrical network, or forces and displacements in a mechanical system). The given variables which appear in the definition are also denoted as 'external variables', to distinguish them from 'internal variables' which are possibly used as auxiliary quantities in a description of the system. The external variables may consist of what are usually called 'inputs' and 'outputs', but, as shown in section 4 of this paper, other interpretations can sometimes also be useful. The family of trajectories is also referred to as a 'behavior' $\mathscr{B}$.

In this approach, there is some flexibility associated with the choice of the function space to which the trajectories that make up the system are supposed to belong. In the study of differential equations, one normally uses function spaces that allow for exponentially growing solutions (such as C^∞, or the space of distributions). In the context of system theory, however, it also makes sense to consider for instance only those trajectories that are square integrable. Different choices of function spaces lead, in this way, to different notions of system; put in another way, they lead to different equivalence relations on system descriptions. More on this will be said below. Willems has shown [68] that, if one interprets 'external variables' as 'inputs and outputs' and uses the classical function spaces alluded to above, the equivalence relation that emerges is in fact different from the equivalence relations that were mentioned above.

It should be noted that the definition of a 'system' as a family of trajectories is not new. Compare, for instance, McMillan's definition of a $2n$-pole:

> The constraints imposed by a general $2n$-pole N on voltages and currents are completely described by the totality of pairs $[\underline{v}, \underline{i}]$ which N admits. We shall *define* a general $2n$-pole, therefore, as
> (i) a collection of n oriented ideal branches, as in 4.11, and
> (ii) a list of pairs $[\underline{v}, \underline{i}]$ of voltages and currents admitted in these branches.
> [44, p. 228]

(The oriented ideal branches in §4.11 of McMillan's paper serve just to define the pairing of the terminals.) In recent work in system theory, the equivalence notion as used by Willems has in fact occurred in several places; see [4, p. 513] ('external equivalence') and [8, p. 92] ('input-output equivalence'). Nevertheless, there is no doubt that the consequences of the acceptance of this intrinsic definition of what a system is have been explored to the fullest in the work of Jan Willems.

3. A ROAD MAP OF REPRESENTATIONS

In this section, we shall review the available representations for a specific class of systems, viz., the class of finite-dimensional, deterministic, time-invariant, real, linear systems in continuous time, without further special structure. (The addition 'without further special structure' refers to the fact that we shall not consider special properties that arise, for instance, for systems defined on a symplectic space.) This is the class that has served as sort of a standard in system theory during the last three decades, except that *causality* is often imposed as an additional requirement. This condition was not included in the list above for two reasons. First of all, we are sometimes interested in external variables that are not to be considered as 'inputs' and 'outputs' (cf. Section 4 of this paper, for instance), and in such cases causality need not be a relevant issue. Secondly, even when we do distinguish inputs and outputs, there are no simple ways to tell, at a general level of representation such as Rosenbrock's system matrix, whether a given system is causal or not [51, p. 51]. Imposing causality as a constraint on such general linear system representations would therefore be awkward.

3.1 Notions of equivalence

When discussing system representations, we will have to specify under which circumstances we shall say that two representations are equivalent in the sense that they correspond to the same system. There are three main options. There is the notion of *strong equivalence*, which boils down to Kalman's concept of equivalence for causal input/output systems in standard state space form. Definitions of this equivalence (by specification of a list of allowed transformations) were given at the level of descriptor systems by Verghese [60] and by Pugh *et al.* [48, 49], and by Anderson *et al.* at the level of the Rosenbrock system matrix [2]. Secondly, for every class of representations that have a given input/output structure and that define a transfer matrix, one has the notion of *transfer equivalence* according to which two representations are equivalent if and only if they define the same transfer matrix. Finally, if one considers representations that define a family of trajectories of the external variables (an 'external behavior' in the sense of [68]), then there is the notion of *external equivalence* according to which two representations are equivalent if and only if they induce the same external behavior.

As noted before, external equivalence can in fact be understood in various ways, depending on the choice of a function space for the trajectories, and on the choice of external variables. There is also some freedom that arises from the interpretation of the external variables. For example, if we allow only permutation transformations on the external variables, this means that these variables are interpreted as quantities which each have there own meaning and are measured on a fixed scale. On the other hand, if we allow general invertible linear transformations, then the implication is that the vector of external variables is understood as an element of a general linear space. It goes without saying that, depending on the problem one has at hand, some of the external variables can be interpreted in one way and others in another way. (The same might be said about the choice of a function space.) The term 'external equivalence' will be used for what might be called the 'classical' interpretation: the function space is such that exponentially growing solutions are admitted (we shall use C^∞ to make life a little bit easier), and only permutation operations will be allowed on the external variables. We call this the 'classical' form because it would seem that the notion of equivalence that is used (often implicitly) in treatments of ordinary differential equations is of this type. If one uses an L_2-space rather than a C^∞-space as a trajectory space, then (cf. [74]) the corresponding notion of external equivalence turns out to be an extension of transfer equivalence, in the sense that it coincides with transfer equivalence on the

class of systems that define a transfer matrix. Suppose now that one has a system of equations in the form

$$\sigma x = Ax + Bu \tag{3.1}$$

$$y = Cx + Du. \tag{3.2}$$

One might propose to take u, y, and x as external variables following C^∞-trajectories, to interpret u and y in a 'classical' sense, and to interpret x as a variable in a general linear space. The resulting concept of equivalence is Kalman's equivalence. It may be suspected that a similar re-interpretation in terms of external equivalence is also possible for strong equivalence.

To keep the presentation manageable, we shall consider transformations under 'classical' external equivalence. For other types of equivalence, the picture will be different but similar. We will discuss special representations for systems equipped with an i/o structure, but the particular representations that are available only for causal systems will be omitted.

3.2 A catalog of representations

We start by listing a number of representations. A number of basic types will be distinguished that are different by appearance; within these, we distinguish subtypes that do not differ notationally but that are subject to more or less severe constraints.

The most unspecific type of representations we shall take into consideration is the AR/MA class. An AR/MA representation is specified by two polynomial matrices $P(s)$ and $Q(s)$, which determine the external behavior consisting of all trajectories w of the external variables for which there exists a trajectory ξ of the internal variables such that

$$\begin{aligned} P(\sigma)\xi &= 0 \\ w &= Q(\sigma)\xi. \end{aligned} \tag{3.3}$$

In the continuous-time interpretation we use here, σ stands for $\mathrm{d}/\mathrm{d}t$. The class is called AR/MA because of the discrete-time interpretation in which σ is the shift: in this case, (3.3) implies that the external variables are expressed as a moving average of the internal variables, which themselves satisfy an autoregressive equation. Every representation of this kind can trivially be rewritten as a 'system with auxiliary variables' [68] (later also called an 'ARMA' representation by Willems [73]), which is defined by an equation of the form

$$P'(\sigma)\xi + Q'(\sigma)w = 0; \tag{3.4}$$

simply take

$$P'(s) = \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}, \qquad Q'(s) = \begin{bmatrix} 0 \\ -I \end{bmatrix}. \tag{3.5}$$

On the other hand, it is also easy to write an AR/MA representation for a system with auxiliary variables, by extending the space of internal variables and writing

$$P(s) = [P'(s) \quad Q'(s)], \qquad Q(s) = [0 \quad I]. \tag{3.6}$$

We see that the AR/MA representation is, in general, less parsimonious in the use of internal variables than the representation as a system with auxiliary variables. Since we are looking for an unspecific representation, this might be construed as an argument against the representation in the form (3.4). Actually, when dealing with systems described by *partial* differential equations, one easily runs into clear-cut cases in which an AR/MA representation appears much more naturally

than a representation with auxiliary variables as in (3.4).

For systems with an i/o structure, another general representation is RSM (Rosenbrock system matrix [51]). An RSM representation is specified by four polynomial matrices $T(s)$, $U(s)$, $V(s)$, $W(s)$, where $T(s)$ is square and invertible. The external behavior defined by an RSM representation consists of the set of all input trajectories u and output trajectories y for which there exists an internal-variable trajectory ξ such that the following equations hold:

$$\begin{aligned} T(\sigma)\xi &= U(\sigma)u \\ y &= V(\sigma)\xi + W(\sigma)u. \end{aligned} \tag{3.7}$$

The third polynomial representation we shall consider is the AR representation [69]. An AR representation is specified by a single polynomial matrix $R(s)$, which should have as many columns as there are external variables. The external behavior it defines is simply the set of all external-variable trajectories w satisfying

$$R(\sigma)w = 0. \tag{3.8}$$

We shall always require $R(s)$ to have full row rank; this simply means that the equations specified by the rows of $R(s)$ are independent. An AR representation given by $R(s)$ will be said to be *minimal* if the sum of the row degrees of $R(s)$ is minimal in the set of all AR representations of the same system. One can show (see for instance [69, Thm. 6]) that a matrix $R(s)$ is minimal in this sense if and only if it is row proper. The class of minimal AR representations will be denoted by $\mathsf{AR}_{\mathsf{min}}$. If the external variable is partitioned into inputs and outputs, the defining matrix $R(s)$ of an AR representation will be divided into two blocks $R_1(s)$ and $R_2(s)$, which correspond to outputs and inputs respectively. If $R_1(s)$ is square and nonsingular, the representation so obtained will be called an LMF representation ('left matrix fraction').

By introducing new internal variables, it is easy to transform an AR/MA representation to a first-order form

$$\begin{aligned} \sigma G\xi &= F\xi \\ w &= H\xi \end{aligned} \tag{3.9}$$

(F, G, and H are constant matrices). This representation, specified by the three matrices F, G, and H, will be called the *pencil* representation ([33]; cf. also [4, 56]), and the corresponding class of representations will be denoted by P. To be complete, one should also indicate the spaces on which the various mappings are defined, and so we shall sometimes also give a P representation as a six-tuple $(F, G, H; Z, X, W)$ where F and G are mappings from the 'internal variable space' Z to the 'equation space' X, and H maps Z into the external variable space W. A descending chain of subclasses can be formed by putting more and more strict requirements on the triple (F, G, H). If G is surjective, the corresponding class will be denoted by P_{dv}, because this class is closely related to the DV representations that will be discussed below. The class of representations which in addition satisfy the condition that $[G^{\mathsf{T}} \quad H^{\mathsf{T}}]^{\mathsf{T}}$ is injective will be denoted by P_{io}; in a representation of this type, one can easily see which partitionings of the external variables into inputs and outputs will lead to a causal i/o structure (cf. [33], Lemma 5.1 and Lemma 6.1). Finally, pencil representations that also satisfy the requirement that $[sG^{\mathsf{T}} - F^{\mathsf{T}} \quad H^{\mathsf{T}}]^{\mathsf{T}}$ has full column rank for all $s \in \mathbb{C}$ form a class that will be denoted by $\mathsf{P}_{\mathsf{min}}$. It has been shown in [33] (Prop. 1.1) that a pencil representation is minimal under external equivalence if and only if it belongs to $\mathsf{P}_{\mathsf{min}}$.

Next in our collection of representations is the DV (driving-variable) representation [4, 68, 69, 73], which, as already mentioned, is closely related to the P_{dv} class. A DV representation

is specified by four constant matrices A, B, C', and D', which determine an external behavior by the equations

$$\begin{aligned} \sigma\xi &= A\xi + B\eta \\ w &= C'\xi + D'\eta \end{aligned} \tag{3.10}$$

(ξ and η are auxiliary variables). The class of DV representations for which the matrix D' is injective will be denoted by $\mathrm{DV}_{\mathrm{io}}$. If also the requirement is imposed that the 'system pencil'

$$\begin{bmatrix} sI - A & B \\ C' & D' \end{bmatrix}$$

has full column rank for all s, then we obtain a class of representations that will be denoted by $\mathrm{DV}_{\mathrm{min}}$. It has been shown in [68] that a $\mathrm{DV}_{\mathrm{min}}$ representation is minimal in the class of DV representations, in the sense that both the length of ξ and the length of η are minimal.

For input/output behaviors, there are further special representations that may be used. A well-known form is the *descriptor* representation [37, 38]. The class of such representations will be denoted by D. A descriptor representation is specified by five constant matrices E, A, B, C, and D, and determines an input/output behavior by the equations

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned} \tag{3.11}$$

The domain of the mappings E and A will be denoted by X_{d} (descriptor space), the codomain will be written as X_{e} (equation space).

Quite a few special properties have been used in the literature in connection with this representation (see for instance [5, 13, 36, 52, 61, 78]. We shall use the following conditions. The representation (3.11) is said to be *controllable at infinity* if

$$\operatorname{im} E + \operatorname{im} B + A(\ker E) = X_{\mathrm{e}}. \tag{3.12}$$

It is said to be *reachable at infinity* if

$$\operatorname{im} E + \operatorname{im} B = X_{\mathrm{e}}. \tag{3.13}$$

It is called *observable at infinity in the sense of Verghese* if

$$\ker E \cap \ker C \cap A^{-1}[\operatorname{im} E] = \{0\} \tag{3.14}$$

and *observable at infinity in the sense of Rosenbrock* if

$$\ker E \cap \ker C = \{0\}. \tag{3.15}$$

The representation (3.11) is said to have *no nondynamic variables* if

$$A(\ker E) \subset \operatorname{im} E. \tag{3.16}$$

These are all properties that relate to the point at infinity. We note that, for representations that satisfy (3.16), there is no difference between controllability and reachability at infinity or between the two notions of observability at infinity. In connection with the finite modes, we shall need the following condition: a representation of the form (3.11) is said to have *no finite unobservable modes* if

$$\operatorname{rank}_{\mathbf{R}} \begin{bmatrix} sE - A \\ C \end{bmatrix} = \operatorname{rank}_{\mathbf{R}(s)} \begin{bmatrix} sE - A \\ C \end{bmatrix} \quad \text{for all } s \in \mathbb{C}. \tag{3.17}$$

In principle, a considerable number of descriptor representation types could be formed by taking combinations of the six conditions mentioned above. We shall consider just four types, which together seem to present a reasonable hierarchy. The most unspecific type is the general descriptor form, for which the symbol D has already been introduced. The symbol D_{ri} will be used for the class of descriptor representations that are reachable at infinity. Descriptor representations that have no nondynamic variables and that are both controllable and observable at infinity will be denoted as D_{mi} representations ('minimal at infinity'). Finally, the class of D_{min} representations consists of the D_{mi} representations that have no finite unobservable modes. It is shown in [34] that a descriptor representation is minimal under external equivalence if and only if it belongs to this class.

3.3 The road map

To indicate the connections between the somewhat vast number of representations introduced above, we shall now present a map. The following organizational principles have been applied:

- polynomial representations are on the left, first-order representations on the right;
- representations that do not distinguish between inputs and outputs are in the middle, i/o representations are on the extremes;
- more specific representations are higher up in the diagram than less specific ones.

Moreover, arrows have been used to indicate known transformation procedures (including the trivial ones, which involve no transformation at all, and the very easy ones, such as the transformation from AR to AR/MA). The organization of the diagram is such that arrows going up represent the heaviest computational loads. The result is shown in Fig. 1 below.

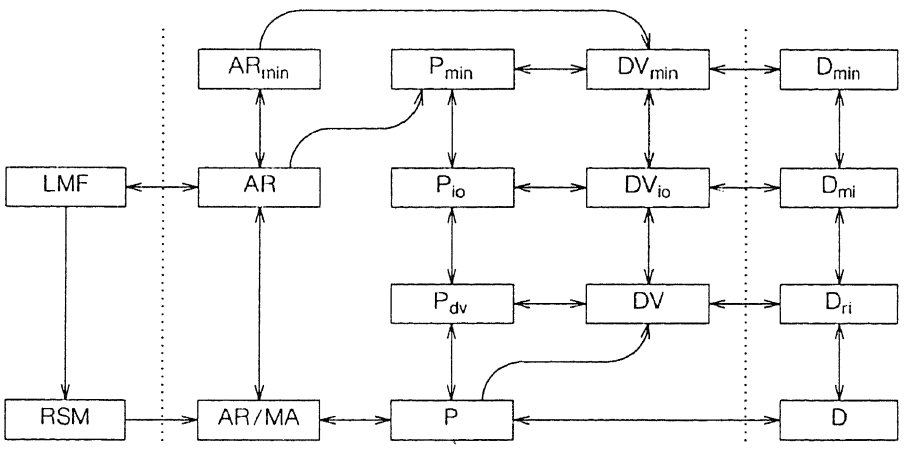

FIGURE 1. Representations and transformations of linear systems.

The arrows going *down* in this diagram all correspond to trivial rewritings or reinterpretations. For instance, an LMF representation is a special case of an RSM representation, obtained by taking $V(s) = I$ and $W(s) = 0$. The connection between LMF and AR is also quite clear. One gets from an RSM representation to an AR/MA representation simply by identifying the inputs with new internal variables. It is quite obvious how to transform the various types of

DV representations to the P representations on the same level, and vice versa. The transformation from AR/MA to P is by the standard trick of replacing higher-order derivatives by new variables. Most of the other transformations require more work, however, and some of the corresponding algorithms will be discussed below.

3.4 Algorithms

We start with the transition from an AR/MA representation to an AR representation. For this, we have the following procedure.

ALGORITHM 1 Let an AR/MA representation be given by $(P(s), Q(s))$. For instance by the algorithm of reduction to Hermite form ([39, pp. 32-33]; see also [29, pp. 375-376] or [12, p. 34]), find a unimodular matrix $U(s)$ such that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix} \tag{3.18}$$

where $T(s)$ has full row rank. Let $R(s)$ be a maximal selection of independent rows from $U_{22}(s)$. Under these conditions, $R(s)$ gives an AR representation that is externally equivalent to the AR/MA representation $(P(s), Q(s))$.

For a proof of this, see [68, Prop. 3.3] or [33, Lemma 4.1]. The algorithm in [68] is actually based on the Smith form; from a computational point of view, this presents a considerable amount of overkill. In the algorithm given above, is is easy to see that $U_{22}(s)$ will automatically have full row rank (so that we simply have $R(s) = U_{22}(s)$) when $P(s)$ has full row rank, which is a natural restriction to impose.

The passage from AR to AR_{min} is just the reduction of a polynomial matrix to row proper form. The standard algorithm to do this is described for instance in [76, pp. 27-29] and in [29, p. 386]. This algorithm essentially requires only operations on constant matrices, and the computational load involved is in general much less than in a transition from AR/MA to AR form.

The steps leading from P to DV, from DV to DV_{io}, and from DV_{io} to DV_{min} are detailed in [56]. These steps can be 'lifted' to the level of P representations, and, in fact, it turns out that they can be derived quite naturally in this context. We shall now explain this in some detail.

First, consider the transition from a general P representations to the P_{dv} representation. From the equation $\sigma G\xi = F\xi$, it follows that any ξ-trajectory satisfying this equation must belong to the subspace $F^{-1}[\text{im } G]$. This implies, of course, that $G\xi$ belongs to $GF^{-1}[\text{im } G]$. From that fact, it follows that any trajectory $\xi(\cdot)$ satisfying $\sigma G\xi = F\xi$ must actually belong to the subspace $F^{-1}[GF^{-1}[\text{im } G]]$, which obviously is contained in $F^{-1}[\text{im } G]$. We can go on in this way; a subspace recursion emerges which can be summarized as follows. Let the space on which G and F act be denoted by Z. Define

$$Q^0 = Z \tag{3.19}$$

and

$$Q^{k+1} = F^{-1}GQ^k. \tag{3.20}$$

We have $Q^{k+1} \subset Q^k$ for all k, and so a limit must be reached after finitely many (in fact, at most $\dim Z$) steps. The limit subspace will be denoted by $Q^*(F, G)$ or simply by Q^* if there is no risk of confusion. We arrive at the following algorithm to obtain a P_{dv} representation from a P representation.

ALGORITHM 2 Let $(F, G, H; Z, X, W)$ be a P representation. Compute the subspace Q^* of Z as the limit of the sequence of subspaces defined by (3.19-3.20). Take $\tilde{Z} = Q^*$, $\tilde{X} = GQ^*$, and define $\tilde{F}$, $\tilde{G}$, and $\tilde{H}$ as the restrictions of the respective mappings to $\tilde{Z}$ and $\tilde{X}$. (Note that, by the definition of Q^*, F does indeed map Q^* into GQ^*.) Under these conditions, a P_{dv} representation that is equivalent to the original P representation is given by $(\tilde{F}, \tilde{G}, \tilde{H}; \tilde{Z}, \tilde{X}, W)$.

Next, we consider the transformation from a P_{dv} to a P_{io} representation. Let $(F, G, H; Z, X, W)$ be a P_{dv} representation, and suppose that $[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ is not injective. We can then split up the internal variable space Z as $Z = Z_1 \oplus Z_2$, where $Z_2 = \ker G \cap \ker H$ is nonzero. With respect to this decomposition, write $G = [G_1 \quad 0]$, $F = [F_1 \quad F_2]$, $H = [H_1 \quad 0]$. The equations $\sigma G\xi = F\xi$, $w = H\xi$ then appear in the following form:

$$\sigma G_1 \xi_1 = F_1 \xi_1 + F_2 \xi_2 \tag{3.21}$$

$$w = H_1 \xi_1. \tag{3.22}$$

Since there are no restrictions on ξ_2, the above equations are equivalent to

$$\sigma T G_1 \xi_1 = T F_1 \xi_1 \tag{3.23}$$

$$w = H_1 \xi_1 \tag{3.24}$$

where T is any map satisfying $\ker T = \operatorname{im} F_2$. It is natural to let T be surjective, and we see that the above transformation achieves a reduction of the dimension of the internal variable space and perhaps also a reduction of the dimension of the equation space. In more geometric terms, what we have done is the following. Define $S^1 = \ker G \cap \ker H$, and let $Z_1 = Z/S^1$, $X_1 = X/FS^1$. With these definitions, the factor mappings $G_1: Z_1 \to X_1$, $F_1: Z_1 \to X_1$, and $H_1: Z_1 \to W$ are all well-defined, and the representation $(F_1, G_1, H_1; Z_1, X_1, W)$ is equivalent to the original representation.

There is no guarantee that, after this step, the reduced representation is of the P_{io} type, and in general the reduction will have to be repeated a number of times. For instance, the reduction in the second step is determined by the subspace

$$\begin{aligned} \ker G_1 \cap \ker H_1 &= \{z \bmod S^1 \mid Gz \in FS^1 \text{ and } Hz = 0\} \\ &= (G^{-1}FS^1 \cap \ker H) \bmod S^1. \end{aligned} \tag{3.25}$$

The subspace recursion that emerges is the following:

$$S^0 = \{0\} \tag{3.26}$$

$$S^{k+1} = G^{-1}FS^k \cap \ker H. \tag{3.27}$$

We have $S^{k+1} \supset S^k$ at every step, and so after finitely many ($\leqslant \dim Z$) steps a limit is reached. The limit subspace will be denoted by $S^*(F, G, H)$ or simply by S^* if the context is clear. The algorithm to go from a P_{dv} to a P_{io} representation can now be formulated as follows.

ALGORITHM 3 Let $(F, G, H; Z, X, W)$ be a P_{dv} representation. Define the subspace S^* of Z as the limit of the sequence defined by (3.26-3.27). Define $\tilde{Z} = Z/S^*$, $\tilde{X} = Z/FS^*$. With these definitions, the factor mappings $\tilde{G}: \tilde{Z} \to \tilde{X}$, $\tilde{F}: \tilde{Z} \to \tilde{X}$, and $\tilde{H}: \to W$ are well-defined, and $(\tilde{F}, \tilde{G}, \tilde{H}; \tilde{Z}, \tilde{X}, W)$ forms a P_{io} representation that is equivalent to the original P_{dv} representation.

The final transformation in this series is the one that leads from P_{io} to $\mathsf{P}_{\mathrm{min}}$ representations.

To achieve this reduction, we note that a redundancy in P_{io} descriptions is associated with subspaces N of the internal variable space Z that satisfy the two properties

$$FN \subset GN \tag{3.28}$$

and

$$N \subset \ker H. \tag{3.29}$$

Indeed, if N is a nonzero subspace having these properties, then we can decompose the internal variable space Z and the equation space X in such a way that $H = [H_1 \quad 0]$ and the mappings G and F take the form

$$G = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}, \qquad F = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}. \tag{3.30}$$

Of course, both G_{11} and G_{22} must be surjective. The equations become

$$\sigma G_{11}\xi_1 = F_{11}\xi_1 \tag{3.31}$$

$$\sigma G_{22}\xi_2 = F_{21}\xi_1 + F_{22}\xi_2 \tag{3.32}$$

$$w = H_1\xi_1. \tag{3.33}$$

Because G_{22} is surjective, the second equation can always be satisfied by a suitable choice of ξ_2; therefore, no constraint is imposed on ξ_1. This means that the second equation as well as the variable ξ_2 may be removed without altering the external behavior. Speaking geometrically, this means that we replace Z by $\tilde{Z} = Z/N$ and X by $\tilde{X} = X/GN$, and that the mappings F, G and H are replaced by the respective factor mappings.

The reduction that is obtained in this way increases with N, and so we are interested in the largest element of the set of subspaces satisfying both (3.28) and (3.29). (The fact that this set indeed has a largest element follows from the fact that the set is closed under subspace addition.) Let us denote this largest element by $N^*(F, G, H)$. The question is, how to compute this subspace. The answer to this is provided by the following equality, which expresses perhaps the most basic result in the geometric theory of linear systems:

$$N^*(F, G, H) = Q^*\left(\begin{bmatrix} G \\ 0 \end{bmatrix}, \begin{bmatrix} F \\ H \end{bmatrix}\right). \tag{3.34}$$

Indeed, this gives us an *algorithm* to compute N^*. The proof of (3.34) is not difficult, and may essentially be found in the standard reference [77, p. 91]. A considerable amount of translation of terms is needed, though, and the reader may find it easier to construct a direct proof. Rewriting the algorithm (3.19-3.20) a little bit to suit the special form which appears in (3.34), we finally obtain the following algorithm.

ALGORITHM 4 Let $(F, G, H; Z, X, W)$ be a P_{io} representation. Define a sequence of subspaces of Z by

$$N^0 = Z \tag{3.35}$$

$$N^{k+1} = F^{-1}GN^k \cap \ker H. \tag{3.36}$$

Denote the limit subspace by N^*, and define $\tilde{Z} = Z/N^*$, $\tilde{X} = X/GN^*$. With these definitions, the factor mappings $\tilde{G}: \tilde{Z} \to \tilde{X}$, $\tilde{F}: \tilde{Z} \to \tilde{X}$, and $\tilde{H}: \tilde{Z} \to W$ (corresponding to G, F, and H

respectively) are well-defined, and the representation $(\tilde{F}, \tilde{G}, \tilde{H}; \tilde{Z}, \tilde{X}, \tilde{W})$ is a P_{min} representation that is equivalent to the given representation.

It has been proved in [56] (using somewhat different terminology) that this algorithm does indeed lead to a minimal representation.

There is a trivial way to pass from a general pencil representation to a general descriptor representation. If (F, G, H) is a P representation, and $H = [H_y^T \quad H_u^T]^T$ is the decomposition of H associated with a given partitioning of the external variables into inputs and outputs, then an equivalent D representation is obviously given by

$$\sigma \begin{bmatrix} G \\ 0 \end{bmatrix} \xi = \begin{bmatrix} F \\ H_u \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -I \end{bmatrix} u \tag{3.37}$$

$$y = H_y \xi. \tag{3.38}$$

The main virtue of this transformation is that it doesn't require computation. A transformation that does a better job at preserving minimality properties is given by the following algorithm.

ALGORITHM 5 Let $(F, G, H; Z, X, W)$ be a pencil representation, and let an i/o structure be given, so that $H = [H_y^T \quad H_u^T]^T$. Decompose the internal variable space Z as $Z_0 \oplus Z_1 \oplus Z_2$ where $Z_1 = \ker G \cap \ker H_u$, and $Z_1 \oplus Z_2 = \ker G$. Accordingly, write

$$G = [G_0 \quad 0 \quad 0], \qquad F = [F_0 \quad F_1 \quad F_2], \tag{3.39}$$

$$H_y = [H_{y0} \quad H_{y1} \quad H_{y2}], \qquad H_u = [H_{u0} \quad 0 \quad H_{u2}]. \tag{3.40}$$

The matrix H_{u2} has full column rank, and by renumbering the u-variables if necessary, we can write

$$H_{u0} = \begin{bmatrix} H_{10} \\ H_{20} \end{bmatrix}, \qquad H_{u2} = \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} \tag{3.41}$$

where H_{22} is invertible (or empty, if $\ker G \subset \ker H_u$). Define descriptor parameters by

$$E = \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} F_0 - F_2 H_{22}^{-1} H_{20} & F_1 \\ H_{10} - H_{12} H_{22}^{-1} H_{20} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & F_2 H_{22}^{-1} \\ -I & H_{12} H_{22}^{-1} \end{bmatrix},$$

$$C = [H_{y0} - H_{y2} H_{22}^{-1} H_{20} \quad H_{y1}], \qquad D = [0 \quad H_{y2} H_{22}^{-1} H_{20}]. \tag{3.42}$$

These parameters define a D representation without nondynamic variables that is externally equivalent to the original P representation. Moreover, if the given representation is of the P_{dv} (P_{io}, P_{min}) type, then the obtained representation is of the D_{ri} (D_{mi}, D_{min}) type.

The proofs of the statements above are given in [34]. At the 'P_{dv}' level and higher, it might be said that the algorithm in fact uses the driving-variable representation as an intermediate step, so that the DV representations fit into the picture as shown in Fig. 1. The converse transformation is obtained as follows.

ALGORITHM 6 Let a D_{ri} representation be given by (E, A, B, C, D) (so that $[E \quad B]$ is surjective). Choose coordinates in such a way that

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

$$C = [C_1 \quad C_2], \qquad D = [D_1 \quad D_2] \tag{3.43}$$

where B_{22} is invertible. Define matrices A', B', C', and D' by

$$\begin{bmatrix} sI - A' & -B' \\ C' & D' \end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 & 0 & B_{12}B_{22}^{-1} \\ 0 & I & 0 & -D_2B_{22}^{-1} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & B_{22}^{-1} \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & -B_{11} \\ C_1 & C_2 & D_1 \\ 0 & 0 & I \\ A_{21} & A_{22} & B_{21} \end{bmatrix}. \tag{3.44}$$

The DV representation given by the four-tuple (A', B', C', D') is externally equivalent to the given D representation. Moreover, if the given descriptor representation is in the D_{mi} ($\mathsf{D}_{\mathsf{min}}$) class, the resulting driving-variable representation is in the $\mathsf{DV}_{\mathsf{io}}$ ($\mathsf{DV}_{\mathsf{min}}$) class.

For a proof of these statements, see again [34]. More refined statements could be made; for instance, it is clear that to obtain a $\mathsf{DV}_{\mathsf{io}}$ representation from the algorithm above, it is sufficient that the D representation we start with is reachable at infinity and observable in the sense of Verghese.

The corresponding reduction to minimal form in the 'DV' branch can be thought of as a reformulation of the above in special coordinates. The details have been worked out in [56]. The reductions take a somewhat different form at the 'D' level. Verghese [60] already gave a simple algorithm to remove nondynamic variables. It has been shown in [33] (Lemma 7.3 and Lemma 7.4) how to reduce a given descriptor representation in case it does not satisfy either one of the conditions '$[E \quad B]$ surjective' or '$[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ injective'. Clearly, by repeating these reduction steps if necessary, it is always possible to arrive at a situation in which these conditions are satisfied. The final passage to $\mathsf{D}_{\mathsf{min}}$ comes down to removing the finite unobservable modes. This might for instance be done via reduction to the Weierstrass canonical form of the pencil $sE - A$ [26] followed by an application of the well-known procedure to remove unobservable modes in standard state space systems.

Finally, we come to the transformation from AR to $\mathsf{P}_{\mathsf{min}}$. This is essentially the Fuhrmann realization [23, 24]. In [33], the transformation is given the following form.

ALGORITHM 7 Let an AR representation be specified by $R(s)$. Consider the following spaces of rational vector functions in a formal parameter λ (π_- denotes projection onto the proper rational functions, W is the space of external variables, k is the number of rows of $R(s)$):

$$X^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid \pi_- R(\lambda)w(\lambda) = 0\} \tag{3.45}$$

$$X_R = \{p(\lambda) \in \mathbb{R}^k[\lambda] \mid \exists w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \text{ s.t. } p(\lambda) = R(\lambda)w(\lambda)\} \tag{3.46}$$

$$N^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid R(\lambda)w(\lambda) = 0\}. \tag{3.47}$$

The following mappings (G and F from $X^R/\lambda^{-1}N^R$ to X_R, H from $X^R/\lambda^{-1}N^R$ to W) are well-defined:

$$G: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)w(\lambda) \tag{3.48}$$

$$F: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)\pi_-(\lambda w(\lambda)) \tag{3.49}$$

$$H: w(\lambda) \bmod \lambda\ {}^{1}N^{R} \mapsto w_{-1}. \tag{3.50}$$

With these definitions, (F, G, H) is a minimal pencil representation that is externally equivalent to the AR representation given by $R(s)$.

This version differs from Fuhrmann's original one in two respects. First, the resulting representation is given in pencil form rather than in standard state space form, so that it becomes possible to consider noncausal systems. (The Fuhrmann realization has been used before in a noncausal context [14, 75], but only by separating finite and infinite frequencies, and under the assumption that a transfer matrix exists.) Secondly, the procedure is presented as one under external equivalence, rather than as one under transfer equivalence.

The transformation algorithm given above is abstract, and may be used very well in theoretical considerations. However, a more computational form can also be given (see [33, §8]). This requires the given representation to be in $\mathrm{AR}_{\mathrm{min}}$ form, and produces a representation in $\mathrm{DV}_{\mathrm{min}}$ form, which explains the arrow between the corresponding boxes in our map of linear system representations.

4. The factor system

In [67], J. C. Willems has pointed out that there is a close connection between the notion of an 'almost controlled invariant subspace' and that of a 'factor system'. Before discussing the connection, let us briefly recall what these two notions mean. To define the factor system, following the development in [65], let first X be a finite-dimensional vector space over $\mathbb{R}$. Also, let A be a linear mapping from X into itself, and let B be a linear mapping ranging in X. The *smooth system* $\Sigma(A, B)$ on X determined by A and B is the following set of C^{∞}-functions from $\mathbb{R}$ into X:

$$\Sigma(A, B) = \{x(\cdot) \in C^{\infty}(\mathbb{R}; X) \mid \dot{x}(t) - Ax(t) \in \operatorname{im} B \text{ for all } t\}. \tag{4.1}$$

Let Σ be a smooth system on X and let K be a subspace of X. Consider the following set of trajectories on the factor space X/K:

$$\Sigma/K := \{x(\cdot) \bmod K \mid x(\cdot) \in \Sigma\}. \tag{4.2}$$

If this set of trajectories is a smooth system on X/K, then Σ/K is called the *factor system* determined by Σ and K.

The notion of an almost controlled invariant subspace can be defined in the same context. So let us assume that a state space X, a state mapping A, and an input mapping B have been given. A subspace K is said to be *almost controlled invariant* [66] if for every $\epsilon > 0$ and for every x_0 in K there exists a trajectory $x(\cdot)$ in $\Sigma(A, B)$ such that $x(0) = x_0$ and $\operatorname{dist}(x(t), K) \leqslant \epsilon$ for all $t \geqslant 0$. This concept has many applications in control theory, of which some are reviewed in the contribution by J. L. Willems to this volume.

Given a smooth system $\Sigma(A, B)$, one would of course like to know under what conditions on K the set Σ/K is a factor system. It is claimed in [67] (Theorem A) that this will hold if and only if K is almost controlled invariant. In the cited paper, only a sketchy proof is provided for the 'if' part of this statement, and the 'only if' part is given without proof. Later on, a detailed proof of the 'if' part has been provided in [59], but a complete proof of the reverse implication is still lacking in the literature. Our goal in this section is to provide a short proof of Theorem A of [67], using a result in [56]. This proof is essentially based on manipulation of representations.

In the previous section, algorithms were presented for the removal of redundancies in pencil representations. In these algorithms, certain subspace recursions played a key role. We will also

need these recursions below, as well as some related recursions which we will introduce now. To the sequence of subspaces S^k defined by (3.26-3.27), another sequence $\hat{S}^k$ can be related by

$$\hat{S}^k = G^{-1}FS^k. \tag{4.3}$$

From (3.26-3.27), we see that this sequence might also be defined by the recursion

$$\hat{S}^0 = \ker G \tag{4.4}$$

$$\hat{S}^{k+1} = G^{-1}F[\hat{S}^k \cap \ker H]. \tag{4.5}$$

Denoting $\lim \hat{S}^k$ by $\hat{S}^*$, we also see from the definitions that $S^* = \hat{S}^* \cap \ker H$. It is furthermore useful to introduce two subspace recursions that do not take place in the 'internal variable space' Z but in the 'equation space' X. The first of these is obtained if we define

$$V^k = GN^k. \tag{4.6}$$

The corresponding recursion is

$$V^0 = X \tag{4.7}$$

$$V^{k+1} = G[F^{-1}V^k \cap \ker H]. \tag{4.8}$$

Similarly, we define

$$T^k = G\hat{S}^k \quad (= FS^k) \tag{4.9}$$

with the corresponding recursion

$$T^0 = \{0\} \tag{4.10}$$

$$T^{k+1} = F[G^{-1}T^k \cap \ker H]. \tag{4.11}$$

The limit subspaces resulting from these recursions will be denoted by V^* and T^*, respectively.

The subspaces that have now been introduced play a role in the characterization of some important system invariants in terms of the parameters in a P_{dv} representation. If $(F, G, H; Z, X, W)$ is a $\mathsf{P}_{\mathsf{min}}$ representation of a behavior $\mathscr{B}$, we define the *degree* of this behavior, to be denoted by $\deg(\mathscr{B})$, as $\dim X$. Also, we define the *order* of $\mathscr{B}$, to be denoted by $\operatorname{ord}(\mathscr{B})$, as $\dim Z$. Since a $\mathsf{P}_{\mathsf{min}}$ representation is determined up to isomorphisms of the internal variable space and the equation space, the degree and the order are clearly independent of the choice of a particular $\mathsf{P}_{\mathsf{min}}$ representation. There are of course many other equivalent characterizations; for instance, the degree is also equal to the sum of the row degrees of the matrix $R(s)$ in any $\mathsf{AR}_{\mathsf{min}}$ representation of $\mathscr{B}$, and to the dimension of the state space in any minimal state space representation of any causal input-output behavior that can be obtained from $\mathscr{B}$ by partitioning the external variables in inputs and outputs. (For a catalog of such results, see [69], Thm. 6.)

From the fact that the internal variable space in a $\mathsf{P}_{\mathsf{min}}$ representation is obtained from the internal variable space in a given P_{dv} representation by successively factoring out the subspaces S^* and N^*, it might be suspected that the degree is given in terms of a P_{dv} representation by $\operatorname{codim}(N^* + S^*)$. It has been established in [56] (Thm. 4.1) that this is indeed the case. The relevant result may be summarized, with some rephrasing, as follows.

PROPOSITION 4.1 *Let a behavior $\mathscr{B}$ be given by a P_{dv} representation $(F, G, H; Z, X, W)$. Define subspaces S^*, N^*, and $\hat{S}^*$ of Z, and subspaces V^* and T^* of X by the recursions (3.26-3.27), (3.35-3.36), (4.4-4.5), (4.7-4.8), and (4.10-4.11) respectively. We then have the following equalities:*

$$\deg(\mathscr{B}) = \operatorname{codim}(N^* + \hat{S}^*) = \operatorname{codim}(V^* + T^*) \tag{4.12}$$

$$\operatorname{ord}(\mathscr{B}) = \operatorname{codim}(N^* + S^*). \tag{4.13}$$

In case $\ker H$ *contains* $\ker G$*, an alternative formula for the order is*

$$\operatorname{ord}(\mathscr{B}) = \operatorname{codim}(V^* + (T^* \cap G(\ker H))). \tag{4.14}$$

Our next concern is to characterize a 'smooth system' in terms of system invariants. This is described in the following lemma.

LEMMA 4.2 *A linear time-invariant behavior* $\mathscr{B}$ *with external variable* w *has a representation in the form*

$$\sigma x = Ax + Bu \tag{4.15}$$

$$w = x \tag{4.16}$$

if and only if $\mathscr{B}$ *has no static constraints (i.e. for all* $w_0 \in W$ *there exists a* $w \in \mathscr{B}$ *such that* $w(0) = w_0$*), and* $\dim W$ *is equal to* $\operatorname{ord}(\mathscr{B})$.

PROOF Consider the 'if' part first. If $\dim W$ equals $\operatorname{ord}(\mathscr{B})$, then there exists a $\Gamma_{\ldots\ldots}$ representation

$$\sigma G\xi = F\xi \tag{4.17}$$

$$w = H\xi \tag{4.18}$$

in which the matrix H is square. From the requirement that $\mathscr{B}$ has no static constraints, it follows that H must be nonsingular. Let G^{-1} denote a right inverse of G, and let $\tilde{F}$ be a mapping satisfying $\operatorname{im}\tilde{F} = \ker G$. The equation (4.17) is then equivalent to

$$\sigma\xi = G^{-1}F\xi + \tilde{F}\eta \tag{4.19}$$

where η is a new internal variable. Using a nonsingular transformation of the ξ-variable, we can replace H by the identity mapping, and then the desired form is reached.

For the 'only if' part, we first note that the behavior defined by (4.15-4.16) has no static constraints. To determine the order of the behavior represented by (4.15-4.16), we have to take into account the fact that this representation is not minimal. Let T be any surjective mapping such that $\ker T = \operatorname{im} B$; then (4.15) is equivalent to

$$\sigma Tx = TAx. \tag{4.20}$$

Moreover, the representation (4.20-4.16) is minimal and we see that $\dim W$ is equal to $\operatorname{ord}(\mathscr{B})$, as claimed.

To obtain the main result of this section we combine the above characterization of smooth systems, the result that gives the order in terms of a P_{dv} representation, and a characterization of almost controlled invariant subspaces in terms of subspace recursions, taken from [66].

THEOREM 4.3 *Let a smooth system* $\Sigma(A, B; X)$ *be given, and let* K *be a subspace of* X*. Under these conditions, the set of trajectories of* Σ *modulo* K*,* Σ/K*, is a smooth system if and only if* K *is almost controlled invariant.*

PROOF Let $C: X \to X/K$ be the factor mapping. Obviously, a representation of the behavior Σ/K is given by

$$\sigma x = Ax + Bu \tag{4.21}$$

$$w = Cx \tag{4.22}$$

and so we have to find the conditions on K under which this is a smooth system. First of all, note that the behavior Σ / K can have no static constraints because otherwise the original system Σ would also have static constraints, which we know is not the case. Therefore, from the above lemma and the proposition we see that Σ / K is a smooth system if and only if

$$\dim X / K = \operatorname{codim}(V^* + (T^* \cap G(\ker H))) \tag{4.23}$$

where everything is taken with respect to the parameters

$$G = [I \quad 0], \qquad F = [A \quad B], \qquad H = [C \quad 0]. \tag{4.24}$$

(Note that indeed $\ker H$ contains $\ker G$, so that the above formula applies.) Rewriting the V^*- and T^*-algorithms for the above special values of the P_{dv} parameters while keeping in mind that $\ker C = K$, we obtain

$$V^0 = X \tag{4.25}$$

$$V^{k+1} = K \cap A^{-1}(V^k + \operatorname{im} B) \tag{4.26}$$

and

$$T^0 = \{0\} \tag{4.27}$$

$$T^{k+1} = A[T^k \cap K] + \operatorname{im} B. \tag{4.28}$$

The algorithm (4.25-4.26) is recognized as the *invariant subspace algorithm* [77, p. 91]. If we define $\hat{T}^k = T^k \cap K$, then the associated recursion is

$$\hat{T}^0 = \{0\} \tag{4.29}$$

$$\hat{T}^{k+1} = K \cap (\hat{A}T^k + \operatorname{im} B) \tag{4.30}$$

and this is recognized as the *controllability subspace algorithm* [77, p. 107], also known as the *almost controllability subspace algorithm* [66]. Noting that $G[\ker H] = \ker C = K$, we see that we always have

$$K \supset V^* + (T^* \cap K) \tag{4.31}$$

so that the condition (4.23) may be rewritten as

$$K = V^* + (T^* \cap K) = V^* + \hat{T}^*. \tag{4.32}$$

But this is exactly the condition given in [66] for a subspace K to be almost controlled invariant with respect to (A, B).

5. Conclusions

It should be emphasized that our 'road map' of system representations covers only a small area in the large field of representation theory. We have only been looking at the 'classical' form of external equivalence, thereby excluding representations such as the matrix fractional form over the ring of proper and stable rational functions, which is one of the main tools in the latest developments in control theory [21, 43]. Also, there are many other classes of systems for which representation theory leads to useful results. This of course includes the generalizations to nonlinear and infinite-dimensional systems, but important new aspects also arise if one considers systems with particular properties. A simple example is provided by the case of linear systems with a Hamiltonian or a gradient structure, such as appear in the modeling of mechanical structures and electrical networks. The problem of setting up state equations for such systems, starting from (higher-order) differential equations and algebraic constraint equations, is in fact a classical one. For a treatment following lines as presented here, see [57]. Of course, the Hamiltonian structure is important in the nonlinear context as well, and the problem of dealing with systems with mixed differential and algebraic equations comes up naturally for instance in setting up models for robots. For general nonlinear systems, the relations between systems of higher-order differential equations on the one hand and the standard state space form on the other have been widely discussed; an early reference is [22], and [15, 20, 55, 58] provide a sample of recent contributions. It has been shown in [54], a nonlinear system of algebraic and differential equations in a DV-type form can be reduced to a minimal representation in standard state space form if and only if certain integrability conditions are satisfied. In the nonlinear case, the partitioning of external variables into inputs and outputs to obtain a causal i/o structure is, in general, a local construction. This could be one of the reasons for interest in a nonlinear version of the pencil form. Such a nonlinear pencil form might be specified by giving a submanifold of the tangent bundle of a manifold of internal variables, plus a mapping from that manifold to the manifold of external variables.

Representation theory for *stochastic* systems is a very well developed subject. The richer structure of stochastic systems allows for a variety of representations, some of which are discussed in the contribution by J. H. van Schuppen to the present volume. However, it seems that not so much study has been made of questions concerning nonminimal representations, such as sometimes appear in modeling problems. As an example, consider an electrical network with linear elements containing some noisy resistors. Writing down network equations in the usual way, one could write down a representation in the form

$$G\dot{\xi} = F\xi + J\eta \tag{5.1}$$

$$w = H\xi \tag{5.2}$$

where η is 'white noise', and w represents the port variables. It requires proof to show that this can be rewritten in the standard form

$$\dot{x} = Ax + Bu + Nv \tag{5.3}$$

$$y = Cx + Du + Mv \tag{5.4}$$

where v is white noise, and w has been partitioned into inputs u and outputs y. Representation of stochastic systems is also the subject of debate in econometric circles (see for instance [3, 17]).

Some aspects of the representation of infinite-dimensional linear systems are discussed in the contribution of R. F. Curtain to this volume. A great deal of effort has been spent by the infinite-dimensional systems community on trying to fit into the standard (A, B, C, D) framework

equations like the following one (the normalized string equation with forces and displacements at both ends as external variables):

$$\frac{\partial^2}{\partial t^2}\phi(x, t) = \frac{\partial^2}{\partial x^2}\phi(x, t) \tag{5.5}$$

$$w(t) = \begin{bmatrix} \phi(0, t) \\ \phi(1, t) \\ \phi'(0, t) \\ \phi'(1, t) \end{bmatrix}. \tag{5.6}$$

(The variable x is used here as the spatial variable, and the prime denotes differentiation with respect to x.) Such an equation would fit more naturally into representations of the pencil type. This advantage doesn't come without a price, however; whereas standard semigroup theory is available for writing down solutions of the equations in (A, B, C, D) form, another route will have to be taken for systems in pencil form. Nevertheless, it would seem to be worth the effort to pursue this direction. It should be noted that a representation which easily incorporates equations like the string equation above has been proposed by D. Salamon under the name 'boundary control systems' [53]; however, this class was introduced by Salamon for specific purposes, and the restrictions he imposes are consequently more severe than one would like to see in a pencil representation.

The theory of system representations can be viewed as a theory of modeling. System-theoretic ideas may be applied to modeling problems as well as to control problems, and it may even be that some problems that are now considered as control problems will eventually be looked at rather as representation problems (model matching might fall in this category). In the process, it may be necessary to abandon some conventional wisdom. This paper has been written as a tribute to Jan Willems, one of the best abandoners of conventional thinking that I know.

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