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*Probability, Networks and Algorithms*

On some conjectural inequalities and their consequences

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# On some conjectural inequalities and their consequences

## ABSTRACT

We discuss some conjectural inequalities involving the sums  $\text{sig}_n(s) := 1^s + 2^s + \dots + n^s$ . Two of our Conjectures assert that both  $a(s) := \text{sig}_{(n+1)}(s)/\text{sig}_n(s)$  and  $a(s) \cdot \text{sig}_{(n+1)}(s)/\text{sig}_{(n+2)}(s)$  are strictly log-convex in  $s$  on all of the real axis. We will also present a common generalization of these two Conjectures. Various applications are described, to existing theorems as well as to some other unproven conjectures.

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*1998 ACM Computing Classification System:* F.2.1

*Keywords and Phrases:* Inequalities; (log-)convexity; sums of powers



# On some conjectural inequalities and their consequences

(on the occasion of Dr N. M. Temme's departure from  
CWI on May 27, 2005)

by

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ABSTRACT. We discuss some conjectural inequalities involving the sums  $\sigma_n(s) := 1^s + 2^s + \dots + n^s$ . Two of our Conjectures assert that both  $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  and  $\frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\sigma_{n+2}(s)}$  are *strictly log-convex* in  $s$  on all of  $\mathbb{R}$ . We will also present a common generalization of these two Conjectures. Various applications are described, to existing theorems as well as to some other unproven conjectures.

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## Part I.

### 1. Statement of the first Conjecture

For  $s \in \mathbb{R}$  and  $n \in \mathbb{N} := \{1, 2, 3, \dots\}$  we define

$$(1.1) \quad \sigma_n(s) := 1^s + 2^s + 3^s + \dots + n^s$$

and

$$(1.2) \quad Q(s) := Q_n(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)}.$$

Our first CONJECTURE, to be discussed in this section, may be formulated in various (obviously) equivalent ways:

(A) For every (fixed)  $n \in \mathbb{N}$  the function  $Q(s)$  is *strictly log-convex* on all of  $\mathbb{R}$ .

(B) For every (fixed)  $n \in \mathbb{N}$  the function  $\log Q(s)$  is *strictly convex* on all of  $\mathbb{R}$ .

$$(C) \quad Q(s) Q''(s) > (Q'(s))^2.$$

$$(D) \quad \frac{Q(s)}{Q'(s)} > \frac{Q'(s)}{Q''(s)}.$$

$$(E) \quad \frac{\sigma_{n+1}(s)\sigma_{n+1}''(s) - (\sigma_{n+1}'(s))^2}{\sigma_{n+1}^2(s)} > \frac{\sigma_n(s)\sigma_n''(s) - (\sigma_n'(s))^2}{\sigma_n^2(s)}.$$

$$(F) \quad \frac{\sigma_{n+1}''(s)}{\sigma_{n+1}(s)} - \left(\frac{\sigma_{n+1}'(s)}{\sigma_{n+1}(s)}\right)^2 > \frac{\sigma_n''(s)}{\sigma_n(s)} - \left(\frac{\sigma_n'(s)}{\sigma_n(s)}\right)^2.$$

It seems that ( even for small  $n \geq 2$  ) this is by no means trivial. The reader is invited to give it a try.

## 2.0 The genesis of the first Conjecture

In [1], [2], [3] and [10] we considered the *monotonicity* of the canonical Riemann Upper and Lower sums corresponding to the elementary integral  $\int_0^1 x^s dx$ , ( $s$  fixed and  $> 0$ )

$$(2.0.1) \quad U_n(s) := \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^s = \frac{\sigma_n(s)}{n^{s+1}} \quad \text{and} \quad L_n(s) := \frac{1}{n} \sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^s = \frac{\sigma_{n-1}(s)}{n^{s+1}}.$$

It was shown there that ( for fixed  $s > 0$  ) the sequences  $(U_n(s))_{n \in \mathbb{N}}$  and  $(L_n(s))_{n \in \mathbb{N}}$  are *strictly monotonic* in  $n$ , ( $U_n(s)$  being *strictly decreasing* and  $L_n(s)$  being *strictly increasing* ):

$$(2.0.2) \quad U_{n+1}(s) < U_n(s) \quad \text{or} \quad \frac{\sigma_{n+1}(s)}{(n+1)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}}, \quad (s > 0)$$

and

$$(2.0.3) \quad L_{n+1}(s) > L_n(s) \quad \text{or} \quad \frac{\sigma_n(s)}{(n+1)^{s+1}} > \frac{\sigma_{n-1}(s)}{n^{s+1}}, \quad (s > 0).$$

It is easily seen that these inequalities may be put together as ( note that  $\sigma_{n+1}(s) = \sigma_n(s) + (n+1)^s$  )

$$(2.0.4) \quad \frac{n^{s+1}(n+1)^s}{(n+1)^{s+1} - n^{s+1}} < \sigma_n(s) < \frac{n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \quad (s > 0).$$

PROOF of (2.0.2). ( Assertion (2.0.3) may be shown in a similar manner. )

Inequality (2.0.2) is easily seen to be true for  $n = 1$  and all  $s > 0$ .

Assume that it is still true for  $n = 1, \dots, N$  and all  $s > 0$ . Then we have

$$(2.0.5) \quad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) > (N+1)^s + \frac{N^{s+1}(N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

$$(2.0.6) \quad (N+1)^s + \frac{N^{s+1}(N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geq \frac{(N+1)^{s+1}(N+2)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

or

$$(2.0.7) \quad (N+1)^s(N+2)^{s+1} - (N+1)^{2s+1} \geq (N+1)^{s+1}(N+2)^s - N^{s+1}(N+2)^s.$$

Writing  $x := 1/(N+1)$  we will be through if we can show that

$$(2.0.8) \quad (1+x)^{s+1} - 1 \geq (1+x)^s - (1-x)^{s+1}(1+x)^s, \quad (0 < x \leq \frac{1}{2})$$

or

$$(2.0.9) \quad \frac{(1+x)^{s+1} - 1}{x} \geq \frac{1 - (1-x^2)^{s+1}}{x^2}, \quad (0 < x \leq \frac{1}{2}).$$

Since for every ( fixed )  $s > 0$  the function  $t^{s+1}$  is convex in  $t$  on  $\mathbb{R}^+$ , it follows that (2.0.9) is true for all  $x \in (0,1)$ , completing the proof of (2.0.2). ■

Various other proofs of (2.0.2) have been given. See [2] and NAvW [1], especially the generalization by van Lint, who proved that if  $f: [a,b] \rightarrow \mathbb{R}$  is either convex or concave on  $[a,b]$ , then the sequence of canonical Riemann Upper / Lower sums corresponding to  $\int_a^b f(x) dx$  is decreasing / increasing ( in  $n$  ).

This should suffice, one would say. Enough is enough !

However, while studying inequality (2.0.2) it was observed that, with

$$(2.0.10) \quad h(s) := h_n(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)} \left( \frac{n}{n+1} \right)^s, \quad (s \in \mathbb{R})$$

we may write (2.0.2) as

$$(2.0.11) \quad h(s) < \frac{n+1}{n}, \quad (s > 0).$$

Since  $h(0) = \frac{n+1}{n}$ , the truth of (2.0.2) would also follow from the assertion:  $h(s)$  is *strictly decreasing* ( in  $s$  ) on all of  $\mathbb{R}$ . In 1995 we succeeded in proving this assertion. Although fully elementary, the proof is *not very simple*, and since it was only published in [4], we will present the full original version here. ( The proof in [4] contains some printing errors. ) Actually we will prove the following slightly stronger

**THEOREM.**

$$(2.0.12) \quad h'(s) < 0 \quad \text{for all } s \in \mathbb{R}.$$

**PROOF** of (2.0.12). After rewriting  $h(s)$  as

$$(2.0.13) \quad h(s) = \left( \frac{n}{n+1} \right)^s + \frac{1}{\sum_{k=1}^n \left( \frac{k}{n} \right)^s}$$

and differentiating with respect to  $s$ , it is easily seen that our claim is equivalent to

$$(2.0.14) \quad \sum_{k=1}^n k^s \log k > \sigma_n(s) \log n - \frac{\sigma_n^2(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right), \quad (n \in \mathbb{N}, s \in \mathbb{R}).$$

We consider a number of cases.

(i) The case  $s \leq -1$ .

The LHS of (2.0.14) is  $\geq 0$  so that we are done if we can show that the RHS is  $< 0$ .

Since  $\sigma_n(s) > 0$  it suffices to show that

$$(2.0.15) \quad \frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right) > \log n \quad \text{or} \quad \frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right)^{n+1} > (n+1) \log n.$$

Since  $\left(1 + \frac{1}{n}\right)^{n+1} > e$  it is enough to show the sharper inequality

$$(2.0.16) \quad \frac{\sigma_n(s)}{(n+1)^s} > (n+1) \log n.$$

For  $s \leq -1$  we have

$$(2.0.17) \quad \frac{\sigma_n(s)}{(n+1)^s} = \sum_{k=1}^n \left(\frac{k}{n+1}\right)^s = \sum_{k=1}^n \left(\frac{n+1}{k}\right)^{-s} \geq \sum_{k=1}^n \frac{n+1}{k} > (n+1) \log n$$

so that we are done with the case  $s \leq -1$ . ■

(ii) The case  $s > -1$ .

In this case we use mathematical induction with respect to  $n$ .

It is easily verified that (2.0.14) is true for  $n = 1$  ( and all  $s \in \mathbb{R}$  ).

Suppose that (2.0.14) still holds for some  $n \in \mathbb{N}$  and all  $s \in \mathbb{R}$ . Then

$$(2.0.18) \quad \begin{aligned} \sum_{k=1}^{n+1} k^s \log k &= \sum_{k=1}^n k^s \log k + (n+1)^s \log(n+1) > \\ &> (n+1)^s \log(n+1) + \sigma_n(s) \log n - \frac{\sigma_n^2(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right) \end{aligned}$$

and we will be through if the RHS of (2.0.18) is

$$(2.0.19) \quad \geq \sigma_{n+1}(s) \log(n+1) - \frac{\sigma_{n+1}^2(s)}{(n+2)^s} \log\left(1 + \frac{1}{n+1}\right).$$

Observing that

$$(2.0.20) \quad \sigma_{n+1}(s) = \sigma_n(s) + (n+1)^s$$

and performing some routine calculations, it is easily seen that we need to show that

$$(2.0.21) \quad \frac{\sigma_{n+1}(s)}{(n+2)^s} \log\left(1 + \frac{1}{n+1}\right) \geq \frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right)$$

or that  $\frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right)$  is increasing in  $n$ .

(iiA) The sub-cases  $-1 < s \leq 0$  and  $s \geq 1$ .

We multiply (2.0.21) by  $n(n+1)$  and observe that  $\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$ , so that it suffices to prove the sharper inequality

$$(2.0.22) \quad n \frac{\sigma_{n+1}(s)}{(n+2)^s} \geq (n+1) \frac{\sigma_n(s)}{(n+1)^s}$$

which may also be written as

$$(2.0.23) \quad \frac{n(n+1)^s}{(n+2)^s} \geq \sigma_n(s) \left( \frac{n+1}{(n+1)^s} - \frac{n}{(n+2)^s} \right).$$

It is easily verified that the factor of  $\sigma_n(s)$  in the RHS is  $> 0$  for  $s \geq -1$ , so that (2.0.23) may also be written as

$$(2.0.24) \quad \sigma_n(s) \leq \frac{n(n+1)^{2s}}{(n+1)(n+2)^s - n(n+1)^s}.$$

Similarly as with (2.0.6), this may be shown by mathematical induction, the ultimate  $x$ -inequality ( compare (2.0.9) ) here being



$$(2.0.25) \quad \frac{(1+2x)^s - (1+x)^s}{x} \leq \frac{(1+2x+x^2)^s - (1+2x)^s}{x^2}.$$

Since, for any fixed  $s \geq 1$  ( as well as for  $s \leq 0$  ), the function  $t^s$  is convex in  $t$  on  $\mathbb{R}^+$ , our claim follows. ■

(iiB) The ( more difficult ) case  $0 < s < 1$ .

Condition (2.0.21) is equivalent to

$$(2.0.26) \quad \sigma_n(s) \left\{ \frac{\log(1+\frac{1}{n})}{(n+1)^s} - \frac{\log(1+\frac{1}{n+1})}{(n+2)^s} \right\} \leq \left( \frac{n+1}{n+2} \right)^s \log\left(1 + \frac{1}{n+1}\right).$$

For  $s > 0$  ( even for  $s \geq -1$  ) the LHS is  $> 0$ , so that we may just as well show that

$$(2.0.27) \quad \sigma_n(s) \leq \frac{(n+1)^{2s} \log(1+\frac{1}{n+1})}{(n+2)^s \log(1+\frac{1}{n}) - (n+1)^s \log(1+\frac{1}{n+1})}.$$

We will prove this by mathematical induction. For  $n = 1$  this reads

$$(2.0.28) \quad 1 \leq \frac{4^s \log \frac{4}{3}}{3^s \log 2 - 2^s \log \frac{3}{2}}$$

or

$$(2.0.29) \quad \frac{\log 2}{\log \frac{3}{2}} \leq \left(\frac{2}{3}\right)^s + \left(\frac{4}{3}\right)^s := f(s).$$

The LHS  $\approx 1.709$  and the RHS is ( log-convex and ) minimal at  $s_0 \approx 0.495$  with minimal value  $f(s_0) \approx 1.971$ . So, our claim holds for  $n = 1$ .

Mathematical induction will be successful if we can show that

$$(2.0.30) \quad (n+1)^s + \frac{(n+1)^{2s} \log(1+\frac{1}{n+1})}{(n+2)^s \log(1+\frac{1}{n}) - (n+1)^s \log(1+\frac{1}{n+1})} \leq \frac{(n+2)^{2s} \log(1+\frac{1}{n+2})}{(n+3)^s \log(1+\frac{1}{n+1}) - (n+2)^s \log(1+\frac{1}{n+2})}$$

which may also be written as

$$(2.0.31) \quad \frac{\left(\frac{n+2}{n+1}\right)^s}{\log \frac{n+2}{n+1}} - \frac{1}{\log \frac{n+1}{n}} \geq \frac{\left(\frac{n+3}{n+2}\right)^s}{\log \frac{n+3}{n+2}} - \frac{1}{\log \frac{n+2}{n+1}}.$$

Now observe that  $\phi(s) := \text{LHS} - \text{RHS}$  of (2.0.31) is increasing in  $s$  for  $s \geq 0$ . Indeed, its derivative

$$(2.0.32) \quad \phi'(s) = \left(\frac{n+2}{n+1}\right)^s - \left(\frac{n+3}{n+2}\right)^s > 0 \text{ for } s > 0.$$

Hence, it suffices to show (2.0.31) for  $s = 0$ , or

$$(2.0.33) \quad \frac{1}{\log \frac{n+2}{n+1}} - \frac{1}{\log \frac{n+1}{n}} \geq \frac{1}{\log \frac{n+3}{n+2}} - \frac{1}{\log \frac{n+2}{n+1}}.$$

Setting  $n = \frac{1}{t}$  it suffices to show that

$$(2.0.34) \quad \frac{1}{\log \frac{1+2t}{1+t}} - \frac{1}{\log(1+t)}$$

is increasing for  $0 < t \leq 1$ . Writing  $u = \frac{t}{t+1}$ , it remains to show that

$$(2.0.35) \quad \psi(u) := \frac{1}{\log(1+u)} + \frac{1}{\log(1-u)}$$

is increasing for  $0 < u \leq \frac{1}{2}$ . So, we will be done if we can show that

$$(2.0.36) \quad \psi'(s) = -\frac{\frac{1}{1+u}}{\log^2(1+u)} - \frac{\frac{-1}{1-u}}{\log^2(1-u)} > 0$$

or

$$(2.0.37) \quad \frac{\frac{1}{1-u}}{\log^2(1-u)} > \frac{\frac{1}{1+u}}{\log^2(1+u)}$$

or

$$(2.0.38) \quad \chi(u) := (1+u) \log^2(1+u) - (1-u) \log^2(1-u) > 0.$$

Since  $\chi(0) = 0$ , we will be through if

$$(2.0.39) \quad \chi'(u) = \log^2(1+u) + 2 \log(1+u) + \log^2(1-u) + 2 \log(1-u) > 0.$$

Since  $\chi'(0) = 0$ , it suffices to show that

$$(2.0.40) \quad \chi''(u) = \frac{2 \log(1+u) + 2}{1+u} - \frac{2 \log(1-u) + 2}{1-u} > 0$$

or

$$(2.0.41) \quad \lambda(u) := (1-u) \log(1+u) - (1+u) \log(1-u) - 2u > 0.$$

Since  $\lambda(0) = 0$  it is enough to show that

$$(2.0.42) \quad \lambda'(u) = -\log(1+u) + \frac{1-u}{1+u} - \log(1-u) + \frac{1+u}{1-u} - 2 > 0$$

or

$$(2.0.43) \quad -\log(1-u^2) + \left( \frac{1-u}{1+u} + \frac{1+u}{1-u} - 2 \right) > 0.$$

But this is trivially true, completing the proof of our claim that  $h'(s) < 0$  for all  $s \in \mathbb{R}$ . ■

Clearly, it would be desirable to have an essentially simpler proof.

While constructing the above proof, several plots of  $h(s)$  were made and these suggested that  $h(s)$  is *strictly convex* in  $s$  ( i.e.  $h''(s) > 0$  ) on all of  $\mathbb{R}$ . Till now, no proof for this has been found ( to our knowledge ). Moreover, these plots suggest that  $h(s)$  is even *strictly log-convex* in  $s$  on all of  $\mathbb{R}$ .

Since  $\left(\frac{n+1}{n}\right)^s$  is log-convex in  $s$  and since the product of log-convex functions is log-convex again, it would follow that  $Q(s) = h(s) \left(\frac{n+1}{n}\right)^s$  is also log-convex in  $s$  on all of  $\mathbb{R}$  ( and conversely ).

We add to say that various ( numerical ) tests have been performed, especially on (D) and (E).

Although these tests strongly suggest that our Conjecture is true, no proof for arbitrary  $n \in \mathbb{N}$  has been found ( yet ).

We add the

CONJECTURE. For every ( fixed )  $n \in \mathbb{N}$  the function  $\frac{\omega_{n+1}(s)}{\omega_n(s)} \left( \frac{2n-1}{2n+1} \right)^s$  is strictly decreasing in  $s$  on all of  $\mathbb{R}$ .

Here

$$(2.0.44) \quad \omega_n(s) := 1^s + 3^s + \dots + (2n-1)^s.$$

A related QUESTION: What is the smallest constant  $a$  for which  $\left( \frac{a+n-1}{a+n} \right)^s \frac{a^s + (a+1)^s + \dots + (a+n)^s}{a^s + (a+1)^s + \dots + (a+n-1)^s}$  is strictly decreasing in  $s$  on all of  $\mathbb{R}$ ? ( Note that the truth of this for  $a \leq \frac{1}{2}$  would yield the previous Conjecture. )

Similarly one may also investigate ( for  $\alpha > 0$  ) the quotients  $q(s) := \frac{t_{n+1,\alpha}(s)}{t_{n,\alpha}(s)}$  of the approximations

$$t_{n,\alpha}(s) := \frac{1}{n^{s+1}} \sum_{k=0}^{n-1} (k + \alpha)^s.$$

It seems that this  $q(s)$  is still ( *monotonic, convex* ) *log-convex* in  $s$  for certain  $\alpha < 1$ .

Some estimates for  $\alpha_{\text{Mono}}$ ,  $\alpha_{\text{Conv}}$  and  $\alpha_{\text{LogConv}}$  are:  $\alpha_M > 0.462$ ,  $\alpha_C > 0.785$ ,  $\alpha_{LC} > 0.803$ .

Note: Similarly, defining ( for  $s > 0$  ) the canonical trapezoidal approximations  $T_n(s)$  of  $\int_0^1 x^s dx$  by

$$T_n(s) := \frac{1}{2} (L_n(s) + U_n(s)),$$

we found experimentally that  $q_n(s) := \frac{T_{n+1}(s)}{T_n(s)}$  is *strictly decreasing* in  $s$  on all of  $\mathbb{R}$ .

No proof of this is available at present.

Similar observations can be made for the various integral approximations ( on page 94, for example ) in Hildebrand's Introduction to Numerical Analysis.

## 2.1. A conjecture of J. P. Lambert

In 1985, J. P. Lambert [8], proposed the following ( unsolved ) problem ( in the Amer. Math. Monthly, 1985, Problem E 3102\* ):

For all integers  $n, s \geq 1$

$$(2.1.1) \quad \frac{\sum_{k=1}^n (2k-1)^{2s}}{\sum_{k=1}^n (2k)^{2s}} < \left( \frac{2n}{2n+1} \right)^{2s+1}.$$

Also see Diamond [9].

First we try to get some idea about the ( possible ) origin of this inequality.

We consider the canonical midpoint approximations of the integral  $\int_{-1}^1 |x|^s dx$ , ( $s > 0$ )

$$(2.1.2) \quad t_n^*(s) := \frac{2}{n} \sum_{k=1}^n \left| -1 + \frac{2k-1}{n} \right|^s = \frac{2}{n^{s+1}} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2k + 1)^s.$$

For  $s$  not too small,  $|x|^s$  will be convex in  $x$  on  $[-1,1]$ , and it stands to reason that  $t_n^*(s)$  might then be *increasing* in  $n$ , i. e.  $t_n^*(s) < t_{n+1}^*(s)$ .

Since

$$(2.1.3a) \quad t_{2n}^*(s) = \frac{2}{(2n)^{s+1}} \sum_{k=1}^n (2n - 2k + 1)^s$$

and

$$(2.1.3b) \quad t_{2n+1}^*(s) = \frac{2}{(2n+1)^{s+1}} \sum_{k=1}^n (2n - 2k + 2)^s$$

the suggested monotonicity may also be written as

$$(2.1.4) \quad t_{2n}^*(s) < t_{2n+1}^*(s) < t_{2n+2}^*(s)$$

or, in full, ( with  $\omega_n(s)$  as above in (2.0.44) )

$$(2.1.5) \quad \frac{\omega_n(s)}{(2n)^{s+1}} < \frac{2^s \sigma_n(s)}{(2n+1)^{s+1}} < \frac{\omega_{n+1}(s)}{(2n+2)^{s+1}}.$$

The first inequality here is Lambert's, while the second one seems to be its natural companion ( compare (2.0.4) ). In order to deal with these inequalities we consider the functions ( with  $s \in \mathbb{R}$  )

$$(2.1.6) \quad f(s) := f_n(s) := \frac{\omega_n(s)}{2^s \sigma_n(s)} \left( \frac{2n+1}{2n} \right)^{s+1} \quad \text{and} \quad g(s) := g_n(s) := \frac{2^s \sigma_n(s)}{\omega_{n+1}(s)} \left( \frac{2n+2}{2n+1} \right)^{s+1}.$$

After extensive testing we arrived at the following

CONJECTURE. The function  $f(s)$  is *strictly decreasing* in  $s$  on all of  $\mathbb{R}$  ( with  $f'(s) < 0$  ) whereas  $g(s)$  is *strictly decreasing* in  $s$  for  $s \geq 2$  ( with  $g'(s) < 0$  ).

It would follow that if  $s > s_0$  then  $f(s) < f(s_0)$  and  $g(s) < g(s_0)$ . For  $s_0 = 2$  this would read

$$(2.1.7a) \quad \frac{\omega_n(s)}{2^s \sigma_n(s)} \left( \frac{2n+1}{2n} \right)^{s+1} < f(2) = 1 - \frac{1}{4} \frac{n + \frac{1}{4}}{(n+1)n^3}$$

and

$$(2.1.7b) \quad \frac{2^s \sigma_n(s)}{\omega_{n+1}(s)} \left( \frac{2n+2}{2n+1} \right)^s < 1 - \frac{1}{4} \frac{n + \frac{3}{4}}{\left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right)^3}$$

the first inequality being an improvement of the original Lambert-inequality.

We expect that the proofs ( if possible ) will be quite demanding. Compare the proof of  $h'(s) < 0$  in Section 2.0.

In relation to Lambert's inequality it was observed ( numerically ) that the function

$$(2.1.8) \quad \phi(s) := \frac{\omega_n(s)}{\sigma_n(s)} \left( \frac{2n+1}{4n} \right)^s, \quad (s \in \mathbb{R})$$

is *strictly decreasing* in  $s$  on all of  $\mathbb{R}$  ( with  $\phi'(s) < 0$  ).

In order to prove this we might just as well show that  $\psi(s) := \log \phi(s)$  is strictly decreasing, or that

$$(2.1.9) \quad L := \frac{\omega'}{\omega} + \log \frac{2n+1}{4n} < \frac{\sigma'}{\sigma} =: R.$$

Since

$$(2.1.10) \quad \lim_{s \rightarrow \infty} L = \log(2n-1) + \log \frac{2n+1}{4n} \quad \text{and} \quad \lim_{s \rightarrow \infty} R = \log n$$

it is clear that (2.1.9) holds if  $s$  is large enough. It is trivial that (2.1.9) holds true for  $s \downarrow -\infty$ .

More precisely, since ( for all  $s \in \mathbb{R}$  )

$$(2.1.11) \quad \frac{\omega'}{\omega} = \frac{3^s \log 3 + 5^s \log 5 + \dots + (2n-1)^s \log(2n-1)}{1 + 3^s + 5^s + \dots + (2n-1)^s} \leq \log(2n-1)$$

and ( for  $s > 0$  )

$$(2.1.12) \quad \frac{\sigma'}{\sigma} = \frac{2^s \log 2 + 3^s \log 3 + \dots + n^s \log n}{1^s + 2^s + \dots + n^s} \geq \frac{n^s \log n}{1^s + 2^s + \dots + n^s} = \frac{\log n}{\sum_{k=1}^n \left(\frac{k}{n}\right)^s} =$$

$$= \frac{\frac{1}{n} \log n}{\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^s} = \frac{\frac{1}{n} \log n}{\int_0^1 x^s dx + \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^s - \int_0^1 x^s dx\right)} > \frac{\frac{1}{n} \log n}{\frac{1}{s+1} + \frac{1}{n}}$$

it follows easily that (2.1.9) holds for  $s \geq n / \left( \frac{\log n}{\log n + \log(1 - \frac{1}{4n^2})} - 1 \right)$ .

In a similar fashion one may show that (2.1.9) is true for

$$(2.1.13) \quad s \leq \log \frac{\frac{1}{n} \log 2}{\log(2n-1)!!} / \log \frac{3}{2}$$

where  $(2n-1)!! := 1 * 3 * 5 * \dots * (2n-3) * (2n-1)$ .

Numerically we observe that (2.1.9) is "most critical" for  $2 < s < 3$ , where it seems to have a ( positive ) local minimum  $> \frac{1}{n^3 \log n}$ .

Even the case  $n = 2$  is not entirely trivial. In this case (2.1.9) is equivalent to

$$(2.1.14) \quad L := 3^s \log \frac{15}{8} < \log \frac{8}{5} + 2^s \log \frac{16}{15} + 6^s \log \frac{16}{15} = : R.$$

Clearly this is true if  $\log \frac{15}{8} \leq 2^s \log \frac{16}{15}$  or  $s \geq \log \frac{\log 15 - \log 8}{\log 16 - \log 15} / \log 2 \approx 3.284$ .

This inequality is also true if  $3^s \log \frac{15}{8} \leq \log \frac{8}{5}$  or  $s \leq \log \frac{\log 8 - \log 5}{\log 15 - \log 8} / \log 3 \approx -0.265$ .

The rest may be completed by a numerical Newton-type procedure ( see Section 5 ). Indeed, note that  $L$  is increasing and  $R$  is increasing and convex.

A slight generalization.

We consider ( for  $0 \leq a \leq \frac{1}{2}$  )

$$(2.1.15) \quad \phi := \phi_{n,a}(s) := \frac{\sum_{k=1}^n (2k-1+a)^s}{\sum_{k=1}^n (2k-a)^s} \left( \frac{2n+1-a}{2n+a} \right)^s.$$

We observe numerically that  $\phi$  is strictly decreasing in  $s$  ( for  $0 \leq a < \frac{1}{2}$  ).

Let  $\psi := \psi(s) := -\frac{\partial}{\partial s} \log \phi(s) = \frac{\sigma'}{\sigma} - \frac{\omega'}{\omega} - \log \frac{2n+1-a}{2n+a}$ . We would like to show that this is  $> 0$ .

In support of this we observe numerically that

$$(2.1.16) \quad \frac{\partial \psi}{\partial a} < 0 \quad \text{for all } s \in \mathbb{R} \quad \text{and all } a \in \left[0, \frac{1}{2}\right].$$

We venture to TRY again and prove (2.1.9) on the basis of our first Conjecture:  $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  is log-convex ( in  $s$  ) on  $\mathbb{R}$ .

As an easy consequence we have that  $\frac{\sigma_{2n}(s)}{\sigma_n(s)}$  is log-convex on  $\mathbb{R}$ . Hence, also  $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)}$  is log-convex on  $\mathbb{R}$ .

It follows that  $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)}$  is convex, so that also  $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)} - 1 = \frac{\omega_n(s)}{2^s \sigma_n(s)} =: h(s)$  is convex.

Note that  $\lim_{s \rightarrow \infty} h(s) = 0$ . So, we must have  $h'(s) < 0$ . In full this reads

$$(2.1.17) \quad h'(s) = \frac{2^s \sigma \omega' - (2^s \sigma \log 2 + 2^s \sigma') \omega}{2^{2s} \sigma^2(s)} < 0$$

which is equivalent to ( the non-trivial )  $\frac{\omega'}{\omega} - \frac{\sigma'}{\sigma} < \log 2$ .

This falls just short of our goal: To prove that  $\frac{\omega'}{\omega} - \frac{\sigma'}{\sigma} < \log\left(\frac{4n}{2n+1}\right) = \log\left(2 - \frac{1}{n+\frac{1}{2}}\right)$ .

So, it may very well be that even our ( strong ) Conjecture 1 is *not strong enough* to prove (2.1.9).

### 3. Relations with some other conjectures

#### 3.1. A conjecture of H. Alzer and A. A. Jagers.

In [5], Alzer and Jagers proposed the following ( unsolved ) problem.

Let

$$(3.1.1) \quad f_n(s) := \left( \frac{\frac{1}{n+1} \sum_{i=1}^{n+1} i^s}{\frac{1}{n} \sum_{i=1}^n i^s} \right)^{\frac{1}{s}}.$$

Prove ( or disprove ) that  $f_n(s)$  is *strictly increasing* in  $s$  on all of  $\mathbb{R}^+$ . ( No full proof was submitted. )

Here we will show that the monotonicity of  $f_n(s)$  on  $\mathbb{R}^+$  ( even on all of  $\mathbb{R}$  ) is a simple consequence of our ( first ) Conjecture in Section 1.

PROOF. We sketch the details only for  $s > 0$ . The remaining case(s) may be treated similarly. It suffices to show that the function

$$(3.1.2) \quad \log f_n(s) = \frac{1}{s} \log\left(\frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)}\right) = \frac{1}{s} \log\left(\frac{n}{n+1} Q(s)\right)$$

is strictly increasing ( in  $s$  ) on all of  $\mathbb{R}^+$ . Sufficient for this is that

$$(3.1.3) \quad \frac{d}{ds} \log f_n(s) = -\frac{1}{s^2} \log \frac{n}{n+1} - \frac{1}{s^2} \log Q(s) + \frac{1}{s} \frac{Q'(s)}{Q(s)} > 0$$

or

$$(3.1.4) \quad \phi(s) := -\log \frac{n}{n+1} - \log Q(s) + s \frac{Q'(s)}{Q(s)} > 0.$$

Since  $\phi(0) = 0$ , it suffices to show that ( for  $s > 0$  )

$$(3.1.5) \quad \phi'(s) = s \frac{d}{ds} \frac{Q'(s)}{Q(s)} > 0 \quad \text{or} \quad (\log Q(s))'' > 0.$$

But this is equivalent with our ( first ) Conjecture in Section 1, proving our claim. ■

#### 3.2. Martins' theorem.

For a given positive sequence  $\{u_n\}_{n \in \mathbb{N}}$ , Martins [6] considered / compared the quotients

$$(3.2.1) \quad \frac{A_{n+1}}{A_n} \quad \text{and} \quad \frac{G_{n+1}}{G_n}$$

where

$$(3.2.2) \quad A_n := \frac{1}{n} \sum_{k=1}^n u_k \quad \text{and} \quad G_n := \left( \prod_{k=1}^n u_k \right)^{\frac{1}{n}}$$

are the progressive Arithmetical and Geometrical means of  $\{u_n\}_{n \in \mathbb{N}}$ .  
 For the special sequence  $u_n := u_n(s) := n^s$  Martins proved that

$$(3.2.3) \quad \frac{G_{n+1}}{G_n} \leq \frac{A_{n+1}}{A_n}, \quad \forall s \in \mathbb{R}.$$

We will show that this inequality is also a consequence of our ( first ) Conjecture in Section 1.

PROOF. Martins' inequality (3.2.3) may be written as

$$(3.2.4) \quad \frac{((n+1)!)^{\frac{s}{n+1}}}{(n!)^{\frac{s}{n}}} \leq \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$$

or

$$(3.2.5) \quad \phi(s) := \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)} (n!)^{\frac{s}{n}} ((n+1)!)^{-\frac{s}{n+1}} \geq 1.$$

Note that  $\phi(0) = 1$ .

Since, in view of Section 1, the function  $\phi(s)$  " is " log-convex on  $\mathbb{R}$ , we will be through if we can show that

$$(3.2.6) \quad \phi'(0) = 0 \quad \text{or} \quad (\log \phi(s))' = \frac{\phi'(s)}{\phi(s)} = 0 \quad \text{at } s = 0.$$

But, this is a simple exercise, proving our claim. ■

Note that it even follows that

$$(3.2.7) \quad \frac{G_{n+1}(s)}{G_n(s)} < \frac{A_{n+1}(s)}{A_n(s)} \quad \text{for all } s \neq 1.$$

### 3.3. Alzer's conjecture.

Alzer [7] has conjectured, as a supplement to Martins' theorem, that

$$(3.3.1) \quad \frac{A_{n+1}(s)}{A_n(s)} \leq \frac{G_{n+1}(s)}{G_n(s)} \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \quad (s > 0).$$

It seems that this inequality has no direct relation to our first Conjecture. We will deal with it in Part II.

## 4. An attempt to prove our first Conjecture.

Some simple observations:

From

$$(4.1) \quad Q_n(s) = \frac{\sigma_{n+1}(s)}{\sigma_n(s)} = \frac{\sigma_n(s) + (n+1)^s}{\sigma_n(s)} = 1 + \frac{(n+1)^s}{\sigma_n(s)} = 1 + \frac{1}{\left(\frac{1}{n+1}\right)^s + \left(\frac{2}{n+1}\right)^s + \dots + \left(\frac{n-1}{n+1}\right)^s + \left(\frac{n}{n+1}\right)^s}$$

it is clear that  $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  is strictly increasing in  $s$  on  $\mathbb{R}$ .

For large  $s$  we have  $Q_n(s) \sim \left(1 + \frac{1}{n}\right)^s$ , which happens to be log-convex in  $s$ .

The first derivative of  $\log Q(s)$  has the following fairly simple form ( easy to show ):

$$(4.2) \quad \frac{d}{ds} \log Q_n(s) = \frac{\sum_{k=0}^{n-1} ((n-k)(n+1))^s \log \frac{n+1}{n-k}}{\sigma_n(s) \sigma_{n+1}(s)} = \frac{\sum_{k=1}^n (k(n+1))^s \log \frac{n+1}{k}}{\sigma_n(s) \sigma_{n+1}(s)}$$

which is clearly  $> 0$ , as it should be.  
 We write this as

$$(4.3) \quad \begin{aligned} \frac{d}{ds} \log Q_n(s) &= \frac{\sum_{k=1}^n (k(n+1))^s \log \frac{n+1}{k}}{\sigma_n(s) \sigma_{n+1}(s)} = \\ &= \frac{n^s (n+1)^s \log \left(\frac{n+1}{n}\right) + \sum_{k=1}^{n-1} (k(n+1))^s \log \frac{n+1}{k}}{\sigma_n(s) \sigma_{n+1}(s)} = \frac{n^s (n+1)^s \log \frac{n+1}{n} + \text{increasing}}{\sigma_n(s) \sigma_{n+1}(s)}. \end{aligned}$$

Now observe that

$$(4.4) \quad \frac{n^s (n+1)^s \log \frac{n+1}{n}}{\sigma_n(s) \sigma_{n+1}(s)} = \frac{\log \frac{n+1}{n}}{\left(\left(\frac{1}{n}\right)^s + \left(\frac{2}{n}\right)^s + \dots + \left(\frac{n}{n}\right)^s\right) * \left(\left(\frac{1}{n+1}\right)^s + \left(\frac{2}{n+1}\right)^s + \dots + \left(\frac{n}{n+1}\right)^s + \left(\frac{n+1}{n+1}\right)^s\right)}$$

which is increasing in  $s$  indeed (with limit  $= \log \frac{n+1}{n}$  as  $s \uparrow \infty$ ).  
 But, this argument is, surprisingly enough, not sufficient for (4.3) to be increasing.

Some brief notes:

- (1) We were able to prove Conjecture 1 for  $s = 0$ .  
 We computed  $(\log Q_n(s))'(0)$ . This seems to be  $> 0$  and *decreasing* in  $n$ .  
 We were able to prove the  $> 0$  part here.  
 The proof by mathematical induction may be reduced to showing that

$$\begin{aligned} n \log^2 n + 2 \log(n+1) + \frac{\log 2 \pi n}{n} \log n! + \frac{1}{6n^3} \log n! &\leq \\ \leq n \log^2(n+1) + \frac{\log 2 \pi(n+1)}{n+1} \log(n+1)! &. \end{aligned}$$

We omit the quite tedious details.

- (2) Numerically we observe that Conjecture 1 is "most critical" for  $2 < s < 4$ .

**A related Open Problem.** For any fixed  $n \in \mathbb{N}$  we define

$$(4.5) \quad q(s) := q_n(s) = \frac{n^s}{1^s + 2^s + 3^s + \dots + n^s} = \frac{n^s}{\sigma_n(s)}, \quad (s \in \mathbb{R}).$$

For  $n \geq 2$  we then have

$$(4.6) \quad \log \frac{\sigma_n(s)}{\sigma_{n-1}(s)} = -\log \frac{\sigma_{n-1}(s)}{\sigma_n(s)} = -\log\left(1 - \frac{n^s}{\sigma_n(s)}\right) = \sum_{k=1}^{\infty} \frac{1}{k} q^k(s).$$

We would be immediately through with our Conjecture 1 if all  $q^k(s)$  would be convex on all of  $\mathbb{R}$ . But, this is not the case!  
 However, there seems to exist an  $s = s_n$ , such that  $q''(s) > 0$  for all  $s < s_n$ .

Defining (for  $m \in \mathbb{N}$ )

$$(4.7) \quad f_m(s) := \sum_{k=1}^m \frac{1}{k} q^k(s)$$



let  $s = r_m$  be the smallest zero of  $f_m''(s)$ . If non extant then we are through !  
 Prove ( or disprove ) that  $r_1 < r_2 < r_3 < \dots$  and that  $\lim_{m \rightarrow \infty} r_m = +\infty$ .

In view of (4.6) it would follow from this that  $f_\infty(s) := \log \frac{\sigma_n(s)}{\sigma_{n-1}(s)}$  is convex, proving our first Conjecture.

Note: It is clear that  $\lim_{s \downarrow -\infty} q(s) = 0$ , and writing  $q(s) = \frac{1}{\left(\frac{1}{n}\right)^s + \left(\frac{2}{n}\right)^s + \dots + \left(\frac{n-1}{n}\right)^s + \left(\frac{n}{n}\right)^s}$  it follows that  $\lim_{s \uparrow \infty} q(s) =$

1. It thus stands to reason that  $q(s)$  is convex " at  $s = -\infty$  " but not " at  $s = +\infty$  ".

Hence:  $s_n$  exists. Similarly:  $r_m$  exists.

### Still another unsolved Problem.

Let

$$(4.8) \quad \phi(s) := \phi_n(s) := (2n)^s - (2n-1)^s + (2n-2)^s - + \dots + 2^s - 1^s .$$

- (a) Prove that  $\phi(s)$  is strictly increasing in  $s$  on all of  $\mathbb{R}$ .
- (b) Determine ( or estimate ) the largest  $\alpha$  such that

$$(4.9) \quad (2n)^s \log^\alpha(2n) - (2n-1)^s \log^\alpha(2n-1) + (2n-2)^s \log^\alpha(2n-2) - + \dots + 2^s \log^\alpha(2) > 0$$

for all  $s \in \mathbb{R}$ .

Numerical work suggests that  $\alpha > \pi$ .

## 5. Numerical verification of our first Conjecture.

When trying to verify our first Conjecture numerically, one may want to use the following simple ( but useful )

NUMERICAL LEMMA ( of Newton-type ).

If  $L(s) < R(s)$  for  $s = a$ , and  $L(s)$  is increasing, and  $R(s)$  is increasing and convex, then  $L(s) < R(s)$  also holds for  $a - \frac{R(a) - L(a)}{R'(a)} \leq s \leq a$ .

Our log-convexity criteria (D), (E) and (F) may be written as  $L < R$  with

$$(5.1) \quad L := \sigma_n \sigma_n'' \sigma_{n+1}^2 + \sigma_n^2 (\sigma_{n+1}')^2 \quad \text{and} \quad R := \sigma_n^2 \sigma_{n+1} \sigma_{n+1}'' + (\sigma_n')^2 \sigma_{n+1}^2 .$$

and these  $L$  and  $R$  satisfy the conditions of the above Lemma.

Now just start with an arbitrary  $a > 0$  ( from above ) and iterate as suggested above.

One may carry out these computations using *Mathematica*, for example.

Quite a time consuming job, though. But it seems to work !

It seems that  $(\log(1 + \frac{(n+1)^s}{\sigma_n(s)}))''$  is critical only for  $0 \leq s \leq 3$  with a positive local minimum.

QUESTION: Is there any point in verifying / proving that  $Q(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  is ( just ) convex in  $s$  ?

Try some examples for small  $n$ .  $Q''(s)$  seems to have a positive local minimum for  $2 < s < 4$ .

*Mathematica* experiments indicate that ( for  $n$  not too small ) this local minimum is  $> \frac{1}{n^2}$ .

## Part II

### 1. Statement of our second Conjecture.

Our second CONJECTURE reads: For any fixed  $n \in \mathbb{N}$ , the function  $\frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\sigma_{n+2}(s)}$  is *log-convex* in  $s$  on all of  $\mathbb{R}$ .

### 2. The genesis of our second Conjecture.

While studying Alzer's inequality ( see (3.1) in Part I )

$$(2.1) \quad \frac{A_{n+1}(s)}{A_n(s)} \leq \frac{G_{n+1}(s)}{G_n(s)} \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \quad (s > 0)$$

or

$$(2.2) \quad \frac{A_{n+1}(s)}{A_n(s)} \frac{G_n(s)}{G_{n+1}(s)} \leq \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \quad (s > 0)$$

it was ( numerically ) observed that for any fixed  $s \in \mathbb{R}$ , the LHS :=  $\frac{A_{n+1}(s)}{A_n(s)} \frac{G_n(s)}{G_{n+1}(s)}$  is *decreasing* in  $n$ . Clearly, this would entail that Alzer's inequality is true for all  $n \in \mathbb{N}$  if it holds for  $n = 3$ . Its validity for  $n = 3$  will be shown below in Section 5.

(\* Testing the monotonicity just mentioned \*)

prec = 25 ;  $s = \frac{1}{4}$  ; (\* Pick your choice \*)

$\sigma n = \text{Sum}[k^s, \{k, n\}]$  ;

$An = \frac{\sigma n}{n}$  ;  $Anp1 = \frac{\sigma n + (n+1)^s}{n+1}$  ;  $Gn = (n!)^{\frac{s}{n}}$  ;  $Gnp1 = ((n+1)!)^{\frac{s}{n+1}}$  ;  $f = \frac{Anp1}{An} * \frac{Gn}{Gnp1}$  ;

For[n = 2, n ≤ 10, n++,

Print["n= ", n, " s= ", N[s], " f(s)= ", N[f, prec]]]

n= 2 s= 0.25	f(s)= 1.002599668283378350809485
n= 3 s= 0.25	f(s)= 1.001939826338575098526561
n= 4 s= 0.25	f(s)= 1.001520157589413043434971
n= 5 s= 0.25	f(s)= 1.001233120792851125950798
n= 6 s= 0.25	f(s)= 1.001026302603682735994488
n= 7 s= 0.25	f(s)= 1.000871326388842402466966
n= 8 s= 0.25	f(s)= 1.000751587656873653937862
n= 9 s= 0.25	f(s)= 1.000656771536769794830343
n= 10 s= 0.25	f(s)= 1.000580158605126596481757

So, we get the *impression* that for any fixed  $s \in \mathbb{R}$

$$(2.3) \quad \frac{A_{n+1}(s)}{A_n(s)} \frac{G_n(s)}{G_{n+1}(s)} < \frac{A_n(s)}{A_{n-1}(s)} \frac{G_{n-1}(s)}{G_n(s)}$$

or

$$(2.4) \quad 1 < \frac{A_n^2(s)}{A_{n-1}(s)A_{n+1}(s)} \frac{G_{n-1}(s)G_{n+1}(s)}{G_n^2(s)} = Q_A(s)Q_G(s), \text{ say.}$$

It is clear that  $Q_G(s)$ , being an exponential function of the simple form  $a^s$ , is log-convex in  $s$  on all of  $\mathbb{R}$ . Further, it is a matter of routine to show that  $(Q_A(s)Q_G(s))' = 0$  at  $s = 0$ .

Now observe that if  $Q_A(s)$  would be log-convex in  $s$  then the whole RHS of (2.4) would be log-convex (and hence convex) in  $s$ , with  $\text{RHS}'(0) = 0$ , proving Alzer's inequality.

In view of

$$(2.5) \quad \frac{A_n^2(s)}{A_{n-1}(s)A_{n+1}(s)} = \frac{n^2 - 1}{n^2} \frac{\sigma_n^2(s)}{\sigma_{n-1}(s)\sigma_{n+1}(s)}$$

where  $n \in \mathbb{N}$  is constant, the above observations led us to our second Conjecture.

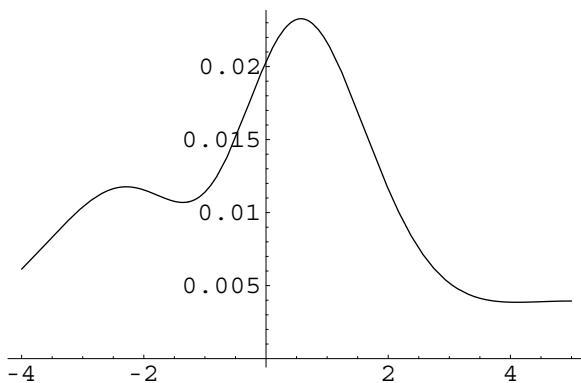
### 3. Verification of the second Conjecture.

Using *Mathematica*, we performed various checks (similar to those in Section 4 of Part I) on our second Conjecture. Needless to say, all these numerical / graphical experiments suggest that the Conjecture is true indeed.

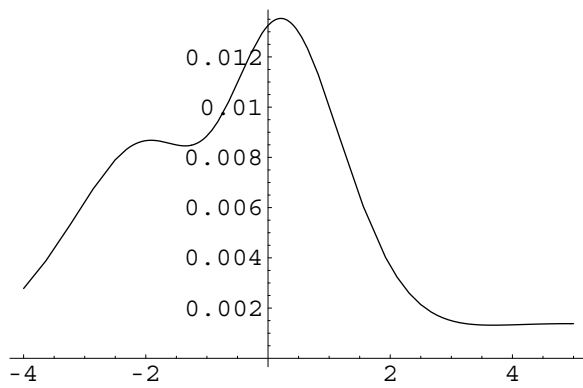
Especially the graphs of the second derivative of  $\log \frac{\sigma_n^2(s)}{\sigma_{n-1}(s)\sigma_{n+1}(s)}$  are very intriguing.

Also compare Section 5 below.

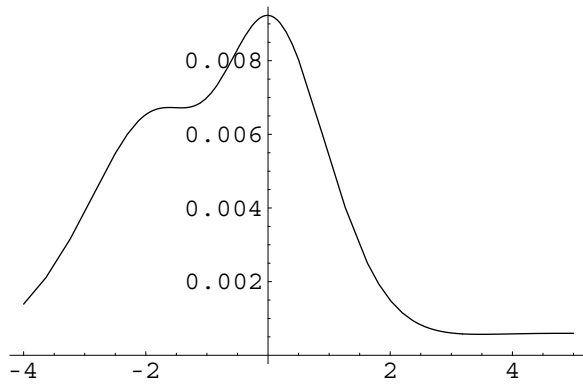
```
Clear[n, s];  $\sigma n = \text{Sum}[k^s, \{k, n\}]; \sigma n m 1 = \sigma n - n^s; \sigma n p 1 = \sigma n + (n + 1)^s;$ 
 $f = \frac{\sigma n^2}{\sigma n m 1 * \sigma n p 1};$  Lf = Log[f]; Lf2 =  $\partial_s \partial_s$  Lf;
nBound = 6; (* say *)
For[n = 3, n ≤ nBound, n++, Plot[Lf2, {s, -4, 5}];
Print["↑ This was n = ", n, " Local Minimum → ", FindMinimum[Lf2, {s, 2, 5}]]]
```



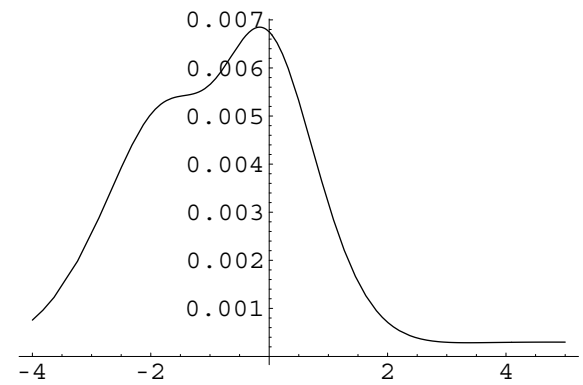
↑ This was n = 3 Local Minimum → {0.00385861, {s → 4.11215}}



↑ This was n=4 Local Minimum → {0.00131938, {s → 3.68589}}



↑ This was n=5 Local Minimum → {0.000569114, {s → 3.49106}}



↑ This was  $n=6$  Local Minimum  $\rightarrow \{0.00028568, \{s \rightarrow 3.38689\}\}$

## 4. A consequence of Conjecture 2.

Writing (2.1) as

$$(4.1) \quad \frac{A_{n+1}(s)}{G_{n+1}(s)} \leq \frac{A_n(s)}{G_n(s)} \frac{2^{-s} + 2^s}{2}, \quad (s > 0)$$

a repeated application of this to itself would result in

$$(4.2) \quad \frac{A_n(s)}{G_n(s)} \leq \left( \frac{2^{-s} + 2^s}{2} \right)^{n-1} \quad \text{for all } n \in \mathbb{N} \text{ and all } s \in \mathbb{R}.$$

For  $n = 1$  this is trivial and for  $n = 2$  it is easily verified.

More generally

$$(4.3) \quad f(s) := (n!)^{\frac{s}{n}} \left( \frac{2^{s/2} + 2^{-s/2}}{2} \right)^{n-1} - \frac{1}{n} * (1^s + 2^s + \dots + n^s)$$

seems to be convex at  $s = 0$ . Moreover,  $f_n''(0)$  ( $> 0$ ) seems to be increasing in  $n$ .

So, (4.2) seems to be "rather easy" at  $s \approx 0$ . However, no full proof of (4.2) is known to us.

## 5. Verification of Alzer's inequality.

It seems that (for small  $n$ ) Alzer's inequality is much easier to verify than our own two Conjectures.

We will perform such a verification for  $s > 0$  (the cases  $s = 0$  and  $n = 2$  are trivial).

So, from now on we assume  $n \geq 3$ .

We start in a very simplistic manner: Alzer claims that

$$(5.1) \quad \frac{n-1}{n} (1^s + 2^s + \dots + n^s) \leq (1^s + 2^s + \dots + (n-1)^s) n^{\frac{s}{n-1}} (n!)^{-\frac{s}{n(n-1)}} \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}.$$

Sharper than this is

$$(5.2) \quad 2 \frac{n-1}{n} n n^s \leq (n-1)^s n^{\frac{s}{n-1}} (n!)^{-\frac{s}{n(n-1)}} 2^{\frac{s}{2}}$$

or

$$(5.3) \quad 2(n-1) \leq \left( \frac{n-1}{n} n^{\frac{1}{n-1}} (n!)^{-\frac{1}{n(n-1)}} \sqrt{2} \right)^s$$

or

$$(5.4) \quad 2(n-1) \leq a^s \quad \text{with } a := \frac{n-1}{n} n^{\frac{1}{n-1}} (n!)^{-\frac{1}{n(n-1)}} \sqrt{2}.$$

LEMMA 1. For  $n \geq 3$  we have  $a > 1$  in (5.4).

PROOF. We have to show that

$$(5.5) \quad n^{\frac{1}{n-1}} \sqrt{2} > \frac{n}{n-1} (n!)^{\frac{1}{n(n-1)}}$$

or

$$(5.6) \quad n^n 2^{\frac{n(n-1)}{2}} > \left(1 + \frac{1}{n-1}\right)^{n(n-1)} n!.$$

Sharper than this is (using Stirling)

$$(5.7) \quad n^n 2^{\frac{n(n-1)}{2}} > e^n n^n e^{-n} \sqrt{2\pi n} e^{\frac{1}{12n}}$$

or

$$(5.8) \quad 2^{n(n-1)} > 2\pi n e^{\frac{1}{6n}} \quad \text{or} \quad 2^{n(n-1)} > 6.65 n$$

which is true indeed for  $n \geq 3$ . ■

Consequence: In view of (5.4), Alzer's inequality holds true for  $s \geq s_0 := \frac{\log(2(n-1))}{\log a}$ . Later we will improve on this. Here is a simple *Mathematica* program computing some  $a$  and  $s_0$ .

```
prec = 30; Clear[n]; a =  $\frac{n-1}{n} * n^{\frac{1}{n-1}} * (n!)^{-\frac{1}{n*(n-1)}} * \sqrt{2}$ ; s0 =  $\frac{\text{Log}[2*(n-1)]}{\text{Log}[a]}$ ;
For[n = 3, n ≤ 10, n++, Print["n= ", n, " a= ", N[a, prec], " s0 = ", N[s0, prec]]]
```

```
n= 3 a= 1.21141372855475977259414117088 s0 = 7.22826251895962701374721299244
n= 4 a= 1.29194885544219791385434161069 s0 = 6.99491214304942877773858189137
n= 5 a= 1.33164487995932388137446564092 s0 = 7.26024143416826576160088643782
n= 6 a= 1.35431467787292939662956751449 s0 = 7.59188542554862154894929685960
n= 7 a= 1.36856293427842082319837273196 s0 = 7.91973756766943950825265923669
n= 8 a= 1.37814358608082555195971619981 s0 = 8.22809441266003199851301550326
n= 9 a= 1.38491735633824125943949924900 s0 = 8.51426342937290467947249668226
n= 10 a= 1.38989618369636924703886291198 s0 = 8.77921222934515474946743418353
```

Now we write Alzer's inequality as

$$(5.9) \quad L(s) := 2 \frac{n-1}{n} \sigma_n(s) (n!)^{\frac{s}{n(n-1)}} 2^{\frac{s}{2}} \leq \sigma_{n-1}(s) n^{\frac{s}{n-1}} (2^s + 1) =: R(s).$$

Note that  $L(s)$  is increasing and that  $R(s)$  is increasing and (log-) convex on all of  $\mathbb{R}$ . By our Newton-type argument we thus find that (5.9) also holds true for all

$$(5.10) \quad s \geq s_0 - \frac{R(s) - L(s)}{R'(s)}.$$

One may iterate this procedure a number of times to extend the domain of validity of (5.9).

However, since  $R'(0) - L'(0) = 0$ , one will experience that only very modest progress is being made.

We improve on this as follows: We want to show that  $f(s) := R(s) - L(s) > 0$  for  $0 < s < s_0$ .

Since  $f(0) = 0$ , it suffices to show that  $f'(s) > 0$  for  $0 < s < s_0$ .

Also  $f'(0) = 0$ , so that it is enough to show that  $f''(s) > 0$  or  $R''(s) - L''(s) > 0$  for  $0 < s < s_0$ .

One may actually verify (numerically) that  $R''(s_0) > L''(s_0)$  for  $3 \leq n \leq 10$ , say. Or, depending on time and patience, for  $3 \leq n \leq 1000$ , say.

Also,  $L(s)$  and  $R(s)$  satisfy the conditions of our numerical Lemma. Iterating, we find, for example, the validity of (5.9) for  $n = 3$  in some 67 steps.

Here is a simple *Mathematica* program.

```

Clear[n, s];  $\sigma = \text{Sum}[k^s, \{k, n\}]; L = 2 * \frac{n-1}{n} * \sigma * (n!)^{\frac{s}{n(n-1)}} * 2^{\frac{s}{2}}; L2 = \partial_s \partial_s L;$ 
 $R = (\sigma - n^s) * n^{\frac{s}{n-1}} * (2^s + 1); R2 = \partial_s \partial_s R; R3 = \partial_s R2; T = \text{Simplify}[\frac{R2 - L2}{R3}];$ 
For[n = 3, n ≤ 10, n++, a =  $N[\frac{n-1}{n} * n^{\frac{1}{n-1}} * (n!)^{-\frac{1}{n(n-1)}} * \sqrt{2}]; s0 = \frac{\text{Log}[2 * (n-1)]}{\text{Log}[a]};$ 
k = 0;
While[s0 > 0, Δ = T /. s → s0; s0 = N[s0 - Δ]; k += 1];
Print["n= ", n, " s0= ", s0, " niter= ", k]

n= 3 s0= -0.00180437 niter= 67
n= 4 s0= -0.00367959 niter= 62
n= 5 s0= -0.00442469 niter= 65
n= 6 s0= -0.00974961 niter= 70
n= 7 s0= -0.0100881 niter= 75
n= 8 s0= -0.000357881 niter= 79
n= 9 s0= -0.0019707 niter= 84
n= 10 s0= -0.00454609 niter= 89

```

We may improve on this as follows: We sharpen (5.9) by means of the trivial estimates

$$(5.11) \quad \sigma_n(s) < \frac{(n+1)^{s+1}}{s+1} \quad \text{and} \quad \frac{(n-1)^{s+1}}{s+1} < \sigma_{n-1}(s)$$

to

$$(5.12) \quad 2(n-1) \frac{(n+1)^{s+1}}{s+1} \leq n n^{\frac{s}{n-1}} (n!)^{-\frac{s}{n(n-1)}} 2^{\frac{s}{2}} \frac{(n-1)^{s+1}}{s+1}$$

or

$$(5.13) \quad 2 \frac{n+1}{n} \leq a^s \quad \text{with} \quad a := \frac{n^{\frac{1}{n-1}} \sqrt{2} (n-1)}{(n+1) (n!)^{\frac{1}{n(n-1)}}}.$$

For  $n \geq 6$  this is better than above. Here are some results:

---

```

Clear[n]; prec = 25; a =  $\frac{n^{\frac{1}{n-1}} * \sqrt{2} * (n-1)}{(n+1) * (n!)^{\frac{1}{n*(n-1)}}$ ; s0 =  $\frac{\text{Log}[2 * \frac{n+1}{n}]}{\text{Log}[a]}$ ;
For[n = 4, n ≤ 10, n++, Print["n= ", n, "   a= ", N[a, prec], "   s0 = ", N[s0, prec]]]

```

n= 4	a= 1.033559084353758331083473	s0 = 27.75943107450229630271386
n= 5	a= 1.109704066632769901145388	s0 = 8.410417598496762008725611
n= 6	a= 1.160841152462510911396772	s0 = 5.681039106244187474052304
n= 7	a= 1.197492567493618220298576	s0 = 4.586801811965574200150807
n= 8	a= 1.225016520960733823964192	s0 = 3.995629039305886521226762
n= 9	a= 1.246425620704417133495549	s0 = 3.624967638233412415065291
n= 10	a= 1.263541985178517497308057	s0 = 3.370644438604987862615722

Starting at these new improved  $s_0$  we find the following:



```

Clear[n, s ];
σ = Sum[ks, {k, n}]; L = 2 *  $\frac{n-1}{n}$  * σ * (n!)  $\frac{s}{n*(n-1)}$  * 2  $\frac{s}{2}$ ; L1 = ∂s L; L2 = ∂s L1;
R = (σ - ns) * n  $\frac{s}{n-1}$  * (2s + 1); R2 = ∂s ∂s R; R3 = ∂s R2; T = Simplify[ $\frac{R2-L2}{R3}$ ];
For[n = 6, n ≤ 10, n++, a = N[ $\frac{n^{\frac{1}{n-1}} * \sqrt{2} * (n-1)}{(n+1) * (n!)^{\frac{1}{n*(n-1)}}$ ]]; s0 =  $\frac{\text{Log}[2 * \frac{n+1}{n}]}{\text{Log}[a]}$ ;
k = 0;
While[s0 > 0, Δ = T /. s → s0; s0 = N[s0 - Δ]; k += 1];
Print["n= ", n, "    a= ", a, "    s0= ", s0, "    # iter= ", k]

n= 6    a= 1.16084    s0= -0.0109249    # iter= 64
n= 7    a= 1.19749    s0= -0.00524811    # iter= 63
n= 8    a= 1.22502    s0= -0.00585335    # iter= 64
n= 9    a= 1.24643    s0= -0.000650036    # iter= 65
n= 10   a= 1.26354    s0= -0.00195887    # iter= 67

```

The upshot of all this is that we need not worry about small  $n$ .

Still better is the following: For  $s > 0$  the Riemann sums  $U_n(s)$  and  $L_n(s)$  are monotonic in  $n$

$$(5.14) \quad U_n(s) < U_{n-1}(s) \quad \text{and} \quad L_{n-1}(s) < L_n(s).$$

Hence

$$(5.15) \quad \frac{A_n}{A_{n-1}} < \left(\frac{n}{n-1}\right)^s$$

and we use this to sharpen (5.9) to

$$(5.16) \quad 2 \leq a^s \quad \text{with} \quad a := \frac{(n-1)n^{\frac{1}{n-1}}\sqrt{2}}{n(n!)^{\frac{1}{n(n-1)}}}.$$

So,  $s \geq \frac{\log 2}{\log a}$  is sufficient for (5.9). We need  $a > 1$ , but this should not be too difficult to prove.

We even observe that  $a$  is increasing in  $n$ . Also this is a matter of routine.

Clearly,  $\lim_{n \rightarrow \infty} a = \sqrt{2}$  so that  $\lim_{n \rightarrow \infty} s_0 = 2$ . Compare the next program.

```

For[n = 3, n ≤ 10, n++, a = N[ $\frac{(n-1) * n^{\frac{1}{n-1}} * \sqrt{2}}{n * (n!)^{\frac{1}{n*(n-1)}}$ ]];

Print["n= ", n, "   a= ", a, "   s0= ",  $\frac{\text{Log}[2]}{\text{Log}[a]}$ ]]

n= 3   a= 1.21141   s0= 3.61413
n= 4   a= 1.29195   s0= 2.706
n= 5   a= 1.33164   s0= 2.42008
n= 6   a= 1.35431   s0= 2.28539
n= 7   a= 1.36856   s0= 2.20915
n= 8   a= 1.37814   s0= 2.16111
n= 9   a= 1.38492   s0= 2.12857
n= 10  a= 1.3899    s0= 2.10536
    
```

From the above it is clear that Alzer's inequality is " only critical " for small  $s$  ( $\approx 0$ ). We therefore restrict ourselves to  $0 < s < 1$ . One more attempt to improve on the above endeavors:

LEMMA 2. For  $0 < s < 1$  we have  $\frac{A_{n+1}(s)}{A_n(s)} < \left(\frac{n+2}{n+1}\right)^s$ .

PROOF.

- (a) We might apply mathematical induction.
  - (b) We *venture* to apply our first log-convexity Conjecture.
- We have to show that

$$(5.17) \quad \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)} < \left(\frac{n+2}{n+1}\right)^s \quad \text{or} \quad \phi(s) := \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)} \left(\frac{n+1}{n+2}\right)^s < 1.$$

Since  $\phi(0) = \phi(1) = 1$ , and  $\phi(s)$  " is " (log-) convex on  $0 \leq s \leq 1$ , we find that  $\phi(s) < 1$  for all  $s \in (0,1)$ . ■

By means of this Lemma Alzer's inequality may be sharpened ( for  $0 < s < 1$  ) to

$$(5.18) \quad \psi(s) := \frac{1}{2} a^s \left(2^{\frac{s}{2}} + 2^{-\frac{s}{2}}\right) > 1 \quad \text{with} \quad a := \frac{n}{n+1} n^{\frac{1}{n-1}} (n!)^{-\frac{1}{n(n-1)}}.$$

Note that  $\psi(0) = 1$  and that  $\psi(s)$  is (log-) convex. Also,  $a$  appears to be increasing ( $\uparrow 1$ ).

The corresponding  $s_0$  may be found by the following *Mathematica* program.

```
For[n = 3, n ≤ 10, n++, a = N[ $\frac{n}{n+1} * n^{\frac{1}{n-1}} * (n!)^{-\frac{1}{n*(n-1)}}$ ];  
Print["n= ", n, "    a= ", a, "    s0= ", FindRoot[a^s * (2s/2 + 2-s/2) == 2, {s, 1}]]]
```

n= 3	a= 0.963674	s0= {s → 0.62086}
n= 4	a= 0.974449	s0= {s → 0.432592}
n= 5	a= 0.980849	s0= {s → 0.322645}
n= 6	a= 0.985006	s0= {s → 0.251892}
n= 7	a= 0.987881	s0= {s → 0.203194}
n= 8	a= 0.989963	s0= {s → 0.168069}
n= 9	a= 0.991526	s0= {s → 0.141775}
n= 10	a= 0.992732	s0= {s → 0.121524}

```
(* The next program relates to the log-convexity in Conjecture 1 *)
$MinPrecision = 25; (* Higher precision ?! *)
Clear[s];
n = 4000; dn = 1000;
While[0 == 0, n += dn;

  σ = Sum[k^s, {k, n}]; q = 1 + (n + 1)^s / σ; Lq = Log[q]; Lq1 = ∂_s q; Lq2 = ∂_s Lq1;

  (* Plot[Lq2, {s, 0, 3}]; *)
  m = FindMinimum[Lq2, {s, 0, 3}, AccuracyGoal → 25, WorkingPrecision → 35];
  Print["n= ", n, " n^2 * min= ", n^2 * m[[1]]]

n= 5000 min= 1.000171304137127467738765

n= 6000 min= 1.000141578311702395389296

n= 7000 min= 1.000120511212464176935035

n= 8000 min= 1.000104821169033464591794

n= 9000 min= 1.000092692619840969988359

n= 10000 min= 1.000083043161272667688070
```

## 6. On the " higher $\sigma$ -quotients ".

So far we discussed the log-convexity in  $s$  of the functions  $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  and  $\frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\sigma_{n+2}(s)}$ . We will now have a look at the " higher  $\sigma$ -quotients ".

With  $\sigma_n := \sigma_n(s) := 1^s + 2^s + 3^s + \dots + n^s$  we found experimentally that all functions  $q_1 := q_1(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$  ( with  $n \in \mathbb{N}$  ) are log-convex in  $s$  on the entire real  $s$ -axis. This is the case  $m = 1$  below.

The next case ( $m = 2$ ) will be  $q_2 := q_2(s) := \frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\sigma_{n+2}(s)}$ . It seems that also all these functions are log-convex in  $s$  on all of  $\mathbb{R}$ .

The next case ( $m = 3$ ) is  $q_3 := q_3(s) := \frac{\sigma_{n+1}^3(s)\sigma_{n+3}}{\sigma_n(s)\sigma_{n+2}^3(s)}$ , and after this we get

$$(m = 4) \quad q_4 := q_4(s) := \frac{\sigma_{n+1}^4(s)\sigma_{n+3}^4(s)}{\sigma_n(s)\sigma_{n+2}^6(s)\sigma_{n+4}(s)}.$$

One will easily recognize the general pattern with binomial coefficients.

In general we get in the  $m$ -th case ( after taking logarithms )

$$(6.1) \quad f := f_m := \log q_m := \log q_m(s) := - \sum_{i=0}^m (-1)^i \binom{m}{i} \log \sigma_{n+i}(s).$$

After performing some experiments we found that all functions  $f := \log q_m(s)$  are convex in  $s$  on all of  $\mathbb{R}$  indeed, *save* for possibly a " few " exceptions for small  $n$ .

- Things seem to be OK, though, for
- $m = 1 \quad n \geq 1$  ( = our first Conjecture )
- $m = 2 \quad n \geq 1$  ( = our second Conjecture )

$m = 3 \quad n \geq 1$  ( hence, no exceptions here either )  
 $m = 4 \quad n \geq 2$   
 $m = 5 \quad n \geq 2$   
 $m = 6 \quad n \geq 3$   
 $m = 7 \quad n \geq 4$   
 $m = 8 \quad n \geq 4$   
 $m = 9 \quad n \geq 5$   
 $m = 10 \quad n \geq 6$   
 $m = 11 \quad n \geq 6$   
 $m = 12 \quad n \geq 7$   
 $m = 13 \quad n \geq 7$   
 $m = 14 \quad n \geq 8$   
 $m = 15 \quad n \geq 9$   
 $m = 16 \quad n \geq 9$   
 $m = 17 \quad n \geq 10$   
 $m = 18 \quad n \geq 10$

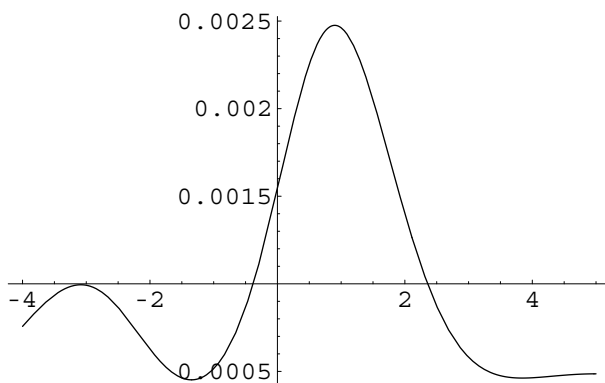
For larger values of  $m$  the computations ( and plots ) become rather fuzzy. So, we stopped the verification here. One may want to experiment further with the following *Mathematica* program. The resulting plots will suggest / reveal / indicate the most critical  $s$ -intervals. One should pay special attention to the interval  $2 < s < 4$  ( for  $n$  not too small ).

```

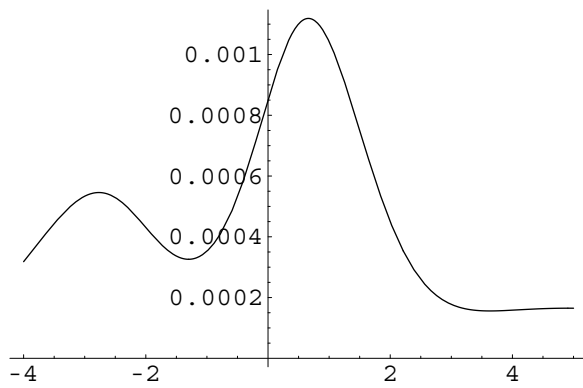
m = 4; (* Here one may take m = 1, 2, 3, 4, ... *)
For[n = 3, n ≤ 6, n++,
  f = - Sum[(-1)^i * Binomial[m, i] * Log[Sum[k^s, {k, n + i}]], {i, 0, m}];
  f1 = ∂s f; f2 = ∂s f1;
  μ = FindMinimum[f2, {s, -2, -1}, AccuracyGoal → 20, WorkingPrecision → 30];
  Print["↓ m = ", m, "      n = ", n, "      μ = ", μ[[1]]];
  Plot[f2, {s, -4, 5}];
(* One should also have a look at other s-intervals ! *)]

```

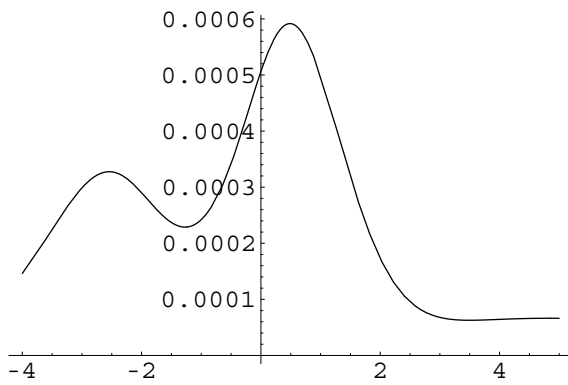
↓ m = 4 n = 3 μ = 0.0004519846914273



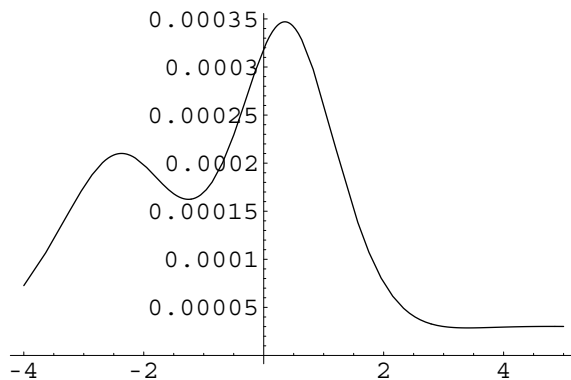
↓ m = 4 n = 4 μ = 0.0003259991033577898142726933



↓ m= 4 n= 5  $\mu=0.0002288355617963047194538695$



↓ m= 4 n= 6  $\mu=0.000162378658911971447310003$



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