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THE NUMBER OF LATTICE POINTS CONTAINED IN CERTAIN CONVEX DOMAINS

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The number of lattice points contained in certain convex domains

by

J. van de Lune

ABSTRACT

For some specific sequences of convex domains D(n), $n \in \mathbb{N}$, the oscillatory behaviour of E(n) = A(n) - P(n) is studied. Here A(n) denotes the area of D(n) whereas P(n) is the number of Gaussian lattice points contained in D(n).

KEY WORDS & PHRASES: *lattice points, uniform distribution*

0. INTRODUCTION

In this note we consider certain sequences of (convex) domains $\{D_n\}_{n=1}^{\infty}$ in the plane $\mathbb{R}^2$. The area of D_n will be denoted by $A(n)$ and the number of (Gaussian) lattice points contained in D_n by $P(n)$.

The "error" $E(n)$ is defined by

$$(0.1)\qquad E(n) = A(n) - P(n).$$

The main purpose of this note is to investigate the frequency (= natural density) of the occurrence

$$(0.2)\qquad E(n) > 0.$$

In addition we will (in some cases) establish upper- and lower bounds for $E(n)$ as a function of n.

1. THE PARABOLIC CASE

Let $\alpha \in \mathbb{R}^+$ be fixed and for $n \in \mathbb{N}$ let the domain D_n be defined by

$$(1.1)\qquad \begin{cases} |x| \le \sqrt{\dfrac{n}{\alpha}}\,, \\ 0 < y \le n - \alpha x^2. \end{cases}$$

Then we have

$$(1.2)\qquad A(n) = 2\int_0^{\sqrt{\frac{n}{\alpha}}} (n-\alpha x^2)\,dx = \frac{4}{3}\, n\sqrt{\frac{n}{\alpha}}$$

and

$$(1.3)\qquad P(n) = n + 2\sum_{k=1}^{r} [n-\alpha k^2]$$

where

(1.4) $$r = r(n) = \left[\sqrt{\frac{n}{\alpha}}\right].$$

Writing

(1.5) $$\Theta = \Theta(n) = \sqrt{\frac{n}{\alpha}} - r$$

we have

(1.6) $$n = \alpha(r+\Theta)^2$$

so that (1.2) may also be written as

(1.7) $$A(n) = \frac{4}{3}\alpha(r+\Theta)^3.$$

Defining $\alpha^* \in [0,1)$ by

(1.8) $$\alpha^* = -\alpha - [-\alpha]$$

we obtain from (1.3) that

(1.9) $$P(n) = n + 2nr + 2\sum_{k=1}^{r}[-\alpha k^2] =$$
$$= n + 2nr + 2\sum_{k=1}^{r}[([-\alpha]+\alpha^*)k^2] =$$
$$= n + 2nr + 2[-\alpha]\sum_{k=1}^{r}k^2 + 2\sum_{k=1}^{r}[\alpha^* k^2].$$

CASE 1. $\alpha^* = 0$ ($\Leftrightarrow \alpha \in \mathbb{N}$).

From (1.9) it is clear that in this case

(1.10) $$P(n) = n + 2nr - 2\alpha\frac{1}{6}r(r+1)(2r+1)$$

so that

(1.11) $$E(n) = \frac{4}{3}\,\alpha(r+\Theta)^3 - \alpha(r+\Theta)^2 - 2r\alpha(r+\Theta)^2 + \frac{\alpha}{3}\,r(r+1)(2r+1) =$$

$$= \alpha r(2\Theta^2-2\Theta+\tfrac{1}{3}) + \alpha(\tfrac{4}{3}\Theta^3-\Theta^2).$$

It follows that the event $E(n) > 0$ is equivalent to

(1.12) $$2\Theta^2 - 2\Theta + \frac{1}{3} > \frac{1}{r}(\Theta^2-\tfrac{4}{3}\Theta^3).$$

Now we recall a theorem of FÉJER (cf. [1; p.89] or [3; p.72, 237]): If the differentiable (real) function f is such that f' is positive and monotonic on $\mathbb{R}^+$ and $f(x) \to \infty$, $f'(x) \to 0$, $xf'(x) \to \infty$ for $x \to \infty$, then the sequence $\{f(n)\}_{n=1}^{\infty}$ is uniformly distributed (mod 1).

It is clear that for any fixed $\sigma \in \mathbb{R}^+$ this theorem applies to $f(x) = \sigma\sqrt{x}$, $x \in \mathbb{R}^+$, so that

(1.13) $$\Theta(n) = \sqrt{\frac{n}{\alpha}} - \left[\sqrt{\frac{n}{\alpha}}\right]$$

is uniformly distributed on the interval $[0,1)$. From this observation and the fact that the right-hand side of (1.12) tends to zero as $n \to \infty$ it is easily seen that the probability of the event $E(n) > 0$ is equal to the probability of the event

(1.14) $$2\Theta^2 - 2\Theta + \frac{1}{3} > 0.$$

The roots of the left-hand side of (1.14) are

(1.15) $$\Theta_1 = \tfrac{1}{2} - \frac{1}{2\sqrt{3}} \text{ and } \Theta_2 = \tfrac{1}{2} + \frac{1}{2\sqrt{3}}$$

so that

(1.16) $$0 < \Theta_1 < \Theta_2 < 1.$$

Since $\Theta = \Theta(n)$ is uniformly distributed on the interval $[0,1)$ it follows that

(1.17) $$\text{Prob}\,\{E(n) > 0\} = \Theta_1 + (1-\Theta_2) = 1 - \frac{1}{\sqrt{3}}$$

which, surprisingly enough, does not depend on α ($\in \mathbb{N}$).

From (1.11) we also obtain that

(1.18) $$\frac{E(n)}{\alpha r} = 2\Theta^2 - 2\Theta + \frac{1}{3} + \frac{1}{r}\left(\frac{4}{3}\Theta^3 - \Theta^2\right)$$

from which it is clear that

(1.19) $$\limsup_{n\to\infty} \frac{E(n)}{\alpha r} = \limsup_{n\to\infty}\left(2\Theta^2 - 2\Theta + \frac{1}{3}\right) = \frac{1}{3}$$

and

(1.20) $$\liminf_{n\to\infty} \frac{E(n)}{\alpha r} = \liminf_{n\to\infty}\left(2\Theta^2 - 2\Theta + \frac{1}{3}\right) = -\frac{1}{6}.$$

Since $r = \sqrt{\frac{n}{\alpha}} + \mathcal{O}(1)$, $(n\to\infty)$ it follows that

(1.21) $$\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \frac{1}{3}\sqrt{\alpha}$$

and

(1.22) $$\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = -\frac{1}{6}\sqrt{\alpha}.$$

<u>CASE 2</u>. α^* is irrational.

From (1.9) we obtain

(1.23) $$P(n) = n + 2nr + 2[-\alpha] \sum_{k=1}^{r} k^2 +$$

$$- 2 \sum_{k=1}^{r} (\alpha^* k^2 - [\alpha^* k^2] - \tfrac{1}{2}) +$$

$$+ 2 \sum_{k=1}^{r} (\alpha^* k^2 - \tfrac{1}{2}) =$$

$$= n + 2nr - r - 2\alpha\frac{1}{6}r(r+1)(2r+1) - 2\Delta$$

where

(1.24) $$\Delta \overset{def}{\equiv} \sum_{k=1}^{r} (\alpha^* k^2 - [\alpha^* k^2] - \tfrac{1}{2}).$$

Using (1.6) we obtain after some simplification

(1.25) $$E(n) = \alpha r\left(2\Theta^2 - 2\Theta + \frac{1}{\alpha} + \frac{1}{3}\right) + \alpha\left(\frac{4}{3}\Theta^3 - \Theta^2\right) + 2\Delta.$$

Hence, the event $E(n) > 0$ is equivalent to

(1.26) $$2\Theta^2 - 2\Theta + \frac{1}{\alpha} + \frac{1}{3} > \frac{1}{r}\left(\Theta^2 - \frac{4}{3}\Theta^3 - \frac{2\Delta}{\alpha}\right).$$

Since α^* is irrational, the sequence $\{k^2\alpha^*\}_{k=1}^{\infty}$ is uniformly distributed (mod 1) (cf. [1; p.95, §4]) so that (cf. [1; p.91, §3])

(1.27) $$\lim_{n\to\infty} \frac{\Delta}{r} = \lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \left(\alpha^* k^2 - [\alpha^* k^2] - \tfrac{1}{2}\right) = \int_0^1 (x - \tfrac{1}{2})dx = 0.$$

It follows that the right-hand side of (1.26) tends to zero as $n \to \infty$. From this and the uniform distribution of $\Theta = \Theta(n)$ on $[0,1)$ it follows easily that

(1.28) $$\text{Prob}\,\{E(n) > 0\} = \text{Prob}\,\{2\Theta^2 - 2\Theta + \frac{1}{\alpha} + \frac{1}{3} > 0\}.$$

The discriminant D of the polynomial in (1.28) is

(1.29) $$D = 4\left(\frac{1}{3} - \frac{2}{\alpha}\right)$$

so that

(1.30) $$\text{Prob}\,\{2\Theta^2 - 2\Theta + \frac{1}{\alpha} + \frac{1}{3} > 0\} = 1 \text{ if } \alpha < 6.$$

If $\alpha > 6$ then the roots of the polynomial in (1.28) are

(1.31) $\quad \Theta_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{3}-\frac{2}{\alpha}}$ and $\Theta_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{3}-\frac{2}{\alpha}}$

so that

(1.32) $\quad 0 < \Theta_1 < \Theta_2 < 1.$

Similarly as before it follows that

(1.33) $\quad \text{Prob}\,\{E(n) > 0\} = \Theta_1 + (1-\Theta_2) = 1 - \sqrt{\frac{1}{3}-\frac{2}{\alpha}}\,, \quad (\alpha>6).$

From (1.25) and (1.27) it also follows that

(1.34) $$\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(\frac{1}{3}+\frac{1}{\alpha}\right)\sqrt{\alpha}$$

and

(1.35) $$\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(-\frac{2}{3}+\frac{1}{\alpha}\right)\sqrt{\alpha}.$$

CASE 3. α^* is rational and $\neq 0$. From the definition of α^* it is clear that $\alpha^* \in [0,1)$ so that we may assume that

(1.36) $\quad \alpha^* = \frac{p}{q}$ with $p,q \in \mathbb{N}$, $p < q$, $(p,q) = 1$.

From (1.9) we obtain

(1.37) $$P(n) = n + 2nr + 2[-\alpha]\sum_{k=1}^{r} k^2 + $$
$$- 2\sum_{k=1}^{r}\left(\alpha^* k^2 - [\alpha^* k^2]\right) + 2\alpha^*\sum_{k=1}^{r} k^2 =$$
$$= n + 2nr - \frac{\alpha}{3}r(r+1)(2r+1) - 2\sum_{k=1}^{r}\left\{k^2\frac{p}{q}\right\}$$

where in the last line $\left\{k^2\frac{p}{q}\right\}$ denotes the fractional part of $k^2\frac{p}{q}$. Similarly as before it follows that

(1.38) $$E(n) = \alpha r\left(2\Theta^2-2\Theta+\frac{1}{3}\right) + \alpha\left(\frac{4}{3}\Theta^3-\Theta^2\right) + 2\sum_{k=1}^{r}\left\{k^2\frac{p}{q}\right\}.$$

Since the sequence $\left\{k^2\frac{p}{q}\right\}$, $k \in \mathbb{N}$, is periodic with period q we have

(1.39) $$\lim_{r\to\infty}\frac{1}{r}\sum_{k=1}^{r}\left\{k^2\frac{p}{q}\right\} = \frac{1}{q}\sum_{k=1}^{q}\left\{k^2\frac{p}{q}\right\}.$$

Defining

(1.40) $$S(p,q) = \frac{1}{q}\sum_{k=1}^{q}\left\{k^2\frac{p}{q}\right\}$$

it follows from (1.38) and the uniform distribution of $\Theta = \Theta(n)$ on $[0,1)$ that

(1.41) $$\text{Prob}\,\{E(n) > 0\} = \text{Prob}\left\{2\Theta^2 - 2\Theta + \frac{1}{3} + \frac{2}{\alpha}S(p,q) > 0\right\}.$$

The discriminant D of the polynomial in (1.41) is

(1.42) $$D = 4\left(\frac{1}{3}-\frac{4}{\alpha}S(p,q)\right).$$

so that

(1.43a) $$D \le 0 \Leftrightarrow S(p,q) \ge \frac{\alpha}{12}$$

and

(1.43b) $$D > 0 \Leftrightarrow S(p,q) < \frac{\alpha}{12}.$$

Hence, if $S(p,q) \ge \frac{\alpha}{12}$ then

(1.44) $$\text{Prob}\,\{E(n) > 0\} = 1.$$

If $S(p,q) < \frac{\alpha}{12}$ then the roots of the polynomial in (1.41) are

(1.45) $$\Theta_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{3}-\frac{4}{\alpha}\,S(p,q)}\right)$$

so that

$$(1.46)\qquad \text{Prob}\,\{E(n) > 0\} = 1 - \sqrt{\frac{1}{3} - \frac{4}{\alpha}\, S(p,q)}\ .$$

From (1.38) it also follows easily that

$$(1.47)\qquad \limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(\frac{1}{3} + \frac{2}{\alpha}\, S(p,q)\right) \sqrt{\alpha}$$

and

$$(1.48)\qquad \liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(-\frac{1}{6} + \frac{2}{\alpha}\ S(p,q)\right) \sqrt{\alpha}.$$

The arithmetical nature of the sums $S(p,q)$ seems to be rather obscure. However, in [4] WILLIAMS discussed the case in which q is prime. One of his results is that if q is a prime such that $q \equiv 1 \pmod 4$ and $(p,q) = 1$ then $S(p,q) = \frac{q-1}{2q}$. He also gives a remarkable formula for $S(p,q)$ in case q is a prime such that $q \equiv 3 \pmod 4$. In the last case $S(p,q)$ appears to depend on the class number $h(-q)$.

For $q \in \mathbb{N}$ let $H(q)$ denote the number of different values of $S(p,q)$ when p runs through all positive integers not exceeding q and such that $(p,q) = 1$. From [4] it follows that if q is prime then

$$(1.49)\qquad H(q) = \begin{cases} 1 \text{ if } q \equiv 1 \pmod 4 \\ 2 \text{ if } q \equiv 3 \pmod 4). \end{cases}$$

We constructed the following table of the arithmetical function H:

n	H(n)	n	H(n)	n	H(n)	n	H(n)	n	H(n)
1	1	8	4	15	4	22	2	29	1
2	1	9	2	16	4	23	2	30	4
3	2	10	1	17	1	24	8	31	2
4	2	11	2	18	2	25	1	32	4
5	1	12	4	19	2	26	1	33	4
6	2	13	1	20	4	27	2	34	1
7	2	14	2	21	4	28	4	35	4

n	H(n)	n	H(n)	n	H(n)	n	H(n)	n	H(n)
36	4	69	4	102	4	135	4	168	16
37	1	70	4	103	2	136	6	169	1
38	2	71	2	104	8	137	1	170	1
39	4	72	8	105	6	138	4	171	4
40	6	73	1	106	1	139	2	172	4
41	1	74	1	107	2	140	6	173	1
42	4	75	4	108	4	141	4	174	4
43	2	76	4	109	1	142	2	175	4
44	4	77	3	110	4	143	4	176	8
45	4	78	4	111	4	144	8	177	4
46	2	79	2	112	6	145	1	178	1
47	2	80	8	113	1	146	1	179	2
48	8	81	2	114	4	147	4	180	8
49	2	82	1	115	4	148	4	181	1
50	1	83	2	116	4	149	1	182	4
51	4	84	8	117	4	150	4	183	4
52	4	85	1	118	2	151	2	184	8
53	1	86	2	119	4	152	8	185	1
54	2	87	4	120	12	153	4	186	4
55	4	88	8	121	2	154	3	187	4
56	8	89	1	122	1	155	4	188	4
57	4	90	4	123	4	156	8	189	4
58	1	91	4	124	4	157	1	190	4
59	2	92	4	125	1	158	2	191	2
60	8	93	4	126	4	159	4	192	8
61	1	94	2	127	2	160	6	193	1
62	2	95	4	128	4	161	4	194	1
63	4	96	8	129	4	162	2	195	6
64	4	97	1	130	1	163	2	196	4
65	1	98	2	131	2	164	4	197	1
66	4	99	4	132	6	165	8	198	4
67	2	100	4	133	3	166	2	199	2
68	4	101	1	134	2	167	2	200	6

The following observations may illustrate the erratic behaviour of H. Since

$$(1.50) \qquad H(3) = 2, \quad H(5) = 1 \quad \text{and } H(15) = 4$$

it follows that H is *not* multiplicative. Since

$$(1.51) \qquad H(40) = 6 \quad (\text{and } H(77)=3)$$

H(n) is *not* always a power of 2.
Although in most cases one has

$$(1.52) \qquad H(u)\cdot H(v) \le H(uv) \text{ if } (u,v) = 1$$

it follows from an example such as

$$(1.53) \qquad H(7) = 2, \quad H(11) = 2 \quad \text{and } H(77) = 3$$

that H does *not* always satisfy (1.52).

More generally one may ask for the arithmetical behaviour of the sums

$$(1.54) \qquad S_a(p,q) = \frac{1}{q}\sum_{k=1}^{q}\left\{k^a \frac{p}{q}\right\}$$

where $a,p,q \in \mathbb{N}$, $p \le q$, $(p,q) = 1$.
The case $a = 1$ is easily dealt with:

$$(1.55) \qquad S_1(p,q) = \frac{q-1}{2q}.$$

2. THE DOUBLE PARABOLIC CASE

Again let $\alpha \in \mathbb{R}^+$ be fixed and for $n \in \mathbb{N}$ define the convex domain D_n by

$$(2.1) \qquad \begin{cases} |x| \le \sqrt{\frac{n}{\alpha}} \\ |y| \le n - \alpha x^2. \end{cases}$$

Defining A(n), P(n), E(n), r(n), Θ(n) and α^* as in section 1 one may verify that we have

CASE 1. $\alpha^* = 0$ ($\Leftrightarrow \alpha \in \mathbb{N}$).

(2.2) $\mathrm{Prob}\,\{E(n) > 0\} = 0 \quad \text{for } \alpha = 1,2,3.$

(2.3) $\mathrm{Prob}\,\{E(n) > 0\} = 1 - \sqrt{\frac{1}{3}+\frac{2}{\alpha}} \quad \text{for } \alpha \geq 4.$

(2.4) $\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(\frac{2}{3} - \frac{2}{\alpha}\right)\sqrt{\alpha} \quad \text{for all } \alpha \in \mathbb{N}.$

(2.5) $\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = -\left(\frac{1}{3} + \frac{2}{\alpha}\right)\sqrt{\alpha} \quad \text{for all } \alpha \in \mathbb{N}.$

CASE 2. α^* is irrational.

(2.6) $\mathrm{Prob}\,\{E(n) > 0\} = 1 - \frac{1}{\sqrt{3}}.$

(2.7) $\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \frac{2}{3}\sqrt{\alpha}.$

(2.8) $\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = -\frac{1}{3}\sqrt{\alpha}.$

CASE 3. $\alpha^* = \frac{p}{q}$, $0 < p < q$, $(p,q) = 1$.

(2.9) $\mathrm{Prob}\,\{E(n) > 0\} = 0 \quad \text{if } S(p,q) \geq \frac{\alpha+6}{12},$

(2.10) $\mathrm{Prob}\,\{E(n) > 0\} = 1 - \sqrt{\frac{1}{3} + \frac{2}{\alpha} - \frac{4}{\alpha}S(p,q)} \quad \text{if } S(p,q) < \frac{\alpha+6}{12}.$

(2.11) $\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}} = \left(\frac{2}{3} - \frac{2}{\alpha} + \frac{4}{\alpha}S(p,q)\right)\sqrt{\alpha}.$

(2.12) $\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}} = -\left(\frac{1}{3} + \frac{2}{\alpha} - \frac{4}{\alpha}S(p,q)\right)\sqrt{\alpha}.$

3. THE CIRCULAR CASE

For any $t \in \mathbb{R}^+$ let D_t be the domain defined by

$$(3.1)\qquad \begin{cases} |x| \le t, \\ |y| \le \sqrt{t^2-x^2}. \end{cases}$$

Denoting the area of D_t by $A(t)$ and the number of lattice points contained in D_t by $P(t)$ we have that the error $E(t) \overset{\text{def}}{=} A(t) - P(t)$ changes sign infinitely often. More precisely, it was shown by HARDY that (cf. [2; p.236, Satz 536])

$$(3.2)\qquad \limsup_{t\to\infty} \frac{E(t)}{\sqrt{t}} > 0$$

and

$$(3.3)\qquad \liminf_{t\to\infty} \frac{E(t)}{\sqrt{t}} < 0.$$

Since all lattice points of the plane lie on circles with radius $\sqrt{k}$ for certain $k \in \mathbb{N} \cup \{0\}$ it seems natural to ask for the natural density of those $n \in \mathbb{N}$ for which, for example, one has $E(\sqrt{n}) > 0$.

We were not able to give a satisfactory answer to this question. However, numerical computations, performed by H.J.J. TE RIELE suggest that the probability of the event $E(\sqrt{n}) > 0$ is *less* than $\frac{1}{2}$.

Another related question is the following: Are there infinitely many $n \in \mathbb{N}$ such that $E(n) < 0$?

Numerical computations reveal that there are 64 values of $n \le 20{,}000$ with the property $E(n) < 0$. We list these values of n in the following table.

All $n \in \mathbb{N}$, $n \leq 20{,}000$ with $E(n) < 0$.

1	489	4771	11456
2	725	4885	11570
3	730	5559	11722
5	1073	5949	12019
10	1310	6203	12024
15	1865	6411	13243
20	1997	7045	14650
35	2480	7084	15857
51	2831	7410	16234
52	3072	7605	17030
85	3424	8931	17306
100	3750	9308	17429
230	3861	9435	17589
247	3921	9646	17970
370	4025	10829	18508
425	4339	10930	19619

4. THE TRUNCATED CIRCULAR CASE

For $t \in \mathbb{R}^+$ let D_t be the domain defined by

$$(4.1) \qquad \begin{cases} x \leq t \\ 0 < y \leq \sqrt{t^2-x^2}. \end{cases}$$

Then

$$(4.2) \qquad A(t) = \tfrac{1}{2}\pi t^2$$

and (cf. [2; p.271, Satz 558])

$$(4.3) \qquad P(t) = \tfrac{1}{2}\pi t^2 - t + \mathcal{O}(t^{2\Theta}), \qquad (t\to\infty)$$

for some $\Theta < \frac{1}{3}$.

It follows that

(4.4) $$E(t) = t + \mathcal{O}(t^{2\Theta}) = t + o(t), \qquad (t\to\infty)$$

so that there exists a t_0 such that

(4.5) $$E(t) > 0 \text{ for all } t > t_0.$$

Numerical computations indicate that one always has $E(\sqrt{n}) > 0$, $(n \in \mathbb{N})$, except for $n = 5$.

5. QUESTION

We conclude this note by proposing the following (unsolved?!)

PROBLEM. Does there exist a *bounded* set V in the plane such that all Euclidian transformations of V contain the same number of (Gaussian) lattice points?

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Addendum.

Continuation of section 3.

During the preparation of this report we extended our computations with respect to the occurrence $E(n) < 0$.

We found 85 values of $n \leq 40{,}000$ such that $E(n) < 0$.

Extension of the table on p.13

20229	34186
20635	35695
21885	36533
22299	36868
23592	36873
24725	37037
24795	37875
26333	38732
28662	38935
31043	39490
32810	