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J. VAN DE LUNE & H.J.J. TE RIELE

ON THE ZEROS OF THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP, III

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On the zeros of the Riemann zeta function in the critical strip, III*)

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

We describe extensive computations which show that Riemann's zeta function $\zeta(s)$ has exactly 300,000,001 zeros of the form $\sigma + it$ in the region $0 < t < 119{,}590{,}809.282$. All these zeros are simple and lie on the line $\sigma = \frac{1}{2}$. (This extends a similar result for the first 200,000,001 zeros, established by Brent, van de Lune, te Riele and Winter in *Math. Comp.*, v. 39, 1982, pp. 681-688.) Counts of the numbers of Gram blocks of various types and the failures of "Rosser's rule" are given, together with some graphs of the function $Z(t)$ near the first observed failures of Rosser's rule.

KEY WORDS & PHRASES: *Gram blocks, Riemann hypothesis, Riemann zeta function, Rosser's rule*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

This paper is a *continuation* of BRENT [1] and BRENT, VAN DE LUNE, TE RIELE & WINTER [2]. We have extended the computations described there to show that *the first* 300,000,001 *zeros of Riemann's zeta function* $\zeta(s)$ *in the critical strip are simple and lie on the line* $\sigma = \frac{1}{2}$. After separating the zeros of Z(t) in the range $[g_{200,000,000}, g_{300,000,000})$ with a slightly changed version of the FORTRAN/COMPASS program described in VAN DE LUNE, TE RIELE & WINTER [4], we ran the computation a little further, and found four Gram blocks (of length 1) in $[g_{300,000,000}, g_{300,000,004})$. By applying Theorem 3.2 of [1] we completed the proof of our claim.

A complete listing of our program may be found in VAN DE LUNE & TE RIELE [5]. We also refer to [4] and [5] for a more detailed description of our slightly adapted strategy (compared with Brent's) for finding the required number of sign changes of Z(t) in Gram blocks of length ≥ 2. It may be noted here that the average number of Z-evaluations needed to separate two successive zeros of Z(t) varied slightly around 1.19. Comparing this with our previous computations, it appears that the oscillatory behaviour of Z(t) has not changed very much.

Besides 126 exceptions to Rosser's rule of length 2, we also found three exceptions *of length 3*. The latter ones appeared as Gram blocks of length 3 containing only 1 zero, followed or preceeded by a Gram block of length 1 with 3 zeros. More details are given in the next section, including some graphs of Z(t) in the neighbourhood of various types of exceptions to Rosser's rule which have emerged. (Previously, two exceptions of length 3 were implicitly given by KARKOSCHKA & WERNER [3], viz. $B_{1,089,751,985}$ and $B_{10,008,051,629}$, although these are far beyond the range covered by our systematic search.) In the next section we also give a number of tables which are (nearly) similar to those given in [1] and [2].

We intend to extend our computations in the near future.

2. STATISTICS

The counts given here are now *exact* (in contrast with the LR & W - counts given in [2]). This was realised by immediately computing $Z_B(g_n)$ in case $|Z_A(g_n)|$ was too small (rather than computing $Z_A(g_n-\delta)$ for a few small values of δ, as *we* did in [2]). We *never* met a value of t for which our method B could *not* determine the sign of Z(t) rigorously, with the bounds given in Section 3 of [2].

In Table 1 we present a list of the 126 exceptions to Rosser's rule of length 2 and the 3 of length 3, in the range $[g_{200,000,000}, g_{300,000,000})$, including their types and the local extreme values of S(t). Note the occurrence of a type 4 - exception (the first one observed) which yields the *second* Gram interval with *four* zeros (viz. $G_{237,516,724}$; the first one is $G_{61,331,768}$ [1]).

Table 2 gives a survey of the various types of exceptions to Rosser's rule observed until now, and their frequencies in $[g_{200,000,000}, g_{300,000,000})$.

Table 3 gives the numbers of Gram blocks of length ≤ 8 in $[g_{200,000,000}, g_{300,000,000})$ for strings of 10^7 successive zeros. The last line gives the totals for the whole range. The average Gram block length for this range is 1.2039, against 1.2003 and 1.1900 for the ranges $[g_{100,000,000}, g_{200,000,000})$ and $[g_0, g_{100,000,000})$, respectively. Note that the number of Gram blocks of length 1 is slowly decreasing, in favour of the numbers of Gram blocks of length ≥ 2 and ≤ 5.

In Table 4 we present the numbers of Gram intervals $G_j=[g_j, g_{j+1})$, $n \leq j < n + 10^7$, which contain exactly m zeros of Z(t), $0 \leq m \leq 4$. Note that the number of Gram intervals with precisely one zero is slightly decreasing in favour of the numbers of Gram intervals with no zeros and those with 2 zeros.

In [1] and [2] we have listed the first occurrences of Gram blocks of various types. Here we have met only one new type of Gram block, viz. the type (7,1) - Gram block $B_{258,779,892}$.

In Table 5 we list the numbers of Gram blocks of type (j,k), $1 \leq j \leq 8$, $1 \leq k \leq j$, in the range $[g_{200,000,000}, g_{300,000,000})$. We also give the numbers of Gram blocks of length 2 with zero-pattern "0 0" and "2 2", and those of length 3 with zero-pattern "0 1 0". The "0 0" - blocks correspond to the 126 length 2 - exceptions to Rosser's rule of types 1 - 6 and the "2 2" - blocks correspond to the length 2 - exceptions of types 5 and 6. The "0 1 0" - blocks correspond to the 3 (newly introduced) length 3 - exceptions of types 1 and 2. The entries in parentheses denote the percentages with respect to the total number of blocks of length j, given in the final column. These percentages are nearly the same as those given in the corresponding table for the range

$[g_{156,800,000}, g_{200,000,000})$ in [2], and we conclude that our strategy of dealing with Gram blocks of length $j \geq 2$ is successful for $2 \leq j \leq 5$.

In order to give the reader an impression of the erratic behaviour of Z(t), we give in Figures 1.1 - 1.8 graphs of Z(t) in the neighbourhood of the first (observed) exceptions to Rosser's rule of length 2 and 3 and of various types. We have plotted $Z(g_x)$ with x as a continuous independent variable. The exceptional Gram block is marked by two arrows pointing *downwards*. The adjacent Gram block where the "missing two" zeros are situated is marked by two arrows pointing *upwards*. A magnification of the latter block is shown in an accompanying graph. Some "critical" values of Z(t) are explicitly mentioned.

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TABLE 1

The 129 *exceptions to Rosser's rule in* $[g_{200,000,000}, g_{300,000,000})$

126 of length 2. Notation: n(type)extreme S(t),

where n is the index of the Gram block $B_n = [g_n, g_{n+2})$ with zero-pattern "0 0"

201007375(1)	-2.002900	231810024(1)	-2.026611	272096379(2)	2.001554
201030606(2)	2.111895	232838063(2)	2.022488	272583009(1)	-2.032037
201184291(2)	2.001518	234389089(2)	2.106429	274190882(2)	2.008416
201685414(5)	-2.016715	235588194(1)	-2.001915	274268747(1)	-2.018420
202762876(2)	2.011439	236645695(1)	-2.089639	275297430(2)	2.014738
202860958(2)	2.018888	236962877(2)	2.023259	275545477(2)	2.032087
203832578(2)	2.063611	237516725(4)	2.108817	275898480(2)	2.005068
205880544(1)	-2.017679	240004911(1)	-2.000249	275953000(1)	-2.007296
206357111(1)	-2.031216	240221307(2)	2.096293	277117197(5)	-2.069283
207159768(2)	2.033954	241549003(1)	-2.036151	277447311(2)	2.058999
207167344(2)	2.029320	241729717(1)	-2.025503	279059658(2)	2.037126
207669541(2)	2.020740	241743685(2)	2.070155	279259145(2)	2.033129
208053426(1)	-2.073357	243780201(2)	2.025648	279513637(2)	2.000375
208110028(2)	2.031212	243801317(1)	-2.020358	279849070(2)	2.048163
209513827(2)	2.023920	244122072(1)	-2.035325	280291419(1)	-2.021221
212623522(1)	-2.010194	244691225(2)	2.018927	281449426(2)	2.000609
213841715(1)	-2.024334	244841577(1)	-2.053021	281507954(2)	2.001841
214012333(1)	-2.010937	245813461(1)	-2.035731	281825600(1)	-2.033191
214073567(1)	-2.009287	246299475(1)	-2.001039	282547094(2)	2.002833
215170601(2)	2.007728	246450177(2)	2.116655	283120964(2)	2.028096
215881040(2)	2.021267	249069349(1)	-2.020698	283323493(1)	-2.032511
216274605(2)	2.052279	250076378(1)	-2.036397	284764536(2)	2.001422
216957121(2)	2.032421	252442158(2)	2.085094	286172640(2)	2.042925
217323208(1)	-2.013607	252904232(2)	2.112235	286688824(1)	-2.046407
218799264(1)	-2.040304	255145220(1)	-2.002286	287222173(2)	2.048065
218803558(2)	2.013448	255285972(2)	2.034861	287235535(2)	2.024894
219735146(1)	-2.026815	256713230(1)	-2.015377	287304862(2)	2.003208
219830063(2)	2.015232	257992082(1)	-2.042307	287433571(1)	-2.021945
219897904(1)	-2.081132	258447957(6)	2.005655	287823551(1)	-2.038399
221205545(1)	-2.014535	259298046(2)	2.091955	287872423(2)	2.016959
223601929(1)	-2.101580	262141503(1)	-2.006009	288766616(2)	2.024072
223907077(2)	2.007094	263681744(2)	2.006016	290122964(2)	2.039001
223970397(1)	-2.028754	266617122(1)	-2.046423	290450849(5)	-2.068090
224874046(6)	2.022804	266628045(2)	2.048158	291426142(2)	2.075533
225291157(1)	-2.152675	267305763(1)	-2.028836	292810354(2)	2.048278
227481734(1)	-2.018298	267388405(2)	2.012716	293109862(2)	2.013978
228006443(2)	2.023042	267441673(2)	2.085691	293398055(2)	2.042772
228357900(1)	-2.022758	267464886(1)	-2.006418	294134427(2)	2.043302
228386399(1)	-2.008899	267554908(2)	2.112706	294216438(1)	-2.005490
228907446(1)	-2.018338	269787480(1)	-2.080890	295367142(2)	2.049246
228984553(2)	2.032004	270881434(1)	-2.026487	297834112(2)	2.022351
229140286(2)	2.000109	270997584(2)	2.021752	299099970(2)	2.030191

3 of length 3. Notation: n(type)extreme S(t),

where n is the index of the Gram block $B_n = [g_n, g_{n+3})$ with zero-pattern "0 1 0"

207482540(2)	2.000431	241389213(1)	-2.010430	266527881(2)	-2.008550

(For the definition of the types in case of length 3 exceptions, see Table 2.)

TABLE 2

Various types of exceptions to Rosser's rule and their frequencies in $[g_{200,000,000}, g_{300,000,000})$

Gram block of length 2 with "0 0" zero-pattern (from g_n to g_{n+2})

LENGTH = 2

$[g_{n-2}, g_{n-1})$	$[g_{n-1}, g_n)$	$[g_n, g_{n+1})$	$[g_{n+1}, g_{n+2})$	$[g_{n+2}, g_{n+3})$	$[g_{n+3}, g_{n+4})$	type	frequency
		0	0	3		1	52
	3	0	0			2	68
		0	0	4	0	3	0
0	4	0	0			4	1
		0	0	2	2	5	3
2	2	0	0			6	2

Gram block of length 3 with "0 1 0" zero-pattern (from g_n to g_{n+3})

LENGTH = 3

$[g_{n-1}, g_n)$	$[g_n, g_{n+1})$	$[g_{n+1}, g_{n+2})$	$[g_{n+2}, g_{n+3})$	$[g_{n+3}, g_{n+4})$	type	frequency
	0	1	0	3	1	2
3	0	1	0		2	1

TABLE 3

Number of Gram blocks of given length

$J'(k,n) := J(k,n+10^7) - J(k,n)$

n	J'(1,n)	J'(2,n)	J'(3,n)	J'(4,n)	J'(5,n)	J'(6,n)	J'(7,n)	J'(8,n)
200,000,000	6,973,019	1,056,242	236,180	44,997	4,838	272	21	0
210,000,000	6,971,273	1,055,810	236,438	45,249	4,976	297	17	2
220,000,000	6,966,636	1,056,779	236,573	45,864	4,921	312	22	0
230,000,000	6,965,176	1,056,494	236,663	46,250	4,979	295	26	0
240,000,000	6,961,469	1,057,120	237,279	46,150	5,156	327	16	0
250,000,000	6,957,609	1,057,612	237,551	46,633	5,145	341	29	1
260,000,000	6,955,568	1,056,920	238,418	46,694	5,300	306	30	2
270,000,000	6,951,895	1,057,635	238,483	47,211	5,257	353	20	0
280,000,000	6,950,241	1,056,940	238,974	47,373	5,420	366	24	0
290,000,000	6,948,974	1,058,297	238,885	47,295	5,254	354	29	0
Totals	69,601,860	10,569,849	2,375,444	463,716	51,246	3,223	234	5

TABLE 4

Number of Gram intervals containing exactly m *zeros*

n	m = 0	m = 1	m = 2	m = 3	m = 4
200,000,000	1,360,273	7,297,177	1,324,827	17,723	0
210,000,000	1,360,333	7,296,878	1,325,245	17,544	0
220,000,000	1,362,066	7,293,463	1,326,876	17,595	0
230,000,000	1,362,561	7,292,733	1,326,852	17,853	1
240,000,000	1,363,647	7,290,305	1,328,449	17,599	0
250,000,000	1,365,010	7,287,678	1,329,614	17,698	0
260,000,000	1,365,698	7,286,632	1,329,642	18,028	0
270,000,000	1,366,741	7,284,300	1,331,176	17,783	0
280,000,000	1,367,037	7,283,866	1,331,158	17,939	0
290,000,000	1,367,806	7,282,080	1,332,422	17,692	0
Totals	13,641,172	72,895,112	13,286,261	177,454	1

Number of Gram blocks of type (j,k), $1 \le j \le 8$, $1 \le k \le j$, *in the range* $[g_{200,000,000}, g_{300,000,000})$

↓ j \ k →	1	2	3	4	5	6	7	8	Totals
1	69,601,860								69,601,860
2	5,285,566 (50)	5,284,152 (50)	126 "0 0" - blocks 5 "2 2" - blocks						10,569,849
3	1,125,098 (47)	124,535 (5)	1,125,808 (47)	3 "0 1 0" - blocks					2,375,444
4	212,286 (46)	19,862 (4)	19,649 (4)	211,919 (46)					463,716
5	20,326 (40)	4,252 (8)	2,177 (4)	4,346 (8)	20,145 (39)				51,246
6	488 (15)	866 (27)	263 (8)	286 (9)	859 (27)	461 (14)			3,223
7	2*)	73	42	9	40	67	1*)		234
8	0	0	2	0	0	3	0	0	5

*) viz. B_n, for n = 258,779,892, 282,307,390 and 299,608,968.

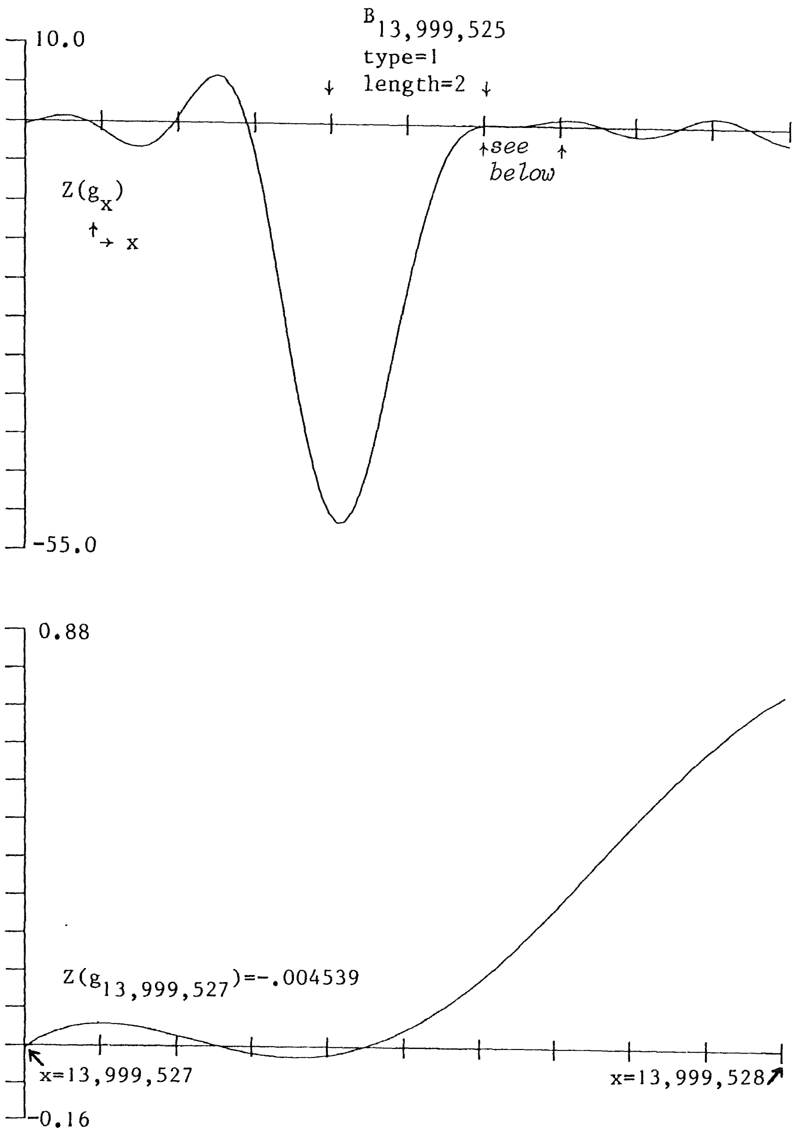

Fig. 1.1 $B_{13,999,525}$

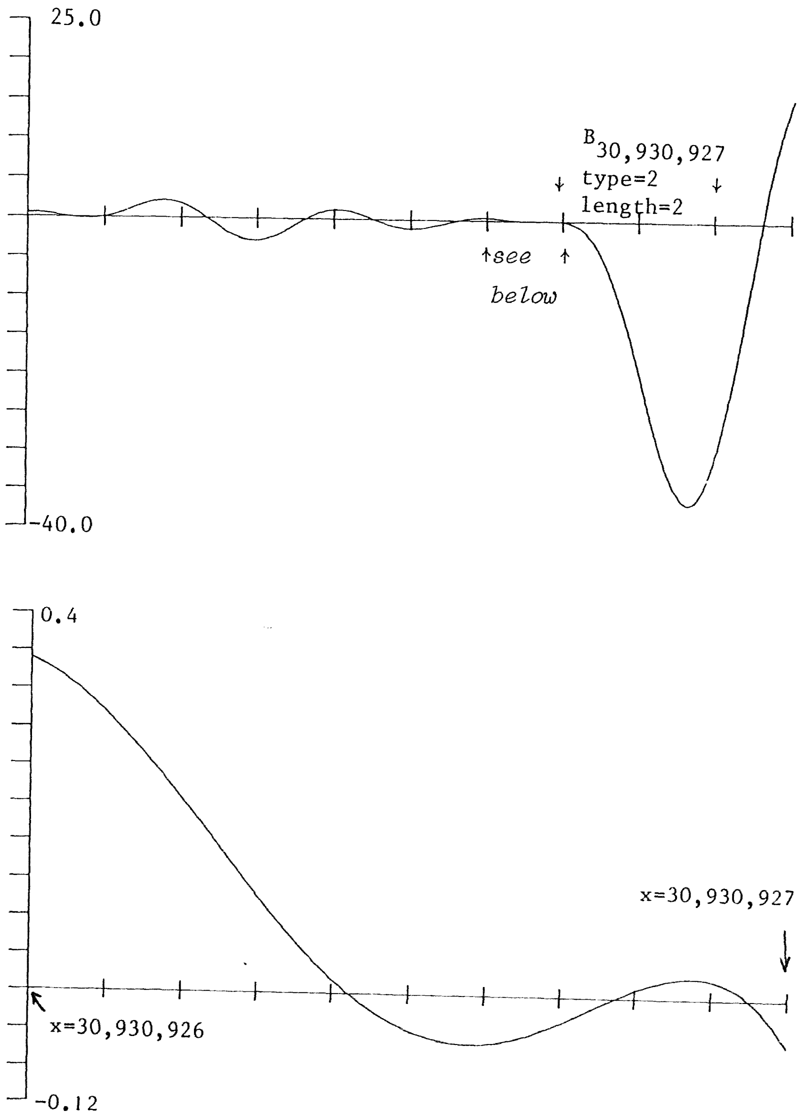

Fig. 1.2 $B_{30,930,927}$

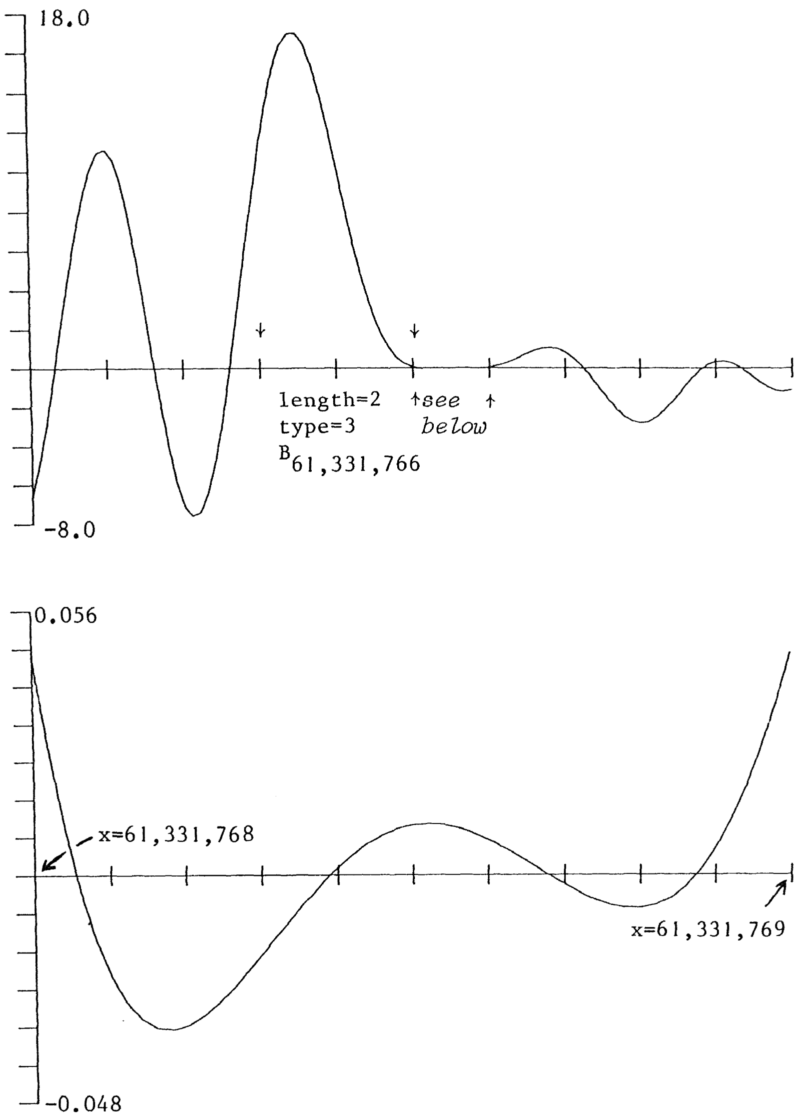

Fig. 1.3 $B_{61,331,766}$

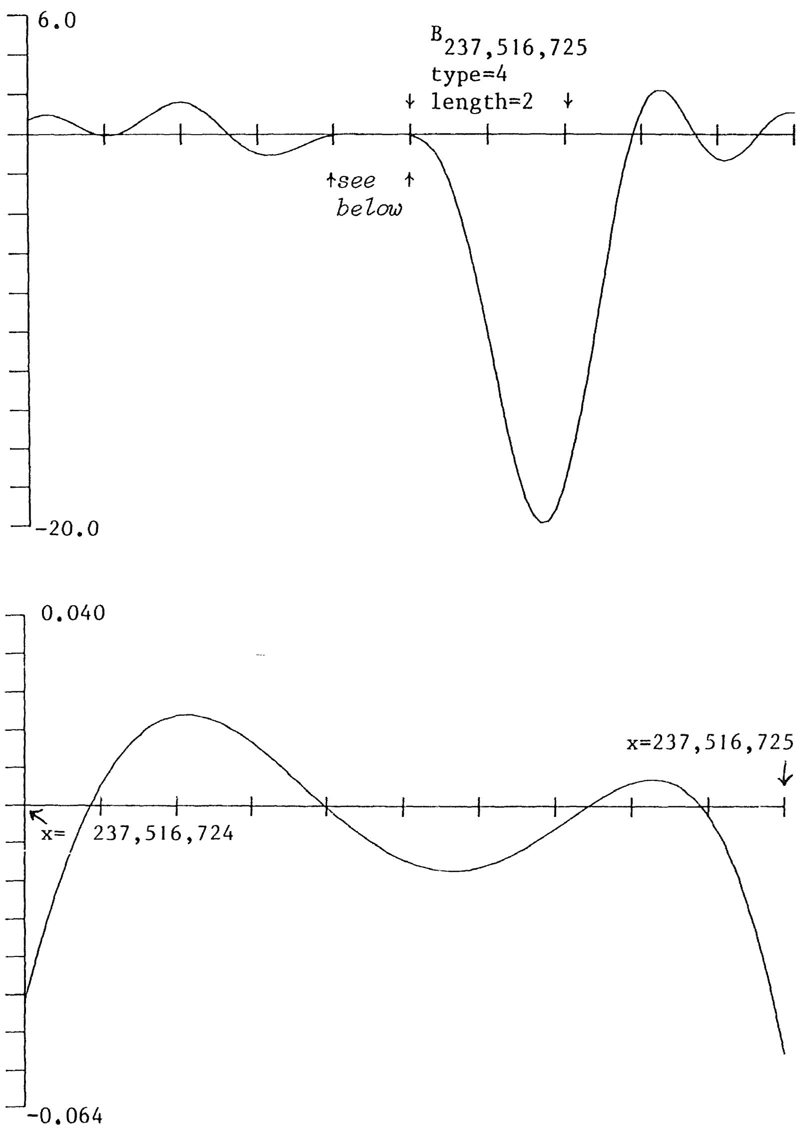

Fig. 1.4 $B_{237,516,725}$

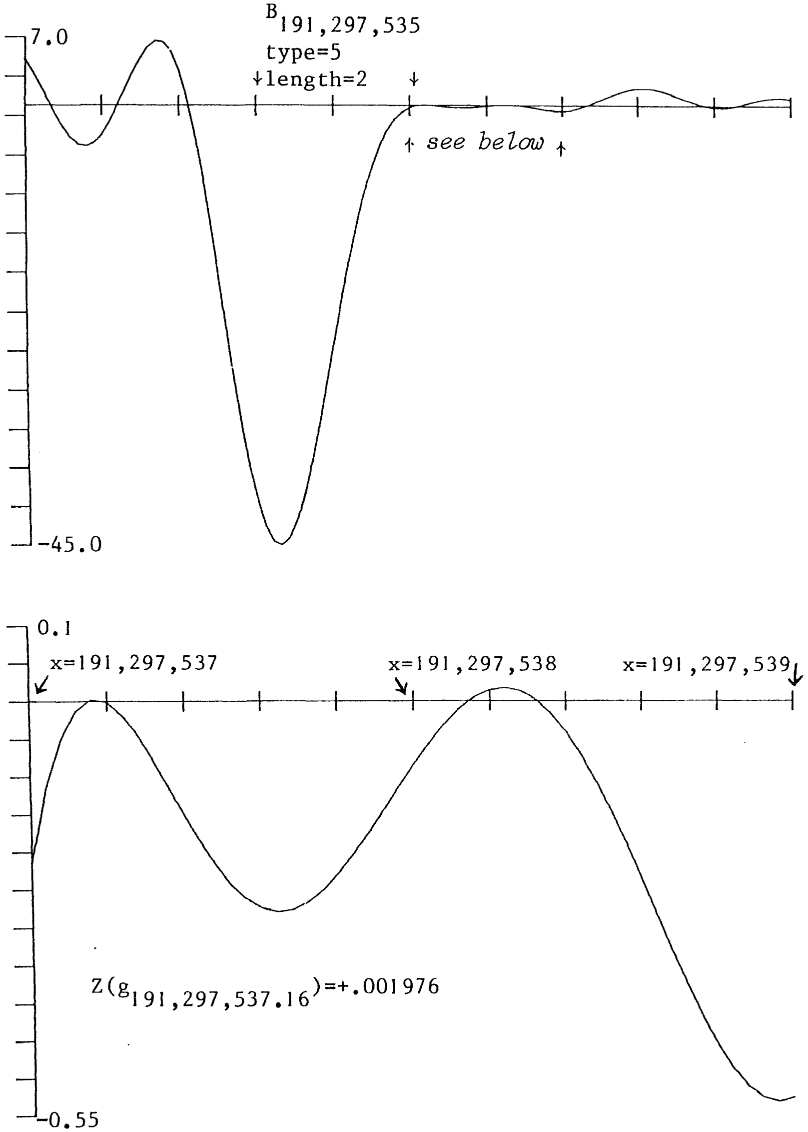

Fig. 1.5 $B_{191,297,535}$

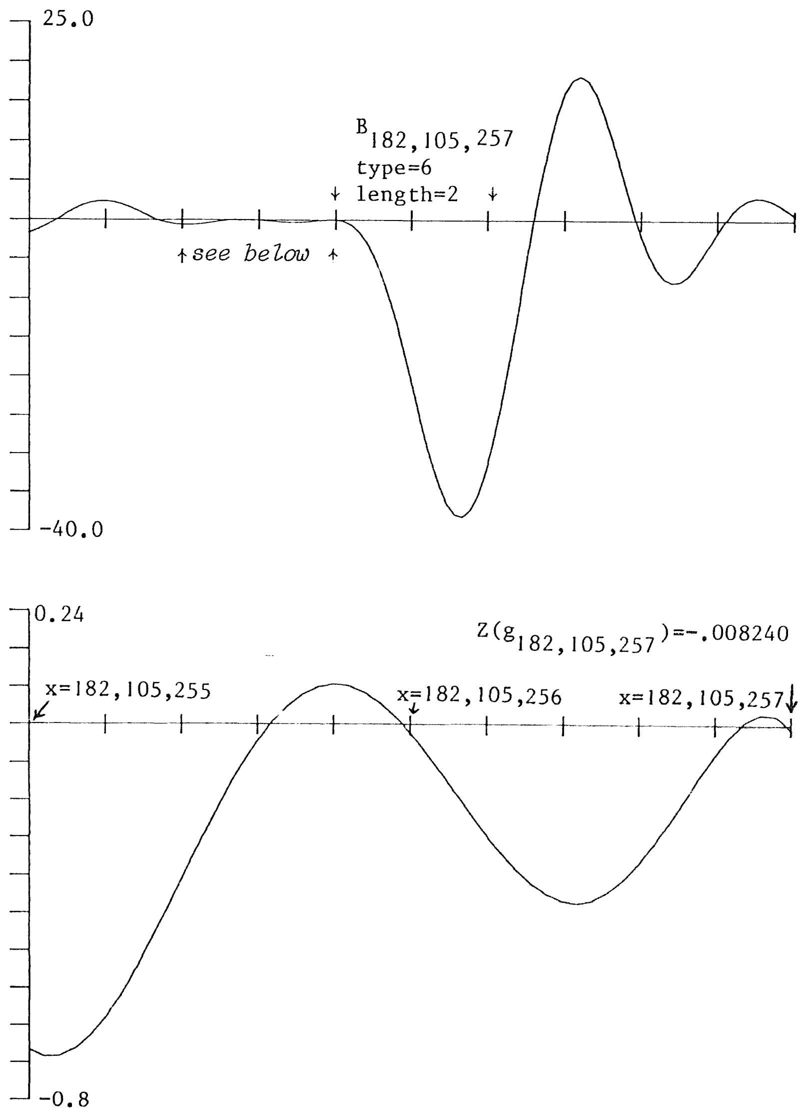

Fig. 1.6 $B_{182,105,257}$

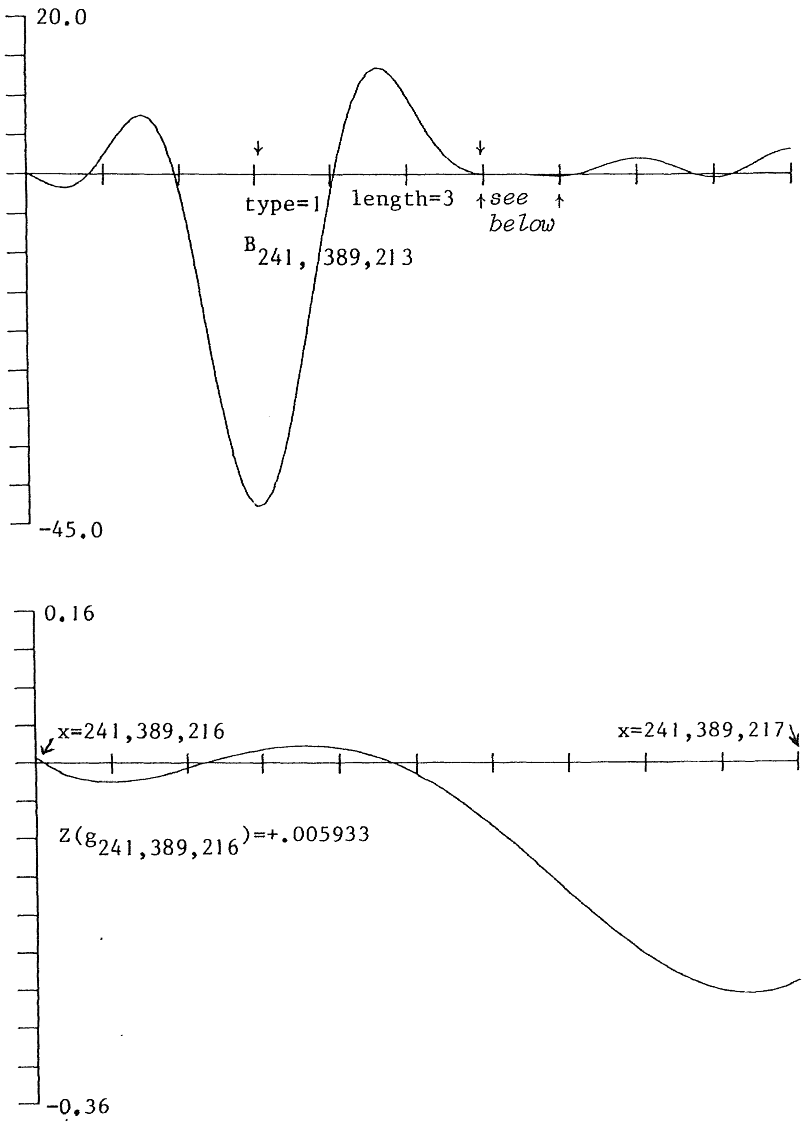

Fig. 1.7 $B_{241,389,213}$

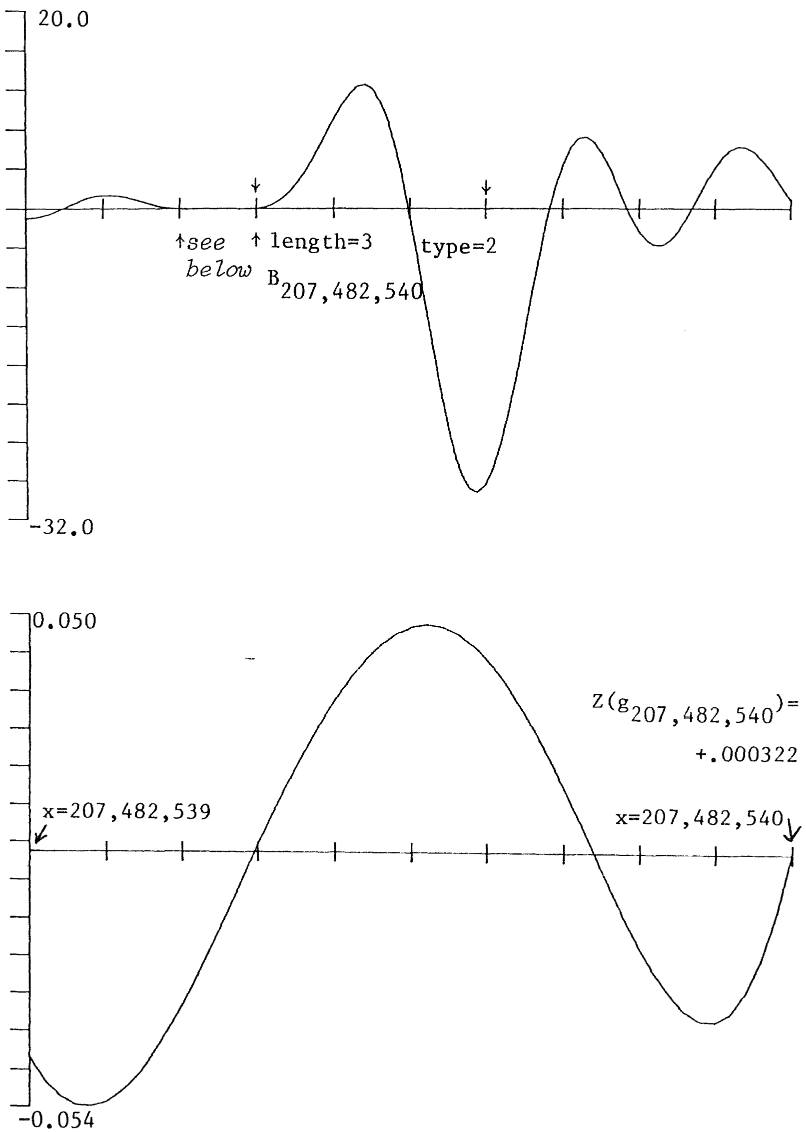

Fig. 1.8 $B_{207,482,540}$