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NELLY KROONENBERG

THE COLLECTION OF ALL Z-SETS IN Q
IS A DENSE G_δ IN THE HYPERSPACE OF Q

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IS A DENSE G_δ IN THE HYPERSPACE OF Q

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The collection of all Z-sets in Q is a dense G_δ in the hyperspace of Q.

In this note it is proved that for any closed subset A of the Hilbert cube Q, the collection of all closed subsets of A which are a Z-set in Q, is a dense G_δ in the hyperspace of A. It is easily seen, as was pointed out to me by Prof. R.D. Anderson, that this collection contains a dense G_δ, e.g. the collection of all subsets of A which are disjoint from the pseudoboundary of Q. Probably the main interest of the theorem lies in the way property Z is approximated.

DEFINITIONS

The *Hilbert cube* Q is the countable product of intervals $\prod_{i=1}^{\infty} [-1,1]$ with the product topology.

A *closed* subset K of Q is a Z-*set* if for every open set O which is non-empty and homotopically trivial, $O\backslash K$ is non-empty and homotopically trivial *) (see [1]). Every finite subset of Q is a Z-set and for every Z-set $K \subset Q$ there exists a homeomorphism $h: Q \xrightarrow{\text{onto}} Q$ such that $h(K) \subset \{1\} \times \prod_{i=2}^{\infty} [-1,1]$.

The *hyperspace* 2^X of a *compact metric* space X with metric d is the collection of all *closed* subsets of X equipped with the metric $\tilde{d}(A,B) = \max(\{d(x,B) \mid x \in A\} \cup \{d(A,y) \mid y \in B\})$. The topology of 2^X does not depend on the metric chosen as long as different metrics are topologically equivalent; furthermore 2^X is compact. For every $K \subset Q$ and $\varepsilon > 0$ there exists a finite set (hence a Z-set) $K' \subset K$ such that $\tilde{d}(K,K') < \varepsilon$. As a consequence, for every closed set $A \subset Q$ the subsets of A which are Z-sets with respect to Q form a dense subset of 2^A.

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*) We employ the definition: X is homotopically trivial if for every n any map from the n-1-sphere S^{n-1} to X can be extended to a map from the n-cell D^n to X. By results of Whitehead [5], for ANR's (e.g. open subsets of Q), this is equivalent to contractibility. If we define $S^{-1} = \emptyset$ and $D^0 = \{0\}$, then homotopic triviality implies non-emptiness.

Closure and interior of $Y \subset Q$ are denoted by $\bar{Y}$ and Y^0 resp.. The set $\{x \mid d(x,Y) < \varepsilon\}$ is denoted by $U_\varepsilon(Y)$ and the set $U_\varepsilon(\{p\})$ by $U_\varepsilon(p)$.
If B is an open subset of Q such that $\bar{B}$ is homotopically trivial, then a closed set K is called a B-δ-Z-set if K has an open neighborhood $O \subset \bar{O} \subset U_\delta(K)$ such that $\bar{B}\setminus O$ is non-empty and homotopically trivial.
$Z(B,\delta)$ denotes the collection of all B-δ-Z-sets.

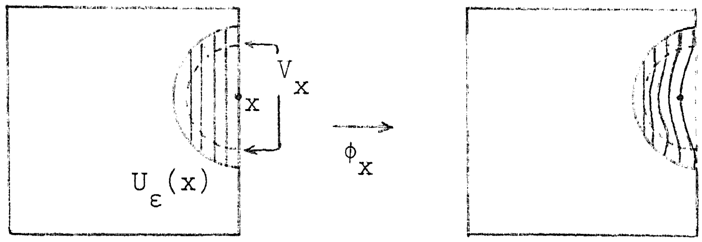
LEMMA 1. *If $A \subset Q$ is closed, B is open and $\bar{B}$ homotopically trivial then, for every $\delta > 0$, $Z(B,\delta) \cap 2^A$ is a dense open subset of 2^A.*

Proof: Suppose K is a B-δ-Z-set. There exists an open set O with $K \subset O \subset \bar{O} \subset U_\delta(K)$ such that $B\setminus O$ is homotopically trivial. Now for some $\delta' < \delta$ and $\varepsilon > 0$, $U_\varepsilon(K) \subset O \subset \bar{O} \subset U_{\delta'}(K)$. If K' is closed and $\tilde{d}(K,K') < \min(\varepsilon,\delta-\delta')$ then $K' \subset O \subset \bar{O} \subset U_\delta(K')$, hence K' is a B-$\delta$-Z-set by virtue of the same set O. □

Let $\mathcal{C}$ be the collection of open subsets B of Q such that $\bar{B} \cong Q$ and for every Z-set K in Q, $K \cap \bar{B}$ is a Z-set in $\bar{B}$.
Let $B \subset Q$ be a product of open subintervals of $[-1,1]$ with at most finitely many factors different from $[-1,1]$. Observing that the topological boundary $\bar{B}\setminus B$ is a Z-set in $\bar{B}$, and writing $K \cap B$ as a countable union of closed sets, one can prove from results on Z-sets that $K \cap \bar{B}$ is a Z-set in $\bar{B}$. Hence all such B are elements of $\mathcal{C}$, and therefore $\mathcal{C}$ contains a (countable) base for the topology of Q.

LEMMA 2. *For $B \in \mathcal{C}$ and $\delta > 0$ every Z-set is a B-δ-Z-set.*

Proof: Let $B \in \mathcal{C}$ and K be a Z-set in Q. Because $\bar{B} \cong Q$ and $K \cap \bar{B}$ is a Z-set in $\bar{B}$, there exists a homeomorphism $h: \bar{B} \xrightarrow{\text{onto}} Q$ mapping K into $W = \{1\} \times \prod_{i=2}^{\infty} [-1,1]$. We construct an embedding $\phi: Q \to Q\setminus h(K\setminus\bar{B})$ which is the identity outside an ε-neighbourhood of $h(K\cap\bar{B})$, where ε is such that $d(x,x') < \varepsilon \Longrightarrow d(h^{-1}(x),h^{-1}(x')) < \delta'$ for a fixed $\delta' < \delta$.



For every $x \in h(K\cap\bar{B})$, let ϕ_x be a motion "to the left" which is the identity outside $U_\varepsilon(x)$ and maps a neighborhood $V_x \subset U_\varepsilon(x)$ of x disjoint from W and changes only the first coordinate of any point. Cover $h(K\cap\bar{B})$ by finitely many sets V_x and let ϕ be the composition (in any order) of the corresponding homeomorphism ϕ_x.
Now ϕ is the identity outside $U_\varepsilon(h(K\cap\bar{B}))$. Furthermore, $h^{-1}\phi h(\bar{B})$ is homeomorphic to $\bar{B}$ and hence homotopically trivial. By choice of ε, $h^{-1}(h(\bar{B})\setminus\phi h(\bar{B})) = \bar{B}\setminus h^{-1}\phi h(\bar{B})$ is contained in $U_{\delta'}(K)$, hence $U_{\delta'}(K)\setminus h^{-1}\phi h(\bar{B})$ is the desired neighborhood of K. □

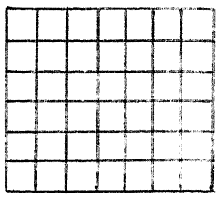
For K closed, B open and $\bar{B}$ homotopically trivial, we call K a B-Z-*set* if $\bar{B}\setminus K$ is non-empty and homotopically trivial. $Z(B)$ is the collection of all B-Z-sets, and, for $\mathcal{B}$ a collection of open sets with homotopically trivial closures, $Z(\mathcal{B})$ denotes the collection of the closed sets K which are a B-Z-set for all $B \in \mathcal{B}$.

LEMMA 3. <u>If</u> K <u>is a</u> B-δ-Z-<u>set for arbitrarily small</u> δ, <u>then</u> K <u>is a</u> B-Z-<u>set</u>.

Proof: Let $f\colon S^{n-1} \to \bar{B}\setminus K$ be given. Then for some δ, $d(f(S^{n-1}),K) > \delta$ (the usual distance; not $\tilde{d}$). Because K is a B-δ-Z-set, there exists a neighborhood $O \subset U_\delta(K)$ of K such that $\bar{B}\setminus O$ is homotopically trivial. Because $f(S^{n-1}) \subset \bar{B}\setminus O$, this provides for an extension $\bar{f}\colon D^n \to \bar{B}\setminus O \subset \bar{B}\setminus K$. □

LEMMA 4. <u>If</u> $\mathcal{B}$ <u>is a base for</u> Q <u>such that for all</u> $B \in \mathcal{B}$, $\bar{B}$ <u>is homotopically trivial, then</u> $Z(\mathcal{B})$ <u>consists of</u> Z-<u>sets</u>.

Proof: Let $K \in \mathcal{Z}(\mathcal{B})$ and let O be open and homotopically trivial. Let also $f: S^{n-1} \to O\setminus K$ be given. We want an extension $\bar{f}: D^n \to O\setminus K$ of f whereas we have an extension $g: D^n \to O$.
Cover $g(D^n)$ by a finite cover $\mathcal{B}_1 \subset \mathcal{B}$ such that $\forall B \in \mathcal{B}_1$, $\bar{B} \subset O$. There exists a closed neighborhood V_1 of $g(D^n)$ which is also covered by $\mathcal{B}_1$. Let ε_1 be a Lebesgue-number for $\mathcal{B}_1$ as covering of V_1 (i.e. each subset of V_1 with diameter less than ε_1 is contained in some element of $\mathcal{B}_1$). Define the <u>mesh</u> $m(\mathcal{A})$ of a collection $\mathcal{A}$ as $\sup\{\text{diameter}(A) \mid A \in \mathcal{A}\}$.
Let $\mathcal{B}_2 \subset \mathcal{B}$ be a covering of $g(D^n)$ with $\cup\mathcal{B}_2 \subset V_1$ and with $m(\mathcal{B}_2) < \frac{\varepsilon_1}{3}$.
There exists a closed neighborhood V_2 of $g(D^n)$ which is also covered by $\mathcal{B}_2$. Again let ε_2 be a Lebesque-number for $\mathcal{B}_2$ as a covering of V_2.
In this way, construct inductively a sequence $\mathcal{B}_1 \subset \mathcal{B}$, $\mathcal{B}_2 \subset \mathcal{B}$, $\ldots$, $\mathcal{B}_n \subset \mathcal{B}$ with Lebesgue-numbers $\varepsilon_1,\ldots,\varepsilon_n$ with respect to closed neighborhoods $V_1,\ldots,V_n$ of $g(D^n)$ and such that $\cup\mathcal{B}_{i+1} \subset V_i$ and $m(\mathcal{B}_{i+1}) < \frac{\varepsilon_i}{3}$.
Because g is uniformly continuous, there exists a $\delta > 0$ such that for x, $x' \in D^n$ and $d(x,x') < \delta$, $d(g(x),g(x')) < \frac{\varepsilon_n}{3}$.



Let P be a cell complex, consisting of a subdivision of D^n in equal subcells of diameter smaller than δ. Let P_i be the i-skeleton of P. Because for every $B \in \mathcal{B}$, $\bar{B}\setminus K$ is non-empty, it follows that K is nowhere dense. Hence there exists a mapping $\bar{f}_0: P_0 \to (\cup\mathcal{B}_n)\setminus K$ with with $d(g|_{P_0},\bar{f}_0) < \frac{\varepsilon_n}{3}$ and $\bar{f}_0|_{P_0\cap S^{n-1}} = f|_{P_0\cap S^{n-1}}$. *) Now for adjacent vertices $p,q \in P_0$, $d(\bar{f}_0(p),\bar{f}_0(q)) \leq d(\bar{f}_0(p),g(p)) + d(g(p),g(q)) + d(\bar{f}_0(q),g(q)) < \varepsilon_n$. Because ε_n is a Lebesgue-number for $\mathcal{B}_n$, $\bar{f}_0$ maps adjacent vertices into a common element of $\mathcal{B}_n$. Now $\{\bar{B}\setminus K \mid B \in \mathcal{B}_n\}$ consists of homotopically trivial sets; therefore we have an extension $\bar{f}_1: P_1 \to O\setminus K$ of $\bar{f}_0$, such that all 1-cells are mapped into an element of $\{\bar{B}\setminus K \mid B \in \mathcal{B}_n\}$. Moreover we may suppose that $\bar{f}_1|_{P_1\cap S^{n-1}} = f|_{P_1\cap S^{n-1}}$.
Furthermore it is easily seen that, if a mapping ϕ maps each face of an n-cell onto a set of diameter smaller than η, then ϕ maps the total boundary of the n-cell onto a set of diameter less than 3η. Observing that $m(\{\bar{B}\setminus K \mid B \in \mathcal{B}_n\}) \leq m(\mathcal{B}_n) < \frac{\varepsilon_n}{3}$, one sees that the boundary of

*) $\phi|_Y$ denotes the restriction of ϕ to the set Y.

every 2-cell of P_2 is mapped onto a set of diameter less than ε_n, which is a Lebesgue-number of $\mathcal{B}_{n-1}$ with respect to V_{n-1}. Since $\bar{f}_1(P_1) \subset \cup\mathcal{B}_n \subset V_{n-1}$, it follows that the image of the boundary of a 2-cell of P_2 is contained in an element of $\mathcal{B}_{n-1}$. Using homotopic triviality of the sets $\bar{B}\setminus K$ with $B \in \mathcal{B}_{n-1}$ one finds an extension $\bar{f}_2 \colon P_2 \to O\setminus K$ of $\bar{f}_1$ such that $\bar{f}_2|_{P_2\cap S^{n-1}} = f|_{P_2\cap S^{n-1}}$ and such that every 2-cell of P_2 is mapped into an element of $\{\bar{B}\setminus K \mid B \in \mathcal{B}_{n-1}\}$. Repeating this procedure, we find eventually the desired extension $\bar{f} = \bar{f}_n$ of f. □

THEOREM. <u>For every closed subset</u> A <u>of</u> Q <u>the collection of all</u> Z-<u>sets in</u> Q <u>intersects</u> 2^A <u>in a dense</u> G_δ.

Proof: Choose in lemma 4 $\mathcal{B} \subset \mathcal{C}$ countable. Then, according to lemma 1, $\bigcap_n \bigcap_{B\in\mathcal{B}} Z(B, \frac{1}{n})$ is a dense G_δ subset of 2^A and, according to lemma 2, 3 and 4 this collection contains exactly all closed subsets of A which are Z-sets in Q. □

PROBLEM. <u>Suppose</u> $A \cong Q$ <u>and</u> $A \subset Q$. <u>Is the set of all mappings from</u> Q <u>to</u> A <u>onto a</u> Z-<u>set in</u> Q (and hence the set of all homeomorphisms onto a Z-set in Q) <u>a dense</u> G_δ <u>in</u> A^Q (the space of all mappings from Q into A, with uniform convergence topology)? *)

A positive answer might lead to a proof of the existence of apparent boundaries in A which consists of Z-sets relative to Q.

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*) see Hurewicz-Wallman [2], page 64, theorem V4. It follows as in theorem V2, page 56, that for compact X the homeomorphisms form actually a dense G_δ.

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