An Eilenberg–like Theorem for Algebras on a Monad

Julian Salamanca[*,1]

1 **CWI Amsterdam, The Netherlands**
salamanc@cwi.nl

Abstract

An Eilenberg–like theorem is shown for algebras on a given monad. The main idea is to explore the approach given by Bojańczyk that defines, for a given monad T on a category $\mathcal{D}$, pseudovarieties of T–algebras as classes of finite T–algebras closed under homomorphic images, subalgebras, and finite products. To define pseudovarieties of recognizable languages, which is the other main concept for an Eilenberg–like theorem, we use a category $\mathcal{C}$ that is dual to $\mathcal{D}$ and a recent duality result between Eilenberg–Moore categories of algebras and coalgebras by Salamanca, Bonsangue, and Rot. Using this duality, we define the concept of a pseudovariety of recognizable languages based on the category $\mathcal{C}$. With this new approach, we can study different kinds of pseudovarieties of algebras as well as different kinds of pseudovarieties of recognizable languages to (re)discover some existing and new Eilenberg–like theorems. By dropping finiteness conditions we also derive an Eilenberg–like theorem for varieties of T–algebras, that is, classes of T–algebras closed under homomorphic images, subalgebras, and (not necessarily finite) products.

Keywords and phrases Eilenberg's variety theorem, language, duality, monad, syntactic algebra, (pseudo)variety of algebras/languages.

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

Eilenberg's variety theorem is an important result in algebraic language theory, stating that there is a one–to–one correspondence between pseudovarieties of regular languages on finite alphabets and pseudovarieties of monoids [8, Theorem 3.4]. The concept of regular language, which is defined in terms of deterministic automata, has an equivalent machine–independent algebraic definition, namely, a finitely recognizable language. Finitely recognizable languages on an alphabet Σ are inverse images of monoid homomorphisms with domain Σ^* and codomain any finite monoid. This algebraic approach allows us to study different kinds of recognizable languages where the notion of homomorphism between algebras is a key ingredient.

The notion of pseudovariety of algebras (also known as variety of finite algebras) has already been studied in universal algebra to get a Birkhoff–like variety theorem for finite algebras [15], Reiterman's theorem. A pseudovariety of algebras is a class of finite algebras of the same type that is closed under homomorphic images, subalgebras, and finite products. At the same time, to prove an Eilenberg-like theorem, one has to define and find the proper notion of pseudovariety of finitely recognizable languages which is, in general, a non–trivial problem.

There are some Eilenberg–like theorems in the literature such as [13] where the algebras considered are ordered monoids/semigroups, the one in [16] for finite dimensional k–algebras,

* The research of this author is funded by the Dutch NWO project 612.001.210.

and [14] for idempotent semirings. Recently, generalizations for Eilenberg–like theorems have been proved by using category theory, such as [1, 4, 19], from which one can derive Eilenberg's variety theorem and other Eilenberg–like theorems by using the categorical framework.

The work in the present paper has its basis in [4] and [18]. We take the main idea given in [4], where all the algebras considered are algebras for a monad T on $\mathcal{D}$, to define the natural notion of (pseudo)variety of T–algebras. On the other hand, to define the notion of (pseudo)variety of (finitely) recognizabe T–languages, we use a category $\mathcal{C}$ that is dual to $\mathcal{D}$ and a result given in [18] that allows us to define a canonical comonad on $\mathcal{C}$ and lift the duality between $\mathcal{C}$ and $\mathcal{D}$ to their corresponding Eilenberg–More categories. With these two ingredients, under mild assumptions, we prove an Eilenberg–like theorem to establish a one–to–one correspondence between pseudovarieties of T–algebras and pseudovarieties of finitely recognizable T–languages. Additionally, all the facts to prove the main theorem are adapted to prove an Eilenberg–like theorem where finiteness conditions are dropped, i.e., a one–to–one correspondence between varieties of T–algebras and varieties of recognizable T–languages. With this we derive Eilenberg–like theorems for varieties of algebras such as [3, Theorem 39].

Related work. We briefly summarize here some works in which categorical approaches to derive Eilenberg–like theorems are used such as [1, 4, 19] (see the Conclusions for a more detailed discussion.) In [1] and [19] predual categories are considered, i.e., categories that are dual on finite objects, whereas in the present paper, we only consider dual categories. In [4] the kind of algebras considered are algebras for a monad T on Set or Set^S, for a fixed set S, but there is no (pre)duality involved, the latter restricts the kind of pseudovarieties of languages that can be studied. The approach in [4] of considering algebras for a monad T is also considered in [19] as well as in the present paper. In the present paper the notion of pseudovariety of languages is defined coalgebraically, by using duality, whose definition avoids the explicit definition of derivatives, in contrast to [1, 4, 19]. In the present paper, we drop finiteness conditions to prove in a similar way an Eilenberg–like theorem for varieties of T–algebras. Categorical approaches such as [1, 4, 19] only consider the finite case.

2 Preliminaries

In this section, we introduce the notation for categories of algebras and coalgebras that we will use in the paper. We assume that the reader is familiar with basic concepts from category theory and (co)algebra, see, e.g., [2, 17].

Given a category $\mathcal{D}$ and a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, we denote by $\mathrm{Alg}(\mathsf{T})$ the category of (Eilenberg–Moore) T-algebras and their homomorphisms. Objects in $\mathrm{Alg}(T)$ are pairs (X, α) where X is an object in $\mathcal{D}$ and $\alpha \in \mathcal{D}(TX, X)$ is a morphism $\alpha : TX \to X$ in $\mathcal{D}$ that satisfies the identities $\alpha \circ \eta_X = id_X$ and $\alpha \circ T\alpha = \alpha \circ \mu_X$. A homomorphism from a T-algebra (X_1, α_1) to a T-algebra (X_2, α_2) is a morphism $h \in \mathcal{D}(X_1, X_2)$ such that $h \circ \alpha_1 = \alpha_2 \circ Th$.

Dually, given a category $\mathcal{C}$ and a comonad $\mathsf{B} = (B, \epsilon, \delta)$ on $\mathcal{C}$, $\mathrm{Coalg}(\mathsf{B})$ denotes the category of (Eilenberg–Moore) B-coalgebras. Objects in $\mathrm{Coalg}(\mathsf{B})$ are pairs (Y, β) where Y is an object in $\mathcal{C}$ and $\beta \in \mathcal{C}(Y, BY)$ satisfies the identities $\epsilon_Y \circ \beta = id_Y$ and $B\beta \circ \beta = \delta_Y \circ \beta$. A homomorphism from a B-coalgebra (Y_1, β_1) to a B-coalgebra (Y_2, β_2) is a morphism $h \in \mathcal{C}(Y_1, Y_2)$ such that $\beta_2 \circ h = Bh \circ \beta_1$.

Each of the categories $\mathrm{Alg}(\mathsf{T})$ and $\mathrm{Coalg}(\mathsf{B})$ has a canonical forgetful functor into the underlying category. For instance, the forgetful functor for $\mathrm{Alg}(\mathsf{T})$ is the functor $U : \mathrm{Alg}(\mathsf{T}) \to \mathcal{D}$ defined as $U(X, \alpha) = X$ and $Uf = f$ for any T-algebra morphism f. We will refer to those forgetful functors without giving them a specific name.

We assume that the categories $\mathcal{D}$ and $\mathcal{C}$ in this paper are concrete categories, i.e., they come with a faithful functor into the category Set of sets and functions. With this assumption we can use set theoretical concepts such as being finite, surjective, and work with elements, which are key properties in this work.

3 Setting the scene

In this section, we set the scene in which we are going to work, state some assumptions we need, and some results we are going to use.

Our main purpose is to abstract the general notions involved in Eilenberg's variety theorem [8, Theorem 3.4], and generalize it to a categorical setting. In order to do this, we summarize Eilenberg's variety theorem and see how we can generalize those concepts to a categorical point of view.

Eilenberg's variety theorem says that there is a one–to–one correspondence between pseudovarieties of finitely recognizable languages and pseudovarieties of monoids. To state this correspondence, we need to define what a language is and what we mean by being finitely recognizable.

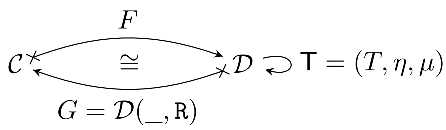
▶ **Definition 1.** Let Σ be a set (*alphabet*) and denote by Σ^* the free monoid with Σ generators, that is, elements in Σ^* are finite words with symbols in Σ, the empty word is denoted as ε. A *language* over Σ is a subset L of Σ^* or, equivalently, a function $L : \Sigma^* \to 2$ from Σ^* into the two element set $2 = \{0, 1\}$. Given a monoid M and a homomorphism of monoids $h : \Sigma^* \to M$, we say that the language $L : \Sigma^* \to 2$ is *recognized* by M through h if there exists a function $L' : M \to 2$ such that $L' \circ h = L$. We say that L is *recognizable* if it is recognized by a monoid and *finitely recognizable* if it is recognized by a finite monoid.

The languages considered in Eilenberg's variety theorem are languages over finite sets Σ, we will make this explicit in the definition of pseudovarieties of languages. Now we can define the concepts of pseudovariety of monoids and pseudovariety of finitely recognizable languages as follows:

- A *pseudovariety of monoids* is a class $\mathcal{A}$ of finite monoids that is closed under homomorphic images, submonoids, and finite products.
- A *pseudovariety of finitely recognizable languages* is an operator $\mathcal{L}$ such that for every finite alphabet Σ we have that $\mathcal{L}\Sigma$ is a set of finitely recognizable languages over Σ satisfying the following properties:
  a) $\mathcal{L}\Sigma$ is a Boolean algebra and it is closed under left and right derivatives. The latter means that for every $L \in \mathcal{L}\Sigma$ and $a \in \Sigma$ we have that ${}_aL, L_a \in \mathcal{L}\Sigma$, where $L_a(w) = L(aw)$ and ${}_aL(w) = L(wa)$, $w \in \Sigma^*$.
  b) $\mathcal{L}$ is closed under morphic preimages: For every finite alphabet Γ, monoid homomorphism $h : \Gamma^* \to \Sigma^*$, and $L \in \mathcal{L}\Sigma$, we have that $L \circ h \in \mathcal{L}\Gamma$.

With these definitions, Eilenberg's variety theorem [8, Theorem 3.4] establishes a one–to–one correspondence between pseudovarieties of finitely recognizable languages and pseudovarieties of monoids. Eilenberg also proved a similar theorem for semigroups and languages not containing the empty word [8, Theorem 3.4s]. To abstract the notion of pseudovariety of algebras we use a similar approach to the one in [4] in which the concept of pseudovariety of monoids is replaced by the concept of pseudovariety of T–algebras for a monad $\mathsf{T} = (T, \eta, \mu)$ on a category $\mathcal{D}$. The approach in the present paper is more general in the sense that the concept of pseudovariety of languages considered will depend on a category $\mathcal{C}$, which is dual to $\mathcal{D}$, thus avoiding the limitation in [4] for which the Boolean algebra requirement in a) above is always present.

We start with a category $\mathcal{D}$ and a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$. A *pseudovariety of* T*–algebras* is a class $\mathcal{A}$ of finite T–algebras that is closed under homomorphic images, subalgebras, and finite products. Now to define the concept of a language we use the monad T and a fixed finite object in $\mathcal{D}$ that we denote by $\mathtt{R}$. Formally, a T*–language* over an object Σ in $\mathcal{D}$ is an element in $\mathcal{D}(T\Sigma, \mathtt{R})$ (for classical languages take $\mathcal{D} = \mathrm{Set}$, $T\Sigma = \Sigma^*$, and $\mathtt{R}$ the two–element set $2 = \{0, 1\}$). Now, as we mentioned, we would like to study T–languages and define the notion of pseudovariety of finitely recognizable T–languages on a different category $\mathcal{C}$. In order to do this a natural assumption is that for every object X in $\mathcal{D}$ we have that $\mathcal{D}(X, \mathtt{R})$ is the underlying set of an object in $\mathcal{C}$. In fact, we assume that we have a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ such that the underlying set of GY is $\mathcal{D}(Y, \mathtt{R})$ and its definition on morphisms is as in the Hom–set functor $\mathcal{D}(\_, \mathtt{R})$. Sometimes we use the notation $\mathcal{D}(\_, \mathtt{R})$ for the functor G. Finally as we would like to exploit the interaction between $\mathcal{D}$ and $\mathcal{C}$ we assume that there is a duality between $\mathcal{D}$ and $\mathcal{C}$ given by the contravariant functor $G = \mathcal{D}(\_, \mathtt{R}) : \mathcal{D} \to \mathcal{C}$ and a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ which leads us to the situation below. Throughout this paper we depict contravariant functors in diagrams with an '×' at the beginning of the arrow.



Notice that most common dualities have this form, e.g. the duality between Set and the category CABA of complete atomic Boolean algebras, the duality between the category Poset of partially ordered sets and the category AlgCDL of algebraic complete distributive lattices, Stone duality, and Priestley duality. In all those cases the underlying set of $\mathtt{R}$ is a two–element 'schizophrenic' object which belongs to the different corresponding categories. Furthermore, since $\mathtt{R}$ is a finite object in $\mathcal{D}$ we can also construct, under certain conditions, dualities for which the contravariant functor $\mathcal{D}(\_, \mathtt{R})$ is part of the duality [6].

To complete our setting we will extend the duality between the categories $\mathcal{C}$ and $\mathcal{D}$ to a duality between Eilenberg–Moore categories, using the following result from [18].

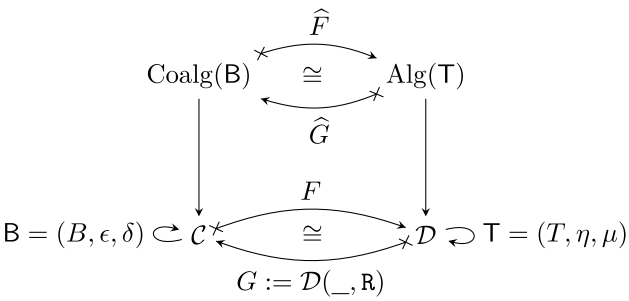
▶ **Proposition 2.** *[18, Proposition 14] Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be contravariant functors that form a duality with natural isomorphisms $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ and $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$. Let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{D}$. Then $\mathsf{B} = (B, \epsilon, \delta)$, where $B = GTF$ and ϵ, δ are defined as:*

$$\epsilon = (GLF \xRightarrow{G\eta_F} GF \xRightarrow{(\eta^{GF})^{-1}} Id_{\mathcal{C}})$$

$$\delta = (GLF \xRightarrow{G\mu_F} GLLF \xRightarrow{GL(\eta^{FG})^{-1}_{LF}} GLFGLF),$$

is a comonad on $\mathcal{C}$. Further, the duality between F and G lifts to a duality between $\widehat{F}$: $\mathrm{Coalg}(\mathsf{B}) \to \mathrm{Alg}(\mathsf{T})$ and $\widehat{G} : \mathrm{Alg}(\mathsf{T}) \to \mathrm{Coalg}(\mathsf{B})$.

We can summarize our setting in the picture below, where the comonad B and the upper part are obtained from the previous proposition.



(1)

4 The main theorem

In this section, we formalize the necessary concepts and formulate the assumptions needed to state the main theorem. As a running example we will consider Eilenberg's classical variety theorem for which $\mathcal{D} = \text{Set}$, $\mathcal{C} = \text{CABA}$ (the category of complete atomic Boolean algebras), $\mathtt{R} = 2 = \{0, 1\}$, and $T\Sigma = \Sigma^*$.

▶ **Definition 3.** Let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{D}$ and let $\mathtt{R}$ be a fixed finite object in $\mathcal{D}$. A T*–language* over an object Σ in $\mathcal{D}$ is an element $L \in \mathcal{D}(T\Sigma, \mathtt{R})$. Let $T\Sigma$ be the algebra $T\Sigma = (T\Sigma, \mu_\Sigma)$ in $\text{Alg}(\mathsf{T})$, a T–language L over Σ is *recognized* by $A = (A, \alpha) \in \text{Alg}(\mathsf{T})$ through a morphism $h \in \text{Alg}(\mathsf{T})(T\Sigma, A)$ if there exists $L' \in \mathcal{D}(A, \mathtt{R})$ such that $L' \circ h = L$. We say that L is *recognizable* if it is recognized by a T–algebra and *finitely recognizable* if it is recognized by a finite T–algebra.

▶ **Example 4.** Let $\mathcal{D} = \text{Set}$ be the category of sets and functions and $\mathtt{R}$ be the two–element set $2 = \{0, 1\}$. Let $\mathsf{T} = (T, \eta, \mu)$ be the monad on Set given by $T\Sigma = \Sigma^*$, where Σ^* is the free monoid on Σ generators with identity given by the empty word ε. Then a T–language over a set (alphabet) Σ is a function $L \in \text{Set}(\Sigma^*, 2)$, i.e., a subset of Σ^*. Algebras in $\text{Alg}(\mathsf{T})$ are monoids, and (finitely) recognizable T–languages are languages recognized by a (finite) monoid in the classical sense, i.e., regular languages. ◀

Now we define the main concepts of pseudovariety of T–algebras and pseudovariety of finitely recognizable T–languages as follows.

▶ **Definition 5.** Consider the situation given in (1). Then:

1) A *pseudovariety of* T*–algebras* is a class $\mathcal{A}$ of finite T–algebras closed under homomorphic images, subalgebras, and finite products[1].
2) A *pseudovariety of finitely recognizable* T*–languages* is an operator $\mathcal{L}$ such that for every finite object Σ in $\mathcal{D}$, $\mathcal{L}\Sigma$ is a set of finitely recognizable T–languages over Σ satisfying the following properties:
   i) For every $L_1, L_2 \in \mathcal{L}\Sigma$ we have that $\langle L_1, L_2 \rangle \subseteq \mathcal{L}\Sigma$, where $\langle L_1, L_2 \rangle$ is the least subcoalgebra in $\text{Coalg}(\mathsf{B})$ of $\widehat{G}(T\Sigma)$ containing $\{L_1, L_2\}$[2].

---

[1] Homomorphic images and subalgebras are defined with respect to the epimorphisms and monomorphisms we are considering, respectively. Here only binary products are considered.

[2] To guarantee the existence of $\langle L_1, L_2 \rangle$ we will assume that $\mathcal{C}$ has generalized pullbacks and that the functor B preserves weak generalized pullbacks, which, by duality, is equivalent to $\mathcal{D}$ having generalized pushouts and T preserving weak generalized pushouts. This assumption will allow the construction of syntactic T–algebras for every language.

ii) Closure under morphic preimages: For every finite object Γ in $\mathcal{D}$, homomorphism $h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$, and $L \in \mathcal{L}\Sigma$, we have that $L \circ h \in \mathcal{L}\Gamma$.

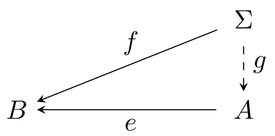
▶ **Example 6.** (Example 4 continued) Let $\mathcal{C} = \mathrm{CABA}$ be the category of complete atomic Boolean algebras with complete Boolean homomorphisms. Then we have that Set is dual to CABA if we consider the contravariant functors $G : \mathrm{Set} \to \mathrm{CABA}$ and $F : \mathrm{CABA} \to \mathrm{Set}$ given by $G = \mathrm{Set}(\_, 2)$ and $F = \mathrm{At}(\_)$, where $\mathrm{At}(B)$ is the set of atoms of a given object B in CABA. In this setting, we have the following:

1) Pseudovarieties of T–algebras are classes of finite monoids that are closed under homomorphic images, submonoids, and finite products.
2) Pseudovarieties of finitely recognizable T–languages are operators $\mathcal{L}$ such that for every finite set (alphabet) $\mathcal{L}\Sigma$ is a set of regular (finitely recognizable) languages over Σ such that:
   i) $\mathcal{L}\Sigma$ is a Boolean algebra and it is closed under left and right derivatives. This property follows from i) in the definition above. Observe that we don't require $\mathcal{L}\Sigma$ to be an object in CABA since in property i) we only require $\langle L_1, L_2\rangle \subseteq \mathcal{L}\Sigma$ for every $L_1, L_2 \in \mathcal{L}\Sigma$.
   ii) $\mathcal{L}$ is closed under morphic preimages: For every finite set (alphabet) Γ and homomorphism of monoids $h : \Gamma^* \to \Sigma^*$, and $L \in \mathcal{L}\Sigma$, we have that $L \circ h \in \mathcal{L}\Sigma$. ◀

To state the main theorem we summarize the list of assumptions needed and some notation, so we can reference them through the paper, as follows:

(A1) (see picture (1)) Let $\mathcal{D}$ be a category with generalized pushouts, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ such that T preserves weak generalized pushouts for which all the arrows are epis. Let $\mathtt{R}$ be a finite object in $\mathcal{D}$ and $\mathcal{C}$ a category that is dual to $\mathcal{D}$ by contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that the underlying set of GY is $\mathcal{D}(Y, \mathtt{R})$ for any object Y in $\mathcal{D}$ and its definition on morphisms is as in the Hom–set functor $\mathcal{D}(\_, \mathtt{R})$. By abuse of notation, we put $G = \mathcal{D}(\_, \mathtt{R})$ to remind the reader about this fact. Notice that the duality between $\mathcal{D}$ and $\mathcal{C}$ can be lifted to a duality between Eilenberg–Moore categories without any extra assumptions by using Proposition 2.

(A2) For every morphism $f \in \mathcal{D}(\Sigma, B)$ and epimorphism $e \in \mathrm{Alg}(\mathsf{T})(A, B)$ there exists a morphism $g \in \mathcal{D}(\Sigma, A)$ such that the following diagram in $\mathcal{D}$ commutes:



(A3) The category $\mathcal{D}$ has epi–mono factorizations.

(A4) Epis in $\mathcal{D}$ are surjective.

(A5) The object $\mathtt{R}$ in (A1) is such that for any finite object X in $\mathcal{D}$ and elements x_1, x_2 in the underlying set of X with $x_1 \neq x_2$, there exists $f \in \mathcal{D}(X, \mathtt{R})$ such that $f(x_1) \neq f(x_2)$. That is, points in X can be separated by a morphism in $\mathcal{D}(X, \mathtt{R})$.

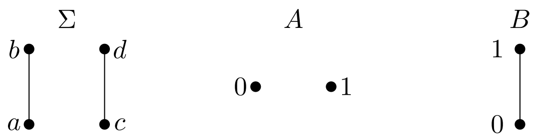
▶ **Example 7.** Some examples for categories $\mathcal{D}$ and $\mathcal{C}$, a monad T on $\mathcal{D}$, and a finite object $\mathtt{R}$ for which the previous assumptions are satisfied include the following:

i) The duality between the category $\mathcal{D} = \mathrm{Set}$ of sets and functions and the category $\mathcal{D} = \mathrm{CABA}$ of complete atomic Boolean algebras with complete Boolean algebra homomorphisms. In this case, $\mathtt{R}$ is the two–element set $\mathtt{R} = 2 = \{0, 2\}$ and the underlying set GX for any set X is $\mathrm{Set}(X, 2)$. Some monads on Set we can consider in this setting include:

- The monad $T\Sigma = \Sigma^+$ of finite nonempty words for which T–algebras are semigroups.
- The monad $T\Sigma = \Sigma^*$ of finite words for which T–algebras are monoids.
- The monad $T\Sigma = \mathfrak{F}(\Sigma)$, where $\mathfrak{F}(\Sigma)$ is the free group on Σ generators, for which T–algebras are groups.
- The monad $T\Sigma = M \times \Sigma$, where M is a fixed monoid, for which T–algebras are monoid actions.

ii) For a fixed finite field $\mathbb{K}$ the duality between $\mathcal{D} = \mathrm{Vec}_{\mathbb{K}}$, the category of $\mathbb{K}$–vector spaces with linear maps, and the category $\mathcal{C} = \mathrm{TBVec}_{\mathbb{K}}$ of topological $\mathbb{K}$–vector spaces whose topology is Boolean (i.e. compact Hausdorff spaces with a basis of clopen sets) and continuous linear maps as morphisms. In this case $\mathtt{R}$ is the finite field $\mathtt{R} = \mathbb{K}$, considered as a $\mathbb{K}$–vector space, and the underlying set of GX is $\mathrm{Vec}_{\mathbb{K}}(X, \mathbb{K})$. For any set S denote by $\mathsf{V}(S)$ the $\mathbb{K}$–vector space with basis S. In this case, we can consider the monad $T(\mathsf{V}(\Sigma)) = \mathsf{V}(\Sigma^*)$.

▶ **Example 8.** [3] Consider the duality between the category $\mathcal{D} = \mathrm{Poset}$ of partially ordered sets with monotone maps and the category $\mathcal{C} = \mathrm{AlgCDL}$ of algebraic complete distributive lattices with complete lattice homomorphisms. In this case, $\mathtt{R}$ is the two–element chain. If we consider the identity monad on Poset then (A1) to (A5) are satisfied except property (A2). In fact, by considering the following posets



the map $f : \Sigma \to B$ such that $f(x) = 1$ iff $x = b$, and the map $e : A \to B$ such that $e(0) = 0$ and $e(1) = 1$, we have that property (A2) is not satisfied.

▶ **Theorem 9.** *Under the assumptions (A1) to (A5), there is a one–to–one correspondence between pseudovarieties of* T*–algebras and pseudovarieties of finitely recognizable* T*–languages.*

▶ Remark. Notice that most of the assumptions can be easily verified except the one that the underlying functor T of the monad T has to preserve weak generalized pushouts for which all the arrows are epis. To verify this property one can use the dual of [11, Theorem 4.5]. The property of T preserving weak generalized pushouts for which all the arrows are epis is needed to guarantee the existence of a syntactic T–algebra for every T–language, Propisition 10. In cases, such as the ones in [8, 13], in which the existence of a syntactic T–algebra has been already established, we can omit the condition of T preserving weak generalized pushouts for which all the arrows are epis and only assume that T preserves epis[4].

5 Proof of the main theorem

In this section, we provide a proof of Theorem 9. A key notion for this proof, as in Eilenberg's variety theorem, is the existence of syntactic T–algebras for every T–language. In order to

[3] Thanks to Henning Urbat who made me aware that property (A2) is not satisfied in this example.
[4] The condition of T preserving epis is used to lift the epi–mono factorization in $\mathcal{D}$ to $\mathrm{Alg}(\mathsf{T})$.

guarantee the existence of a syntactic algebra for every T–language we need to assume that the category $\mathcal{D}$ has generalized pushouts and that the functor T preserves weak generalized pushouts.

▶ **Proposition 10.** *Assume (A1). Then, for every* T*–language* L *over* $\Sigma \in \mathcal{D}$ *there exists a syntactic algebra* $S_L \in \mathrm{Alg}(\mathsf{T})$ *and a canonical epimorphism* $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ *such that* S_L *recognizes* L *through* e_L *and they satisfy the following property:*

- *For any* $A \in \mathrm{Alg}(\mathsf{T})$ *recognizing* L *through an epimorphism* $e \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$*, there exists* $g \in \mathrm{Alg}(\mathsf{T})(A, S_L)$ *such that* $e_L = g \circ e$*. (observe that* g *is necessarily an epimorphism).*

The T–algebra S_L in the previous proposition associated with a T–language is called the *syntactic* T*–algebra* of L and the epimorphism e_L it is the *canonical epimorphism associated to* L.

Proof. *(sketch)* Given a T–language L over an object Σ in $\mathcal{D}$, consider the collection $\{T\Sigma \xrightarrow{e_i} A_i\}_{i\in I}$ of epimorphisms $e_i \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A_i)$ recognizing L. Notice that I is not empty since the identity epimorphism $id_{T\Sigma} \in \mathrm{Alg}(\mathsf{T})(T\Sigma, T\Sigma)$ is one of them. Let $\{A_i \xrightarrow{q_i} Q\}_{i\in I}$ be the generalized pushout of $\{T\Sigma \xrightarrow{e_i} A_i\}_{i\in I}$ in $\mathcal{D}$. Every q_i is epi since every e_i is also epi. As T preserves weak generalized pushouts for which all the arrows are epis then $\{TA_i \xrightarrow{Tq_i} TQ\}_{i\in I}$ is a weak generalized pushout of the family $\{TT\Sigma \xrightarrow{Te_i} TA_i\}_{i\in I}$ and from this we can define a map $\alpha : TQ \to Q$ such that $Q = (Q, \alpha) \in \mathrm{Alg}(\mathsf{T})$. Take $S_L = (Q, \alpha)$ and $e_L = q_i \circ e_i$, for some $i \in I$. ◀

As shown in [4, Example 2], there are T–languages for the monad $\mathsf{T} = (T, \eta, \mu)$ on Set given by $TA = A^\infty = A^+ \cup A^\omega$ that have no syntactic T–algebra. From that fact we get the following.

▶ **Corollary 11.** *Let* T *be the* Set *endofunctor such that for any set* A*,* $TA = A^+ \cup A^\omega$ *and for any function* $f : A \to B$*,* $(Tf)(g) = f \circ g$*. Then* T *does not preserve weak generalized pushouts.*

▶ Remark. We can equivalently construct the syntactic T–algebra by using duality. In this case, as $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ we construct the least subcoalgebra $\langle L\rangle$ in $\mathrm{Coalg}(\mathsf{B})$ of $\widehat{G}(T\Sigma)$ containing L by using generalized pullbacks in $\mathcal{C}$ and the fact that B preserves weak generalized pullbacks. Then we get a canonical monomorphism $m \in \mathrm{Coalg}(\mathsf{B})(\langle L\rangle, \widehat{G}(T\Sigma))$ and take $S_L = \widehat{F}\langle L\rangle$ and $e_L = Fm$.

Given a pseudovariety of T–algebras $\mathcal{A}$ and a pseudovariety of finitely recognizable T–languages $\mathcal{L}$ we define the operator $\mathcal{L}(\mathcal{A})$ and the class of T–algebras $\mathcal{A}(\mathcal{L})$ as:

- For every finite object Σ in $\mathcal{D}$ define $(\mathcal{L}(\mathcal{A}))(\Sigma)$ as the set of T–languages over Σ that are recognized by an algebra in $\mathcal{A}$.
- $\mathcal{A}(\mathcal{L})$: finite T–algebras that recognize only languages in $\mathcal{L}$.

Notice that every T–language in $(\mathcal{L}(\mathcal{A}))(\Sigma)$ is finitely recognizable since all the algebras in $\mathcal{A}$ are finite.

We will prove that if $\mathcal{L}$ is a pseudovariety of finitely recognizable T–languages, then $\mathcal{A}(\mathcal{L})$ is a pseudovariety of finite T–algebras. A key observation is that for every finite Σ in $\mathcal{D}$ the set $\mathcal{L}\Sigma$ is a set of coequations in the following sense (cf. [18, Section 3]).

We have that the T–algebra $T\Sigma = (T\Sigma, \mu_\Sigma)$ is the free T–algebra on Σ generators. Then, by duality, $\widehat{G}(T\Sigma)$ is the cofree B–coalgebra on $G\Sigma$ colours. A set of *coequations* on $G\Sigma$ colours is a subset S of $GT\Sigma$. Given $Y = (Y, \beta) \in \mathrm{Coalg}(\mathsf{B})$, we say that Y satisfies the set of coequations S, denoted as $Y \Vdash S$ if for every colouring $f \in \mathcal{C}(Y, G\Sigma)$, the unique morphism $f^\flat \in \mathrm{Coalg}(Y, \widehat{G}(T\Sigma))$ such that $f = G\eta_\Sigma \circ f^\flat$, which is given by the cofreeness of $\widehat{G}(T\Sigma)$, is such that $\mathrm{Im}(f^\flat) \subseteq S$.

▶ **Proposition 12.** *Assume (A1). Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages and A a finite T–algebra. Then $A \in \mathcal{A}(\mathcal{L})$ if and only if for every finite object Σ in $\mathcal{D}$ we have that $\widehat{G}A \Vdash \mathcal{L}\Sigma$.*

The previous proposition says that the class of finite algebras $\mathcal{A}(\mathcal{L})$, where $\mathcal{L}$ is a pseudovariety of finitely recognizable T–languages, is the class of finite algebras A such that its corresponding coalgebra $\widehat{G}A \in \mathrm{Coalg}(\mathsf{B})$ satisfies the coequations $\mathcal{L}\Sigma$ for every finite Σ. Now, by assuming (A2) and using duality, we get the following.

▶ **Corollary 13.** *Assume (A1) and (A2). Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages. Then $\mathcal{A}(\mathcal{L})$ is a pseudovariety of T–algebras.*

Now, we have that if $\mathcal{A}$ is a pseudovariety of T–algebras then $\mathcal{L}(\mathcal{A})$ is a pseudovariety of finitely recognizable T–languages.

▶ **Proposition 14.** *Assume (A1) and (A3). Let $\mathcal{A}$ be a pseudovariety of T–algebras, then $\mathcal{L}(\mathcal{A})$ is a pseudovariety of finitely recognizable T–languages.*

Finally, to finish the proof of the main theorem we have to prove that the operator $\mathcal{A}$, which takes pseudovarieties of finitely recognizable T–languages to pseudovarieties of T–algebras, and the operator $\mathcal{L}$, which takes pseudovarieties of T–algebras to pseudovarieties of finitely recognizable T–languages, are inverses of each other.

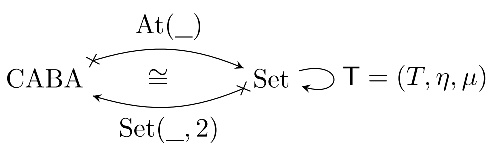
▶ **Proposition 15.** *Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages and let $\mathcal{A}$ be a pseudovariety of T–algebras. Then:*

i) Assume (A1), (A2), and (A4), then $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.
ii) Assume (A1) to (A5), then $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$.

6 Examples

In this section, we apply our main theorem to obtain specific examples of Eilenberg–like theorems.

▶ **Example 16.** (Example 6 Continued) Consider the setting



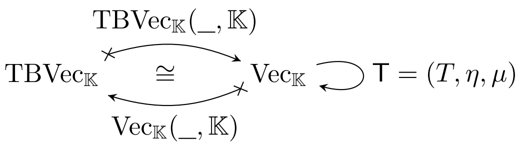
Then we can get the following results according to the monad T we consider:

i) Consider the monad $T\Sigma = \Sigma^*$, where Σ^* is the free monoid with Σ generators, then we get Eilenberg's variety theorem for monoids [8, Theorem 3.4].
ii) Consider the monad $T\Sigma = \Sigma^+$, where Σ^+ is the free semigroup with Σ generators, then we get Eilenberg's variety theorem for semigroups [8, Theorem 3.4s].
iii) Consider the monad $T\Sigma = \mathfrak{F}(\Sigma)$, where $\mathfrak{F}(\Sigma)$ is the free group on Σ generators, then we get an Eilenberg–like theorem for pseudovarieties of groups.

iv) Consider the monad $T\Sigma = M \times \Sigma$, where M is a fixed monoid, then we get an Eilenberg–like theorem for pseudovarieties of monoid actions.

Notice that we can also get commutative versions of i), ii), and iii). ◀

▶ **Example 17.** For a fixed finite field $\mathbb{K}$ consider the setting



where for every object $\mathsf{V}(\Sigma) \in \mathrm{Vec}_{\mathbb{K}}$ we define $T(\mathsf{V}(\Sigma)) = \mathsf{V}(\Sigma^*)$. Then the category $\mathrm{Alg}(\mathsf{T})$ is the category of $\mathbb{K}$–algebras with $\mathbb{K}$–algebra morphisms. With this setting we get [16, Théorème III1.1.] for the case of finite fields.

7 Eilenberg-like theorem for varieties of T–algebras and varieties of recognizable T–languages

In this section we show that we can also get an Eilenberg–like theorem if we drop the finiteness assumption in the definition of pseudovarieties of T–algebras and pseudovarieties of finitely recognizable T–languages to get the following definitions:

▶ **Definition 18.** Consider the situation given in (1). Then:

1) A *variety of* T*–algebras* is a class $\mathcal{A}$ of T–algebras closed under homomorphic images, subalgebras, and products.
2) A *variety of recognizable* T*–languages* is an operator $\mathcal{L}$ such that for every object Σ in $\mathcal{D}$:
   i) $\mathcal{L}\Sigma$ is a subcoalgebra of $\widehat{G}(T\Sigma)$.
   ii) Closure under morphic preimages: For every object Γ in $\mathcal{D}$, homomorphism $h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$, and $L \in \mathcal{L}\Sigma$, we have that $L \circ h \in \mathcal{L}\Gamma$.

With this definition of varieties of T–algebras and varieties of recognizable T–languages we get the following Eilenberg–like theorem whose proof is made in a similar way as in the proof of Theorem 9.

▶ **Theorem 19.** *Under the assumptions (A1) to (A5), there is a one–to–one correspondence between varieties of* T*–algebras and varieties of recognizable* T*–languages.*

▶ **Example 20.** We can apply Theorem 19 to the settings in the previous section. Notice that in the case of Example 16 i) we get the Eilenberg–like theorem [3, Theorem 39].

8 Conclusions

An Eilenberg–like theorem was proved using a categorical approach in which the kind of algebras considered are algebras on a monad T on a category $\mathcal{D}$. In this setting, we defined the natural notion of pseudovariety of T–algebras. To define the notion of pseudovariety of finitely recognizable T–languages we used a category $\mathcal{C}$ that is dual to $\mathcal{D}$, which under certain conditions allow us to define this concept coalgebraically, i.e., using duality. By dropping finiteness conditions we also derive an Eilenberg–like theorem for varieties of T–algebras, i.e., classes of T–algebras closed under homomorphic images, subalgebras and (not necessarily finite) products.

Some other Eilenberg–like theorems that use a categorical setting exist in the literature, such as [1], [4], and recently [19], in those works only the finite case is considered. In [1] and [19], whose work was inspired by [10], one considers predual categories $\mathcal{C}$ and $\mathcal{D}$, i.e., $\mathcal{C}$ and $\mathcal{D}$ are dual on finite objects. In the present paper $\mathcal{C}$ and $\mathcal{D}$ are dual, which is a stronger condition. This (pre)duality allows one to define pseudovarieties of algebras on one category, say $\mathcal{D}$, and pseudovarieties of recognizable languages on the other category, say $\mathcal{C}$. In [1] the kind of algebras considered are $\mathcal{D}$–monoids which can be seen as a limitation on the kind of algebras considered (e.g. it is not possible to derive the semigroup version [8, Theorem 3.4s] or versions for other algebraic structures that don't have a monoid structure). Later, in [4], this kind of limitation was avoided by considering algebras on a monad T on Set or on Set^S, where S is a fixed set, but then there was a limitation on the kind of pseudovarieties of languages considered since every $\mathcal{L}\Sigma$ is a Boolean algebra (Eilenberg–like theorems such as the ones in [13, 14, 16] could not be derived with in this case). The approach given in [1] is generalized in [19] now considering algebras for a monad, as in [4].

The work of the present paper is more general than [4]. In [4] there are two kinds of derivatives: syntactic derivatives and polynomial derivatives. Polynomial derivatives are special cases of syntactic derivatives but not vice versa, in some cases those two notions coincide. In the present paper, the treatment of derivatives is avoided and it is implicitely included in property i) of Definition 5. Those derivatives are syntactic derivatives (Lemma 24 in the Appendix). To obtain the setting given in [4] it is enough to consider the duality between $\mathcal{C} = \mathrm{CABA}$ and $\mathcal{D} = \mathrm{Set}$. Some of the facts here are similar to the ones in [4] in which similar proofs were adapted according to the definition of pseudovariety of finitely recognizable T–languages. It is worth mentioning that our present construction of the syntactic algebra for a given T–language is simpler than the construction given in [4] and in [19] (in [1] syntactic algebras are not considered). Here, syntactic algebras are constructed abstractly by using (weak) generalized pushouts thereby avoiding the notion of congruences and polynomials given in [4]. In [19] the construction of syntactic algebras depends on finding a set of polynomials (unary operations) that satisfy certain properties (i.e., to find a unary representation, [19, Definition 3.7]), which is a parameter in the theorem.

The notion of (pseudo)varieties of (finitely) recognizable languages given in [1] is closely related to the classical Eilenberg theorem, and in [19] it is a parameter in the theorem (one has to find the correct notion of derivatives in order to get the theorem, i.e., to find a unary representation). Here the notion of derivatives is automatic and it is obtained via duality by using coalgebras. In contrast, in [1, 19] one can derive Eilenberg-like theorems for ordered algebras which cannot be derived in the present paper (see Example 8), this limitation could be possibly avoided if we restrict the class of alphabets considered, i.e., to consider finite discrete posets as the only alphabets.

The present paper presents an abstract approach in the sense that proofs, constructions, and definitions are mainly categorical (e.g. construction of syntactic T–algebras by using weak generalized pushouts and the definition of pseudovarieties of finitely recognizable T–languages in coalgebraic terms). The previous fact has the advantage that it allows us to derive an Eilenberg–like theorem by omitting finiteness conditions, Theorem 19. From this, we derived new Eilenberg–like theorems and less known Eilenberg–like theorems like the ones in Example 20 which includes [3, Theorem 39] as a particular case. Categorical approaches such as [1, 4, 19] only consider the finite case.

As future work it is worth to explore the following:

- The setting in which we consider predual categories and a monad on one of the categories. This can possibly be general enough to include the work presented here as well as [1, 4, 19].

- To derive Eilenberg–like theorems for ordered algebras such as [13]. This could possibly be done if we consider the duality between Poset and AlgCDL by restricting the alphabets to finite discrete posets or, equivalently, by considering finite objects given by the left adjoint $F \dashv U$ where U : Poset → Set is the forgetful functor, which is the technique used in [19].
- Find more examples and applications of the main theorem by considering dualities such as Stone duality, nominal Stone duality [9], or general dualities such as the ones in [6].
- Study the relation of this work with profinite equations [10], profinite monads [5] and Reiterman's theorem [15]. In this case we conjecture, see Proposition 12, that (pseudo)varieties of T–languages are the same as coequational (pseudo)theories which, by duality, give us equational (pseudo)theories that axiomatize (pseudo)varieties of T–algebras and they are all in one–to–one correspondence.

Acknowledgements: I would like to thank to Marcello Bonsangue and Jan Rutten for their support, comments, and suggestions during the writing of this paper. I also thank Henning Urbat who pointed out a mistake in an earlier version of this paper.

References

1 Adámek, J., S. Milius, R. Myers, and H. Urbat, *Varieties of Languages in a Category,* Logic in Computer Science, LICS 2015.
2 Awodey., S., "Category theory," Oxford University Press, 2006.
3 Ballester-Bolinches, A., E. Cosme-Llópez, and J.J.M.M. Rutten, *The dual equivalence of equations and coequations for automata,* Information and Computation **244** (2015), 49-75.
4 Bojańczyk., M., *Recognisable Languages over Monads,* LNCS **9168** (2015), 1–13.
5 Chen, L-T., J. Adámek, S. Milius, and H. Urbat, *Profinite Monads, Profinite Equations, and Reiterman's Theorem,* arXiv:1511.02147v2 [cs.FL] (2016).
6 Clark, D., and B. Davey, "Natural Dualities for the Working Algebraist," Cambridge studies in advanced mathematics 57, 1998.
7 Davey, B., and H. Priestley, "Introduction to lattices and order, 2nd edition," Cambridge University Press, 2002.
8 Eilenberg., S., "Automata, Languages, and Machines, Volume B.," Pure and Applied Mathematics. Academic Press, 1976.
9 Gabbay, M., L. Litak, and D. Petrisan, *Stone duality for nominal Boolean algebras with 'new': topologising Banonas,* LNCS **6859** (2011), 192–207.
10 Gehrke, M., S. Grigorieff, and J.E. Pin, *Duality and equational theory of regular languages,* LNCS **5126** (2008), 246–257.
11 Gumm, H., and T. Schröder, *Coalgebraic structure from weak limit preserving functors,* Electronic Notes in Theoretical Computer Science **33** (2000), 111–131.
12 Hermida, C., and B. Jacobs, *Structural induction and coinduction in a fibrational setting,* Information and Computation **145 (2)** (1998), 107–152.
13 Pin, J.E., *A variety theorem without complementation,* Russian Mathematics (Izvestija vuzov.Matematika) **39** (1995), 80–90.
14 Polák, L., *Syntactic Semiring of a Language,* LNCS **2136** (2001), 611–620.
15 Reiterman, J., *The Birkhoff theorem for finite algebras,* Alg. Universalis **14** (1982), 1–10.
16 Reutenauer, C., *Séries formelles et algèbres syntactiques,* J. of Alg. **66** (1980), 448–483.
17 Rutten, J., *Universal coalgebra: a theory of systems,* Theoretical Computer Science **249 (1)** (2000), 3–80.
18 Salamanca, J., M. Bonsangue, and J. Rot. *Duality of Equations and Coequations via Contravariant Adjunctions,* To appear in Proceedings of CMCS 2016.
19 H. Urbat, J. Adámek, L-T. Chen, and S. Milius. One Eilenberg Theorem to Rule Them All. arXiv:1602.05831v1 [cs.FL] , 2016.

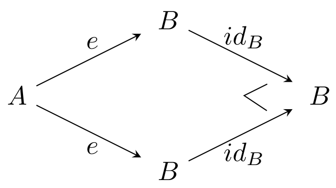
9 Appendix

9.1 Proof of Proposition 10

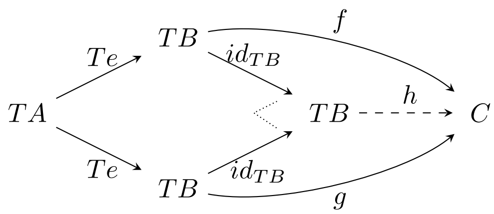
First we prove the following.

▶ **Lemma 21.** *Let T be an endofunctor on a category $\mathcal{D}$ such that T preserves weak pushouts. Then T preserves epis.*

Proof. Let $e \in \mathcal{D}(A, B)$ be an epimorphism and $f, g \in \mathcal{D}(TB, C)$ be morphisms such that $f \circ Te = g \circ Te$. We have to prove that $f = g$. In fact, as e is epi then we have the following pushout:



Since T preserves weak pushouts, then we have the following commutative diagram:

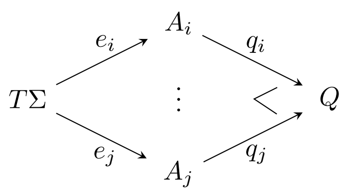


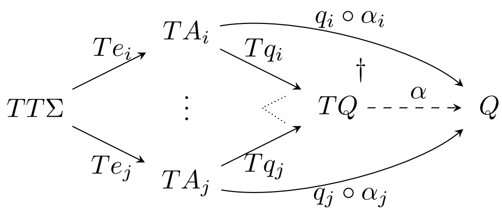
Where the arrow h was obtained from the property that the square is a weak pushout. But then $f = h \circ id_{TB} = g$. Therefore, Te is epi. ◀

Proposition 10. *Assume (A1). Then, for every T–language L over $\Sigma \in \mathcal{D}$ there exists a syntactic algebra $S_L \in \mathrm{Alg}(\mathsf{T})$ and an epimorphism $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ such that S_L recognizes L through e_L and they satisfy the following property:*

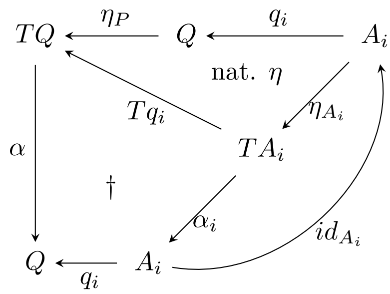
- *For any $(A, \alpha) \in \mathrm{Alg}(\mathsf{T})$ recognizing L through an epimorphism $e \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$, there exists $g \in \mathrm{Alg}(\mathsf{T})(A, S_L)$ such that $e_L = g \circ e$. (observe that g is necessarily an epimorphism).*

Proof. Given a T–language L over an object Σ in $\mathcal{D}$, consider the collection $\{T\Sigma \xrightarrow{e_i} A_i\}_{i\in I}$ of epimorphisms $e_i \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A_i)$ recognizing L, $A_i = (A_i, \alpha_i) \in \mathrm{Alg}(\mathsf{T})$. Notice that I is not empty since the identity epimorphism $id_{T\Sigma} \in \mathrm{Alg}(\mathsf{T})(T\Sigma, T\Sigma)$ is one of them. Let $\{A_i \xrightarrow{q_i} Q\}_{i\in I}$ be the generalized pushout of $\{T\Sigma \xrightarrow{e_i} A_i\}_{i\in I}$ in $\mathcal{D}$. Every q_i is epi since every e_i is also epi. As T preserves weak generalized pushouts then $\{TA_i \xrightarrow{Tq_i} TQ\}_{i\in I}$ is a weak generalized pushout of the family $\{TT\Sigma \xrightarrow{Te_i} TA_i\}_{i\in I}$. Since the family $\{TA_i \xrightarrow{q_i \circ \alpha_i} Q\}_{i\in I}$ is such that $q_i \circ \alpha_i \circ Te_i = q_j \circ \alpha_j \circ Te_j$, $i, j \in I$, there exists $\alpha \in \mathcal{D}(TQ, Q)$ such that $\alpha \circ Tq_i = q_i \circ \alpha_i$. That is we have the following situation:

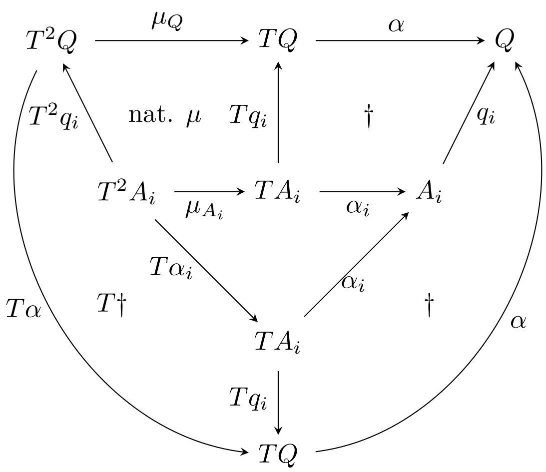




We prove that $Q = (Q, \alpha) \in \mathrm{Alg}(\mathsf{T})$. In fact, from the commutative diagram:

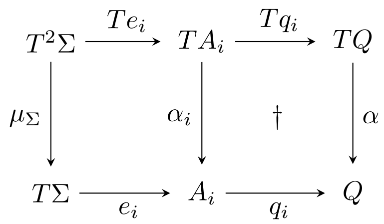


We conclude that $\alpha \circ \eta_P = id_Q$ since q_i is epi. Now, from the commutative diagram:



We conclude that $\alpha \circ \mu_Q = \alpha \circ T\alpha$ (start at Q following the external arrows and then compose with $T^2 q_i$. Then use the fact that $T^2 q_i$ is epi by the previous lemma since p_i is epi). This concludes the proof that $Q = (Q, \alpha) \in \mathrm{Alg}(\mathsf{T})$.

To finish the proof, take $S_L = (Q, \alpha)$ and $e_L = q_i \circ e_i$, for some $i \in I$ (remember that $q_i \circ e_i = q_j \circ e_j$ since $\{A_i \xrightarrow{q_i} Q\}_{i \in I}$ be the generalized pushout of $\{T\Sigma \xrightarrow{e_i} A_i\}_{i \in I}$). The fact that S_L recognizes L follows from the fact that every A_i recognizes L and the property of $\{A_i \xrightarrow{q_i} Q\}_{i \in I}$ being the generalized pushout of $\{T\Sigma \xrightarrow{e_i} A_i\}_{i \in I}$. Also, we have that $e_L = q_i \circ e_i \in \mathrm{Alg}(\mathsf{T})(T\Sigma, Q)$ since we have the following commutative diagram:

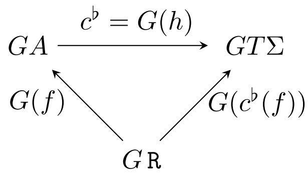


◄

9.2 Proof of Proposition 12

Proposition 12. *Assume (A1). Let $\mathcal{L}$ be a pseudovariety of T–languages and $A = (A, \alpha)$ a finite T–algebra. Then $A \in \mathcal{A}(\mathcal{L})$ if and only if for every finite object Σ in $\mathcal{D}$ we have that $\widehat{G}A \Vdash \mathcal{L}\Sigma$.*

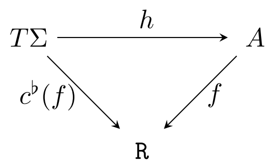
Proof. ($\Rightarrow$): Let A be a finite T–algebra such that $A \in \mathcal{A}(\mathcal{L})$. Let Σ be a finite object in $\mathcal{D}$ and $c \in \mathcal{C}(GA, G\Sigma)$ be any colouring. By cofreeness of $\widehat{G}(T\Sigma)$ there is a unique morphism $c^\flat \in \mathrm{Coalg}(\mathsf{B})(\widehat{G}A, \widehat{G}(T\Sigma))$ such that $G\eta_\Sigma \circ c^\flat = c$. Because of the duality we have that $c^\flat = G(h) = \mathcal{D}(h, \mathtt{R})$ for some $h \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$. To show that $\widehat{G}A \Vdash \mathcal{L}\Sigma$ we have to show that $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$. In fact, let $f \in GA = \mathcal{D}(A, \mathtt{R})$ then we have the following commutative diagram in $\mathcal{C}$:



This follows because for every $k \in G\,\mathtt{R}$ we have that

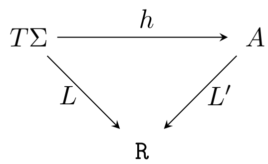
$$(G(h) \circ G(f))(k) = k \circ f \circ h = k \circ (G(h)(f)) = k \circ c^\flat(f) = G(c^\flat(f))(k).$$

Hence by applying F to the previous diagram we get the diagram:

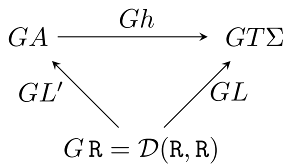


Which means that $c^\flat(f)$ is recognized by A and as $A \in \mathcal{A}(\mathcal{L})$ it follows that $c^\flat(f) \in \mathcal{L}\Sigma$. Therefore, $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$.

($\Leftarrow$): Assume that $\widehat{G}A \Vdash \mathcal{L}\Sigma$ for every finite object Σ in $\mathcal{D}$. Let Σ be a finite object in $\mathcal{D}$, a morphism $h \in \mathrm{Alg}(T\Sigma, A)$ and morphisms $L \in \mathcal{D}(T\Sigma, \mathtt{R})$ and $L' \in \mathcal{D}(A, \mathtt{R})$ such that the following diagram in $\mathcal{D}$ commutes:



To prove that $A \in \mathcal{A}(\mathcal{L})$ we have to prove that $L \in \mathcal{L}\Sigma$. By applying G to the previous diagram we get the following commutative diagram:



As $\widehat{G}A \Vdash \mathcal{L}\Sigma$ then we have that $\mathrm{Im}(Gh) \subseteq \mathcal{L}\Sigma$, which implies that $(Gh \circ GL')(id_{\mathtt{R}}) \in \mathcal{L}\Sigma$, but

$$(Gh \circ GL')(id_{\mathtt{R}}) = (GL)(id_{\mathtt{R}}) = \mathcal{D}(L,\mathtt{R})(id_{\mathtt{R}}) = id_{\mathtt{R}} \circ L = L$$

i.e., $L \in \mathcal{L}\Sigma$. ◀

9.3 Proof of Corollary 13

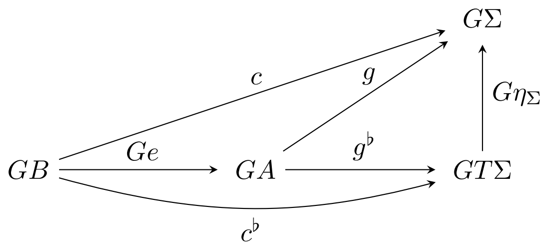
Corollary 13. *Assume (A1) and (A2). Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages. Then $\mathcal{A}(\mathcal{L})$ is a pseudovariety of T–algebras.*

Proof. By Proposition 12 we have that

$$\mathcal{A}(\mathcal{L}) = \{A \in \mathrm{Alg}(\mathsf{T}) \mid A \text{ is finite and } \forall\ \Sigma \in \mathcal{D} \text{ finite } \widehat{G}A \Vdash \mathcal{L}\Sigma\}$$

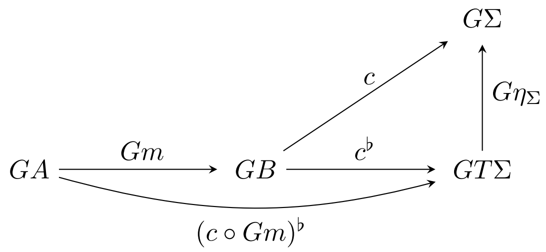
We need to prove that $\mathcal{A}(\mathcal{L})$ is closed under homomorphic images, subalgebras, and finite products. In fact,

i) Assume that $A \in \mathcal{A}(\mathcal{L})$ and let $e \in \mathrm{Alg}(\mathsf{T})(A,B)$ be an epimorphism. Clearly B is finite. Now let Σ be a finite object in $\mathcal{D}$ and $c \in \mathcal{C}(GB, G\Sigma)$ be any colouring. Then the unique morphism $c^\flat \in \mathrm{Coalg}(\mathsf{B})(\widehat{G}B, \widehat{G}(T\Sigma))$ such that $G\eta_\Sigma \circ c^\flat = c$ is given by $c^\flat = g^\flat \circ Ge$, where $g \in \mathcal{C}(GA, G\Sigma)$ is a morphism such that $g \circ Ge = c$ given by the dual of (A2). That is, we have the following commutative diagram in $\mathcal{C}$ where the three lower arrows are morphisms in $\mathrm{Coalg}(\mathsf{B})$:



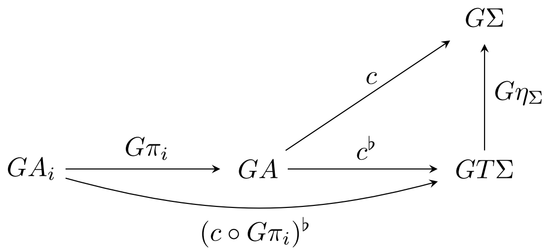
Hence we have that $\mathrm{Im}(c^\flat) \subseteq \mathrm{Im}(g^\flat) \subseteq \mathcal{L}\Sigma$, where the last inclusion follows from the fact that $A \in \mathcal{A}(\mathcal{L})$. Therefore, $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$, i.e., $\widehat{G}B \Vdash \mathcal{L}\Sigma$.

ii) Assume that $A \in \mathcal{A}(\mathcal{L})$ and let $m \in \mathrm{Alg}(\mathsf{T})(B,A)$ be a monomorphism. Clearly B is finite. Now let Σ be a finite object in $\mathcal{D}$ and $c \in \mathcal{C}(GB, G\Sigma)$ be any colouring. Let $c^\flat \in \mathrm{Coalg}(\mathsf{B})(\widehat{G}B, \widehat{G}(T\Sigma))$ be the unique morphism such that $G\eta_\Sigma \circ c^\flat = c$. Then we have that $(c \circ Gm)^\flat = c^\flat \circ Gm$. That is, we have the following commutative diagram in $\mathcal{C}$ where the three lower arrows are morphisms in $\mathrm{Coalg}(\mathsf{B})$:



Hence we have that $\mathrm{Im}(c^\flat) = \mathrm{Im}((c \circ Gm)^\flat) \subseteq \mathcal{L}\Sigma$, where the equality follows from the fact that Gm is epi and the inclusion follows from the fact that $A \in \mathcal{A}(\mathcal{L})$. Therefore, $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$, i.e., $\widehat{G}B \Vdash \mathcal{L}\Sigma$.

iii) Assume that $A_i \in \mathcal{A}(\mathcal{L})$, $i = 0, 1$, and let $A = A_0 \times A_1 \in \mathrm{Alg}(\mathsf{T})$ be the product of A_0 and A_1 with projections $\pi_i : A \to A_i$. Clearly A is finite. Now let Σ be a finite object in $\mathcal{D}$ and $c \in \mathcal{C}(GA, G\Sigma)$ be any colouring. Let $c^\flat \in \mathrm{Coalg}(\mathsf{B})(\widehat{G}A, \widehat{G}(T\Sigma))$ be the unique morphism such that $G\eta_\Sigma \circ c^\flat = c$. Then we have that $(c \circ G\pi_i)^\flat = c^\flat \circ G\pi_i$. That is, we have the following commutative diagram in $\mathcal{C}$ where the three lower arrows are morphisms in $\mathrm{Coalg}(\mathsf{B})$:



Hence we have that $\mathrm{Im}(c^\flat) = \mathrm{Im}((c \circ G\pi_0)^\flat) \cup \mathrm{Im}((c \circ G\pi_1)^\flat) \subseteq \mathcal{L}\Sigma$, where the equality follows from the fact that $GA = GA_0 + GA_1$ and the inclusion follows from the fact that $A_i \in \mathcal{A}(\mathcal{L})$. Therefore, $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$, i.e., $\widehat{G}A \Vdash \mathcal{L}\Sigma$.

◀

9.4 Proof of Proposition 14

To prove this proposition we use the following lemma.

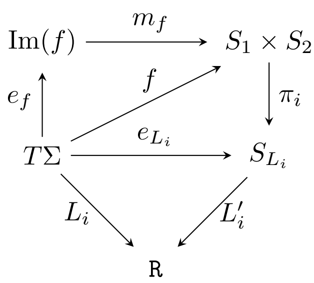
▶ **Lemma 22.** *Assume (A1) and (A3). Let $\mathcal{A}$ be a pseudovariety of T–algebras and Σ a finite object in $\mathcal{D}$. Then $S_L \in \mathcal{A}$ for any $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$.*

Proof. Let $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$, i.e., there exists an algebra $A = (A, \alpha) \in \mathcal{A}$, a morphism $h \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$ and $L' \in \mathcal{D}(A, \mathtt{R})$ such that $L' \circ h = L$. Let $h = m \circ e$ be the epi–mono factorization for h, i.e., $e \in \mathrm{Alg}(\mathsf{T})(T\Sigma, \mathrm{Im}(h))$ is an epimorphism and $m \in \mathrm{Alg}(\mathsf{T})(\mathrm{Im}(h), A)$ is a monomorphism. Then we have that $\mathrm{Im}(h)$ is a subalgebra of A which is in $\mathcal{A}$, since $\mathcal{A}$ is a pseudovariety of T–algebras, and it recognizes L since $L = L' \circ h = (L' \circ m) \circ e$. Now as $\mathrm{Im}(h)$ is a T–algebra recognizing L through the epimorphism e, by Proposition 10, there exists an epimorphism $g \in \mathrm{Alg}(\mathsf{T})(\mathrm{Im}(h), S_L)$ such that $e_L = g \circ e$. As g is epi and $\mathrm{Im}(h) \in \mathcal{A}$ then $S_L \in \mathcal{A}$. ◀

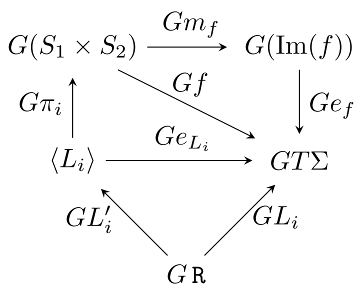
Proposition 14. *Assume (A1) and (A3). Let $\mathcal{A}$ be a pseudovariety of T–algebras, then $\mathcal{L}(\mathcal{A})$ is a pseudovariety of finitely recognizable T–languages.*

Proof. We have to prove properties i) and ii) of Definition 5. In fact, let Σ be a finite object in $\mathcal{D}$, then:

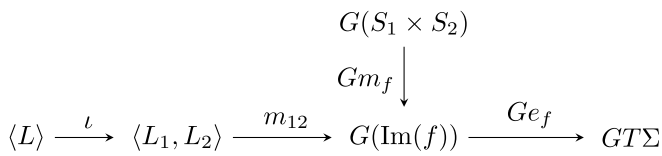
i) Assume that $L_1, L_2 \in (\mathcal{L}(\mathcal{A}))(\Sigma)$, we are going to prove that $\langle L_1, L_2 \rangle \subseteq (\mathcal{L}(\mathcal{A}))(\Sigma)$. From the previous lemma we have that $S_{L_i} \in \mathcal{A}$, $i = 1, 2$. As S_{L_i} recognizes L_i through e_{L_i} there exists $L'_i \in \mathcal{D}(S_{L_i}, \mathtt{R})$ such that $L'_i \circ e_{L_i} = L_i$. Let $\pi_i \in \mathrm{Alg}(\mathsf{T})(S_1 \times S_2, S_i)$ be the i-th projection. Then by the universal property of the product there is a morphism $f \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_1 \times S_2)$ such that $\pi_i \circ f = e_{L_i}$. That is we have the following commutative diagram in $\mathcal{D}$:



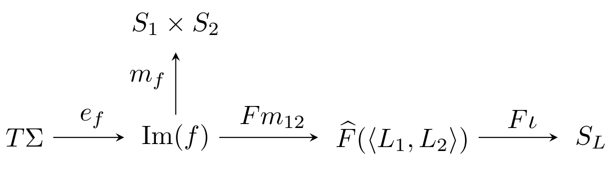
where $f = m_f \circ e_f$ is the epi–mono factorization for f. If we apply $\widehat{G}$ to the previous diagram we get the following commutative diagram:



From this diagram, by taking the identity map $id_{\mathtt{R}} \in G\,\mathtt{R} = \mathcal{D}(\mathtt{R}, \mathtt{R})$ and using the fact that Ge_f is mono, we have that $L_1, L_2 \in G(\mathrm{Im}(f))$ and hence $G(\mathrm{Im}(f))$ contains $\langle L_1, L_2 \rangle$, so there exists a monomorphism $m_{12} \in \mathrm{Coalg}(\mathsf{B})(\langle L_1, L_2 \rangle, G(\mathrm{Im}(f)))$. Now, for any $L \in \langle L_1, L_2 \rangle$ we have that $\langle L \rangle \subseteq \langle L_1, L_2 \rangle$. Let $\iota \in \mathrm{Coalg}(\mathsf{B})(\langle L \rangle, \langle L_1, L_2 \rangle)$ be the inclusion monomorphism. Then we have the following situation in $\mathrm{Coalg}(\mathsf{B})$:



Where ι, m_{12}, and Ge_f are monomorphisms and Gm_f is an epimorphism. By applying $\widehat{F}$ to the previous diagram we get:



Where e_f, Fm_{12} and $F\iota$ are epimorphisms and m_f is a monomorphism. Hence, as $S_1 \times S_2 \in \mathcal{A}$ then $S_L \in \mathcal{A}$ which implies that $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$ because L is recognized by S_L. Therefore $\langle L_1, L_2 \rangle \subseteq (\mathcal{L}(\mathcal{A}))(\Sigma)$.

ii) To prove closure under morphic preimages, consider a finite object Γ in $\mathcal{D}$ and a homomorphism $h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$. Assume that $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$, i.e. there exists an algebra A in $\mathcal{A}$, $g \in \mathrm{Alg}(\mathsf{T})(T\Sigma, A)$, and $L' \in \mathcal{D}(A, \mathtt{R})$ such that $L' \circ g = L$. Hence by composing h to the last equality on the right we get $L' \circ g \circ h = L \circ h$ which means that $L \circ h$ is in $(\mathcal{L}(\mathcal{A}))(\Gamma)$.

◀

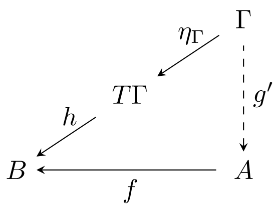
9.5 Proof of Proposition 15

We will use the following lemmas.

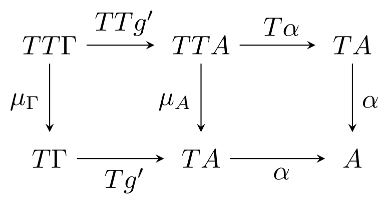
▶ **Lemma 23.** *([4, Lemma 4.8.]) Assume (A2). Let Γ be an object in $\mathcal{D}$, $f \in \mathrm{Alg}(\mathsf{T})(A, B)$, and $h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, B)$. If f is surjective, then there is $g \in \mathrm{Alg}(\mathsf{T})(T\Gamma, A)$ such that $f \circ g = h$.*

The proof of this lemma is the same as in [4, Lemma 4.8.] if we assume (A2). We reproduce its proof here.

Proof. Consider the following situation for which the morphism g' is obtained using (A2):



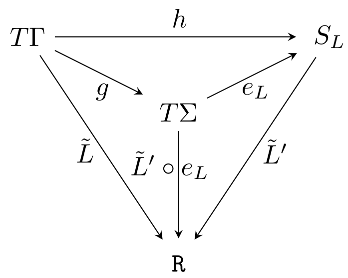
Put $A = (A, \alpha)$ and take $g = \alpha \circ Tg'$. Then, the property $g \in \mathrm{Alg}(\mathsf{T})(T\Gamma, A)$ follows from the following commutative diagram:



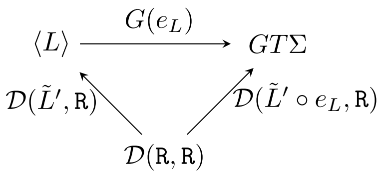
◀

▶ **Lemma 24.** *Assume (A1), (A2), and (A4). Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages. Then $S_L \in \mathcal{A}(\mathcal{L})$ for every finite $\Sigma \in \mathcal{D}$ and $L \in \mathcal{L}\Sigma$.*

Proof. We have to show that for every finite Γ in $\mathcal{D}$, the set $\mathcal{L}\Gamma$ contains all the languages over Γ that are recognized by S_L, the syntactic algebra of L. In fact, let $\tilde{L}$ be a T–language over Γ that is recognized by S_L, i.e., there exists $h \in \mathrm{Alg}(\mathsf{T})(T\Gamma, S_L)$ and $\tilde{L}' \in \mathcal{D}(S_L, \mathtt{R})$ such that $\tilde{L}' \circ h = \tilde{L}$. Now as $e_L \in \mathrm{Alg}(\mathsf{T})(T\Sigma, S_L)$ is an epimorphism then, by the previous lemma, there exists $g \in \mathrm{Alg}(\mathsf{T})(T\Gamma, T\Sigma)$ such that $e_L \circ g = h$. So we have the following commutative diagram in $\mathcal{D}$:



If we apply $G = \mathcal{D}(\_, \mathtt{R})$ to the lower right triangle we get:



So by considering the identity map $id_{\mathtt{R}} \in \mathcal{D}(\mathtt{R}, \mathtt{R})$ we get that $\tilde{L}' \circ e_L \in \langle L \rangle$. Therefore $\tilde{L}' \circ e_L \in \mathcal{L}\Sigma$, which from the first diagram implies that $\tilde{L} \in \mathcal{L}\Gamma$ since $\mathcal{L}$ is closed under morphic preimages. Observe that S_L is finite since L is a finitely recognizable T–language. ◀

Proposition 15. *Let $\mathcal{L}$ be a pseudovariety of finitely recognizable T–languages and let $\mathcal{A}$ be a pseudovariety of T–algebras. Then:*

i) Assume (A1), (A2), and (A4), then $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.
ii) Assume (A1) to (A5), then $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$.

Proof. i) Fix a finite object Σ in $\mathcal{D}$, we have to prove that $(\mathcal{L}(\mathcal{A}(\mathcal{L})))(\Sigma) = \mathcal{L}\Sigma$.

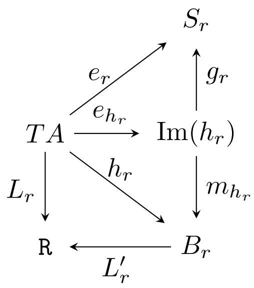
($\subseteq$): Assume that $L \in (\mathcal{L}(\mathcal{A}(\mathcal{L})))(\Sigma)$. Then there exists a T–algebra A in $\mathcal{A}(\mathcal{L})$ that recognizes L. Then as $A \in \mathcal{A}(\mathcal{L})$ and it recognizes L, we have that $L \in \mathcal{L}\Sigma$.

($\supseteq$): Assume that $L \in \mathcal{L}\Sigma$. Then by the previous lemma $S_L \in \mathcal{A}(\mathcal{L})$ which means that $L \in (\mathcal{L}(\mathcal{A}(\mathcal{L})))(\Sigma)$ since S_L recognizes L.

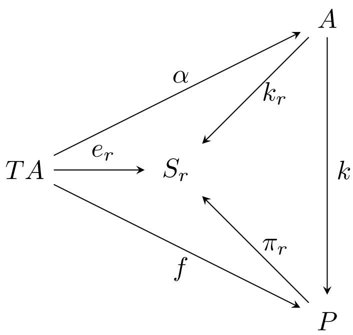
ii) Let $A = (A, \alpha)$ be a finite T–algebra.

($\supseteq$): Assume that $A \in \mathcal{A}$, then all the T–languages over a finite object Σ in $\mathcal{D}$ that are recognized by A are in $(\mathcal{L}(\mathcal{A}))(\Sigma)$, but this means that $A \in \mathcal{A}(\mathcal{L}(\mathcal{A}))$.

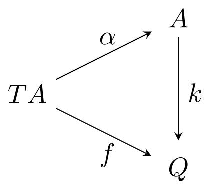
($\subseteq$): We have to prove that if for every finite object Σ in $\mathcal{D}$ and every T–language L over Σ recognized by A we have that L is recognized by some algebra B_L in $\mathcal{A}$, then $A \in \mathcal{A}$. For every $r \in \mathcal{D}(A, \mathtt{R})$ let S_r be the syntactic algebra for the T–language $L_r := r \circ \alpha$ over A and $e_r \in \mathrm{Alg}(\mathsf{T})(TA, S_r)$ its corresponding canonical epimorphism. As L_r is recognized by A through α then there exists an algebra B_r in $\mathcal{A}$ recognizing L_r, i.e., there exists $h_r \in \mathrm{Alg}(\mathsf{T})(TA, B_r)$ and $L'_r \in \mathcal{D}(B_r, \mathtt{R})$ such that $L'_r \circ h_r = L_r$. Now let $h_r = m_{h_r} \circ e_{h_r}$ be the epi–mono factorization of h_r, where $e_{h_r} \in \mathrm{Alg}(\mathsf{T})(TA, \mathrm{Im}(h_r))$ is an epimorphism and $m_{h_r} \in \mathrm{Alg}(\mathsf{T})(\mathrm{Im}(h_r), B_r)$ is a monomorphism. As e_{h_r} is an epimorphism recognizing L_r, there exists $g_r \in \mathrm{Alg}(\mathsf{T})(\mathrm{Im}(h_r), S_r)$ such that $g_r \circ e_{h_r} = e_r$. In summary, we have the following commutative diagram:



As $B_r \in \mathcal{A}$, m_{h_r} is mono, and g_r is epi, then $S_r \in \mathcal{A}$ since $\mathcal{A}$ is a pseudovariety of T–algebras.
Now as $A \in \mathrm{Alg}(\mathsf{T})$, we have that $\alpha \in \mathrm{Alg}(\mathsf{T})(TA, A)$ is an epimorphism, since $\alpha \circ \eta_A = id_A$. By Proposition 10 there exists $k_r \in \mathrm{Alg}(\mathsf{T})(A, S_r)$ such that $k_r \circ \alpha = e_r$. Let $P = \prod_{r \in \mathcal{D}(A,\mathtt{R})} S_r$ then, as $\mathcal{D}(A, \mathtt{R})$ is finite since A is also finite, we have that $P \in \mathcal{A}$. Let $f \in \mathrm{Alg}(\mathsf{T})(TA, P)$ be the morphism given by the universal property of the product P such that $\pi_r \circ f = e_r$ for every $r \in \mathcal{D}(A, \mathtt{R})$, where $\pi_r \in \mathrm{Alg}(\mathsf{T})(P, S_r)$ is the r–th projection. Similarly, let $k \in \mathrm{Alg}(\mathsf{T})(A, P)$ such that $\pi_r \circ k = k_r$ for every $r \in \mathcal{D}(A, \mathtt{R})$. That is, we have the following commutative diagram in $\mathrm{Alg}(\mathsf{T})$:



As α is epi and $k \circ \alpha = f$ we have that $\mathrm{Im}(f) = \mathrm{Im}(k)$ and as $P \in \mathcal{A}$ then $Q := \mathrm{Im}(f) = \mathrm{Im}(k)$ is also in $\mathcal{A}$. Hence we have the following commutative diagram in $\mathrm{Alg}(\mathsf{T})$:



We have that k in the last diagram is epi by definition. To finish the proof we are going to prove that k is mono [5], which means that $A \cong Q$ and hence $A \in \mathcal{A}$. In fact, let $a_1, a_2 \in A$ such that $a_1 \neq a_2$. As α is epi there exist $w_i \in TA$ such that $\alpha(w_i) = a_i$, $i = 1, 2$. Clearly $w_1 \neq w_2$.
We have that $f(w_1) \neq f(w_2)$. In fact, as $a_1 \neq a_2$ there exists $r' \in \mathcal{D}(A, \mathtt{R})$ such that $r'(a_1) \neq r'(a_2)$. Then we have that $(\pi_{r'} \circ f)(w_1) \neq (\pi_{r'} \circ f)(w_2)$, since otherwise we have that $e_{r'}(w_1) = e_{r'}(w_2)$ because $\pi_{r'} \circ f = e_{r'}$, and as $S_{r'}$ recognizes $L_{r'} := r' \circ \alpha$

[5] We'll prove this by considering elements.

through $e_{r'}$ then we have that $L_{r'}(w_1) = L_{r'}(w_2)$ from which we get:

$$r'(a_1) = (r' \circ \alpha)(w_1) = L_{r'}(w_1) = L_{r'}(w_2) = (r' \circ \alpha)(w_2) = r'(a_2)$$

which is a contradiction. Therefore, $f(w_1) \neq f(w_2)$.
Finally, $k(a_1) = (k \circ \alpha)(w_1) = f(w_1) \neq f(w_2) = (k \circ \alpha)(w_2) = k(a_2)$, i.e., k is mono.
◀

9.6 Details for Section 6

We need to prove that the endofunctors T in Example 16 preserve weak generalized pullbacks where all the arrows are epis (surjective). We will do this for the endofunctor $T\Sigma = \Sigma^*$ considered in i). The endofunctors considered in ii) and iii) have a similar proof. The endofunctor considered in iv) automatically preserves weak generalized pullbacks since it is a left adjoint and hence it preserves colimits.

Let $\{A \xrightarrow{e_i} A_i\}_{i\in I}$ be a collection of epimorphisms with common domain and let $\{A_i \xrightarrow{q_i} Q\}_{i\in I}$ be its pushout. Let $\{A_i^* \xrightarrow{p_i} P\}_{i\in I}$ be the pushout of $\{A^* \xrightarrow{e_i^*} A_i^*\}_{i\in I}$. We have to show that $\{A_i^* \xrightarrow{q_i^*} Q^*\}_{i\in I}$ is a weak generalized pushout of $\{A^* \xrightarrow{e_i^*} A_i^*\}_{i\in I}$. Using the dual of [11, Theorem 4.5], it is enough to find a function $\delta : Q^* \to P$ such that $\delta \circ q_i^* = p_i$, $i \in I$.

In order to prove this we are gonna fix some notation. We have that $Q = \coprod_{i\in I} A_i/\sim$ where $\sim$ is the least equivalence relation such that $\{\iota_j \circ e_j(a) \mid j \in I\}$ is contained in an equivalence class, $a \in A$, where $\iota_l : A_l \to \coprod_{i\in I} A_i$ is the inclusion function, $l \in I$. Similarly, $P = \coprod_{i\in I} A_i^*/\approx$ where $\approx$ is the least equivalence relation such that $\{\imath_j \circ e_j^*(w) \mid j \in I\}$ is contained in an equivalence class, $w \in A^*$, where $\imath_l : A_l^* \to \coprod_{i\in I} A_i^*$ is the inclusion function, $l \in I$.

First we are going to define $\delta : Q^* \to P$. In fact, let $[\iota_\ell(c_1)]_\sim[\iota_\ell(c_2)]_\sim \cdots [\iota_\ell(c_n)]_\sim \in Q^*$, where $c_k \in A_\ell$, for a fixed $\ell \in I$. We can always find $c_1, c_2, \ldots, c_n$ since all the e_i's are surjective. Let

$$\delta\left([\iota_\ell(c_1)]_\sim[\iota_\ell(c_2)]_\sim \cdots [\iota_\ell(c_n)]_\sim\right) = [\imath_\ell(c_1)\imath_\ell(c_2)\cdots\imath_\ell(c_n)]_\approx$$

We will show that δ is well–defined, i.e., does not depend on the choice of the c_k's and $\ell \in I$. In fact, assume that $\iota_\ell(c_1) \sim \iota_\ell(c_1')$, i.e., there exist $d_0, d_1, \ldots, d_m$, with $d_l \in A_{i_l}$, $d_0 = c_1$, $d_m = c_1'$, $i_0 = i_m = \ell$, and there exist $a_1, \ldots, a_m \in A$ such that:

$$e_{i_l}(a_{l+1}) = d_l, \quad e_{i_{l+1}}(a_{l+1}) = d_{l+1} \qquad (0 \le l < m)$$

Let $b_1, \ldots, b_m \in A$ such that $e_\ell(b_l) = c_l$. Then

$$\begin{aligned}
\imath_\ell(c_1)\imath_\ell(c_2)\cdots\imath_\ell(c_n) &= \imath_\ell(e_\ell(b_1))\imath_\ell(e_\ell(b_2))\cdots\imath_\ell(e_\ell(b_n)) \\
&= (\imath_\ell^* \circ e_\ell^*)(b_1b_2, \cdots b_n) \\
&= (\imath_{i_0}^* \circ e_{i_0}^*)(a_1b_2, \cdots b_n) \\
&\approx (\imath_{i_1}^* \circ e_{i_1}^*)(a_1b_2, \cdots b_n) \\
&= (\imath_{i_1}^* \circ e_{i_1}^*)(a_2b_2, \cdots b_n) \\
&\approx (\imath_{i_2}^* \circ e_{i_2}^*)(a_2b_2, \cdots b_n) \\
&\;\;\vdots \\
&\approx (\imath_{i_m}^* \circ e_{i_m}^*)(a_mb_2, \cdots b_n) \\
&= \imath_\ell(c_1')\imath_\ell(c_2)\cdots\imath_\ell(c_n)
\end{aligned}$$

Which proves that δ is independent of the choice of the c_k's. A similar argument shows that δ is independent of the index $\ell \in I$. Now to prove that $\delta \circ q_i^* = p_i$, assume that $a_1, a_2, \ldots, a_n \in A_i$, $n \in \mathbb{N}$, then

$$\begin{aligned}(\delta \circ q_i^*)(a_1 a_2 \cdots a_n) &= \delta(q_i(a_1) q_i(a_2) \cdots q_i(a_n)) \\ &= \delta([\iota_i(a_1)]_\sim [\iota_i(a_2)]_\sim \cdots [\iota_i(a_n)]_\sim) \\ &= [\imath_i(a_1) \imath_i(a_2) \cdots \imath_i(a_n)]_\approx \\ &= [\imath_i(a_1 a_2 \cdots a_n)]_\approx \\ &= p_i(a_1 a_2 \cdots a_n).\end{aligned}$$

◀

▶ Remark. Notice that the preservation of weak generalized pushouts of epis is only needed to show the existence of a syntactic T–algebra S_L for every language L. In some cases it is easier to show directly the existence of S_L instead of showing that T preserves weak generalized pullbacks of epis. In the examples we show, this can be done by finding the largest congruence that saturates L (if any), i.e., the largest congruence such that L is the union of equivalence classes. Some of the constructions for syntactic algebras are already known in the literature [8, 13].

For Example 17 the existence of the syntactic T–algebra is shown in [16].

10 Proof of Theorem 19.

Theorem 19. *Under the assumptions (A1) to (A5), there is a one–to–one correspondence between varieties of* T*–algebras and varieties of recognizable* T*–languages.*

The proof of this theorem is similar to the one of Theorem 9 and all the facts hold by replacing 'pseudovarieties of T–algebras' by 'varieties of T–algebras' and 'pseudovarieties of finitely recognizable T–languages' by 'varieties of T–languages'. We will only provide part of the proofs of some of the facts we proved in Section 5. Proposition 10 was proved for any T–language over $\Sigma \in \mathcal{D}$ so we can use it as it is.

As in Section 5, given a variety of T–algebras $\mathcal{A}$ and a variety of recognizable T–languages $\mathcal{L}$ we define the operator $\mathcal{L}(\mathcal{A})$ and the class of T–algebras $\mathcal{A}(\mathcal{L})$ as:

- For every object Σ in $\mathcal{D}$ define $(\mathcal{L}(\mathcal{A}))(\Sigma)$ as the set of T–languages over Σ that are recognized by an algebra in $\mathcal{A}$.
- $\mathcal{A}(\mathcal{L})$: T–algebras that recognize only languages in $\mathcal{L}$.

The proof of Proposition 12 is easily adapted to get its following corresponding version:

▶ **Proposition 25** (cf. Proposition 12). *Assume (A1). Let $\mathcal{L}$ be a variety of* T*–languages and $A = (A, \alpha)$ a* T*–algebra. Then $A \in \mathcal{A}(\mathcal{L})$ if and only if for every object Σ in $\mathcal{D}$ we have that $\widehat{G}A \Vdash \mathcal{L}\Sigma$.*

Proof. Similar to the proof of Proposition 12 by omiting finiteness conditions. ◀

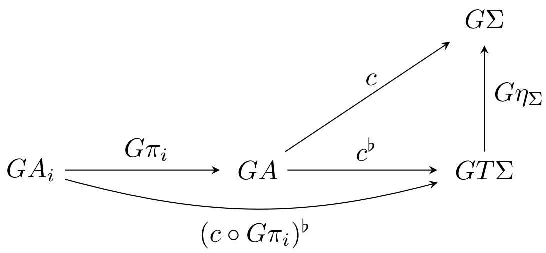
▶ **Corollary 26** (cf. Corollary 13). *Assume (A1) and (A2). Let $\mathcal{L}$ be a variety of recognizable* T*–languages. Then $\mathcal{A}(\mathcal{L})$ is a variety of* T*–algebras.*

Proof. By Proposition 25 we have that

$$\mathcal{A}(\mathcal{L}) = \{A \in \mathrm{Alg}(\mathsf{T}) \mid \forall\, \Sigma \in \mathcal{D}\ \widehat{G}A \Vdash \mathcal{L}\Sigma\}$$

We need to prove that $\mathcal{A}(\mathcal{L})$ is closed under homomorphic images, subalgebras, and products. Closure under homomorphic images and subalgebras is proved as in Proposition 13. Let's prove closure under products.

Assume that $A_i \in \mathcal{A}(\mathcal{L})$, $i \in I$, and let $A = \prod_{i\in I} A_i \in \mathrm{Alg}(\mathsf{T})$ be the product of the A_i's with projections $\pi_i : A \to A_i$. Now let Σ be an object in $\mathcal{D}$ and $c \in \mathcal{C}(GA, G\Sigma)$ be any colouring. Let $c^\flat \in \mathrm{Coalg}(\mathsf{B})(\widehat{G}A, \widehat{G}(T\Sigma))$ be the unique morphism such that $G\eta_\Sigma \circ c^\flat = c$. Then we have that $(c \circ G\pi_i)^\flat = c^\flat \circ G\pi_i$. That is, we have the following commutative diagram in $\mathcal{C}$ where the three lower arrows are morphisms in $\mathrm{Coalg}(\mathsf{B})$:



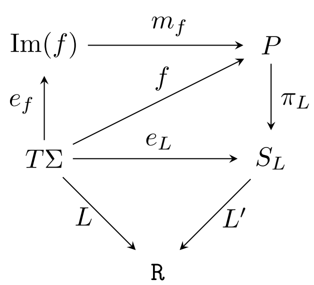
Hence we have that $\mathrm{Im}(c^\flat) = \bigcup_{i\in I} \mathrm{Im}((c \circ G\pi_i)^\flat) \subseteq \mathcal{L}\Sigma$, where the equality follows from the fact that $GA = \coprod_{i\in I} GA_i$ and the inclusion follows from the fact that $A_i \in \mathcal{A}(\mathcal{L})$. Therefore, $\mathrm{Im}(c^\flat) \subseteq \mathcal{L}\Sigma$, i.e., $\widehat{G}A \Vdash \mathcal{L}\Sigma$. ◀

To prove the following version of Proposition 14 we use Lemma 22 for the case that $\mathcal{A}$ is a variety of T–algebras whose proof is made in a similar way.

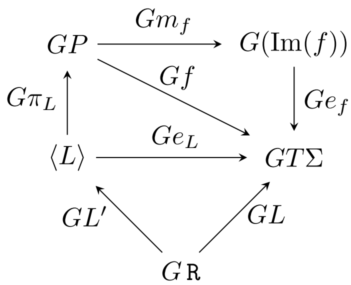
▶ **Proposition 27** (cf. Proposition 14). *Assume (A1) and (A3). Let $\mathcal{A}$ be a variety of T–algebras, then $\mathcal{L}(\mathcal{A})$ is a variety of recognizable T–languages.*

Proof. We have to prove properties i) and ii) of Definition 18. In fact, let Σ be an object in $\mathcal{D}$, then:

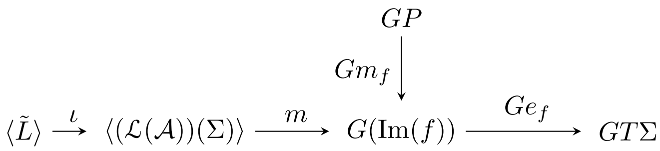
i) We will prove that $(\mathcal{L}(\mathcal{A}))(\Sigma) = \langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle$. The inclusion $\subseteq$ is obvious. To prove the other inclusion put $P = \prod_{L\in(\mathcal{L}(\mathcal{A}))(\Sigma)} S_L \in \mathrm{Alg}(\mathsf{T})$. We have that $P \in \mathcal{A}$ since every $S_L \in \mathcal{A}$ by Lemma 22 for the case of varieties of T–algebras.
Let $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$. As S_L recognizes L through e_L, there exists $L' \in \mathcal{D}(S_L, \mathtt{R})$ such that $L' \circ e_L = L$. Let $\pi_L \in \mathrm{Alg}(\mathsf{T})(P, S_L)$ be the L–th projection then, by the universal property of P, there exists $f \in \mathrm{Alg}(\mathsf{T})(T\Sigma, P)$ such that $\pi_L \circ f = e_L$. Let $m_f \circ e_f = f$ be the epi–mono factorization of f. That is, we have the following commutative diagram:



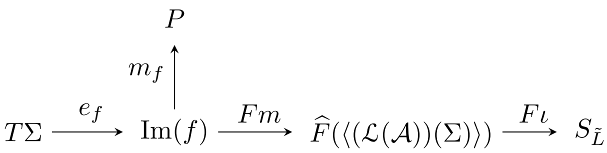
If we apply $\widehat{G}$ to the previous diagram we get the following commutative diagram:



From this diagram, by taking the identity map $id_{\mathtt{R}} \in G\,\mathtt{R} = \mathcal{D}(\mathtt{R},\mathtt{R})$ and using the fact that Ge_f is mono, we have that $L \in G(\mathrm{Im}(f))$, $L \in (\mathcal{L}(\mathcal{A}))(\Sigma)$, and hence $G(\mathrm{Im}(f))$ contains $\langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle$, so there exists a monomorphism $m \in \mathrm{Coalg}(\mathsf{B})(\langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle, G(\mathrm{Im}(f)))$. Now, for any $\tilde{L} \in \langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle$ we have that $\langle\tilde{L}\rangle \subseteq \langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle$, hence there exists a monomorphism $\iota \in \mathrm{Coalg}(\mathsf{B})(\langle\tilde{L}\rangle, \langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle)$. Then we have the following situation in $\mathrm{Coalg}(\mathsf{B})$:



Where ι, m, and Ge_f are monomorphisms and Gm_f is an epimorphism. By applying $\widehat{F}$ to the previous diagram we get:



Where e_f, Fm and $F\iota$ are epimorphisms and m_f is a monomorphism. Hence, as $P \in \mathcal{A}$ then $S_{\tilde{L}} \in \mathcal{A}$ which implies that $\tilde{L} \in (\mathcal{L}(\mathcal{A}))(\Sigma)$ because $\tilde{L}$ is recognized by $S_{\tilde{L}}$. Therefore $\langle(\mathcal{L}(\mathcal{A}))(\Sigma)\rangle \subseteq (\mathcal{L}(\mathcal{A}))(\Sigma)$.

ii) Closure under morphic preimages is proved as in Proposition 14.

◀

To prove the corresponding version of Proposition 15 we will make use of Lemma 24 for the case of varieties of recognizable T–languages whose proof is similar and therefore it is omited.

▶ **Proposition 28** (cf. Proposition 15). *Let $\mathcal{L}$ be a variety of recognizable T–languages and let $\mathcal{A}$ be a variety of T–algebras. Then:*

i) Assume (A1), (A2), and (A4), then $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.
ii) Assume (A1) to (A5), then $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$.

Proof. Similar to proof of Proposition 15. ◀