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CLINES IN THE PRESENCE OF ASYMMETRIC MIGRATION

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Clines in the presence of asymmetric migration*)

by

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ABSTRACT

If a population, which consists of individuals having genetic variation at one locus, with two alleles A and a, evolves under the influence of migration and selection, gradients in the distribution of alleles may arise. We consider the effect of asymmetry in the migration, and spatial dependence of the selection process, upon the emergence and stability of such gradients.

KEY WORDS & PHRASES: *clines, population genetics, nonlinear diffusion problems, stability*

*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

Consider a population, distributed over a habitat containing genetic variation at one locus with two alleles, A and a. Then the possible genotypes are AA, Aa and aa. If the individuals of one genotype enjoy a selective advantage in one part of the habitat and a disadvantage elsewhere, it may happen that due to the combined effects of migration and selection a gradient in the frequency with which one allele occurs in the population is established. Huxley [13] first used the word *cline* for a gradient in phenotype. In the present context we shall use this word for a gradient of the frequency of alleles.

The occurrence of clines in habitats in which selection varies from one part to another, was first studied mathematically by Haldane [12]. His work was based on a model proposed by Fisher [10], to describe the effects of migration and selection on the evolution of a population.

Following Haldane, we assume that the habitat Ω is effectively one-dimensional (as may be the case along a river bank), and we define the position of a point in the habitat by a scalar variable $x \in \Omega$. Let $u(x,t)$ denote the fraction of alleles of type a amongst the total number of alleles in the population at the point x in the habitat, at time t. Then it was shown by Haldane that if the migration is independent of the genotype the evolution of u with time can be described by the equation

$$(1.1) \qquad u_t = u_{xx} + f(x,u) \qquad x \in \Omega,\ t > 0,$$

in which subscripts denote differentiation. The change in u due to the migration of individuals is represented by the term u_{xx}. It is obtained by drawing an analogy between the movement of individuals and the movement of particles in a diffusion process. The change in u due to selection is given by the function f, which is derived from the deathrates of the three genotypes; the variable x reflects the inhomogeneity of the habitat.

In deriving (1.1) Haldane assumed that migration was random and without preferential direction, and therefore symmetric in x. However, situations exist in which this is not so. For instance, a gradient in the suitability of the habitat due to the availability of food or existing temperatures may cause migration to be asymmetric. For plants, asymmetry in the migration may be caused by prevailing winds or the movement of pollinating insects, etc. It was shown by Nagylaki [16] that if migration is allowed to be asymmetric, but still independent of genotype, equation (1.1) must be modified as follows:

$$(1.2) \qquad u_t = u_{xx} + mu_x + f(x,u) \qquad x \in \Omega,\ t > 0.$$

Here m is a constant, which can be regarded as a measure for the asymmetry in the migration.

Since Haldane, equation (1.1) has been studied by Nagylaki [15,16], Conley [5], Fleming [11], Fife and Peletier [8,9], Anderson [1], Peletier [14] and Saut and Scheurer [20]. This has led to an understanding of the conditions on Ω and f which ensure the existence, monotonicity, uniqueness and stability of clines in the presence of symmetric migration. In this paper we shall extend a number of these results to situations where migration is not symmetric. In order to focus attention on the rôle played by m and f we shall only consider clines in a habitat which is unbounded at both ends, i.e. we choose $\Omega = \mathbb{R}$.

The first mathematical study of clines in the presence of asymmetric migration was carried out by Nagylaki [16]. He investigated the ability of an environmental pocket - an area where one genotype enjoys an advantage, in an otherwise hostile environment - to sustain a cline.

To interpret our results, it is instructive first to consider the effects of selection and *symmetric* migration in a spatially *uniform* habitat. For this situation, (1.1) becomes

$$(1.3) \qquad u_t = u_{xx} + f(u) \qquad x \in \mathbb{R},\ t > 0$$

where f(0) = f(1) = 0. It is well known that this equation may have a solution of the form

$$(1.4) \qquad u(x,t) = \phi(x-ct),$$

where the function $\phi(z)$ increases monotonically from $\phi = 0$ at $z = -\infty$ to $\phi = 1$ at $z = \infty$, and $c = c^*(f)$, where c^* is a number associated with f. This solution represents a wave with a constant profile, which moves with a constant speed c through the habitat. Ahead of the wave, the population consists entirely of one of the homozygotes, and behind it, it consists exclusively of the other.

If we now consider *asymmetric* migration, and add the term mu_x to the right hand side of (1.3), the function $\phi(x-ct)$ is still a solution, but now the wave speed c has changed into

$$c = c^*(f) - m,$$

i.e. the wave moves forward if $c^* > m$ and backward if $c^* < m$. Thus, the selection process- which determines c^*- and the drift in the migration, measured by m, can be regarded as being in competition with each other.

Let us finally return to a habitat in which the selection process depends on the location. We shall usually assume that

$$(1.5) \qquad \lim_{x \to -\infty} f(x,u) = f^-(u) \quad \text{and} \quad \lim_{x \to \infty} f(x,u) = f^+(u).$$

Suppose now that

$$(1.6) \qquad c^*(f^+) < m < c^*(f^-).$$

Then far to the left a wave like the one given by (1.4) would move forward and far to the right, such a wave would move backward (Fig.1).

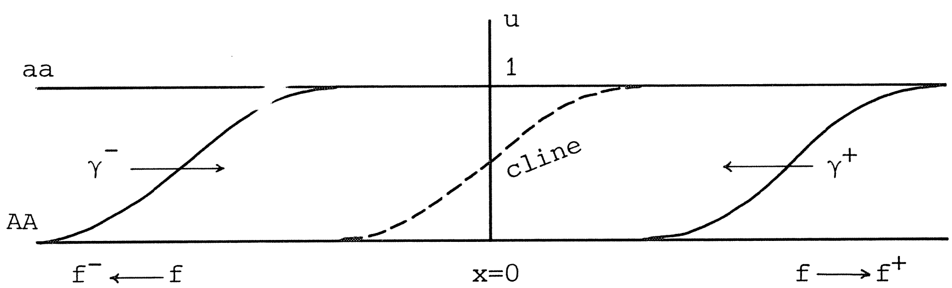

Fig. 1. $\gamma^- = c^*(f^-)-m$; $\gamma^+ = c^*(f^+)-m$.

We shall find that (1.5) and (1.6) are essentially the conditions which insure the existence of a cline. In addition it will appear that for a large class of functions f, the condition on m is both necessary and sufficient.

It is interesting to observe here that the sign of $c^*(f^-)$ and $c^*(f^+)$ does not enter into the conditions for existence. Thus it can happen that although one allele is favoured in the entire habitat ($c^*(f^-)$ and $c^*(f^+)$ have the same sign), a cline is established as a consequence of the asymmetry of the migration.

The plan of the paper is the following. In section 2 we shall collect and, where necessary, derive some results about equilibrium solutions of equation (1.2), when f does not depend explicitly on x. These results are used in section 3 to obtain conditions for the existence, monotonicity and uniqueness of clines. This section closes with a theorem giving conditions on f and m, which exclude the existence of clines.

Finally, in section 4 we turn to the question of stability. Let $\phi(x)$ be the cline we found in section 3, and let

$$u = u(x,t;\psi) \qquad x \in \mathbb{R},\ t \geq 0$$

be the frequency profile, which evolves from a given initial profile

$$u(x,0;\psi) = \psi(x) \qquad x \in \mathbb{R}.$$

Then we shall obtain criteria for $\psi(x)$, which guarantee that

$$u(x,t;\psi) \to \phi(x) \text{ as } t \to \infty,\ x \in \mathbb{R}.$$

In particular, we shall show that if $|\psi(x) - \phi(x)|$ is sufficiently small for every $x \in \mathbb{R}$ then $u(x,t;\psi)$ converges exponentially towards $\phi(x)$ as $t \to \infty$ for every $x \in \mathbb{R}$. Thus, if for some reason, the frequency profile of the cline is slightly disturbed, the mechanisms of migration and selection, as described in this model, will tend to restore the profile of the original cline.

The presentation of the results, in particular their accessibility to biologists, owes much to the careful reading of the manuscript by Dr. M.S. Knaap. It is a pleasure to thank her.

2. PRELIMINARIES

In this section we consider the autonomous problem

$$\text{(I)}\quad \begin{cases} u'' + cu' + f(u) = 0 & x \in \mathbb{R} \quad (2.1a) \\ u(-\infty) = 0,\ u(+\infty) = 1 & (2.1b) \end{cases}$$

in which we shall make the following assumptions about the function $f:[0,1] \to \mathbb{R}$:

A1. $f \in C^1([0,1])$;

A2. $f(0) = 0$ and $f(1) = 0$;

A3. f satisfies one of the following sets of conditions:

I. $\exists a \in (0,1)$ such that

$$f(u) < 0 \text{ on } (0,a), \quad f(u) > 0 \text{ on } (a,1),$$

$$f'(0) < 0 \quad \text{and} \quad f'(1) < 0.$$

The set of functions f which satisfy these conditions will be denoted by F_1.

II. $f(u) > 0$ on $(0,1)$ and $f'(0) > 0$, $f'(1) < 0$.

The set of such functions f will be denoted by F_2.

III. $f(u) < 0$ on $(0,1)$ and $f'(0) < 0$, $f'(1) > 0$.

The set of such functions f will be denoted by F_3.

Note that if in Problem I, $f \in F_3$, then we can transform Problem I by replacing u by $1-\tilde{u}$, x by $-\tilde{x}$ and $-f(1-\tilde{u})$ by $\tilde{f}(\tilde{u})$ to one in which $\tilde{f} \in F_2$.

Problem I has been the subject of many studies. Below we shall summarize, adapt and generalize a few results, drawing mainly upon the work of Aronson and Weinberger [2,3], Fife and McLeod [7] and Fife [6]. In particular, we shall discuss the dependence of the solution of Problem I, and the set of values of c for which a solution exists, on the function f.

Equation (2.1a) can be written as the system

$$\begin{cases} \dfrac{du}{dx} = p \\ \\ \dfrac{dp}{dx} = -cp - f(u) \end{cases}$$

and a solution u of Problem I can be viewed as an orbit in the (u,p)-plane connecting the critical points (0,0) and (1,0). Since any solution of Problem I is strictly increasing [7], we may introduce u as an independent variable. This leads to the problem

$$(2.2) \qquad (I_0) \quad \begin{cases} p' + \dfrac{f(u)}{p} + c = 0 & 0 < u < 1 \\ \\ p(0) = 0. \end{cases}$$

It can be shown that the problem: find a solution p(u) of Problem I_0 such that p(1) = 0 is equivalent with Problem I [7].

Following Aronson and Weinberger [3] we consider the solution $p_c(u;\nu)$ of the regular problem

$$(I_\nu) \quad \begin{cases} p' + \dfrac{f(u)}{p} + c = 0 & u > 0 \\ \\ p(0) = \nu > 0 \end{cases}$$

in a neighbourhood of the point $(0,\nu)$. We can continue $p_c(u;\nu)$ for $u > 0$ as long as $p_c(u;\nu) > 0$. This defines p_c on an interval $[0,u_{c,\nu})$. Since we wish to define p_c on $[0,1]$, we set $p_c(u;\nu) = 0$ on $[u_{c,\nu},1]$ in the event that $u_{c,\nu} \le 1$. Now we let $\nu \downarrow 0$. Then for each $u \in [0,1]$, $p_c(u;\nu)$ decreases monotonically. Moreover $p_c(u;\nu) \ge 0$ for all $\nu > 0$ and $u \in [0,1]$. Hence the limit

$$p_c(u) = \lim_{\nu \downarrow 0} p_c(u;\nu) \qquad 0 \le u \le 1$$

exists. It can be shown that if $p_c(u) > 0$ on $(0,\alpha)$ for some $\alpha \in (0,1]$, then $p_c(u)$ is the maximal solution of Problem I_0. Still following Aronson and Weinberger [3] we now define the set

$$T_c = \{(u,p): 0 < u < 1,\ p > 0,\ \ p = p_c(u)\}.$$

Clearly it may happen that $p_c(u) \equiv 0$ on $[0,1]$. In that case $T_c = \emptyset$. Define

$$D_0(f) = \{c \in \mathbb{R} : T_c \neq \emptyset).$$

One obtains from the standard theory of ordinary differential equations that if $f \in \mathcal{F}_1 \cup \mathcal{F}_3$, $D_0(f) = \mathbb{R}$, and that if $f \in \mathcal{F}_2$, $D_0(f) = (-\infty, -2\{f'(0)\}^{\frac{1}{2}}]$.

Next, we define the set

$$K_0(f) = \{c \in D_0(f): p_c(u) > 0 \text{ on } (0,1]\}.$$

This set is nonempty because $c \in K_0(f)$ if $-c$ is large enough [3], and it is bounded above as we shall show later. In addition we define

$$c_0(f) = \sup\{c: c \in K_0(f)\}.$$

If $f \in \mathcal{F}_2$, $D_0(f)$ is a proper subset of $\mathbb{R}$ and we have two possibilities:

(a) $c_0(f) < -2\{f'(0)\}^{\frac{1}{2}}$. In this case $p_{c_0}(u)$ vanishes at some point $u \in (0,1]$.

(b) $c_0(f) = -2\{f'(0)\}^{\frac{1}{2}}$. In this case $p_{c_0}(u)$ may or may not vanish at some point $u \in (0,1]$.

We shall denote the set of functions $f \in \mathcal{F}_2$, for which possibility (a) holds by $\mathcal{F}_{2a}$ and the set for which possibility (b) holds by $\mathcal{F}_{2b}$.

LEMMA 2.1. (i) $(-\infty, c_0(f)) \subset K_0(f)$, (ii) $p_c(u)$ *depends continuously on* c *for* $c \in K_0(f)$.

PROOF. (i) Let $\tilde{c} \in K_0(f)$ and let $c < \tilde{c}$. Then $c \in D_0(f)$ and, because $p_{\tilde{c}}(u) > 0$ on $(0,1]$, it follows from Lemma 2.5 of [7] that

$$p_c(u) \geq p_{\tilde{c}}(u) \quad \text{for } 0 \leq u \leq 1.$$

Thus $p_c(u) > 0$ on $(0,1]$, whence $c \in K_0(f)$, i.e. $(-\infty, \tilde{c}] \subset K_0(f)$. The result now follows from the definition of c_0.

(ii) This can be proved as in Proposition 4.5 of [3]

LEMMA 2.2. *If* $c = c_0(f)$, *Problem I has a strictly increasing solution.*

PROOF. By definition there exists a sequence $\{c_n\} \subset K_0(f)$ such that $c_n \uparrow c_0$ as $n \to \infty$. If $c_n \leq c_m$, then by Lemma 2.5 of [7]

$$p_{c_n}(u) \geq p_{c_m}(u) \quad \text{for } 0 \leq u \leq 1,$$

whence $\{p_{c_n}\}$ is a nonincreasing sequence. Since $p_{c_n} \geq 0$ for all $n \geq 1$ on $[0,1]$, we may define a function

$$q(u) = \lim_{n \to \infty} p_{c_n}(u) \quad \text{for } 0 \leq u \leq 1.$$

Moreover q is a solution of the problem

$$q' + \frac{f(u)}{q} + c_0 = 0 \qquad q(0) = 0$$

on any right-neighbourhood of $u = 0$ in which $q > 0$.

We distinguish two cases:

(i) $f \in \mathcal{F}_1 \cup \mathcal{F}_{2a} \cup \mathcal{F}_3$. By the definition of c_0, and the continuous dependence of $p_c(u)$ on c, for $c \in K_0(f)$, there exists an $\alpha \in (0,1]$ such that

$$q(\alpha) = 0, \quad q(u) > 0 \text{ for } 0 < u < \alpha.$$

As in the proof of Theorem 4.1 of [3] one can show that $\alpha = 1$.

(ii) $f \in \mathcal{F}_{2b}$. Since $p_{c_n}(u) \geq 0$ on $[0,1]$ for all $n \geq 1$, it follows that

$$q(u) \geq 0 \quad \text{for } 0 \leq u \leq 1.$$

If $q(\alpha) = 0$ for some $\alpha \in (0,1]$, we can complete the proof as in case (i).

Thus, assume that

$$q(u) > 0 \text{ for } 0 < u \leq 1.$$

Then the unique orbit $(\tilde{u},\tilde{p})$ which approaches the singular point (1,0) through the set

$$S = \{(u,p): 0 < u < 1,\ p > 0\}$$

can only enter S from the singular point (0,0). This establishes the existence of an orbit connecting (0,0) and (1,0) and hence, of a solution of Problem I.

<u>REMARK</u>. Let $\phi(x)$ be a solution of Problem I, in which $c = c_0$. Then the function $u(x,t) \equiv \phi(x-c_0t)$ is a travelling wave solution of the equation

$$u_t = u_{xx} + f(u)$$

with wave speed c_0. If $f \in F_1 \cup F_3$ then $D_0(f) = \mathbb{R}$ and if $f \in F_2$, then $D_0(f) = (-\infty, 2\{f'(0)\}^{\frac{1}{2}}]$. Following Stokes [21] we say that if $c_0 < -2\{f'(0)\}^{\frac{1}{2}}$, i.e. $f \in F_{2a}$, the corresponding wave ϕ if a *pushed* wave, and if $c_0 = -2\{f'(0)\}^{\frac{1}{2}}$, i.e. $f \in F_{2b}$, ϕ is a *pulled* wave.

Next, we consider the problem

$$(\bar{I}_0) \qquad p' + \frac{f(u)}{p} + c = 0 \qquad\qquad p(1) = 0$$

and define the function $\bar{p}_c(u)$ on [0,1] as the limit of solutions $\bar{p}_c(u;\nu)$ of the problem

$$(\bar{I}_\nu) \quad p' + \frac{f(u)}{p} + c = 0 \qquad p(1) = \nu,$$

where $\nu > 0$. Then $\bar{p}_c(u)$ is the maximal solution of Problem $\bar{I}_0$. Proceeding as in Problem I_0, we define

$$\bar{T}_c = \{(u,p): 0 < u < 1,\ \ p > 0,\ \ p = \bar{p}_c(u)\}$$

$$D_1(f) = \{c \in \mathbb{R} : \bar{T}_c \neq \emptyset\}$$
$$K_1(f) = \{c \in D_1(f) : \bar{p}_c(u) > 0 \text{ on } [0,1)\}$$
$$c_1(f) = \inf\{c : c \in K_1(f)\}.$$

If $f \in F_1 \cup F_2$, then $D_1(f) = \mathbb{R}$, but if $f \in F_3$, $D_1(f) = [2\{f'(1)\}^{\frac{1}{2}},\infty)$, and we can distinguish two possibilities:

(a) $c_1(f) > 2\{f'(1)\}^{\frac{1}{2}}$. In this case $\bar{p}_{c_1}(u)$ vanishes at some point $u \in [0,1)$.

(b) $c_1(f) = 2\{f'(1)\}^{\frac{1}{2}}$. In this case $\bar{p}_{c_1}(u)$ may or may not vanish at some point $u \in [0,1)$.

In case (a), we say that $f \in F_{3a}$ and in case (b) we say that $f \in F_{3b}$.

In a manner, entirely analogous to the one used to prove Lemma's 2.1 and 2.2 we prove

LEMMA 2.3. (i) $(c_1(f),\infty) \subset K_1(f)$, (ii) $\bar{p}_c(n)$ *depends continuously on* c *for* $c \in K_1(f)$.

LEMMA 2.4. *If* $c = c_1(f)$, *Problem I has a strictly increasing solution.*

Finally, we relate the two wave speeds c_0 and c_1. Suppose $d \in K_0(f)$. Then, because trajectories in the phase portrait cannot intersect, $d \notin K_1(f)$. Thus, $K_0(f) \cap K_1(f) = \emptyset$, whence, by Lemma's 2.1 and 2.3:

$$c_0 \leq c_1.$$

In fact $c_0 = c_1$. This is the content of the next Lemma.

LEMMA 2.5. $c_0 = c_1$.

PROOF. If $f \in F_1$, the equality for c_0 and c_1 follows from the uniqueness of the traveling wave solution [7].

Suppose $f \in F_3$. For convenience we write $p_{c_0} = p_0$ and $\bar{p}_{c_1} = p_1$. Then

$$p_0(0) = 0, \qquad p_0(1) = 0$$

and

$$p_1(0) \geq 0, \qquad p_1(1) = 0.$$

Suppose that $c_0 < c_1$. Then if $p_1(0) = 0$, it follows from Lemma 2.2 [7] that $p_0(u) > p_1(u)$ on (0.1). However, a local analysis near the singular point (1,0) reveals that $p_0(u) < p_1(u)$ in a left neighbourhood of $u = 1$, whence we have a contradiction.

On the other hand, if $p_1(0) > 0$, then $f \in \mathcal{F}_{3b}$, i.e. $c_1 = 2\{f'(1)\}^{\frac{1}{2}}$. In order to have an orbit (u,p_0) connecting (0,0) and (1,0), (1,0) must be a node. Since $c_0 < c_1$ this is only possible if $c_0 \leq -2\{f'(1)\}^{\frac{1}{2}}$. However, for such values of c, the principal directions of (1,0) do not point into the set S, whence we have a contradiction.

Next, suppose $f \in \mathcal{F}_2$. By defining the variables $\tilde{u} = 1-u$, $\tilde{p}(\tilde{u}) = p(1-\tilde{u})$, $\tilde{f}(\tilde{u}) = -f(1-\tilde{u})$, $\tilde{c} = -c$, equation (2.2) becomes

$$\tilde{p}' + \frac{\tilde{f}(\tilde{u})}{\tilde{p}} + \tilde{c} = 0.$$

Note that $\tilde{f} \in \mathcal{F}_3$ and

$$d \in K_0(f) \Longleftrightarrow -d \in K_1(\tilde{f})$$
$$d \in K_1(f) \Longleftrightarrow -d \in K_0(\tilde{f}).$$

Hence

$$c_0(\tilde{f}) = -c_1(f) \text{ and } c_1(\tilde{f}) = -c_0(f).$$

i.e. $c_0(f) < c_1(f)$ implies $c_0(\tilde{f}) < c_1(\tilde{f})$. Thus, we are back at the case handled above. This completes the proof.

Henceforth we shall write

$$c_0(f) = c_1(f) = c^*(f).$$

LEMMA 2.6a. *Let p_1 and p_2 be the maximal solutions of the problems*

$$(2.3) \qquad p' + \frac{f_i(u)}{p} + c_i = 0 \quad p(0) = 0 \qquad i = 1,2$$

and let $p_i > 0$ *on* $(0,1)$. *Suppose* $f_1 \le f_2$ *on* $(0,1)$ *and* $c_1 \le c_2$. *Then if* $f_2 \in F_1 \cup F_2$,

(i) $p_1 \ge p_2$ *on* $[0,1]$;

(ii) *if* $f_1 < f_2$ *on* $(\alpha,\beta) \subset (0,1)$, *then* $p_1 > p_2$ *on* $(\alpha,1]$;

(iii) *if* $c_1 < c_2$, *then* $p_1 > p_2$ *on* $(0,1]$.

PROOF. (i) Let $\nu > 0$ and let $p_{1,\nu}$ be the solution of the problem

$$p' + \frac{f_1(u)}{p} + c_1 - \nu = 0, \qquad p(0) = \nu.$$

Then by Lemma 4.1 of [3]

$$p_{1,\nu} \ge p_2 \text{ on } [0,1].$$

Now let $\nu \downarrow 0$. Then $p_{1,\nu} \downarrow p_1$ and hence

$$p_1 \ge p_2 \text{ on } [0,1].$$

(ii) and (iii). Write $z = p_1 - p_2$. Then

$$z' - \frac{f_2}{p_1p_2} z = \frac{f_2 - f_1}{p_1} + c_2 - c_1. \tag{2.4}$$

Choose $u_0 \in (0,1)$, and define

$$\Phi(u) = z(u)\exp \int_{u_0}^{u} \{-f_2(t)/p_1(t)p_2(t)\}dt. \tag{2.5}$$

Then

$$\Phi'(u) = \left[\frac{f_2(u) - f_1(u)}{p_1(u)} + c_2-c_1\right] \exp \int_{u_0}^{u} \{-f_2(t)/p_1(t)p_2(t)\}dt.$$

By (i)

$$\Phi' \ge 0 \text{ on } (0,1)$$

and

$$\Phi' > 0 \text{ on } (\alpha,\beta) \qquad \text{in case (ii)},$$

$$\Phi' > 0 \text{ on } (0,1) \qquad \text{in case (iii)}.$$

Thus

$$\Phi(u) > 0 \text{ on } (\alpha,1].$$

Since $f_2(u) > 0$ near $u = 1$, the factor of $z(u)$ in (2.5) is bounded, whence $z(u) > 0$ on $(\alpha,1]$.

LEMMA 2.6b. *Let* p_1 *and* p_2 *be the maximal solutions of the problems*

$$p' + \frac{f_i(u)}{p} + c_i = 0 \qquad p(1) = 0 \qquad i = 1,2.$$

and let $p_i > 0$ *on* $(0,1)$. *Suppose* $f_1 \le f_2$ *on* $(0,1)$ *and* $c_1 \le c_2$. *Then if* $f_1 \in \mathcal{F}_1 \cup \mathcal{F}_3$,

(i) $p_1 \le p_2$ *on* $[0,1]$;

(ii) *if* $f_1 < f_2$ *on* $(\alpha,\beta) \subset (0,1)$ *then* $p_1 < p_2$ *on* $[0,\beta)$;

(iii) *if* $c_1 < c_2$ *then* $p_1 < p_2$ *on* $[0,1)$.

PROOF. The transformation $\tilde{u} = 1-u$, $\tilde{p}(\tilde{u}) = p(1-\tilde{u})$, $\tilde{f}_1(\tilde{u}) = -f_i(1-\tilde{u})$, yields

$$\tilde{p}' + \frac{\tilde{f}_i(\tilde{u})}{\tilde{p}} - c_i = 0 \qquad \tilde{p}(0) = 0.$$

Since $f_1 \in \mathcal{F}_1 \cup \mathcal{F}_2$, we may now apply Lemma 2.6a to obtain the desired result.

LEMMA 2.7. *Suppose* $f_1 \le f_2$ *on* $[0,1]$ *and* $f_1 < f_2$ *on some interval* $(\alpha,\beta) \subset (0,1)$.

(a) *Let* $f_i \in F_1 \cup F_2$, *then*

(a1) *if* $f_2 \notin F_{2b}$, $c^*(f_1) > c^*(f_2)$;

(a2) *if* $f_2 \in F_{2b}$, $c^*(f_1) \ge c^*(f_2)$, *and, if in addition,* $f_1'(0) < f_2'(0)$, *we have strict inequality.*

(b) *Let* $f_i \in F_1 \cup F_3$, *then*

(b1) *if* $f_1 \notin F_{3b}$, $c^*(f_1) > c^*(f_2)$;

(b2) *if* $f_1 \in F_{3b}$, $c^*(f_1) \ge c^*(f_2)$, *and, if in addition,* $f_1'(1) > f_2'(1)$, *we have strict inequality.*

(c) *Let* $f_1 \in F_3$ *and* $f_2 \in F_2$. *Then* $c^*(f_1) > c^*(f_2)$.

PROOF. (a1). Let p_1 and p_2 be the maximal solutions of Problem I_0, corresponding to respectively f_1 and f_2 and $c^*(f_1)$ and $c^*(f_2)$. Then if $f_1 \notin F_{2b}$, $p_1(1) = p_2(1) = 0$. However by Lemma 2.6a (ii), $p_1(1) > p_2(1)$, i.e. we have a contradiction. If $f_1 \in F_{2b}$, we have by definition

$$c^*(f_1) = -2\{f_1'(0)\}^{\frac{1}{2}} \ge -2\{f_2'(0)\}^{\frac{1}{2}} > c^*(f_2).$$

(a2) In this case

$$c^*(f_1) = -2\{f_1'(0)\}^{\frac{1}{2}} \ge -2\{f_2'(0)\}^{\frac{1}{2}} \ge c^*(f_2).$$

Clearly, if $f_1'(0) < f_2'(0)$, we have strict inequality. Parts (b) and (c) are proved similarly.

LEMMA 2.8. *Let* $\{f_n\} \subset C^1([0,1])$ *be a nonincreasing (nondecreasing) sequence of functions satisfying* A1 - A3, *which converges in* $C^1([0,1])$ *to a function* $f \in C^1([0,1])$ *which also satisfies* A1 - A3. *Then* $c^*(f_n) \uparrow c^*(f)$ $(c^*(f_n) \downarrow c^*(f))$ *as* $n \to \infty$.

PROOF. Suppose that $\{f_n\}$ is nonincreasing. Then by Lemma 2.7,

$$c^*(f_n) \le c^*(f_{n+1}) \le c^*(f) \text{ for all } n \ge 1.$$

Thus, $c^*(f_n) \uparrow \bar{c}$, where $\bar{c} \le c^*(f)$, and it remains to prove that $\bar{c} = c^*(f)$.

Suppose to the contrary that $\bar{c} < c^*(f)$. If $f \in F_1 \cup F_2$ it follows from Lemma 2.1 that $\bar{c} \in K_0(f)$, whence the maximal solution $p_{\bar{c}}(u,f)$ of Problem I_0, belonging to $\bar{c}$ and f satisfies

(2.6) $$p_{\bar{c}}(u,f) > 0 \text{ for } 0 < u \le 1.$$

As in Proposition 4.5 of [3] it can be shown that $p_c(u,f)$ depends continuously on f in the C^1-topology and on c. Hence (2.6) implies that for n large enough,

(2.7) $$p_{c^*(f_n)}(u,f_n) > 0 \text{ for } 0 < u \le 1.$$

Suppose $f \in F_1 \cup F_{2a}$. Then we assert that $f_n \in F_1 \cup F_{2a}$ for n large enough. If $f \in F_1$, this is plainly true. If $f \in F_{2a}$, it follows from the fact that if, to the contrary there exists a sequence $\{f_\mu\} \subset \{f_n\}$ such that $f_\mu \in F_{2b}$ for every $\mu \ge 1$, then on the one hand

$$c^*(f_\mu) = -2\{f_\mu'(0)\}^{\frac{1}{2}} \to -2\{f'(0)\}^{\frac{1}{2}} \text{ as } \mu \to \infty,$$

but on the other:

$$c^*(f_\mu) \uparrow \bar{c} \le c^*(f) < -2\{f'(0)\}^{\frac{1}{2}}.$$

This contradiction proves the assertion. But if $f_n \in F_1 \cup F_{2a}$,

$$p_{c^*(f_n)}(1,f_n) = 0$$

which contradicts (2.7).

Next suppose that $f \in F_{2b}$. We now distinguish two cases:

(i) $f_n \in F_{2b}$ for n large enough;

(ii) there exists a sequence $\{f_\mu\} \subset \{f_n\}$ such that $f_\mu \in F_{2a}$ for all $\mu \ge 1$.

In case (i)

$$\bar{c} = \lim_{n\to\infty} c^*(f_n) = -2 \lim_{n\to\infty} \{f_n'(0)\}^{\frac{1}{2}} = -2\{f'(0)\}^{\frac{1}{2}} = c^*(f)$$

which contradicts the assumption $\bar{c} < c^*(f)$.

In case (ii)

$$p_{c^*(f_\mu)}(1,f_\mu) = 0 \qquad \text{for all } \mu \geq 1,$$

which contradicts (2.7).

If $f \in F_3$, we find that the solution q of the problem

$$q' + \frac{g(u)}{q} + c = 0 \qquad q(0) = 0$$

is positive on (0,1] if $g = f$ and $c = \bar{c}$, and depends continuously on g and c, when c is near $\bar{c}$. Thus, if $g = f_n$, $c = c_n$ and n is large enough, the corresponding solution q_n is also positive on (0,1]. However, $q_n(1) = 0$ for all $n \geq 1$, i.e. we have a contradiction.

3. EXISTENCE, UNIQUENESS AND MONOTONICITY OF CLINES

We now turn to the study of equilibrium solutions of equation (1.2). Specifically we consider the problem

$$\text{(3.1a)} \quad \text{(II)} \begin{cases} u'' + mu' + f(x,u) = 0 & x \in \mathbb{R} \\ u(-\infty) = 0, \quad u(+\infty) = 1. & \end{cases} \quad \text{(3.1b)}$$

About the function $f: \mathbb{R} \times [0,1] \to \mathbb{R}$ in this problem we shall make the following hypotheses.

H1. f has continuous derivatives f_x and f_u and is, together with f_x and f_u, uniformly bounded in $\mathbb{R} \times [0,1]$.

H2. $f(x,0) = f(x,1) = 0$ for all $x \in \mathbb{R}$.

H3. There exist functions $f^+, f^-: [0,1] \to \mathbb{R}$ which satisfy the assumptions A1 - A3, and a constant $N > 0$ such that

$$f(x,u) \le f^-(u) \qquad \text{for } x \le -N, \quad 0 \le u \le 1,$$

$$f(x,u) \ge f^+(u) \qquad \text{for } x \ge N, \quad 0 \le u \le 1.$$

We shall write

$$c^- = c^*(f^-) \text{ and } c^+ = c^*(f^+).$$

<u>THEOREM 3.1</u>. *Let* f *satisfy the hypotheses* H1 - H3, *and let* $c^+ < c^-$. *Then for each* $m \in (c^+,c^-)$ *there exists a solution of Problem II.*

<u>REMARK</u>. Observe that in [8], where $m = 0$, the conditions on f were such that functions f^+ and f^- could be found such that $c^+ < 0 < c^-$.

<u>PROOF</u>. As in [8] we establish the existence of a solution by constructing a super solution $\bar{u}$ and a sub solution $\underline{u}$ such that $\bar{u} > \underline{u}$ on $\mathbb{R}$.

Since $m < c^-$, it follows from Lemma 2.1 that $m \in K_0(f^-)$. Hence there exists a function ϕ: $(-\infty,-N] \to (0,1]$ such that

$$\begin{cases} \phi'' + m\phi' + f^-(\phi) = 0 & -\infty < x < -N \\ \phi(-\infty) = 0, \quad \phi(-N) = 1 \end{cases}$$

and $\phi'(x) > 0$ on $(-\infty,-N)$. Define

$$\bar{u}(x) = \begin{cases} \phi(x) & \text{for} \quad -\infty < x < -N \\ 1 & \text{for} \quad -N \le x < \infty . \end{cases}$$

Then on $(-\infty,-N)$ we have, in view of H3:

$$\bar{u}'' + m\bar{u}' + f(x,\bar{u}) \le \phi'' + m\phi' + f^-(\phi) = 0$$

and on $(-N,\infty)$, $\bar{u}'' + m\bar{u}' + f(x,\bar{u}) = 0$. Finally, $\bar{u}$ has a concave corner at $x = -N$. Thus $\bar{u}$ is a super solution of Problem II.

Next, since $m > c^+$, it follows from Lemma 2.3 that $m \in K_1(f^+)$. Therefore we can find a function ψ: $[N,\infty) \to [0,1)$ such that

$$\begin{cases} \psi'' + m\psi' + f^+(\psi) = 0, & N < x < \infty, \\ \psi(N) = 0, \quad \psi(\infty) = 1, \end{cases}$$

and $\psi'(x) > 0$ on (N,∞). It is now easily verified that the function

$$\underline{u}(x) = \begin{cases} 0 & \text{for } -\infty < x \le N \\ \psi(x) & \text{for } \; N < x < \infty \end{cases}$$

is a subsolution of Problem II. Clearly $\bar{u} > \underline{u}$, whence the existence of a solution of Problem II follows from [19].

If $f(x,u)$ tends to a limit in $C^1([0,1])$ as $x \to +\infty$ and as $x \to -\infty$, we may obtain a result in terms of the limit functions. We replace H3 by:

H3*. There exist functions $f^+, f^-: [0,1] \to \mathbb{R}$ which satisfy the assumptions A1 - A_3, and in addition, if $f^\pm \in \mathcal{F}_1$, then $df^\pm/du > 0$ as $u = a$, such that

$$\lim_{x\to\infty} f(x,u) = f^+(u)$$

$$\lim_{x\to-\infty} f(x,u) = f^-(u)$$

in the $C^1([0,1])$ norm. We write again

$$c^- = c^*(f^-) \text{ and } c^+ = c^*(f^+).$$

THEOREM 3.2. *Let* f *satisfy the hypotheses* H1, H2 *and* H3*, *and let* $c^+ < c^-$. *Then for each* $m \in (c^+,c^-)$, *there exists a solution of Problem II.*

PROOF. Define

$$\bar{f}(\xi,u) = \max\,\{f(x,u): x \le \xi\}$$

$$\underline{f}(\xi,u) = \min\,\{f(x,u): x \ge \xi\}.$$

Then by H3*, as $\xi \to \infty$,

$$\bar{f}(-\xi,u) \downarrow f^-(u) \text{ and } \underline{f}(\xi,u) \uparrow f^+(u).$$

Moreover, for ξ large enough, $\bar{f}(-\xi,u)$ and $\underline{f}(\xi,u)$ satisfy assumptions A1 - A3. Hence, by Lemma 2.8, as $\xi \to \infty$,

$$c^*(\bar{f}(-\xi,\cdot)) \uparrow c^- \text{ and } c^*(\underline{f}(\xi,\cdot)) \downarrow c^+$$

Thus, for ξ large enough

$$c^*(\underline{f}(\xi,\cdot)) < m < c^*(\bar{f}(-\xi,\cdot)).$$

By construction,

$$f(x,u) \geq \underline{f}(\xi,u) \text{ for } x \geq \xi, \quad 0 \leq u \leq 1$$

and

$$f(x,u) \leq \bar{f}(-\xi,u) \text{ for } x \leq -\xi, \quad 0 \leq u \leq 1.$$

Therefore the conditions of Theorem 3.1 are satisfied, and we may conclude that a cline exists.

If f does not depend explicitly on x, it is well known that a solution of Problem II, if it exists, is strictly increasing. We shall show that this property is preserved if $f_x \geq 0$. The proof proceeds in two steps.

<u>LEMMA 3.3</u>. *Suppose* f *satisfies* H1 *and* $f_x(x,u) \geq 0$ *on* $\mathbb{R} \times [0,1]$. *If* u *is a solution of Problem II, it has the following properties:*
(a) *Suppose there exists an interval* (a,b), $a \geq -\infty$, $b < \infty$, *such that* $u'(x) > 0$ *on* (a,b) *and* $u'(b) = 0$. *Then* (i) $u''(b) < 0$ *and* (ii) *if* $m < 0$, *there exists a* $c > b$ *such that* $u(c) < u(a)$.
(b) *Suppose there exists an interval* (b,c), $b > -\infty$, $c \leq \infty$ *such that* $u'(x) > 0$ *on* (b,c) *and* $u'(b) = 0$. *Then* (i) $u''(b) > 0$ *and* (ii) *if* $m > 0$ *there exists an* $a < b$ *such that* $u(a) > u(c)$.

<u>PROOF</u>. (a) Set $a = x_1$, $b = x_2$ and $u(x_i) = u_i$, $i = 1,2$. Since $u' > 0$ on (x_1,x_2) we can define the inverse function $y_1:[u_1,u_2] \to [x_1,x_2]$ by $y_1(u(x)) = x$.

If we multiply (3.1a) by $2e^{2mx}u'$ and integrate we obtain

$$\{e^{mx}u'(x)\}^2 = 2\int_{u(x)}^{u_2} e^{2my_1(s)} f(y_1(s),s)ds \qquad x \in (x_1,x_2)$$

and hence, if $m \neq 0$

$$\frac{1}{m}(e^{-mx}-e^{-mx_2}) = \int_{u(x)}^{u_2} \left\{2\int_t^{u_2} e^{2my_1(s)} f(y_1(s),s)ds\right\}^{-\frac{1}{2}} dt.$$

As in [8] it can now be shown that if $u''(x_2) = 0$

$$\frac{1}{m}(e^{-mx}-e^{-mx_2}) > \infty$$

which is not possible. If $m = 0$, the situation is as in [8] and again $u''(x_2) < 0$.

Since $u''(x_2) < 0$, there exists a right neighbourhood of x_2 in which $u'(x) < 0$. Let (x_2,x_3) be the maximal interval in which $u' < 0$. Since $u(x) \to 1$ as $x \to \infty$, $x_3 < \infty$. We shall show that $u_3 = u(x_3) < u_1$ if $m < 0$.

Suppose to the contrary that $u_3 \geq u_1$, and let $\xi \in [x_1,x_2)$ be such that $u(\xi) = u_3$. Then

$$-\{u'(\xi)\}^2 + 2m\int_{\xi}^{x_3}\{u'(x)\}^2dx + 2\int_{u_1}^{u_2}\{f(y_1(s),s) - f(y_2(s),s)\}ds = 0, \tag{3.2}$$

where y_2 is the inverse of u on $[x_2,x_3]$. Since $f_x \geq 0$, $f(y_1(s),s) \leq f(y_2(s),s)$. Hence the first term and the third are nonpositive. If $m < 0$, the second term is negative, and we have a contradiction. It follows that $u_3 < u_1$.

The second part of the lemma can be proved in an entirely analogous manner.

THEOREM 3.4. *Suppose* f *satisfies* H1 *and* $f_x(x,u) \geq 0$ *on* $\mathbb{R} \times [0,1]$. *Let* u *be a solution of Problem II. Then* $u'(x) > 0$ *for all* $x \in \mathbb{R}$.

PROOF. Let $u_0 \in (0,1)$, and let x_0 be the largest value of x such that $u(x) = u_0$. Then $u(x) > u_0$, for all $x > x_0$.

Suppose there exists a $\xi \in (x_0,\infty)$ such that $u'(\xi) = 0$. Then one can distinguish two cases:

(i) There exists an $x_1 > x_0$ such that $u'(x) > 0$ on (x_0, x_1).

(ii) There exists a sequence $\{\xi_n\} \subset (x_0, \infty)$ such that $\xi_n \to x_0$ and $u'(\xi_n) = 0$ for all $n \geq 1$.

In case (i) it follows from Lemma 3.3 that if $m < 0$, there exists an $x_2 > x_1$ such that $u(x_2) < u(x_0) = u_0$, which contradicts the definition of x_0. In case (ii) it follows from Lemma 3.3 that x_0 is the limit from the right of a sequence of points $\{\tilde{\xi}_n\}$ at which u attains a local minimum. Moreover, if $m < 0$, $u(\tilde{\xi}_{n+1}) \geq u(\tilde{\xi}_n)$ for $n \geq 1$, and hence, by the continuity of u, $u(\tilde{\xi}_n) \leq u(x_0) = u_0$. Since $\tilde{\xi}_n > x_0$ this contradicts the definition of x_0. Thus we have a contradiction if $m < 0$. Because u_0 was arbitrary this implies that $u'(x) > 0$ for all $x \in \mathbb{R}$, provided $m < 0$.

Next, let y_0 be the smallest value of x such that $u(x) = u_0$. Then $u(x) < u_0$ for all $x < y_0$. Proceeding as above, we find that $u'(x) > 0$ for all $x \in \mathbb{R}$, provided $m > 0$.

Finally, if $m = 0$, we are back at the case treated in [8]. This completes the proof.

Next we turn to the question of uniqueness. We begin with two preliminary lemmas.

LEMMA 3.5. *Let* f *satisfy* H1 *and let* $f_x(x,u) \geq 0$ *on* $\mathbb{R} \times [0,1]$. *Suppose there exist two solutions* u_1 *and* u_2 *of Problem II, with* $u_1 < u_2$ *on an interval* (a,b), *where* $-\infty \leq a < b \leq \infty$, *and* $u_1(a) = u_2(a)$, $u_1(b) = u_2(b)$. *Then* $a = -\infty$ *and* $b = +\infty$.

PROOF. To begin with we shall show that a and b cannot both be finite. For suppose that $-\infty < a < b < \infty$, then

$$u_1'(a) < u_2'(a), \quad u_1'(b) > u_2'(b). \tag{3.3}$$

By Theorem 3.4, u_1 and u_2 are both strictly increasing. Hence we can define the inverse functions y_i:

$$y_i(u_i(x)) = x.$$

Next define

$$f_i(u)=f(y_i(u),u) \qquad u_1(a) \le u \le u_1(b).$$

Then the function $p_i = u_1'$ satisfies

$$p' + \frac{f_i(u)}{p} + m = 0 \quad \text{on } (\alpha,\beta),$$

where $\alpha = u_1(a) = u_2(a)$ and $\beta = u_1(b) = u_2(b)$. Moreover, by (3.3)

$$p_1(\alpha) < p_2(\alpha).$$

Since in addition $f_1(u) \ge f_2(u)$ on (α,β), because $y_1 \ge y_2$, it follows from an easy refinement of Lemma 4.1 of [3] that

$$p_1(u) \le p_2(u) \quad \text{on} \quad [\alpha,\beta]$$

and in particular

$$p_1(\beta) \le p_2(\beta),$$

or

$$u_1'(b) \le u_2'(b),$$

which contradicts (3.3).

Thus, the function $u_1 - u_2$ can have at most one finite zero. We shall denote it by c. Suppose

$$u_1 > u_2 \text{ for } x < c \text{ and } u_1 < u_2 \text{ for } x > c,$$

where we may have to relabel the two functions. Then $0 < u_1'(c) < u_2'(c)$ and it follows from Lemma 4.1 of [3] that

(3.4) $\qquad p_1(u) \le p_2(u)$ on $(\gamma,1)$, where $\gamma = u_i(c)$.

Suppose $m \leq 0$. Then if we multiply (3.1a) for p_i by $2p_i$ and integrate from γ to 1, we obtain after subtraction

$$2[p_2^2(\gamma) - p_1^2(\gamma)] + 2m \int_\gamma^1 (p_1-p_2)du + 2 \int_\gamma^1 (f_1-f_2)du = 0.$$

Since the first term is positive and the second and third are nonnegative, we have a contradiction.

Next, suppose $m \geq 0$. Then proceeding as before, but integrating over $(0,\gamma)$ we also obtain a contradiction. This completes the proof.

Let $u:\mathbb{R} \to (0,1)$ be a solution of Problem II. Then, by Theorem 3.4, $u'(x) > 0$ for all $x \in \mathbb{R}$. Hence we can define the inverse $y:(0,1) \to \mathbb{R}$ of u by $y(u(x)) = x$, and the function

(3.5) $\qquad g(u) = f(y(u),u) \qquad u \in (0,1).$

In addition we define

(3.6) $\qquad g(0) = 0, \quad g(1) = 0.$

In the following lemma we derive a number of properties of the function g defined by (3.5) and (3.6). However, to obtain sufficient smoothness near $u = 0$ and $u = 1$, we need to strengthen the hypothesis H1.

H1*. f satisfies H1 and there exist constants $N > 0$ and $\nu > 0$ such that $f_{xxu}(\cdot,0)$ is continuous and bounded in $(-\infty,-N) \cup (N,\infty)$ and f_{xuu} is continuous and bounded in $(-\infty,-N) \times [0,\nu)$ and $(N,\infty) \times (1-\nu,1]$.

LEMMA 3.6. *Let* u *be a solution of Problem II in which* f *satisfies* H1*, H2, H3*. *Then the function* g *defined by* (3.5) *and* (3.6) *has the properties* (i) $g \in C^1([0,1])$ *and* (ii) $g'(0) = f_u^-(0)$, $g'(1) = f_u^+(1)$.

PROOF. Clearly $g \in C(0,1)$. Moreover, since $u(x) \to 0$ as $x \to -\infty$,

$$\lim_{u\to 0} g(u) = \lim_{x\to -\infty} g(u(x)) = \lim_{x\to -\infty} f(x,u(x)) = f^-(0) = 0,$$

where we have used H3*. Similarly, since $u(x) \to 1$ as $x \to \infty$,

$$\lim_{u\to 1} g(u) = \lim_{x\to\infty} g(u(x)) = f^{+}(1) = 0.$$

Hence, by (3.6), $g \in C([0,1])$.

Next,

$$g'(u) = f_x(y(u),u)y'(u) + f_u(y(u),u),$$

whence $g' \in C(0,1)$. Thus, it remains to investigate the behaviour of $g'(u)$ as $u \downarrow 0$ and as $u \uparrow 1$. Observe that

$$g'(u(x)) = \frac{f_x(x,u(x))}{u(x)} \cdot \frac{u(x)}{u'(x)} + f_u(x,u(x)).$$

By H3*

$$\lim_{x\to -\infty} f_u(x,u(x)) = f_u^{-}(0).$$

Since $g \in C([0,1])$, $\lim_{x\to -\infty} u(x)/u'(x)$ exists ([4], p. 371). Also by the mean value theorem

$$f_x(x,u(x)) = f_{xu}(x,v(x))u(x),$$

where $0 < v(x) < u(x)$, and, using the mean value theorem again,

$$f_{xu}(x,v(x)) = f_{xuu}(x,w(x))v(x) + f_{xu}(x,0),$$

where $0 < w(x) < v(x)$. Since f_{xuu} is bounded for $-x$ large and $w(x)$ small, the first term vanishes as $x \to -\infty$. Since $f_u(x,0)$ is decreasing as x decreases, and bounded below by $f_u^{-}(0)$, the boundedness of $f_{xxu}(x,0)$ implies that $f_{xu(x,0)} \to 0$ as $x \to -\infty$. Thus

$$\lim_{x\to -\infty} f_{xu}(x,v(x)) = 0$$

and

$$\lim_{x \to -\infty} g'(u(x)) = f_u^-(0).$$

Similarly

$$\lim_{x \to +\infty} g'(u(x)) = f_u^+(1).$$

To ensure uniqueness of solutions of Problem II, it is not enough just to require that $f_x(x,u) \geq 0$ in $\mathbb{R} \times [0,1]$. For instance, if $f_x(x,u) \equiv 0$ in $\mathbb{R} \times [0,1]$, and a solution exists, then any of its translates is also a solution. A less trivial example of nonuniqueness is given in [8]. To remove this possibility, we introduce the following hypothesis.

H4. $f_x(x,u) \geq 0$ for $x \in \mathbb{R}$ and $u \in [0,1]$, and one of the following statements holds.

(i) There exists an interval $I \subset \mathbb{R}$ such that

$$f_x(x,u) > 0 \text{ for } x \in I \text{ and } u \in (0,1).$$

(ii) There exists an interval $J \subset (0,1)$ such that

$$f_x(x,u) > 0 \text{ for } x \in \mathbb{R} \text{ and } u \in J.$$

In addition, if $f^- \in \mathcal{F}_2$ or $f^+ \in \mathcal{F}_3$, we need to restrict the class of admissible solutions of Problem II to insure uniqueness.

DEFINITION. *Suppose* $f^- \in \mathcal{F}_2$ *and* $m < -2\{f_u^-(0)\}^{\frac{1}{2}}$. *Then we shall denote by* $\mathcal{S}^-$ *the set of functions* $\zeta: \mathbb{R} \to (0,1)$ *with the property:*

$$\limsup_{x \to -\infty} e^{-\ell^- x}\zeta(x) < \infty,$$

where

$$\ell^- > -\tfrac{1}{2}m - \tfrac{1}{2}\{m^2 - 4f_u^-(0)\}^{\frac{1}{2}}.$$

Similarly, if $f^+ \in \mathcal{F}_3$ *and* $m > 2\{f_u^+(1)\}^{\frac{1}{2}}$, *we shall denote by* $\mathcal{S}^+$ *the set of* *functions* $\zeta\colon \mathbb{R} \to (0,1)$ *with the property*

$$\lim_{x\to+\infty} \sup e^{\ell^+ x}\{1-\zeta(x)\} < \infty,$$

where

$$\ell^+ > \tfrac{1}{2}m - \tfrac{1}{2}\{m^2-4f_u^+(1)\}^{\frac{1}{2}}.$$

Suppose u(x) is a solution of Problem II, then

$$u'' + mu' + g(u) = 0,$$

where g is defined by (3.5) and (3.6). Since $g \in C^1([0,1])$ by Lemma 3.6 it follows that, if $m < -2\{g'(0)\}^{\frac{1}{2}} = -2\{f_u^-(0)\}^{\frac{1}{2}}$, then

$$\lim_{x\to-\infty} \frac{u'(x)}{u(x)} = L \in \{L_0^+, L_0^-\},$$

where

$$L_0^{\pm} = -\tfrac{1}{2}m \pm \tfrac{1}{2}\{m^2-4f_u^-(0)\}^{\frac{1}{2}}.$$

If it is given that $u \in S^-$, L cannot be equal to L_0^-, whence $L = L_0^+$. This means in particular that the orbit (u(x),p(x)) associated with u(x) is maximal. In a similar manner, solutions u belonging to S^+ correspond to the maximal orbit entering (1,0) from S.

THEOREM 3.7. *Let* f *satisfy the hypotheses of Theorem 3.2 and* $c^+ < m < c^-$. *Then*

(i) *if* $f^- \in F_2$, *Problem II has a solution in* S^-

(ii) *if* $f^+ \in F_3$, *Problem II has a solution in* S^+.

PROOF. Let $f^- \in F_2$. We shall show that the solution constructed in Theorem 3.2 actually belongs to S^-. Note that because $m < c^- \le -2\{f_u^-(0)\}^{\frac{1}{2}}$, S^- is well defined.

Recall that in the proof of Theorem 3.2, a supersolution $\bar{u}(\cdot;\xi)$ was constructed, which satisfied

$$\begin{cases} \bar{u}'' + m\bar{u}' + f(\xi,\bar{u}) = 0 \text{ on } (-\infty,\xi) \\ \bar{u}(-\infty;\xi) = 0, \quad \bar{u}(\xi;\xi) = 1 . \end{cases}$$

Here $\bar{u}$ corresponds to the maximal orbit entering the singular point (0,0) in the (u,u') plane. For this orbit we know that

$$\lim_{x\to-\infty} \frac{\bar{u}'(x;\xi)}{\bar{u}(x;\xi)} = -\tfrac{1}{2}m + \tfrac{1}{2}\{m^2-4f_u(\xi,0)\}^{\frac{1}{2}}$$

$$> -\tfrac{1}{2}m - \tfrac{1}{2}\{m^2-4f_u^-(0)\}^{\frac{1}{2}}$$

for $-\xi$ large enough. Since $u(\cdot) < \bar{u}(\cdot,\xi)$ for some large value of $-\xi$, it follows that $u \in S^-$.

The proof of part (ii) is similar.

We are now in a position to discuss the uniqueness of solutions of Problem II.

THEOREM 3.8. *Let* f *satisfy the hypotheses* H1*, H2, H3* *and* H4.
(a) *If* $f^- \in F_1 \cup F_3$ *and* $f^+ \in F_1 \cup F_2$, *Problem II has a unique solution.*
(b) *If* $f^- \in F_2$ *and* $m < c^-$, *Problem II has at most one solution in* S^-.
(c) *If* $f^+ \in F_3$ *and* $m > c^+$, *Problem II has at most one solution in* S^+.

PROOF. Let u_1 and u_2 be two different solutions of Problem II. Then by Lemma 3.5, we may assume, without loss of generality, that $u_1 < u_2$ on $\mathbb{R}$. Let y_i be the inverse of u_i and let $g_i(u) = f(y_i(u),u)$ for $0 \le u \le 1$. Then by H4

(3.7) $\qquad g_1(u) \ge g_2(u)$ for $0 \le u \le 1$,

and there exists an interval $(\alpha,\beta) \subset (0,1)$ such that

(3.8) $\qquad g_1(u) > g_2(u)$ for $\alpha < u < \beta$.

The functions u_1 and u_2 satisfy the system of equations

$$\begin{cases} u' = p \\ p' = -\,mp - g_i(u) \end{cases}$$

or

(3.9) $$p' + \frac{g_i(u)}{p} + m = 0 \qquad 0 < u < 1 \qquad i = 1,2$$

with

(3.10) $$p(0) = p(1) = 0.$$

We shall denote the solutions of (3.19) and (3.10), which correspond with u_1 and u_2, by p_i and p_2.

Suppose $f^- \in F_1 \cup F_3$ and $f^+ \in F_1 \cup F_2$. Then $g_i \in F_1$ and p_1 and p_2 are clearly maximal solutions entering (0,0). The same is true if $f^- \in F_2$, $m < c^-$ and $u_i \in S^-$. This means that we can apply Lemma 2.6a to conclude from (3.7) and (3.8) that $p_2(1) > p_1(1)$, which contradicts (3.10).

Finally suppose that $f^- \in F_3$. Then either $f^+ \in F_1 \cup F_2$, in which case $g_i \in F_1$ and p_1 and p_2 are evidently maximal solutions entering (1,0), or $f^+ \in F_3$ and the fact that p_1 and p_2 are maximal is ensured by the fact that $m > c^+$ and $u_i \in S^+$. The result follows now from an application of Lemma 2.6b.

To conclude this section we explore the existence of clines when $m \notin (c^+,c^-)$, and of clines which do not belong to S^- (when $f^- \in F_2$) or S^+ (when $f^+ \in F_3$).

<u>THEOREM 3.9</u>. *Let* f *satisfy hypotheses* H1*, H2, H3* *and* H4. *Then Problem II does not have a solution if one of the following sets of conditions is satisfied.*

(a) $f^- \in F_1 \cup F_3$, $f^+ \in F_1 \cup F_2$ *and* $m \notin (c^+,c^-)$;

(b) $f^- \in F_2$ *and* $m > c^-$;

(c) $f^+ \in F_3$ *and* $m < c^+$.

<u>PROOF</u>. Suppose to the contrary that u is a solution, y its inverse and

$g(u) = f(y(u),u)$. In view of hypothesis H4, we have

$$(3.12) \qquad f^-(u) \le g(u) \le f^+(u) \qquad 0 \le u \le 1,$$

where for some values of u, the inequalities are strict.
(a) It follows from (3.11) that $g \in F_1$. Hence $m = c^*(g)$ [6], and therefore, by Lemma 2.7,

$$c^+ = c^*(f^+) < m < c^*(f^-) = c^-,$$

which contradicts the assumption about m.
(b) Now we may conclude from (3.12) that $g \in F_2$. Hence by Theorem 2.4 of [7] and Theorem 4.15 of [6] $m \le c^*(g)$. Thus, using Lemma 2.7 again, we find that

$$m \le c^*(g) \le c^*(f^-) = c^-,$$

which contradicts the assumption about m.
The case (c) is proved similarly.

REMARKS. We may deduce from the monotonicity hypothesis H4 that
(a) if $f^- \in F_1 \cup F_3$ and $f^+ \in F_1 \cup F_2$, then

$$c^+ < c^-;$$

(b) if $f^- \in F_2$ or $f^+ \in F_3$, then

$$c^+ \le c^-.$$

PROOF. By H4, $f^- \le f^+$ and $f^- < f^+$ on some interval $(\alpha,\beta) \subset (0,1)$. Thus Lemma 2.7 may be applied, and it follows that $c^+ \le c^-$.
(a) Suppose $f^- \in F_1$. Then if $f^+ \in F_1$, the result follows from part (a1) of Lemma 2.7 and if $f^+ \in F_2$, it follows from part (a2) because $f_u^-(0) < 0 < f_u^+(0)$.
Next, suppose that $f^- \in F_3$.

(b) To see that we cannot expect a better result than $c^+ \leq c^-$, we consider the function

$$f(x,u) = u(1-u)\{1 + s(x)u(1-u)\}$$

where $s \in C^1(\mathbb{R})$, $s' > 0$ and

$$s(-\infty) = 0, \qquad s(\infty) \in (0,1),$$

Then $f(x,\cdot) \in F_{2b}$ for any $x \in \mathbb{R}$ and

$$c^*(f(x,\cdot)) = -2, \text{ for all } x \in \mathbb{R}.$$

If $f^- \in F_2$, Theorems 3.8 and 3.9 still leave the possibility of solutions of Problem II, which do not belong to S^- if $m < c^-$. Similarly if $f^+ \in F_3$, Theorems 3.8 and 3.9 do not exclude the existence of solutions of Problem II, which do not belong to S^+ if $m > c^+$. In the next theorem we shall show that if $m < c^-$ ($f^- \in F_2$) or $m > c^+$ ($f^+ \in F_3$), an *infinite* number of clines exists which do not belong to S^- or S^+, respectively.

THEOREM 3.10. *Let* f *satisfy the hypothesis* H1*, H2 *and* H3* *and suppose* $f_x > 0$. *Suppose one of the following sets of conditions is satisfied.*
(a) $f^- \in F_2$ *and* $m < c^-$;
(b) $f^+ \in F_3$ *and* $m > c^+$;
Then Problem II has an infinite number of solutions.

COROLLARY 3.11. *Of the solutions constructed in Theorem 3.10 only one can belong to* S^- *(case(a)) and only one can belong to* S^+ *(case b)).*

PROOF. To begin with we consider the problem

$$(P) \quad \begin{cases} u'' + mu' + f(x,u) = 0, & x \in (M,\infty), \\ u(M) = \alpha,\ u(\infty) = 1, \end{cases}$$

where $\alpha \in (0,1)$ is so chosen that

(3.12) $f_u(x,u) < 0$ on $\mathbb{R} \times (\alpha,1]$

and $-M$ is so large that

(3.13) $m < c^*(f(M,\cdot))$.

We shall first show that Problem P has a solution. Let u_1 and u_2 be functions which satisfy

$$u_1'' + mu_2' + f^+(u_1) = 0,$$

$$u_2'' + mu_2' + f(M,u_2) = 0,$$

$$u_1(M) = u_2(M) = \alpha,\ u_1(\infty) = u_2(\infty) = 1.$$

Let $\gamma_i = \{(u_i(x),\ u_i'(x)): x \in \mathbb{R}\}$ be the orbits which enter the point $(1,0)$ in the phase plane, coming from the region $S = \{(u,p): 0 < u < 1, p > 0\}$. Since both $f^+(u) > 0$ as $f(M,u) > 0$ for $0 < u < 1$, γ_i can only enter S through the half-line $\{(0,p)|p \geq 0\}$. This implies that for some value $x_i \in \mathbb{R}$, $u_i(x_i) = \alpha$. By an appropriate shift of the variable we thus obtain the function u_1 and u_2.

Because $f(M,u) \leq f(x,u) \leq f^+(u)$ for $x \geq M$, u_1 is a supersolution of Problem P and u_2 is a subsolution. Moreover, it follows from (3.12) and the maximum principle that $u_1(x) > u_2(x)$ for all $x > M$. Thus we may conclude that Problem P has a solution ϕ_α. Since at stationary points $\phi_\alpha'' < 0$, ϕ_α can have no minima and thus $\phi_\alpha' > 0$ for $x \geq M$.

Denote by y the inverse of the function ϕ_α while $\phi_\alpha' > 0$ and introduce $p_\alpha(u) = \phi_\alpha'(y(u))$. Then p_α satisfies

(3.14) $$p' + \frac{g(u)}{p} + m = 0, \qquad p(1) = 0,$$

where $g(u) = f(y(u),u)$. Let γ_α denote the orbit $\{(\phi_\alpha(x),\phi_\alpha'(x)): x \in \mathbb{R}\}$ Clearly γ_α enters S from $(1,0)$ and cannot leave S through the line-segment $\{(u,0): 0 < u < 1\}$ because $f(M,u) > 0$ on $(0,1)$.

To bound γ_α above, we consider the maximal solution $\hat{p}$ of the problem

(3.15) $$\hat{p}' + \frac{f(M,u)}{\hat{p}} + m = 0, \qquad \hat{p}(0) = 0 .$$

Since $m < c^*(f(M,\cdot))$, $m \in K_0(f(M,\cdot))$ and hence $\hat{p}(1) > 0$. Let us set $\hat{\gamma} = \{(u,\hat{p}(u)) : 0 \le u \le 1\}$.

Since the orbits γ_α enter $(1,0)$ at an angle, which does not depend on α, it is possible to choose α so close to 1 that $\phi_\alpha'(M) < \hat{p}(\alpha)$. Thus if γ_α intersects $\hat{\gamma}$, it must do so at some $u^* \in (0,\alpha)$, and hence at some $x^* < M$. At u^*, we would have $\hat{p}' \ge p_\alpha'$. However,

$$f(x,u) < f(M,u) \quad \text{for } x < M.$$

Hence, by (3.14) and (3.15), we would have, on the contrary, that

$$\hat{p}'(u^*) < p_\alpha'(u^*).$$

This proves part (a). Part (b) is proved in a similar manner.

4. STABILITY

In this section we shall investigate the stability of the family of clines ϕ which belong to S^- if $f^- \in F_2$ and to S^+ if $f^+ \in F_3$. Under the assumption that f satisfies H1*, H2, H3* and H4, and that $c^+ < m < c^-$, we are assured that a cline ϕ exists, and that it is uniquely determined by f and m.

Thus in this section we consider the problem

(4.1) $$\text{(III)} \begin{cases} u_t = u_{xx} + mu_x + f(x,u) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \psi(x) & x \in \mathbb{R} \end{cases}$$

where $\psi \in C(\mathbb{R})$ takes on values in the interval [0,1]. The assumptions on f guarantee that this problem has a unique classical solution, which exists for all time $t \ge 0$ and takes on values in the interval [0,1] [14].

To emphasize its dependence on ψ, we shall write it as

$$u = u(x,t;\psi).$$

In Theorem 3.2, we obtained ϕ by constructing appropriate sub - and supersolutions of Problem II. It is well known that this construction implies that ϕ is stable - in some sense - provided ϕ is unique. Thus, whereas the stability of ϕ is not really in question, we shall direct our attention to two problems:

1. Find conditions on ψ such that

$$(4.2) \qquad \|u(\cdot,t;\psi) - \phi\| \to 0 \text{ as } t \to \infty,$$

where $\|\cdot\|$ denotes the supremum norm on $C(\mathbb{R})$.

2. Obtain an estimate for the rate of convergence in (4.2).

We begin with a simple result.

THEOREM 4.1. *Let* $u(x,t;\psi)$ *be the solution of Problem III in which* f *satisfies* H1*, H2, H3*, H4 *and* $c^+ < c^-$, *and let* $m \in (c^+,c^-)$. *Suppose there exists a number* $h \in \mathbb{R}^+$ *such that*

$$(4.3) \qquad \phi(x-h) \le \psi(x) \le \phi(x+h) \qquad x \in \mathbb{R}.$$

Then

$$\|u(\cdot,t;\psi) - \phi\| \to 0 \text{ as } t \to \infty .$$

PROOF. As in [8] it can be shown that if $h > 0$, $\phi_{-h}(x) = \phi(x-h)$ is a subsolution and $\phi_{+h}(x) = \phi(x+h)$ is a supersolution of Problem III. Hence, by a monotonicity argument due to Aronson and Weinberger [3]

$$u(\cdot,t;\phi_{-h}) \uparrow \phi_1 \;, \quad u(\cdot,t;\phi_{+h}) \downarrow \phi_2 \quad \text{as } t \to \infty$$

where ϕ_1 and ϕ_2 are both solutions of Problem II. Clearly

$$\phi_{-h} < \phi_1 \le \phi_2 < \phi_{+h}.$$

Suppose $f^- \in F_2$. Then $\phi_{+h} \in S^-$ and hence $\phi_1,\phi_2 \in S^-$. Similarly, if $f^+ \in F_3$, $\phi_{-h} \in S^+$ and hence $\phi_1,\phi_2 \in S^+$. Therefore by Theorem 3.8, $\phi_1 = \phi_2$.

Since by the maximum principle

$$u(\cdot,t;\phi_{-h}) \le u(\cdot,t;\psi) \le u(\cdot,t;\phi_{+h}) \qquad \text{for all } t \ge 0$$

it follows that

$$\|u(\cdot,t;\psi) - \phi\| \to 0 \text{ as } t \to \infty,$$

where we have written $\phi_1 = \phi_2 = \phi$.

The conditions imposed on ψ in Theorem 4.1 are quite severe, and can be considerably weakened. This will be done in the next theorem. Define

$$a^- = \max\{u \in [0,1]: f^-(s) \le 0 \text{ for } s \in [0,u]\}$$
$$a^+ = \min\{u \in [0,1]: f^+(s) \ge 0 \text{ for } s \in [u,1]\}.$$

Thus $a^- \in (0,1)$ if $f^- \in F_1$, $a^- = 0$ if $f^- \in F_2$, $a^- = 1$ if $f^- \in F_3$, and similarly for f^+.

THEOREM 4.2. *Let* $u(x,t;\psi)$ *be the solution of Problem III in which* f *satisfies* H1*, H2, H3*, H4 *and* $c^- < c^+$, *and let* $m \in (c^+,c^-)$. *Suppose that* ψ *satisfies the following conditions:*

(i)

(4.4) $$\limsup_{x \to -\infty} \psi(x) < a^- \qquad \textit{if } a^- > 0$$

or

$$\psi(x) \le \phi_h(x) \textit{ for some } h \in \mathbb{R} \textit{ if } a^- = 0$$

(ii)

(4.5) $$\liminf_{x \to +\infty} \psi(x) > a^+ \qquad \textit{if } a^+ < 1$$

or

$$\psi(x) \geq \phi_h(x) \textit{ for some } h \in \mathbb{R} \qquad \textit{if } a^+ = 1.$$

Then

$$\|u(\cdot,t;\psi) - \phi\| \to 0 \qquad \textit{as } t \to \infty.$$

REMARK. Suppose $a^- = 0$. Then $f^- \in \mathcal{F}_2$ and hence $c^- \leq -2\{f_u^-(0)\}^{\frac{1}{2}}$. Thus, since $m < c^-$, the set S^- is well defined. Similarly, if $a^+ = 1$ the set S^+ is well defined.

PROOF. If $a^- > 0$ and $a^+ < 1$, the proof proceeds entirely along the lines of the proof of Theorem 4 in [8], so we need not give it here.

Thus let us consider first the case $a^- = 0$. In view of the assumption on ψ, it immediately follows from the proof of Theorem 4.1 that

(4.6) $$\limsup_{t \to \infty} u(x,t;\psi) \leq \phi(x) \qquad \text{for } x \in \mathbb{R}.$$

Therefore it is sufficient to prove that

(4.7) $$\liminf_{t \to \infty} u(x,t;\psi) \geq \phi(x) \qquad \text{for } x \in \mathbb{R}.$$

Since $f^- \in \mathcal{F}_2$ and $f_x \geq 0$, $f^+ \in \mathcal{F}_2$ as well and hence $a^+ = 0$. Therefore, by (4.5)

$$\liminf_{x \to \infty} \psi(x) = 2\nu > 0$$

and there exists a constant $\xi_1 > 0$ such that

$$\min\{\phi(x),\psi(x)\} > \nu \qquad \text{if } x > \xi_1.$$

By Lemma 2.8 we can choose a $\xi_2 > 0$ such that

$$c_1(f(\xi_2,\cdot)) < m.$$

We now define $\xi_0 = \max\{\xi_1,\xi_2\}$. Then by Lemma 2.7, $c_1(f(\xi_0,\cdot)) < m$, whence $m \in K_1(f(\xi_0,\cdot))$. This implies that the problem

$$\begin{cases} u'' + mu' + f(\xi_0,u) = 0 \\ u(0)=0, u(\infty) = 1 \end{cases}$$

has a unique solution $v(x)$. Following [7] we now show that there exist positive functions $\xi(t)$ and $q(t)$ such that

$$z(x,t) = \max\{0, v(x-\xi(t)) - q(t)\}$$

is a subsolution of (4.1).

Choose $q(0) = 1-\nu$ and $\xi(0) = \xi_0$. Then

$$z(x,0) \le \psi(x) \quad \text{on } \mathbb{R}. \tag{4.8}$$

Since $f_u(\xi_0,1) < 0$, there exist constants $\delta,\mu > 0$ such that

$$f(\xi_0,u-q) - f(\xi_0,u) \ge \mu q$$

if $1-\delta \le u \le 1$ and $0 \le q \le 1-\nu$. Hence, if $z > 0$,

$$\begin{aligned} L(z) &\equiv z_{xx} + mz_x + f(x,z) - z_t \\ &\ge v'' + mv' + f(\xi_0,v-q) + v'\xi' + q' \\ &= -f(\xi_0,v) + f(\xi_0,v-q) + v'\xi' + q' \\ &\ge \mu q + v'\xi' + q' \end{aligned}$$

provided $1-\delta \le v \le 1$ and $0 \le q \le 1-\nu$. If $\xi' \ge 0$ and $q(t) = q(0)e^{-\mu t}$ we obtain when $z > 0$

$$L(z) \geq 0$$

i.e. z is a subsolution if $1-\delta \leq v \leq 1$.

If $0 \leq v \leq 1-\delta$, there exists a constant $\beta > 0$ such that $v'(x) \geq \beta$. Let $f_u(\xi_0,u) \geq -k$ for all $u \in [0,1]$. Then if $z > 0$

$$L(z) \geq \beta\xi' - (k+\mu)q.$$

Hence, if we choose

$$\xi(t) = \xi(0) + \frac{\mu+k}{\mu\beta} q(0)(1-e^{-\mu t})$$

we achieve that $L(z) \geq 0$ when $z > 0$. Clearly, $L(z) = 0$ if $z = 0$. Therefore, with $\xi(t)$ and $q(t)$ as defined above, z is a subsolution of equation (4.1). In view of (4.8) this implies that

$$\liminf_{t \to \infty} u(x,t;\psi) \geq v(x-\xi(\infty)). \tag{4.9}$$

Thus at large values of x, the function $z(x,t)$ lifts $u(x,t;\psi)$ up to 1.

This action of z enables us to construct a new subsolution under $u(x,t;\psi)$ at some sufficiently large time t. The function w, which we choose for this purpose, satisfies

$$w'' + mw' + f(\xi_0,w) = 0$$

on an interval (x_1,x_2), where $x_1 < 0 < x_2$, and

$$w(x_1) = w(x_2) = 0, \quad w(x) > 0 \text{ on } (x_1,x_2), \; w(0) = 1-2\rho.$$

Because $m > c_1(f(\xi_0,\cdot))$ there exists a constant ρ_0 such that if $0 < \rho < \rho_0$, such a function exists (see [2], p.31).

Fix $\rho \in (0,\rho_0)$ so that $\rho \leq 1-\nu$ and define $t_1 > 0$ such that

$$\rho = q(t_1).$$

Then for η large enough,

$$w(x-\eta) < z(x,t_1).$$

The function

$$\underline{w}(x) = \max\{0,w(x-\eta)\} \qquad x \in \mathbb{R}$$

is a subsolution of (4.1). Hence $u(x,t;\underline{w})$ is strictly increasing. Since it is bounded above by ϕ it converges to a solution ϕ^* of (3.1a) and

$$\text{(4.10)} \qquad \underline{w}(x) \le \phi^*(x) \le \phi(x).$$

Because $\phi \in S^-$, $\phi^*(-\infty) = 0$ and $\phi^* \in S^-$. Also $f(x,\phi^*(x)) > 0$ for $x \in \mathbb{R}$ which implies, together with (4.10) that $\phi^{*\prime}(x) > 0$ and hence, using (4.10) again that $\phi^*(\infty) = 1$. Thus ϕ^* is a solution of Problem II which belongs to S^-. Since ϕ is also such a solution, and, by Theorem (3.8), there exists only one, we have $\phi^* = \phi$.

Finally, since

$$u(x,t_1;\psi) \ge z(x,t_1) > \underline{w}(x),$$

we find that

$$\text{(4.11)} \qquad \liminf_{t \to \infty} u(x,t;\psi) \ge \phi(x),$$

which we wanted to prove.

Thus, we have proved that

$$\lim_{t\to\infty} u(x,t;\psi) = \phi(x).$$

It is not difficult to see that the limit in Lemma 4.2 and in (4.11) is uniform with respect to $x \in \mathbb{R}$. This completes the case $a^- = 0$.

The remaining case: $a^+ = 1$ (which implies $a^- = 1$) can be handled in an identical manner.

We now turn to an investigation of the rate of convergence. To begin with, we show that under suitable conditions on f, the cline ϕ is linearly stable.

We write

$$u(x,t;\psi) = \phi(x) + v(x,t)$$

and substitute into (4.1). This yields

$$v_t = Lv + h(x,v), \tag{4.12}$$

where

$$Lv = v_{xx} + mv_x + f_u(x,\phi(x))v$$

and

$$h(x,v) = f(x,\phi(x)+v) - f(x,\phi(x)) - f_u(x,\phi(x))v.$$

If we assume instead of H1*:

H1**. f satisfies H1* and f_{uu} is continuous and bounded in $\mathbb{R} \times [0,1]$

It is readily shown that

$$\sup_{\mathbb{R}}|h(x,v)| \leq M\|v\|^2,$$

where

$$M = \tfrac{1}{2}\sup\{|f_{uu}(x,u)| : x \in \mathbb{R},\ u \in [0,1]\}.$$

We shall construct a function z(x,t) such that

(i) $z > 0$ in $\mathbb{R} \times [0,\infty)$.

(ii) z is a supersolution and -z is a subsolution of (4.12).

(iii) $z(\cdot,t) \to 0$ as $t \to \infty$ exponentially, in *some sense*.

Given this function z, we shall have shown that if

$$|\psi(x) - \phi(x)| < z(x,0) \qquad \text{for } x \in \mathbb{R}$$

then

$$u(\cdot,t;\psi) \to \phi \qquad \text{as } t \to \infty$$

exponentially in the same sense as $z(\cdot,t) \to 0$ as $t \to \infty$.

As in [8] we begin with the equation

$$(L-\lambda)\tilde{y} = 0$$

but transform to a symmetric form by introducing the new dependent variable $y(x) = \exp(\tfrac{1}{2}mx).\tilde{y}(x)$. This yields

$$(M-\lambda)y = 0\,, \tag{4.13}$$

where

$$My = y'' + \{f_u(x,\phi(x)) - \tfrac{1}{4}m^2\}y\,.$$

For convenience we shall write

$$q(x) = f_u(x,\phi(x)) - \tfrac{1}{4}m^2.$$

Note that by Lemma 3.6

$$q^- = \lim_{x\to-\infty} q(x) = f_u^-(0) - \tfrac{1}{4}m^2$$
$$q^+ = \lim_{x\to\infty} q(x) = f_u^+(1) - \tfrac{1}{4}m^2\,.$$

We assert that $q^- < 0$ and $q^+ < 0$. If $f_u^-(0) < 0$ it is obvious that $q^- < 0$, thus assume that $f_u^-(0) > 0$. Then $c^- \le -2\{f_u^-(0)\}^{\frac{1}{2}}$. By assumption $m < c^-$. Hence

$$m < -2\{f_u^-(0)\}^{\frac{1}{2}}$$

and therefore

$$m^2 > 4f_u^-(0)$$

i.e. $q^- < 0$. That $q^+ < 0$ follows in a similar fashion. Thus

$$q^* = \max\{q^-, q^+\} < 0. \tag{4.14}$$

We now consider (4.13) as an eigenvalue problem in $L^2(\mathbb{R})$, and denote its spectrum by $\sigma(M)$. Since M is symmetric $\sigma(M) \subset \mathbb{R}$ and, following [8], one can show that

$$\lambda_0 = \max\{\lambda : \lambda \in \sigma(M)\} < 0.$$

Choose $\lambda \in (\lambda_0, 0)$ and a function $g \in C_0^\infty(\mathbb{R})$ such that $g \geq 0$ on $\mathbb{R}$ and $g(x) \not\equiv 0$. Then the equation

$$(M-\lambda)w = -g$$

has a unique solution $w \in H^1(\mathbb{R})$ such that $w(x) > 0$ for all $x \in \mathbb{R}$ [8]. We adjust g so that $\sup_{\mathbb{R}} w(x) = 1$.

We distinguish three cases

(a) $f_u^-(0) < 0$ and $f_u^+(1) < 0$;

(b) $f_u^-(0) > 0$ (and hence $f_u^+(1) < 0$);

(c) $f_u^+(1) > 1$ (and hence $f_u^-(0) < 0$).

Here we have assumed that $f_x \geq 0$.

Case (a). Define

$$z(x,t) = \beta\{e^{-\frac{1}{2}mx}w(x) + \gamma\}e^{-\mu t}.$$

Note that in view of the asymptotic behaviour of w as $|x| \to \infty$, $e^{-\frac{1}{2}mx}w(x) \to 0$ as $|x| \to \infty$, if $|\lambda|$ is chosen sufficiently small. Proceeding as in [8] we can find constants β, γ and μ so that z is a supersolution of equation (4.1). Thus, there exist constants δ, $K > 0$ such that if $\|\psi-\phi\| < \delta$, then

$$\|u(\cdot,t:\psi) - \phi\| \le Ke^{-\mu t} \qquad \text{for } t \ge 0.$$

Case(b). Observe that in this case $f(x,u) > 0$ in $\mathbb{R} \times (0,1)$ and hence $m < 0$. Let $q_0 \in (q^*,0)$. Then there exists a constant $\xi > 0$ such that $q(x) \le q_0$ if $x \le -\xi$. Define $\tilde{w}(x) = e^{-\frac{1}{2}mx}w(x)$ and

$$z(x,t) = \begin{cases} \beta\{\tilde{w}(x) + \gamma e^{-\frac{1}{2}mx}\}e^{-\mu t} & \text{if } x < -\xi \\ \beta\{\tilde{w}(x) + \gamma e^{+\frac{1}{2}m\xi}\}e^{-\mu t} & \text{if } x \ge -\xi\,. \end{cases}$$

Note that $z(x,t)$ is continuous at $x = -\xi$.

(i) $x < -\xi$. We obtain upon substitution

$$\begin{aligned} Nz &= Lz + h(x,z) - z_t \\ &\le \beta e^{+\frac{1}{2}m\zeta-\mu t}\{(\lambda+\mu) + \gamma(\mu+q_0) + \beta M(1+\gamma)^2\}. \end{aligned} \tag{4.15}$$

Choose $0 < \mu < \min\{-\lambda,-q_0\}$, and $\beta \le \beta_1$, where β_1 is defined by

$$\gamma(q_0+\mu) + \beta_1 M(1+\gamma)^2 = 0.$$

Then

$$Nz \le 0 \quad \text{if } x < -\xi \text{ and } t > 0.$$

Let $-\alpha \in (f_u^+(1),0)$. Then there exists a constant $\eta > 0$ such that $f_u(x,\phi(x)) < -\alpha$ if $x > \eta$.

(ii) $x > \eta$. We now obtain

$$Nz = \beta e^{\frac{1}{2}m\xi-\mu t}\{(\lambda+\mu)\tilde{w}e^{-\frac{1}{2}m\xi} + \gamma(-\alpha+\mu) + \beta M(\|\tilde{w}\|+\gamma)^2\}$$

and we choose $\mu \in (0,\alpha)$ and $\beta \le \beta_2$, where

$$\gamma(-\alpha+\mu) + \beta_2 M(\|\tilde{w}\|+\gamma)^2 = 0.$$

(iii) $-\xi < x < \eta$. Since $\tilde{w}$ is positive and continuous on $\mathbb{R}$,

$$\nu = \min\{\tilde{w}(x): -\xi \le x \le \eta\} > 0 .$$

Therefore

$$Nz \le \beta e^{-\mu t}\{(\lambda+\mu)\nu + \gamma(\kappa+\mu) + \beta M(\|\tilde{w}\|+\gamma)^2\},$$

where

$$\kappa = \max\{f_u(x,\phi(x)): x \in \mathbb{R}\} .$$

Choose

$$\beta = \gamma \frac{\min(\alpha,-q_0) - \mu}{M(\|\tilde{w}\|^2+\gamma)^2} .$$

Then $\beta \le \min\{\beta_1,\beta_2\}$ and

$$Nz \le \beta e^{-\mu t}[(\lambda+\mu)\nu + \gamma\{\kappa + \min(\alpha,-q_0)\}].$$

Thus, if we choose

$$\gamma = -(\lambda+\mu)\{\kappa+\min(\alpha,q_0)\}^{-1},$$

we have achieved that

$$Nz \le 0 \text{ if } -\xi < x < \eta, \qquad t > 0.$$

Finally, observe that z has a concave corner at $x = -\xi$. Hence z is indeed a supersolution of equation (4.12). Similarly $-z$ is a subsolution of (4.12). Thus if

(4.16) $$|\psi(x) - \phi(x)| \le z(x,0) \qquad x \in \mathbb{R}$$

then

$$(4.17) \qquad |u(x,t;\psi) - \phi(x)| \le z(x,t) \qquad x \in \mathbb{R}\,,\ t \ge 0.$$

At this point it is convenient to introduce the following weighted norm (see also ROTHE [18]). Let

$$\rho(x) = \max\{1, e^{\frac{1}{2}mx}\}.$$

Then we define

$$(4.18) \qquad \|h\|_\rho = \sup\{\rho(x)|h(x)| : x \in \mathbb{R}\}.$$

Let

$$(4.19) \qquad \|\psi-\phi\|_\rho \le \delta,$$

where $\delta = \beta\gamma e^{\frac{1}{2}m\xi}$. Then we shall see that (4.16) is satisfied. To verify this we inspect the intervals $(-\infty,-\xi)$, $[-\xi,0]$ and $(0,\infty)$ in turn.

(i) If $x \in (-\infty,-\xi)$ we have by definition

$$|\psi(x)-\phi(x)| \le \beta\gamma e^{\frac{1}{2}m(\xi-x)} < \beta\gamma e^{-\frac{1}{2}mx} < z(x,0).$$

(ii) If $x \in [-\xi,0]$,

$$|\psi(x) - \phi(x)| \le \beta\gamma e^{\frac{1}{2}m(\xi-x)} \le \beta\gamma e^{\frac{1}{2}m\xi} < z(x,0).$$

(iii) If $x \in (0,\infty)$,

$$|\psi(x) - \phi(x)| \le \beta\gamma e^{\frac{1}{2}m\xi} < z(x,0).$$

Thus, we may conclude that (4.17) is satisfied. This means that

(i) if $x \in (-\infty,-\xi)$,

$$e^{\frac{1}{2}mx}|u(x,t;\psi) - \phi(x)| \le e^{\frac{1}{2}mx}z(x,t) \le \beta(1+\gamma)e^{-\mu t};$$

(ii) if $x \in [-\xi, 0]$,

$$e^{\frac{1}{2}mx}|u(x,t;\psi) - \phi(x)| \le \beta(w(x) + \gamma e^{\frac{1}{2}m(x+\xi)})e^{-\mu t}$$
$$\le \beta(1+\gamma)e^{-\mu t};$$

(iii) if $x > 0$

$$|u(x,t;\psi) - \phi(x)| \le \beta\{\tilde{w}(x) + \gamma\}e^{-\mu t}$$
$$< \beta(\|\tilde{w}\| + \gamma)e^{-\mu t}.$$

Thus, (4.17) implies that

$$\|u(\cdot,t;\psi) - \phi\|_\rho \le Ke^{-\mu t} \qquad t \ge 0, \tag{4.20}$$

where

$$K = \beta[\max\{1, \|\tilde{w}\|\} + \gamma].$$

Case(c). In this case $f(x,u) < 0$ for all $x \in \mathbb{R}$ and $u \in (0,1)$, and therefore $m > 0$. Arguing exactly as in case (b) it is possible to find positive constants δ and K such that (4.19) implies (4.20).

Thus we have proved the following result.

<u>THEOREM 4.3</u>. *Let* $u(x,t;\psi)$ *be the solution of Problem III in which* f *satisfies the hypotheses* H1**, H2, H3*, H4 *and* $c^+ < c^-$, *and let* $m \in (c^+, c^-)$.

(i) *Suppose* $f_u^-(0) < 0$ *and* $f_u^+(1) < 0$. *Then there exist positive constants* μ, δ *and* K *such that* $\|\psi-\phi\| < \delta$ *implies*

$$\|u(\cdot,t;\psi) - \phi\| \le Ke^{-\mu t} \quad \textit{for } t \ge 0.$$

(ii) *Suppose either* $f_u^-(0) > 0$ *or* $f_u^+(1) > 0$. *Then there exist positive constants* μ, δ *and* K *such that* $\|\psi-\phi\|_\rho < \delta$ *implies*

$$\|u(\cdot,t;\psi) - \phi\|_\rho \le Ke^{-\mu t} \qquad \text{for } t \ge 0,$$

in which $\|\cdot\|_\rho$ *has been defined in* (4.18).

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