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AFDELING MATHEMATISCHE BESLISKUNDE
(DEPARTMENT OF OPERATIONS RESEARCH)

BW 113/79 SEPTEMBER

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FACTORIZATION INDICES AT INFINITY FOR RATIONAL MATRIX FUNCTIONS

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1980): 93B25, 47B35

Factorization indices at infinity for rational matrix functions*)

by

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ABSTRACT

The paper studies the Wiener-Hopf factorizations at infinity of a transfer function and relates the corresponding factorization indices to the reachability indices of a feedback equivalent, feedback irreducible system.

KEY WORDS & PHRASES: *Wiener-Hopf factorization indices, feedback equivalence, feedback reducibility.*

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*) This report will be sumitted for publication in Integral Equations and Operator Theory.

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1. Introduction

In the last few years there has been an increasing interest from the side of operator theorists in the area of multivariable linear system theory. This interest was originally mainly motivated by generalization to infinite dimensional systems, but it has recently become apparent that also the structure theory of finite dimensional linear systems may have applications, for example in the theory of integral equations.

In this note we point out some intersting relations between the so-called factorization indices of operator theory and the reachability and observability indices of linear system theory. The factorization indices originally came up in the theory of systems of singular integral equations [5, 6]. They play also a role in partial differential equations and the classification of holomorphic vector bundles on the Riemann sphere. That there is a relation between the two sets of indices has been demonstrated for the first time in [7] where there is an analysis of the nonsingular polynomial matrix case which makes an implicit connection. In this paper we remove the restriction on nonsingular polynomial matrices. Moreover, by using purely algebraic methods, we can carry the analysis over an arbitrary field. However, in contrast with [7] where general contours are treated, we will only do the 'global' case, namely that of the factorization indices at infinity. Given a strictly proper rational matrix function G, we construct a minimal realization $\Sigma = (A,B,C)$ of G. We identify the reachability indices of (A,B) with the left factorization indices of the denominator matrix in a right coprime factorization of G. The factorization indices of G itself are associated with the reachability indices of an associated feedback irreducible system feedback equivalent to G.

2. Preliminaries and Notation

Throughout, our approach is purely algebraic which allows us to work over an arbitrary field F. Let F^n denote the vector space of all n-vectors, with elements from F and $F^{p\times m}$ the $p\times m$ matrices with elements from F.

If V is any vector space over F then $V[\lambda]$ denotes the module over $F[\lambda]$, consisting of all polynomials with coefficients in V, and $V[[\lambda^{-1}]]$ the set of all formal power series in λ^{-1} with coefficients in V. The subset of those formal power series with vanishing constant term will be denoted by $\lambda^{-1}V[[\lambda^{-1}]]$, and $V((\lambda^{-1}))$ is the set of all truncated Laurent series with coefficients in V. Hence we have

$$(2.1) \qquad V((\lambda^{-1})) = V[\lambda] \oplus \lambda^{-1}V[[\lambda^{-1}]].$$

Clearly, $V((\lambda^{-1}))$ is a module over $F[\lambda]$ with $V[\lambda]$ as a submodule, and $\lambda^{-1}V[[\lambda^{-1}]]$ can be identified with the quotient module $V((\lambda^{-1}))/V[\lambda]$. We define two projections in $V((\lambda^{-1}))$ by

$$(2.2) \qquad \pi_- \sum_{-\infty<k\le k_0} v_k\lambda^k = \sum_{-\infty<k<0} v_k\lambda^k$$

and

$$(2.3) \qquad \pi_+ = I - \pi_-$$

The sets $F^{n\times n}[\lambda]$ and $F^{n\times n}[[\lambda^{-1}]]$ are rings. An invertible element in $F^{n\times n}[\lambda]$ is called <u>unimodular</u> whereas an invertible element of $F^{n\times n}[[\lambda^{-1}]]$ is called a <u>bicausal isomorphism</u>. It is easy to see that $A \in F^{n\times n}[\lambda]$ is unimodular if and only if $\det A$ is a nonzero constant whereas $B \in F^{n\times n}[[\lambda^{-1}]]$ is a bicausal isomorphism if and only if the constant term in its power series is nonsingular.

Given $A \in F^{p\times m}((\lambda^{-1}))$ then A induces a multiplication operator, called a Laurent operator, $L_A: F^m((\lambda^{-1})) \to F^p((\lambda^{-1}))$ defined by

$$(2.3) \qquad (L_A f)(\lambda) = g(\lambda) = \sum_k g_k \lambda^k$$

with

$$(2.4) \qquad g_k = \sum_j A_{k-j} f_j$$

Clearly the sum in (2.4) is well defined as it contains only a finite number of nonzero elements. The Hankel operator $H_A: F^m[\lambda] \to \lambda^{-1}F^p[[\lambda]]$ and the Toeplitz operator $T_A: P^m[\lambda] \to F^p[\lambda]$ are derived from L_A by

$$(2.5) \qquad H_A = \pi_- L_A \mid F^m[\lambda]$$

and

$$(2.6) \qquad T_A = \pi_+ L_A \mid F^m[\lambda].$$

3. Factorization Indices at Infinity

Let $G \in F^{p\times m}((\lambda^{-1}))$ be rational. A left (Wiener Hopf) factorization at infinity is a factorization of G of the form

$$(3.1) \qquad G = G_- D G_+$$

with $G_+ \in F^{m\times m}[\lambda]$ unimodular, $G_- \in F^{p\times p}[[\lambda^{-1}]]$ a bicausal isomorphism, and $D(\lambda) = \begin{pmatrix} \Delta(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$ where $\Delta(\lambda) = \mathrm{diag}(\lambda^{\kappa_1}, \dots, \lambda^{\kappa_r})$. The integers κ_i, assumed decreasingly ordered, are called the left factorization indices at infinity and will be denoted by $\kappa_R = (\kappa_1, \dots, \kappa_r)$. A right factorization and the right factorization indices are analogously defined with the plus and minus signs in (3.1) reversed.

The basic properties of actorizations have been derived by Gohberg and Krein [6]. We indicate an approach which stresses the notion of column properness that is widely used in system theory.

Given $D \in F^{p\times m}[\lambda]$, let $d^{(i)}(\lambda)$ denote the i-th column of $D(\lambda)$. The degree of $d^{(i)}(\lambda)$, i.e. the degree of the highest degree element in $d^{(i)}(\lambda)$, is called the i-th column degree. Let us denote it by $\delta_i = \deg d^{(i)}(\lambda)$. The coefficient vector of λ^{δ_i} is called the i-th leading coefficient column vector and is denoted by $[d^{(i)}]_c$. We let $[D]_c$ be the matrix of leading coefficient column vectors. A matrix D is called <u>column proper</u> if rank $[D]_c$ = rank D. In an analogous way we define row properness.

The basic result on column properness is the following [15]:

<u>THEOREM 3.1</u>. Let $D \in F^{p\times m}[\lambda]$. Then there exists a unimodular matrix $U \in F^{m\times m}[\lambda]$ such that $DU = (D_1\ 0)$ with $D_1 \in F^{p\times r}[\lambda]$ a full column tank, column proper matrix with column degrees $\kappa_1,\dots,\kappa_r$ decreasingly ordered. The column degrees are uniquely determined, although U is not, and are called the column indices of D.

This theorem yield immediately the existence of factorizations.

<u>THEOREM 3.2</u>. Let $G \in F^{p\times m}((\lambda^{-1}))$ be rational. Then there exist left and right factorizations of G.

<u>PROOF</u>. Assume to begin with that $G \in F^{p\times m}[\lambda]$. By the previous theorem there exists a unimodular V such that $G(\lambda)V(\lambda) = (G_1(\lambda)\ 0)$ with G_1 p×r column proper with column indices $\kappa_1 \geq \dots \geq \kappa_r$. The column properness of G_1 implies the left invertibility of $[G_1]_c$ and we denote by E_0 the left inverse of $[G_1]_c$. Let E be any invertible p×p matrix whose first r rows coincide with E_0. Then $E(G_1(\lambda)\ 0)$ has the λ^{κ_j} terms with unit coefficients on the diagonal and all other terms in the j-th column of lower degree. So we can write

$$E(G_1(\lambda)\ 0) = \begin{pmatrix} \Omega_{11}(\lambda) & 0 \\ \Omega_{21}(\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Delta(\lambda) & 0 \\ 0 & I_{m-r} \end{pmatrix}$$

$$= \begin{pmatrix} \Omega_{11}(\lambda) & 0 \\ \Omega_{21}(\lambda) & I_{p-r} \end{pmatrix} \begin{pmatrix} \Delta(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\Omega_{11}(\lambda) = I + \Omega'_{11}(\lambda)$$

with

$$\Omega'_{11} \in \lambda^{-1}F^{r\times r}[[\lambda^{-1}]],$$

$$\Omega_{21} \in F^{(p-r)\times r}[[\lambda^{-1}]]$$

and

$$\Delta(\lambda) = \mathrm{diag}(\lambda^{\kappa_1},\ldots,\lambda^{\kappa_r}) \in F^{r\times r}[\lambda].$$

Since Ω_{11} is a bicausal isomorphism it has an inverse Γ_{11}. Define $\Gamma_0 \in F^{p\times p}[[\lambda^{-1}]]$ by

$$\Gamma_0 = \begin{pmatrix} \Gamma_{11} & 0 \\ \Omega_{21}\Gamma_{11} & I \end{pmatrix}$$

then

$$\Gamma_0 \begin{pmatrix} \Omega_{11} & 0 \\ \Omega_{21} & I \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}.$$

Altogether we have $(\Gamma_0 E)GV = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}$ and hence the factorization $G = G_- D G_+$ follows with $G_- = (\Gamma_0 E)^{-1}$, $G_+ = V^{-1}$ and $D = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}$.

In the general case there exists, by the assumption of the rationality of G, a nonzero polynomial $g(\lambda) = \lambda^n + g_{n-1}\lambda^{n-1}+\ldots+g_0$ such that gG is a polynomial matrix. Let $gG = H_- D\, H_+$ be a right factorization. Since $g(\lambda)^{-1} = \lambda^{-n}\gamma(\lambda)$ and

$$\gamma(\lambda) = \left(1 + \frac{g_{n-1}}{\lambda} + \ldots + \frac{g}{\lambda^n}\right)$$

it follows that a right factorization of G exists with $G_- = \gamma H_-$,

$G_+ = H_+$ and $D_1 = \lambda^{-n}D$.

To obtain a left factorization of G we transpose a right factorization of G.

It is obvious from the construction that for $G \in F^{p\times m}[\lambda]$ the factorization indices of G are nonnegative. Also if $G \in F^{p\times m}[[\lambda^{-1}]]$ is rational then its factorization indices are nonpositive.

It is clear that if G is singular then the right and left factorizations of G are not unique. However, even in the nonsingular case we do not have uniqueness. This has been studied by Gohberg and Krein [6] and we note a special case of their result, adapted to our circumstances.

THEOREM 3.3. Let $G \in F^{m\times m}((\lambda^{-1}))$ be nonsingular and let $G = G_-\Delta G_+ = G'_-\Delta' G'_+$ be two left factorizations. Then $\Delta = \Delta'$ and there exists a unimodular matrix $U \in P^{m\times m}[\lambda]$ satisfying

$$(3.2)\qquad \begin{array}{ll} u_{ij} = 0 & \text{if } \kappa_i > \kappa_j \\ \text{degree } u_{ij} \le \kappa_j - \kappa_i & \text{if } \kappa_j \ge \kappa_i \end{array}$$

for which $G_+ = UG_+$ and $G_- = G_-\Delta U^{-1}\Delta^{-1}$.

Clearly the set of all unimodular matrices $U \in F^{m\times m}[\lambda]$ satisfying condition (3.2) forms a multiplicative group called the left factorization group. An analogous result naturally holds for right factorizations.

Actually the proof of [5] can be adapted to the singular case to yield the unqiueness of the factorization indices for any rational metrix. However, the freedom of the factors G_+ and G_- is greater in this case.

4. Feedback Equivalence

In this section we indicate some relations between factorizations and certain problems of system theory.

The proper rational matrix $G \in F^{p\times m}([\lambda^{-1}])$ introduced in the previous section has various representations. It may be written as $G(\lambda) = D + C(\lambda I - A)^{-1}B$ with $A \in F^{n\times n}$, $B \in F^{n\times m}$, $C \in F^{p\times n}$ and $D \in F^{p\times m}$. The quadruple (A,B,C,D) is called a realization of G_ε in the sense that G is the transfer function of the linear system

$$(4.1) \qquad \Sigma: \begin{array}{l} x_{n+1} = Ax_n + Bu_n \\ \\ y_n = Cx_n + Du_n. \end{array}$$

We will use the terminology reachability, observability, minimality, McMillan degree, reachability indices etc. in the sense it is used in this context [10, 15].

Like every rational function, G admits factorizations of the form

$$(4.2) \qquad G(\lambda) = N_r(\lambda)D_r(\lambda)^{-1} = D_\ell(\lambda)^{-1}N_\ell(\lambda)$$

with N_r, $N_\ell \in F^{p\times m}[\lambda]$, $D_r \in F^{m\times m}[\lambda]$ and $D_\ell \in F^{p\times p}[\lambda]$, where we can assume that $N_rD_r^{-1}$ is a right coprime factorizarion and $D_\ell^{-1}N_\ell$ a left coprime factorization [13]. In a right coprime factorization D_r and N_r are determined up to a right unimodular factor. The extra freedom provided by the unimodular factor can be used to make D_r satisfy some extra condition like column properness, as in Theorem 3.1.

Consider the set of transformations defined on quadruples, with m, n and p being fixed, by $(A,B,C,D) \to (R^{-1}(A + BK)R, R^{-1}BP, (C + DK)R, DP)$ with $\det P \neq 0$ and $\det R \neq 0$. The set of all such transformations forms a group called the feedback group. Two quadruples (A_i,B_i,C_i,D_i), $i = 1,2$, are called feedback equivalent if there is an element of the feedback group mapping one into the other. Two transfer functions G_1 and G_2 are feedback equivalent if they have the same McMillian degree and if they have feedback equivalent canonical realizations. This is clearly an equivalence relation on the rational element in $F^{p\times m}[[\lambda^{-1}]]$.

Feedback has been studied using coprime factorizations by Hautus and Heymann [8] as well as by one of the authors [4]. The main result can be stated as follows:

THEOREM 4.1. Let G be a proper rational function in $F^{p\times m}[[\lambda^{-1}]]$

and let $G = ND^{-1}$ be a right coprime factorization. Then the transfer function G_1 of a system feedback equivalent to a canonical realization of G has the representation $G_1 = G\Gamma$ where Γ is a bicausal isomorphism.

Actually in this case $\Gamma = P(D+Q)$ with P a constant nonsingular matrix and $Q \in F^{m\times m}[\lambda]$ such that $QD^{-1} \in \lambda^{-1}F^{m\times m}[[\lambda^{-1}]]$. In particular this guarantees that the McMillian degree of G_1 is not greater than that of G.

The result of Hautus and Heymann allows us to clarify the connection of factorizations to system theory. From the right coprime factorization $G(\lambda) = N_rD_r^{-1} = \Sigma_{i=0}^{\infty} G_i\lambda^{-1}$ one obtains easily a canonical realization following the procedure outlined in [2, 3]. We define a projection $\pi_{D_r} : F^m[\lambda] \to F^m[\lambda]$ by

(4.3) $$\pi_{D_r} f = D_r\pi_- D_r^{-1} f$$

and denote its range by K_{D_r}. Clearly K_{D_r} is isomorphic to the quotient module $F^m[\lambda]/D_rF^m[\lambda]$ as Ker $\pi_{D_r} = D_rF^m[\lambda]$. Let

(4.4) $$S_{D_r} f = \pi_{D_r}\lambda f \qquad \text{for } f \in K_{D_r}$$

Define now the quadruple (A,B,C,D) by $A = S_{D_r}$, $B\xi = \pi_{D_r}\xi$ for $\xi \in F^m$, $Cf = (N_rD_r^{-1}f)_{-1}$, and $D = G_0$ then this is a canonical realization of G. In particular (S_{D_r}, π_{D_r}) is a pair isomorphic to (A,B) in any canonical realization of G. Thus the reachability indices are derivable from D_r and it is well known that actually they are equal to the column indices of D_r [8, 4]. This allows us to state the following

THEOREM 4.2. Let $G \in F^{p\times m}[[\lambda^{-1}]]$ be a proper rational function admitting the coprime factorizations (4.2) and let (A,B,C,D) be any canonical realization og G. Then the reachability indices of the realization are equal to the left factorization indices of D_r and the observability indices are equal to the right factorization indices of D_ℓ.

PROOF. By the state space isomorphism theorem the pair (A,B) is isomorphic to the pair (S_{D_r}, π_{D_r}). By Theorem 3.1 there exists a unimodular matrix U such that D_rU is column proper with column indices $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$. Clearly

(4.5) $\qquad D_r U = \Gamma\Delta$

with $\Delta(\lambda) = \text{diag}(\lambda^{\kappa_1},\ldots,\lambda^{\kappa_m})$ and Γ a bicausal isomorphism. This implies that (S_{D_r},π_{D_r}) and (S_Δ,π_Δ) are feedback equivalent pairs. However, the reachability indices of (S_Δ,π_Δ) are easily seen to be equal to $\kappa_1,\ldots,\kappa_m$. Finally (4.5) can be rewritten as

(4.6) $\qquad D_r = G_-\Delta G_+$

with $G_- = \Gamma$ and $G_+ = U^{-1}$; i.e., we have a left factorization of D_r. The statement concerning observability indices follows by duality. □

The pair (S_Δ,π_Δ) should be considered as the polynomial way of writing the Brunovsky canonical form [15]. Indeed, since $K_\Delta = K_{\lambda^{\kappa_1}} \oplus \ldots \oplus K_{\lambda^{\kappa_m}}$, the natural choice of basis in K_Δ, namely $\{\lambda^i e_j \mid j = 1,\ldots,m,\ i = 0,\ldots,\kappa_j-1\}$ with $e_1,\ldots,e_m$ the standard basis for F^m, yields the better known matrix representation. In view of this (4.6) is just a shorthand notation for reducing a reachable pair (A,B) to Brunovsky canonical form. Here G_+ provides a similarity transformation whereas G_- provides the feedback and change of basis in the input space.

Given a reachable pair (A,B) then the set of all elements of the feedback group that leave (A,B) invariant is a subgroup called the stabilizer at (A,B). This subgroup has been studied previously by Brockett [1] using the Brunovsky form and Münzner and Prätzel-Wolters [12] using polynomial methods. Our approach is based on the previous analysis. Clearly the stabilizers at feedback equivalent pairs are isomorphic so we may as well study the stabilizer at the Brunovsky canonical form. It is immediately evident that the structure of the stabilizer depends only on the reachability indices, $\kappa_1,\ldots,\kappa_m$, of the pair (A,B).

THEOREM 4.3. Let (A,B) be a reachable pair and let $H(\lambda)D(\lambda)^{-1}$ be a right coprime factorization of $(\lambda I - A)^{-1}B$. Then the stabilizer of (A,B) is isomorphic to the left factorization group of D.

PROOF. The pair (A,B) is isomorphic to (S_D,π_D) and in turn to the

Brunovsky form (S_Δ,π_Δ) with $\Delta(\lambda) = \mathrm{diag}(\lambda^{\kappa_1},\ldots,\lambda^{\kappa_m})$. It suffices therefore to study the stabilizer at (S_Δ,π_Δ). This is equivalent to finding all solutions of the equation

$$(4.7) \qquad \Gamma\Delta = \Delta U$$

with $U \in F^{m\times m}[\lambda]$ unimodular and $\Gamma \in F^{m\times m}[[\lambda^{-1}]]$ a bicausal isomorphism. Equation (4.7) is equivalent to $\gamma_{ij}\lambda^{\kappa_j} = \lambda^{\kappa_i}u_{ij}$ which in turn implies

$$(4.8) \qquad u_{ij} = \begin{cases} 0 & \text{if } \kappa_i > \kappa_j \\ \text{of degree} \le \kappa_j - \kappa_i & \text{if } \kappa_j \ge \kappa_i. \end{cases}$$

Conversely, if U is unimodular and satisfies (4.8) then it is easily seen that equation (4.7) is solvable with a bicausal isomorphism Γ.

Thus the unimodular solutions of (4.7) have a block triangular structure. This structure is reflected in Γ. If we let $\Gamma(\lambda) = \Sigma_{\nu=0}^{\infty} \Gamma^{(\nu)}\lambda^{-\nu}$ then $\Gamma^{(0)} = U(0)$ and

$$(4.9) \qquad \gamma_j^{(\nu)} = 0 \qquad \text{if } \nu > \kappa_j \text{ or } \kappa_i > \kappa_j. \qquad \square$$

The results of this section can be taken as an alternative derivation of the factorization of Section 3. Indeed Theorem 3.2, in the nonsingular case, is equivalent to the existence of Brunovsky's canonical form, the uniqueness of the diagonal matrix Δ in Theroem 3.3 is equivalent to the uniqueness of the reachability indices, and finally the nonuniqueness of the factorizations in Theroem 3.3 is determined through the stabilizer.

5. Feedback Irreducibility and Factorization Indices

Given a rational G in $F^{p\times m}[[\lambda^{-1}]]$ with the right coprime factorization $G = ND^{-1}$ we were able to identify the left factorization indices at infinity of D with the reachability indices of any canonical realization of G. However, by Theorem 3.2, G itself has a right factorization and corresponding right factorization indices and one wants a system theoretic interpretation of these.

This is closely connected to problems of feedback irreducibility which we now proceed to describe.

In the set of rational elements of $F^{p\times m}[[\lambda^{-1}]]$ we introduce a partial order. We say G_1 is <u>feedback reducible</u> to G_2, and write $G_1 \succ G_2$, if G_2 is the transfer function of a sysetm (state) feedback equivalent to a canonical realization of G_1. If $\delta(G)$ denotes the McMillan degree of G we clearly have that $G_1 \succ G_2$ implies $\delta(G_1) \geq \delta(G_2)$. It is easy to see that feedback reducibility is a reflexive and transitive relation. Also feedback equivalence of G_1 and G_2, as defined in the previous section, holds if and only if $G_1 \succ G_2$ and $G_2 \succ G_1$. Thus the definition of feedback equivalence is in agreement with Morse's [11]. A rational function G is <u>feedback irreducible</u> or <u>minimal</u> if whenever $G \succ G'$ we have $\delta(G') = \delta(G)$.

The study of the irreducible transfer functions is intimately related to some questions of geometric control theory [15]. Given a system (A,B,C) we say a subspace V of the state space is (A,B)-invariant if $AV \subset V + \mathcal{B}$ where $\mathcal{B}$ = Range B. Let now $G \in F^{p\times m}[[\lambda^{-1}]]$ and we assume for simplicity that G is strictly proper. With each of the following coprime factorizations of G, namely ND^{-1} and $T^{-1}U$ is associated a state space model with the state spaces being respectively K_D and K_T. For the appropriate definitions we refer to [2, 3]. In terms of these factorizations we have the following characterizations of (A,B)-invariant subspaces that are included in Ker C. The details of the proof will be published elsewhere [16]:

<u>THEOREM 5.1</u>. A subspace V og K_D is an (A,B)-invariant subspace of K_D if and only if

$$(5.1) \qquad V = T_{DD_1^{-1}}(E_1 K_{F_1})$$

where DD_1^{-1} is a bicausal isomorphism, $T_{DD_1^{-1}}$ the Toeplitz operator induced by DD_1^{-1}, $D_1 = E_1 F_1$, and $N = N_1 F_1$.

<u>THEOREM 5.2</u>. A subspace V of K_T is an (A,B)-invariant subspace in Ker C if and only if

$$(5.2) \qquad V = U_0 K_{E_0}$$

where $U = U_0E_0$ is a factorization of U with $E_0 \in P^{m\times m}[\lambda]$ nonsingular and $U_0 \in F^{p\times m}[\lambda]$.

As an immediate corollary of Theorem 5.1 we have the following.

THEOREM 5.3. Let $G \in F^{p\times m}[[\lambda^{-1}]]$ be rational and let (A,B,C,D) be a canonical realization of G. Then G is feedback irreducible if and only if there is no nontrivial (A,B)-invariant subspace in Ker C.

We recall that the characterization of feedback irreducibility given by this theorem actually serves as the definition in [11].

LEMMA 5.4. Let $G \in F^{n\times m}[[\lambda^{-1}]]$ have the right factorization $G = U\nabla\Gamma$ with $U \in F^{p\times p}[\lambda]$ unimodular and $\Gamma \in F^{m\times m}[[\lambda^{-1}]]$ a bicausal isomorphism. Let $N \in F^{p\times m}[\lambda]$ be left invertible then NG has a factorization $NG = U_1\begin{pmatrix}\Delta\\0\end{pmatrix}\Gamma$ with $U_1 \in F^{p\times p}[\lambda]$ unimodular. In particular G and NG have the same right factorization indices.

PROOF. Since N is left invertible, the Smith form of N is $\begin{pmatrix}I\\0\end{pmatrix}$ and so there exist unimodular matrices V and W such that $N = V\begin{pmatrix}I\\0\end{pmatrix}W$. Hence

$$NG = V\begin{pmatrix}I\\0\end{pmatrix}WU\Delta\Gamma = V\begin{pmatrix}WU\\0\end{pmatrix}\Delta\Gamma = V\begin{pmatrix}WU & 0\\0 & I\end{pmatrix}\begin{pmatrix}\Delta\\0\end{pmatrix}\Gamma = U_1\begin{pmatrix}\Delta\\0\end{pmatrix}\Gamma$$

where

$$U_1 = V\begin{pmatrix}WU & 0\\0 & I\end{pmatrix}. \qquad \square$$

We are now ready to interpret the factorization indices at infinity of a rational function as reachability indices.

THEOREM 5.5. Let $G \in F^{p\times m}[[\lambda^{-1}]]$ be rational. Then the right factorization indices at infinity are equal to the negatives of the reachability indices of any canonical realization of any feedback irreducible G' that satisfies $G \succ G'$.

PROOF. In view of Theorem 4.1 if $G \succ G'$ then G and G' have the same right factorization indices. So without loss of generality we may assume G is feedback irreducible. Also for simplicity we assume G is strictly proper.

To begin with we consider that G is of full column rank as a matrix over the rational functions. In this case we claim that if $G = ND^{-1}$ is a right coprime factorization of G then the feedback irreducibility of G is equivalent to the left invertibility of N. Indeed, if G is feedback reducible, then for some bicausal isomorphism Γ we have ND_1^{-1}, with $D_1 = \Gamma^{-1}D$, a factorization with a nontrivial common right factor, say F_0. Thus $N = N_0F_0$ and $D_1 = E_0F_0$. Since N is left invertible it follows that for some $M \in F^{m\times p}[\lambda]$ we have $MN = I$ and hence $MN = (MN_0)F_0 = I$. This contradicts the nontriviality of F_0, hence F_0 is unimodular.

Conversely assume N is not left invertible. There exists therefore a factorization $N = N_0F_0$ with $N_0 \in F^{p\times m}[\lambda]$ invertible and F_0 a nonsingular nonunimodular element in $F^{m\times m}[\ ]$. By Lemma 5.4 we have that $G = ND^{-1}$ and F_0D^{-1} have the same right factorization indices, which are necessarily all negative.

Let $F_0D^{-1} = U\Delta^{-1}$ be a right factorization of F_0D^{-1} with $\Delta(\lambda) = \mathrm{diag}(\lambda^{\kappa_1},\dots,\lambda^{\kappa_m})$ and $\kappa_i > 0$. It follows that $D_1 = \Delta U^{-1}F_0 = \Gamma D$. But $E_0 = \Delta U^{-1}$ is in $F^{n\times n}[\lambda]$ and hence G is feedback reducible.

Therefore we conclude that G is injective and ND^{-1} is a coprime factorization of G then, by Lemma 5.4, the factorization indices of G are equal to the factorization indices of D^{-1} which are the negatives of the reachability indices of any canonical realization of G.

If G is not injective then there exists a becausal isomorphism Γ such that $G\Gamma = (G_0\ 0)$ with G_0 injective. If $G_0 = N_0D_0^{-1}$ is a right coprime facotrization of G_0 then

$$(N_0\ 0)\begin{pmatrix} D_0 & 0 \\ 0 & I \end{pmatrix}^{-1}$$

is a right coprime factorization of $G\Gamma$ and G is feedback irreducible if and only if G_0 is and we can apply the first part of the proof. □

One should point out that a very closely related characterization of feedback irreducibility has been given by Heymann in [9].

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