Coequations and Eilenberg-type Correspondences

Julian Salamanca

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Contents

Introduction

The notion of an equation is one of the basic concepts that allows us to classify algebraic structures, also called *algebras*, which are nonempty sets with finitary operations (finitary operations may be constants, unary operations, binary operations, ternary operations and so on). For example, an equation such as $x \cdot y = y \cdot x$, which formally stands for the formula $\forall x \forall y (x \cdot y = y \cdot x)$, is satisfied by the structure $(\mathbb{N}, \cdot)$ of the natural numbers with the usual multiplication, but it is not satisfied by the structure $(\mathcal{M}_{2\times 2}(\mathbb{Z}), \cdot)$ of 2×2 matrices with entries on the integers with the usual multiplication of matrices. In spite of its simplicity, there is a whole research area which focuses on the study of (equationally defined) algebras and which is called *universal algebra*. Commonly known texbooks in this area include [27, 45] and also a specialized journal: *Algebra Universalis*.

Algebras of the same type, i.e., having the same kind of finitary operations, that are equationally defined are called *equational classes*. Classical algebras that form an equational class include: semigroups, monoids, groups, lattices, vector spaces and Boolean algebras. A nonexample is the case of fields, since the axiom of existence of a multiplicative inverse, $\forall x (x \neq 0 \Rightarrow \exists y (x \cdot y = 1))$, is not an equation (we are restricting the domain of x and we are using an existencial quantifier, or, in other words, the multiplicative inverse operation is not total since 0 has no multiplicative inverse), even though all the remaining axioms that define a field are equations.

Another important concept for algebras, and mathematical structures in general, is the notion of homomorphism. A homomorphism between two algebras is a function that preserves all the operations, e.g., a homomorphism between $(\mathbb{N}, +_1)$ and $(\mathbb{Z}, +_2)$ is a function $h : \mathbb{N} \to \mathbb{Z}$ that preserves $+$, i.e., $h(x +_1 y) = h(x) +_2 h(y)$ for all $x, y \in \mathbb{N}$ (e.g., $h(x) = 2x$ defines a homomorphism but $h(x) = 2^x$ does not). This notion of a homomorphism, together with the notion of an algebra, allows us to use categorical methods for the study of algebras.

In order to define a category, two ingredients are needed: its objects and its morphisms. Additionally, certain axioms are needed in order to have a category, such as compositionality of morphisms, associativity of composition and existence of an identity morphism for each object which works as an identity element with respect to composition. This branch of mathematics whose main purpose is the study of categories is called *category theory*. Books in which its general theory is studied include [9, 66].

The main advantage of categorical methods is their generality. There are categorical concepts that allow us to define a category of algebras of the same type together with its homomorphisms. This can be done by considering the algebras for the monad (or the polynomial functor) associated to a given type. The study of the concept of an equation can also be made from a categorical point of view [52, 65, 50, 54, 38, 25].

Categorical concepts usually come in pairs, once a concept is defined we have its dual concept which is obtained by reversing the direction of the morphisms, and therefore we reverse the order in which compositions are made. For instance, the concept of a monomorphism, which is that of an injective function in the category `Set` of sets and functions, is defined as follows:

- $f : X \to Y$ is a *monomorphism* if for every $g, h : Z \to X$, the equality $f \circ g = f \circ h$ implies $g = h$.

Now, if we reverse the arrows, and therefore the order of composition, we get the dual concept of a monomorphism, which is called an epimorphism and it is defined as follows:

- $f : Y \to X$ is an *epimorphism* if for every $g, h : X \to Z$, the equality $g \circ f = h \circ f$ implies $g = h$.

Epimorphisms in the category `Set` are exactly surjective functions. That is, injective and surjective are dual concepts in the category `Set`. In this thesis we will mainly study the concepts of algebras and equations and their duals, which are called coalgebras and coequations, respectively.

Coalgebras and coequations

The study of coalgebras has been a very active field during the last two decades. Some of the main subjects studied in this area include: automata theory, dynamical systems, labeled transition systems, modal logic, language semantics, coequations and covarieties. Some references for the general theory include [76, 61]. As in the case of equations and algebras where equations allow us to classify and study algebraic structures, coequations allow us to classify and study coalgebraic structures. The use of coequations is less known.

When we study algebras we are mainly interested in "constructing" new elements from other elements, e.g., we can construct 4 from 1 and 3 by using the binary operation $+$ as $4 = 1 + 3$. Here the algebraic operation is a binary operation $+ : X \times X \to X$. Now if we reverse the arrow of the operation and we think of a pair (X, f) where $f : X \to X \times X$ we get a dual concept, i.e., a specific kind of coalgebra. Such pair (X, f), where $f : X \to X \times X$ is a particular instance of an automaton on the alphabet $2 = \{0, 1\}$, since $X \times X = X^2$. In this case, given $x \in X$, the element $f(x)(0) \in X$ is the state we reach from x by processing the symbol 0, similarly, $f(x)(1)$ is the state we reach from x by processing the symbol 1.

More generally, we can consider any alphabet A and consider coalgebras given by pairs $(X, \langle c, f \rangle)$, where $\langle c, f \rangle : X \to 2 \times X^A$ is the pairing of the two functions $c : X \to 2$ and $f : X \to X^A$ which is defined as $\langle c, f \rangle(x) = (c(x), f(x))$. This kind of coalgebra is an automaton on A whose function c represents the accepting states of the automaton. Categorical methods allow us, for each such pair $(X, \langle c, f \rangle)$, to define a map $o : X \to 2^{A^*}$ such that for each $x \in X$ the function $o(x) : A^* \to 2$ is the language accepted by the state x [13]. This element $o(x)$ is also called the *behaviour* of x. In this particular case, the concept of a coequation is a specific kind of subset S of 2^{A^*} and a coalgebra $(X, \langle c, f \rangle)$ satisfies S if for every $x \in X$ the element $o(x)$ is in S. That is, $(X, \langle c, f \rangle)$ satisfies S if S contains all the behaviours of the states in $(X, \langle c, f \rangle)$.

The general notion of a coequation and that of a coalgebra satisfying a coequation comes from what is done on the algebraic side by using duality. On the algebraic side there is a well–known theorem, Birkhoff's theorem [18], that characterizes equational classes as classes of algebras of the same type that satisfy three closure properties. By duality, a similar result is obtained in the coalgebraic case, usually called coBirkhoff's theorem. That is, we define the concept of a *coequational class* as a class of coalgebras of the same type that is defined by some coequations. Then, coBirkhoff's theorem characterizes coequational classes as classes of coalgebras of the same type that satisfy three closure properties, see, e.g., [76, Theorem 17.5].

Birkhoff's theorem and its dual

The study of mathematical structures that share common properties, such as satisfying a given family of equations in our case of interest, allows us to classify and study the structures of interest by proving properties they have in common. An important question in this respect is if there are equivalent ways of knowing when a class of algebraic structures is equational. That is, given a class of algebraic structures, do they all satisfy a common family of identities and those identities exactly describe the given class? An answer to this question is given by Birkhoff's theorem. Birkhoff's theorem [18] is a celebrated theorem in universal algebra that characterizes classes of algebras of the same type that are equational. In order to state this, we need to describe three different ways of obtaining new algebras from old ones.

i) The *product* of a family of algebras is defined by taking its cartesian product and operations defined componentwise, e.g., the product $(\mathbb{N}, +_1) \times (\mathbb{Z}, +_2)$ between $(\mathbb{N}, +_1)$ and $(\mathbb{Z}, +_2)$ is defined as $(\mathbb{N} \times \mathbb{Z}, +)$ where the operation $+$ is defined componentwise, i.e., $(a, b) + (x, y) := (a +_1 x, b +_2 y)$. Note that the example we just showed is the case of the product of two algebraic structures, but infinite products can also be considered. A class of algebras $\mathcal{K}$ of the same type is *closed under products* if for every family of algebras in $\mathcal{K}$ their product is also in $\mathcal{K}$.

ii) An algebra $\mathbf{B}$ is a *homomorphic image* of the algebra $\mathbf{A}$ if there exists a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$. For example, for any $n \geq 1$, the algebra $(\mathbb{Z}_n, +_n)$, where $+_n$ is the usual addition modulo n, is a homomorphic image of $(\mathbb{Z}, +)$, which can be witnessed by the surjective homomorphism $h : \mathbb{Z} \to \mathbb{Z}_n$ such that $h(x) = x \pmod n$. Homomorphic images are also called *quotients*. A class of algebras $\mathcal{K}$ of the same type is *closed under homomorphic images* if homomorphic images of elements in $\mathcal{K}$ are also in $\mathcal{K}$.

iii) An algebra $\mathbf{B}$ is a *subalgebra* of the algebra $\mathbf{A}$ if there exists an injective homomorphism from $\mathbf{B}$ to $\mathbf{A}$. For example, for any $n \geq 1$, the algebra $(n\mathbb{Z}, +)$, where $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ and $+$ is the usual addition, is a subalgebra of $(\mathbb{Z}, +)$, which can be witnessed by the injective homomorphism $h : n\mathbb{Z} \to \mathbb{Z}$ such that $h(nx) = nx$, $x \in \mathbb{Z}$. A class of algebras $\mathcal{K}$ of the same type is *closed under subalgebras* if subalgebras of elements in $\mathcal{K}$ are also in $\mathcal{K}$.

A class of algebras of the same type is a *variety* if it is closed under homomorphic images, subalgebras and products. Now, Birkhoff's theorem states that a class of algebras of the same type is a variety if and only it it is an equational class [18]. A similar version for Birkhoff's theorem is also obtained for the finite case [14, 73, 37]. That is, for varieties of finite algebras, also called pseudovarieties. A *pseudovariety* is a class of finite algebras of the same type that is closed under homomorphic images, subalgebras and finite products. In this case, the kind of "equations" that define a pseudovariety are of a more general kind by using topological methods. This finite version is also known as Reiterman's theorem.

There are also coalgebraic versions of Birkhoff's theorem [10, 2, 46]. In this case, the definition of a *covariety* is that of a class of coalgebras of the same type that are closed under homomorphic images, subcoalgebras and sums. The coalgebraic construction for a sum is, in most classical cases, the disjoint union of coalgebras. By duality, the dual of Birkhoff's theorem, also called *coBirkhoff's theorem*, states that a class of coalgebras of the same type is a covariety if and only if it is a coequational class.

Birkhoff's theorem states that, for a class of algebras of the same type, the "semantic" property of being closed under subalgebras, quotients and products is equivalent to the "syntactic" property of being defined by equations. As a consequence, satisfaction of equations is preserved under subalgebras, quotients and products, and any class of algebras of the same type closed under subalgebras, quotients and products has an axiomatization by means of equations. A similar but less explored argument applies for the case of coalgebras of the same type, by coBirkhoff's theorem, where coequations (also called "behaviours") are considered instead of equations and the construction of sums is considered instead of products.

Eilenberg–type correspondences

There exist other kinds of variety theorems in the literature, most of them for finite algebras, that are known as *Eilenberg–type correspondences*. They are stated as one–to–one correspondences between varieties of algebras and varieties of languages. The cases of varieties of finite monoids and varieties of finite semigroups was proved by Eilenberg [36].

Other kinds of Eilenberg–type correspondences have been proved, e.g., [70] for pseudovarieties of ordered monoids and ordered semigroups, the one in [74] for pseudovarieties of finite dimensional $\mathbb{K}$–algebras, [72] for pseudovarieties of idempotent semirings and [13, Theorem 39] for varieties of monoids. In each of those cases, the definition of a variety of languages had to be modified in order to prove the desired result. Most of them followed the same recipe and proof idea as the one used by Eilenberg, which was by means of syntactic algebras. For instance, for the case of monoids, the syntactic algebra of a language L on Σ is a homomorphic image of Σ^* that satisfies a universal property described in terms of L.

What are varieties of languages?

A *language* on an alphabet Σ is a subset of $L \subseteq \Sigma^*$. This concept of a language, which is usualy studied by using automata, can be also studied from an algebraic point of view. In fact, we have the following notion of a language recognized by a monoid:

- A language $L \subseteq \Sigma^*$ is *recognized* by a monoid M if there exists a monoid homomorphism $h : \Sigma^* \to M$ and a subset $N \subseteq M$ such that $L = h^{-1}(N)$. We say that L is *recognizable* if it is recognized by a finite monoid.

A well–known fact is that recognizable languages are the same as languages accepted by finite automata.

Now, varieties of languages are usually defined as operators $\mathscr{L}$ on a certain collection of objects, finite sets in most cases, satisfying certain properties. For instance, the definition of a variety of languages that corresponds to the case of varieties of finite monoids is the following:

- A *variety of languages* is an operator $\mathscr{L}$ on finite sets, such that for every finite set Σ:

  i) $\mathscr{L}(\Sigma)$ is a Boolean algebra where each element is a recognizable language on Σ (here the operations for the Boolean algebra are the union, intersection and complement),

  ii) $\mathscr{L}(\Sigma)$ is closed under left and right derivatives. That is, ${}_aL, L_a \in \mathscr{L}(\Sigma)$ for every $L \in \mathscr{L}(\Sigma)$ and $a \in \Sigma$, where ${}_aL = \{w \in \Sigma^* \mid wa \in L\}$ and $L_a = \{w \in \Sigma^* \mid aw \in L\}$, and

  iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every finite set Γ, homomorphism of monoids $h : \Gamma^* \to \Sigma^*$ and $L \in \mathscr{L}(\Sigma)$, we have that the inverse image $h^{-1}(L)$ of L under h is in $\mathscr{L}(\Gamma)$.

From the previous definition other kinds of varieties of languages were similarly defined. For instance:

a) For the case of varieties of finite semigroups we only need to replace Σ^* by Σ^+ and change property iii) above for the case of homomorphisms of semigroups.

b) For the case or varieties of monoids we only need to change “Boolean algebra” in i) above by “complete atomic Boolean algebra” and omit the word “recognizable”.

c) For the case of varieties of finite ordered monoids we only need to change “Boolean algebra” in i) above by “(distributive) lattice”, that is, $\mathscr{L}(\Sigma)$ is not necessarily closed under complements.

d) For the case of varieties of finite groups we have to replace Σ^* by the free group on Σ, add a new special kind of derivative in ii) above, and change property iii) above for the case of homomorphisms of groups.

Most of the known Eilenberg–type correspondences have been proved separately in which the corresponding definition for a variety of languages has been an ad hoc definition without fully explaining where it comes from. Many questions can emerge in this respect for which there has been no formal explanation, such as:

1. Why do we get a Boolean algebra in i) above and why is each language recognizable?

2. Where do derivatives come from? How many kinds of derivatives are needed?

3. Why closure under homomorphic images?

4. What should be the conditions if one wants an Eilenberg–type correspondence for varieties of finite distributive lattices? And more generally...

5. What is a variety of languages?

One of the main purposes of this thesis is to give an answer to this kind of questions and to fully explain what Eilenberg–type correspondences are. The main idea is to understand the relation between equational classes, i.e., varieties of algebras, and its dual, i.e., coequational classes. Varieties and equations are related by Birkhoff’s theorem, each variety is defined by some equations and each family of equations define a variety. The previous fact can be stated as a one–to–one correspondence if we require that the family of equations is deductively closed, i.e., if we have an equational theory. Now, by duality, each equational theory will define a coequational theory and it turns out that this coequational theory is exactly the variety of languages that defines a given variety of algebras. In this sense, we will explain and justify the fact that “varieties of languages = coequational theories".

It is worth mentioning that the concept of variety of languages has been studied for varieties of algebras and also for pseudovarieties of algebras, in the first case we call them *varieties of languages* and in the latter case we call them *pseudovarieties of languages*. In this sense, we will also explain and justify that "pseudovarieties of languages = pseudocoequational theories".

Approach

Eilenberg–type correspondences are one–to–one correspondences between varieties of algebras and varieties of languages, the latter being a particular instance of coequations. The concept of a coequation is a rather less known concept than that of an equation. In this thesis we will start by studying particular cases of coequations and its relationship with equations. We will do this for the case of deterministic automata and weighted automata. Then we develop the general categorical approach of coequations and equations and a general duality result between them which will help us to understand how coequations are one of the main ingredients in Eilenberg–type correspondences.

In *Chapter 1*, we introduce some preliminaries to fix the notation used in this thesis and most of the main general concepts that we need in subsequent chapters. Here we recall some categorical definitions such as covariant and contravariant functors, particularly the covariant and contravariant hom–set functors, natural transformations, initial and final object, free and cofree objects, algebras and coalgebras for an endofunctor, adjunctions, contravariant adjunctions and duality. Classical Birkhoff's theorem will also be stated in this chapter.

In *Chapter 2*, we study equations and coequations for deterministic automata. Here we recall the basic definitions of equations and coequations for deterministic automata given in [13]. We illustrate satisfaction of equations and coequations with some examples and explain what exactly means that a deterministic automaton satisfies some given (co)equations. We illustrate how those concepts can be captured categorically by using arrows and commutative diagrams. Based on that, we illustrate how this categorical approach allows us to obtain the duality result between equations and coequations in [13] and show some applications. Based on this duality, we show that regular varieties of automata can be defined either by equations or coequations.

In *Chapter 3*, we study equations and coequations for weighted automata. Here we introduce some concepts that are needed to define weighted automata such as semirings and semimodules. Then we introduce equations and coequations for weighted automata and a duality result between them. All of this was motivated by the case of deterministic automata. We also study a more general kind of equations called linear equations.

Chapter 4 is one of the most important chapters in this work where we introduce the abstract theory of equations and coequations from a categorical point of view. We show the basic definitions of equations and coequations over a given object for a given endofunctor. Equations are defined on the algebraic side and coequations are defined on the coalgebraic side. We show how a contravariant adjunction between two categories can be lifted to a contravariant adjunction between a category of algebras and a category of coalgebras on the base categories. This fact is obtained from [51]. From this, we add still another layer and establish results that allow us to lift a duality to a duality between categories of equations and coequations. We show how satisfaction of equations on one side is equivalent to satisfaction of coequations on the other side if we assume that the contravariant adjunction is a duality. This is a key fact to understand the interaction between equations and coequations. We do a similar work by considering monads and comonads instead of endofunctors, i.e., by considering Eilenberg–Moore categories.

In *Chapter 5*, we summarize and present a categorical version of Birkhoff's theorem. The main purpose is to apply and use the categorical definition of equations in order to obtain a categorical version of Birkhoff's theorem for algebras over a monad. Birkhoff's theorem is a well–known theorem for which some categorical versions are already proved in the literature, such as [10, 15, 17], but we present a different version which takes into account algebras over a monad and a new concept of an equational theory. We include this chapter for the following reasons:

- To make this thesis self–contained.

- To show a version of Birkhoff's theorem for algebras over a monad.

- To show a version of Birkhoff's theorem as a one–to–one correspondence between varieties of algebras and equational theories. The latter is a new categorical concept introduced in this work, which is based on the classical definition of an equational theory [27, Definition II.14.16] and some ideas in the paper [78]. A similar work is made for pseudovarieties of algebras and also for local varieties and local pseudovarieties. (A local variety is a class of algebras in which all the algebras are a quotient of a given free algebra on a fixed set of generators, say X, and it is closed under quotients, subalgebras that are quotients of the free algebra on X and subdirect products, which is a special subalgebra of a product. Local pseudovarieties are defined in a similar way by considering finite algebras and finite subdirect products).

- To derive Eilenberg–type correspondences in subsequent chapters.

From this version of Birkhoff's theorem, we will easily obtain Eilenberg–type correspondences in the subsequent chapters. The main contributions and differences with other categorical approaches are:

- A definition of an equational theory and a pseudoequational theory.

- A version of Birkhoff's theorem for algebras for a monad as a one–to–one correspondence between varieties of algebras and equational theories. Similar versions are also proved for pseudovarieties, local varieties and local pseudovarieties.

- A proof of Birkhoff's theorem for finite algebras without using topology and profinite methods.

Most of the proofs and the idea on how to prove the finite version are obtained from [15, 14].

In *Chapter 6*, we unveil Eilenberg–type correspondences. That is, we show that:

- Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality.
- "Varieties of languages" = duals of equational theories.

To this end, we use the theory that was developed in Chapters 4 and 5. We obtain general categorical versions of Eilenberg–type correspondences for the following cases: varieties of algebras, pseudovarieties of algebras, local varieties of algebras and local pseudovarieties of algebras. Additionally, we also discuss the subject of syntacic algebras, which is a common concept that has been used for classical proofs of Eilenberg–type correspondences. There are some important contributions in this respect:

- Syntactic algebras are not needed to obtain Eilenberg–type correspondences.
- Syntactic algebras are generalized pushouts.
- The syntactic algebra of a language is the dual of the least coalgebra generated by the language.

We also derive, as an example, an Eilenberg–type correspondence for varieties of monoids, which is an Eilenberg–type correspondence that has been established in [13], but in our case, we obtain a more simplified definition of a variety of languages compared to the one in [13]. We show that the two versions are equivalent.

In *Chapter 7*, we show applications of the categorical Eilenberg–type correspondence theorems to derive specific Eilenberg–type correspondences for some varieties and pseudovarieties. Some of them were already proved in the literature and some others appear to be new, especially the ones where finiteness of algebras is not required, i.e., the case of varieties of algebras.

Main results and contributions

Eilenberg–type correspondences have been studied in the literature during the last forty years since its first version due Eilenberg [36, Theorem 34]. Even though its popularity, which can be witnessed by the numerous Eilenberg–type correspondences shown in the literature such as [70, 74, 72, 13], and besides some categorical generalizations, such as [22, 1, 5, 89, 78], Eilenberg–type correspondences had not been fully understood. The main contribution of this thesis is to fully explain how Eilenberg–type correspondences are obtained.

Historically, one of the key concepts used to obtain Eilenberg–type correspondences was the concept of a syntactic algebra for a language. Syntactic algebras were used by Eilenberg himself in order to prove his theorem and similar strategies were used in [70, 74, 72, 22, 89, 78]. This could possibly mean that in order to get an Eilenberg–type correspondence one needs to use syntactic algebras. An important observation and contribution in this thesis is that syntactic algebras are not needed in order to get Eilenberg–type correspondences. In fact, syntactic algebras can work as the building blocks to generate a variety of algebras but there is a more direct way to build a variety, namely, via equations by Birkhoff's theorem [18]. In this thesis, we establish Eilenberg–type correspondences without the use of syntactic algebras, but we show categorical properties of syntactic algebras and its relationship with known concepts in Section 6.4.

The two main concepts that are involved in an Eilenberg–type correspondence are: varieties of algebras and varieties of languages. The study and definition of a variety of algebras is fully understood and has been broadly studied in universal algebra. On the other hand, the concept of a variety of languages had not been fully understood and the different definitions of a variety of languages given in the literature do not fully explain where its defining properties come from and they are usually defined to each particular case without any explanaition. In this thesis we fully explain what a variety of languages is and give a simple and easy to understand picture on how Eilenberg–type correspondences are obtained. In fact, varieties of languages are duals of equational theories.

Our main result for Eilenberg–type correspondences is shown in Chapter 6: Proposition 143, 153, 158 and 161. In order to obtain this result, which we called an abstract Eilenberg–type correspondence, we needed to obtain a categorical version of Birkhoff's theorem as a one–to–one correspondence. For this purpose, a categorical approach to equations and coequations is given by focusing in their relationship via duality. From this, the notion of an equational theory is defined which will be one of the key concepts that will lead us to establish our main result and derive new and known Eilenberg–type correspondences.

Published work

Here we summarize the published work made by the author during his PhD period and relate it with the contents of this thesis. The following is the list of papers published by the autor in chronological order:

[79] Julian Salamanca, Adolfo Ballester-Bolinches, Marcello Bonsangue, Enric Cosme-Llópez, and Jan Rutten. *Regular Varieties of Automata and Coequations*, pages 224–237. Springer International Publishing, 2015.

[81] Julian Salamanca, Marcello Bonsangue, and Jan Rutten. *Equations and Coequations for Weighted Automata*, pages 444–456. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015.

[80] Julian Salamanca, Marcello Bonsangue, and Jurriaan Rot. *Duality of Equations and Coequations via Contravariant Adjunctions*, pages 73–93. Springer International Publishing, Cham, 2016.

[78] Julian Salamanca. *An Eilenberg–like theorem for algebras on a monad*. CWI Technical Report, (FM–1602), March 2016.

[82] Julian Salamanca Tellez. *Unveiling Eilenberg–type correspondences: Birkhoff's theorem for (finite) algebras + duality*. CWI Technical Report, (FM–1604), December 2016. (submitted to a journal)

Chapter 2 is based on the results given in [79], which is based on the work on equations and coequations initiated by Rutten et al. [13], whose main contribution is how equations and coequations can equivalently describe regular varieties of automata.

Chapter 3 is based on the paper [81] which is the study of equations and coequations for weighted automata and a duality result between them.

Chapter 4 is based on the paper [80] about presenting duality results between equations and coequations. The paper is based on results that allow liftings of contravariant adjunctions to categories of algebras and coalgebras and then liftings to categories of equations and coequations are added for the case that the contravariant adjunction is a duality. Definitions of categories of equations and coequations are given. Liftings of adjunctions to categories of algebras is made in [51] and a similar version is proved for the case of Eilenberg–Moore categories in [80].

Chapters 5, 6 and 7 are based on the papers [78, 82] whose main idea is to obtain categorical versions for general Eilenberg–type correspondences. Early ideas of this work appear in [78] while a complete and improved version is presented in [82]. Particular versions of Birkhoff's theorem were needed for this purpose, which are presented in Chapter 5. General and abstract Eilenberg–type correspondences are presented in Chapter 6 and applications are shown in Chapter 7.

Related work

Here we summarize some of the main related work for this thesis. We will make a brief discussion here, but a more detailed discussion that includes more references will be made through the thesis and especially at the end of each chapter.

Equations and coequations for deterministic automata have been studied in [13], from which most of the concepts for Chapter 2 are obtained. In the same chapter, we study the connection between equations, coequations and regular varieties. Regular varieties and their defining equations were already studied in [71, 88, 44]. In Chapter 3, we imitate what it is done in Chapter 2 in order to obtain similar results for the case of weighted automata.

With respect to the study of equations and coequations from a categorical point of view, one of the starting points in this direction was inspired in obtaining categorical versions of Birkhoff's theorem such as [10, 15, 17] and its first version for coalgebras in [76] in which the concept of coequations is not mentioned but used under the name of a subsystem. In [15], subcategories that are in some sense equational are defined and the role of equations is played by regular epimorphisms with regular–projective domain. From this, dual versions can be easily obtained and hence the idea of defining coequations as a special kind of monomorphisms. In [10], coequations are defined as regular subobjects of a cofree coalgebra, i.e., a special kind of monomorphism. In [61], coequations are called modal rules or modal formulas, and they are represented by morphisms in M, usually monomorphisms, for a given $(\mathscr{E}, M)$–category [61, Definition 2.4.1]. In [25], equations are presented as pairs of arrows, left–hand side and right–hand side, and the definition of satisfaction is in terms of coequalizing those two arrows, this property can be presented in terms of their coequalizer, when it exists, and hence equations are a special kind of epimorphism. A similar idea of defining equations with left–hand side and right–hand side is explored in [38]. Lifting adjunctions to categories of algebras was studied in [51].

The study of Birkhoff's theorem [18] has been an important research area, especially their categorical versions and dual versions such as [10, 15, 17, 76]. The first categorical approach in this direction is in [15] and a first coalgebraic version is in [76].

Eilenberg–type correspondences have been widely studied since Eilenberg's variety theorem in [36]. Some of the particular instances include the work in [70] for pseudovarieties of ordered monoids and ordered semigroups, the one in [74] for pseudovarieties of finite dimensional $\mathbb{K}$–algebras, [72] for pseudovarieties of idempotent semirings and [13, Theorem 39] for varieties of monoids. General categorical approaches, in which neither the coalgebraic nor the coequational points of view are explored, include [22, 1, 5, 89]

Chapter 1

Preliminaries

In this chapter, we introduce some notation, some definitions and some general facts that will be used. In this thesis, we assume that the reader is familiar with basic concepts from category theory, see, e.g., [9, 66].

We denote arbitrary categories by calligraphic letters such as $\mathcal{C}$, $\mathcal{D}$, etc. We use the notation $X \in \mathcal{C}$ to say that X is an object of the category $\mathcal{C}$. If $X, Y \in \mathcal{C}$, we denote the collection of morphisms in $\mathcal{C}$ with domain X and codomain Y as $\mathcal{C}(X, Y)$. Collections of morphisms of the form $\mathcal{C}(X, Y)$ are called *hom–sets*. If the category $\mathcal{C}$ is clear from the context we use the notation $f : X \to Y$ for a morphism $f \in \mathcal{C}(X, Y)$. If $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, we denote their composition as $g \circ f$, which is an element in $\mathcal{C}(X, Z)$. For every object $X \in \mathcal{C}$ we denote the identity morphism on X as id_X, which is an element in $\mathcal{C}(X, X)$.

Given a category $\mathcal{C}$, we denote by $\mathcal{C}^{op}$ its *dual category* which has the same objects as $\mathcal{C}$ and whose arrows are reversed, that is, $f \in \mathcal{C}^{op}(X, Y)$ if and only if $f \in \mathcal{C}(Y, X)$. In this case, given $f \in \mathcal{C}^{op}(X, Y)$ and $g \in \mathcal{C}^{op}(Y, Z)$, their composition $g \circ^{op} f$ in $\mathcal{C}^{op}$ is defined as $f \circ g \in \mathcal{C}(Z, X)$ in $\mathcal{C}$, which is an element in $\mathcal{C}^{op}(X, Z)$.

We denote functors by capital letters F, G, etc. A *covariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$, denoted as $F : \mathcal{C} \to \mathcal{D}$, is an assignment on objects and morphisms of $\mathcal{C}$ which satisfies the following properties, for every $X, Y, Z \in \mathcal{C}$, $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$:

$$F(X) \in \mathcal{D},\ \ F(f) \in \mathcal{D}(F(X), F(Y)),\ \ F(id_X) = id_{F(X)} \ \text{ and } \ F(g \circ f) = F(g) \circ F(f).$$

Functors in which the domain and codomain are the same category, say $\mathcal{C}$, are called *endofunctors* on $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *faithful* if for every $X, Y \in \mathcal{C}$ and $f, g \in \mathcal{C}(X, Y)$, $F(f) = F(g)$ implies $f = g$. We say that F is *full* if for every $X, Y \in \mathcal{C}$ and $g \in \mathcal{D}(F(X), F(Y))$ there exists $f \in \mathcal{C}(X, Y)$ such that $F(f) = g$.

Another kind of functors we will consider are the contravariant ones, which reverse the direction of the arrows in the following sense. A *contravariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$, denoted as $F : \mathcal{C} \rightarrowtimes \mathcal{D}$, is a functor $F : \mathcal{C} \to \mathcal{D}^{op}$.

We say that a category $\mathcal{C}$ is *locally small* if for every $X, Y \in \mathcal{C}$ the collection $\mathcal{C}(X, Y)$ of morphisms with domain X and codomain Y is a set. We say that $\mathcal{C}$ is *small* if it is locally small and the collection of objects in $\mathcal{C}$ is a set. If $\mathcal{C}$ is a locally small category then we can define the following hom–set functors from $\mathcal{C}$ into the category `Set` of sets and functions:

i) The *covariant hom–set functor* for an object $X \in \mathcal{C}$ is the functor $\mathcal{C}(X, \_) : \mathcal{C} \to \mathtt{Set}$ which is defined as $\mathcal{C}(X, \_)(Y) := \mathcal{C}(X, Y)$ for any $Y \in \mathcal{C}$ and $\mathcal{C}(X, \_)(f) := \mathcal{C}(X, f) : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ for $f \in \mathcal{C}(Y, Z)$, where $\mathcal{C}(X, f)$ is defined as $\mathcal{C}(X, f)(g) = f \circ g$ for every $g \in \mathcal{C}(X, Y)$.

ii) The *contravariant hom–set functor* for an object $X \in \mathcal{C}$ is the functor $\mathcal{C}(\_, X) : \mathcal{C} \times\!\!\!\longrightarrow \mathtt{Set}$ which is defined as $\mathcal{C}(\_, X)(Y) := \mathcal{C}(Y, X)$ for any $Y \in \mathcal{C}$ and $\mathcal{C}(\_, X)(f) := \mathcal{C}(f, X) : \mathcal{C}(Z, X) \to \mathcal{C}(Y, X)$ for $f \in \mathcal{C}(Y, Z)$, where $\mathcal{C}(f, X)$ is defined as $\mathcal{C}(X, f)(g) = g \circ f$ for every $g \in \mathcal{C}(Z, X)$.

A *concrete category* is a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and $U : \mathcal{C} \to \mathtt{Set}$ is a faithful functor. We also say that a category $\mathcal{C}$ is *concrete* if there exists a faithful functor $U : \mathcal{C} \to \mathtt{Set}$ such that $(\mathcal{C}, U)$ is a concrete category.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* ϑ from F to G, denoted as $\vartheta : F \Rightarrow G$, is an assignment on objects of $\mathcal{C}$ to morphisms of $\mathcal{D}$ such that $\vartheta_X \in \mathcal{D}(F(X), G(X))$ for each object X in $\mathcal{C}$, and for every $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X, Y)$ the following diagram commutes[1]:

$$\begin{tikzcd} F(X) \arrow[r, "\vartheta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\vartheta_Y"'] & G(Y) \end{tikzcd}$$

that is, $G(f) \circ \vartheta_X = \vartheta_Y \circ F(f)$.

An object $X \in \mathcal{C}$ is an *initial object* if for any object $Y \in \mathcal{C}$ there is a unique morphism in $\mathcal{C}(X, Y)$, which we denote as $!_Y$. Dually, an object $X \in \mathcal{C}$ is a *final object* if for any object $Y \in \mathcal{C}$ there is a unique morphism in $\mathcal{C}(Y, X)$, which we also denote as $!_Y$. It will be clear from the context to which $!_Y$ we are refering to.

Definition 1. Let $\mathcal{C}, \mathcal{D}$ be categories, $U : \mathcal{C} \to \mathcal{D}$ a functor and $X \in \mathcal{D}$. The *free U–object over X* is an object $\mathfrak{F}(X) \in \mathcal{C}$ together with a morphism $\eta_X \in \mathcal{D}(X, U(\mathfrak{F}(X)))$, called the *unit morphism on X*, that satisfies the following (universal) property:

(UP) For every $A \in \mathcal{C}$ and every morphism $f \in \mathcal{D}(X, U(A))$ there exists a unique morphism $f^\sharp \in \mathcal{C}(\mathfrak{F}(X), A)$ such that the following diagram commutes:

[1] A *commutative diagram* is a diagram such that any two paths, with same sourse and target, their compositions are the same.

$$\begin{tikzcd} U(\mathfrak{F}(X)) \arrow[r, "U(f^\sharp)"] & U(A) \\ X \arrow[u, "\eta_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

The morphism $f^\sharp$ is called the *extension* of f.

The most common cases of free U–objects we deal with are the cases in which U is a faithful (forgetful) functor. For example, if $U : \texttt{Mon} \to \texttt{Set}$ denotes the forgetful functor from the category Mon of monoids and monoid homomorphisms into the category Set of sets and functions, which sends every monoid $(M, \cdot, e) \in \texttt{Mon}$ to its underlying set $U(M, \cdot, e) = M \in \texttt{Set}$ and U acts as the identity on morphisms, then, for every $X \in \texttt{Set}$, the free U–object over X is the *free monoid* on X generators which is given by $X^* = (X^*, \cdot, \varepsilon)$, where X^* is the set of all words with symbols on X, $\cdot$ is the concatenation operation defined as $u \cdot v = uv$, and ε is the *empty word*.

By dualizing the previous concept we obtain the following definition.

Definition 2. Let $\mathcal{C}, \mathcal{D}$ be categories, $U : \mathcal{C} \to \mathcal{D}$ a functor and $Y \in \mathcal{D}$. The *cofree U–object over Y* is an object $\mathfrak{C}(Y) \in \mathcal{C}$ together with a morphism $\epsilon_Y \in \mathcal{D}(U(\mathfrak{C}(Y)), Y)$ that satisfies the following (universal) property:

(UP) For every $A \in \mathcal{C}$ and every morphism $f \in \mathcal{D}(U(A), Y)$ there exists a unique morphism $f^\flat \in \mathcal{C}(A, \mathfrak{C}(Y))$ such that the following diagram commutes:

$$\begin{tikzcd} U(A) \arrow[r, "U(f^\flat)"] \arrow[dr, "f"'] & U(\mathfrak{C}(Y)) \arrow[d, "\epsilon_Y"] \\ & Y \end{tikzcd}$$

We assume that the reader is familiar with different kinds of morphisms in a category including the following: epimorphisms (also known as epis), monomorphisms (also known as monos), isomorphisms (also known as isos). We also assume the knowledge of limits and colimits, especially products and coproducts in a category. See, e.g., [66, 4].

Let $f : X \to Y$ be a function. Define the *kernel* $\ker(f)$ of f and the *image* $\mathrm{Im}(f)$ of f as follows:

$$\ker(f) = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} \quad \mathrm{Im}(f) = \{y \in Y \mid \exists x \in X\ f(x) = y\}.$$

1.1 Algebras for an endofunctor

In this section, we define the general notion of algebras for an endofunctor. We will fix some notation and show some examples. See, e.g., [9] for more details and examples on this subject.

We start by defining the key concept of algebras for an endofunctor and its corresponding category, which can be seen as the categorical generalization of algebraic structures.

Definition 3. Let $\mathcal{D}$ be a category and $F : \mathcal{D} \to \mathcal{D}$ an endofunctor on $\mathcal{D}$. We define the category $\mathtt{alg}(F)$ of F– *algebras* as the category whose objects are pairs $\mathbf{X} = (X, \alpha_X)$ such that $X \in \mathcal{D}$ and $\alpha_X \in \mathcal{D}(F(X), X)$ and a morphism $\mathtt{alg}(F)(\mathbf{X}, \mathbf{Y})$, with $\mathbf{Y} = (Y, \alpha_Y)$, is a morphism $f \in \mathcal{D}(X, Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Pairs (X, α_X) as above are called F*–algebras*.

As examples of algebras for an endofunctor we have the following.

Example 4 (Deterministic automata as F–algebras)**.** Let A be a set of symbols, i.e., an *alphabet*. Consider the functor $F : \mathtt{Set} \to \mathtt{Set}$ defined on objects as $F(X) := A \times X$ and on morphisms as $F(f) := id_A \times f$, i.e., for any $f \in \mathtt{Set}(X, Y)$, $a \in A$ and $x \in X$ we have $F(f)(a, x) = (id_A, f)(a, x) = (a, f(x))$. A *deterministic automaton* on A is an F–algebra, that is, a pair (X, α_X) such that $X \in \mathtt{Set}$ and $\alpha_X \in \mathtt{Set}(A \times X, X)$. The function α_X is called the *transition function* of (X, α_X) and X its *states*. In this case, $\alpha_X(a, x)$ is the state we reach from $x \in X$ with the symbol $a \in A$. A *homomorphism of automata* is an F–algebra morphism, that is, a homomorphism from (X, α_X) to (Y, α_Y) is a function $f : X \to Y$ such that for every $a \in A$ and $x \in X$ we have that $f(\alpha_X(a, x)) = \alpha_Y(a, f(x))$. That is, the state we reach from $f(x) \in Y$ with the symbol $a \in A$ is the image under f of the state we reach from $x \in X$ with the symbol a. □

Example 5. Let τ be a *type of algebras*, also called a *signature*, i.e., τ is a set of *function symbols* such that for every $f \in \tau$ there is $n_f \in \mathbb{N}$ which is called the arity of f. An *algebra of type* τ is a pair $(A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$. An *algebra homomorphism* from an algebra $(A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$ to an algebra $(B, \{f_B : B^{n_f} \to B\}_{f \in \tau})$ is a function $h : A \to B$ such that for every $f \in \tau$ and $a_1, \ldots, a_{n_f} \in A$ we have that $h(f_A(a_1, \ldots, a_{n_f})) = f_B(h(a_1), \ldots h(a_{n_f}))$.

Now, for a given type τ, define the functor $F_\tau : \mathtt{Set} \to \mathtt{Set}$ as $F_\tau(\_) = \coprod_{f \in \tau} \mathtt{Set}(n_f, \_)$. Then we have that F_τ–algebras are exactly algebras of type τ and algebra homomorphisms are exactly F_τ–algebra morphisms[2]. For instance, if

[2] If we consider τ as a discrete category, i.e., its objects are the elements in τ and its only morphisms are the identity morphisms, then, if we consider the functor $ar : \tau \to \mathtt{Set}$ such that $ar(f) = n_f$ and the functor $i : \tau \to \mathtt{Set}$ such that $i(f) = \{f\}$, then we have that F_τ is the left Kan extension $\mathtt{Lan}_{ar}(i)$ of i along ar.

$\tau = \{\cdot, e\}$, where $\cdot$ is a binary function symbol and e is a nullary function symbol, then an F_τ algebra is a pair (A, α_A) where $\alpha_A : \{\cdot\} \times A^2 + \{e\} \times 1 \to A$ is a function. From α_A we can define the functions $\cdot_A : A^2 \to A$ and $e_A : 1 \to A$ as $\cdot_A(a_1, a_2) = \alpha_A(\cdot, a_1, a_2)$ and $e_A(0) = \alpha_A(e, 0)$ to obtain the algebra $(A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$ of type τ. Conversely, given an algebra $(A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$ of type τ we can define the function $\alpha_A : \{\cdot\} \times A^2 + \{e\} \times 1 \to A$ as $\alpha_A(\cdot, a_1, a_2) = \cdot_A(a_1, a_2)$ and $\alpha_A(e, 0) = e_A(0)$ to obtain the F_τ–algebra (A, α_A). Examples of such algebras include monoids, but not every such algebra is a monoid since we do not necessarily have that the operation $\cdot$ is associative or that e is the neutral element for $\cdot$. Note that the notions of algebra homomorphism and F_τ–algebra morphism coincide under the previous correspondence. Also note that deterministic automata on a set A are the same as algebras of the type $\tau = A$, where each symbol in A is a unary function symbol. □

If $\mathcal{D}$ is a category that has a final object $X \in \mathcal{D}$, then also $\mathtt{alg}(F)$ has a final object, namely $(X, !_{F(X)})$. In the case of the previous example, the final deterministic automaton on A is the automaton $(1, !_{A\times 1})$ where $1 = \{0\}$ and $!_{A\times 1} : A \times 1 \to 1$ is defined as $!_{A\times 1}(a, 0) = 0$ for every $a \in A$. We now define the concept of a free algebra.

Definition 6. Let $\mathcal{D}$ be a category, $F : \mathcal{D} \to \mathcal{D}$ an endofunctor on $\mathcal{D}$ and $X \in \mathcal{D}$. The *free F–algebra over X generators* is the free U–object over X (see, Definition 1), where $U : \mathtt{alg}(F) \to \mathcal{D}$ is the *forgetful functor* defined on objects as $U(X, \alpha_X) = X$ for every $(X, \alpha_X) \in \mathtt{alg}(F)$ and $U(f) = f$ for every morphism f in $\mathtt{alg}(F)$.

The following illustrates the free deterministic automaton over 1 generator, which will be a key concept for defining equations for deterministic automata in the next chapter.

Example 7 (Example 4 continued). For a given set A, let $F : \mathtt{Set} \to \mathtt{Set}$ be the functor defined as $F(X) = A \times X$ on objects and $F(f) = id_A \times f$ on morphisms. The free F–algebra over the set $1 = \{0\}$ is given by the F–algebra $\mathbf{A}^* = (A^*, \varrho)$ where $\varrho : A \times A^* \to A^*$ is defined as $\varrho(a, w) = aw$, for every $a \in A$ and $w \in A^*$, and the function $\eta_1 \in \mathtt{Set}(1, A^*)$ is defined as $\eta_1(0) = \varepsilon$. Now, given an F–algebra $\mathbf{X} = (X, \alpha_X)$ and a function $f \in \mathtt{Set}(1, X)$, there is a unique F–algebra morphism $f^\sharp \in \mathtt{alg}(F)(\mathbf{A}^*, \mathbf{X})$ such that $f^\sharp \circ \eta_1 = f$, which is inductively defined as:

$$f^\sharp(w) = \begin{cases} f(0), & \text{if } w = \varepsilon \\ \alpha_X(a, f^\sharp(u)) & \text{if } w = au. \end{cases}$$

That is, $f^\sharp(w)$ is the state we reach in the automaton X from the state $f(0)$ by processing the word w from right to left.

Note that the F–algebra (A^*, ϱ') where $\varrho' : A \times A^* \to A^*$ is defined as $\varrho'(a, w) = wa$ for every $a \in A$ and $w \in A^*$ is also a free F–algebra over 1. In this case, for any deterministic automaton $\mathbf{X} = (X, \alpha_X)$ and $f \in \mathtt{Set}(1, X)$, the function $f^\sharp \in \mathtt{alg}(F)((A^*, \varrho'), \mathbf{X})$ is the function such that for every $w \in A^*$, $f^\sharp(w)$ is the state we reach from the state $f(0)$ by processing the word w from left to right. Also, we have that (A^*, ϱ) and (A^*, ϱ') are isomorphic under the function $\varphi : A^* \to A^*$ which sends every word $w \in A^*$ to its reversal $w^R =: \varphi(w)$. $\square$

Note that, in general, for any endofunctor $F : \texttt{Set} \to \texttt{Set}$, the free F–algebra over $X \in \texttt{Set}$ is essentially the same as the initial F_X–algebra, where F_X is the functor $F_X(\_) := X + F(\_)$. For instance, in the setting of the previous example, $(A^*, [\eta_1, \varrho])$ is the initial F_1–algebra, where $[\eta_1, \varrho] : 1 + A^* \to A^*$ is obtained from η_1 and ϱ from the universal property of the coproduct $1 + A^*$.

Another observation regarding the previous example is how the induction principle is related with free algebras. In fact, the definition of the $f^\sharp$ above is an inductive one. The previous fact is represented by the slogan "induction=use of initiality for algebras", see [55].

1.2 Birkhoff's theorem

In this section, we recall some basic definitions and state Birkhoff's theorem, see, e.g., [18, 27]. The study of algebras and classes of algebras is a main subject of study in universal algebra. A well–known theorem in this area is Birkhoff's variety theorem, which states that a class of algebras of a given type is defined by a set of equations if and only if it is a variety, i.e., it is closed under homomorphic images, subalgebras, and products.

We already defined type of algebras (also called a signature), algebras of a given type and algebra homomorphism in Example 5. Now we proceed to define the concepts of homomorphic image (also called quotient), subalgebra, product, terms and equations.

Given two algebras $\mathbf{A} = (A, \{f_A : A^{n_f} \to A\}_{f\in\tau})$ and $\mathbf{B} = (B, \{f_B : B^{n_f} \to B\}_{f\in\tau})$ of type τ, a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $h : A \to B$ such that for every $f \in \tau$ and $a_1, \ldots, a_{n_f} \in A$ we have that $h(f_A(a_1, \ldots, a_{n_f})) = f_B(h(a_1), \ldots, h(a_{n_f}))$. The algebra $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$ if there exists an injective homomorphism from $\mathbf{A}$ to $\mathbf{B}$. The algebra $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$, or $\mathbf{B}$ is a *quotient of* $\mathbf{A}$, if there exists a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$. The algebra $\mathbf{A}$ is isomorphic to $\mathbf{B}$ if there exists a bijective homomorphism $h : A \to B$. Finally, given a set I and algebras $\mathbf{A}_i = (A_i, \{f_{A_i} : A_i^{n_f} \to A_i\}_{f\in\tau})$ of type τ, we define the *product* $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ as the algebra $\mathbf{A} = (\prod_{i\in I} A, \{f_A : A^{n_f} \to A\}_{f\in\tau})$ such that for each $i \in I$, $f \in \tau$ and $a_1, \ldots, a_{n_f} \in A$ we have that $f_A(a_1, \ldots, a_{n_f})(i) = f_{A_i}(a_1(i), \ldots, a_{n_f}(i))$, i.e., operations in $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ are defined componentwise. We say that the product is *finite* if the index set I is finite.

One way of restricting the kind of algebras we want to study is by using equations. Given a set X of *variables*, we define the set $T_\tau(X)$ of *terms of type* τ *over* X as the least set, with respect to inclusion, such that:

i) $X \subseteq T_\tau(X)$.

ii) $f \in T_\tau(X)$ for every nullary function symbol $f \in \tau$.

iii) $f(t_1, \ldots, t_{n_f}) \in T_\tau(X)$ for every function symbol $f \in \tau$ or arity $n_f \geq 1$ and $t_1, \ldots t_{n_f} \in T_\tau(X)$.

Note that the set $T_\tau(X)$ of terms of type τ over X it is an algebra of type τ in which each operation $f_{T_\tau(X)} = f$. Furthermore, we have that the algebra $T_\tau(X)$ is the free U–object over X, where $U : \mathtt{alg}(F_\tau) \to \mathtt{Set}$ is the forgetful functor. In fact, given any F_τ–algebra $\mathbf{A} = (A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$ and a function $g : X \to A$, the unique extension $g^\sharp : T_\tau(X) \to A$ in $\mathtt{alg}(F_\tau)(T_\tau(X), \mathbf{A})$ is defined by induction as:

i) $g^\sharp(x) = g(x)$ for every $x \in X$.

ii) $g^\sharp(f) = f_A$ for every nullary function symbol $f \in \tau$.

iii) $g^\sharp(f(t_1, \ldots, t_{n_f})) = f_A(g^\sharp(t_1), \ldots, g^\sharp(t_{n_f}))$ for every function symbol $f \in \tau$ of arity $n_f \geq 1$ and $t_1, \ldots, t_{n_f} \in T_\tau(X)$.

An *equation of type τ over X* is a pair $(t_1, t_2) \in T_\tau(X) \times T_\tau(X)$. The equation (t_1, t_2) is also denoted as $t_1 \approx t_2$. An algebra $\mathbf{A} = (A, \{f_A : A^{n_f} \to A\}_{f \in \tau})$ of type τ satisfies the equation $t_1 \approx t_2$, denoted as $\mathbf{A} \models t_1 \approx t_2$, if for every function $g \in \mathtt{Set}(X, A)$ we have that $(t_1, t_2) \in \ker(g^\sharp)$, i.e., $g^\sharp(t_1) = g^\sharp(t_2)$. If E is a set of equations we say that $\mathbf{A}$ satisfies E, denoted as $\mathbf{A} \models E$, if for every $t_1 \approx t_2 \in E$ we have that $\mathbf{A} \models t_1 \approx t_2$.

Example 8. Let $\tau = \{\cdot, e\}$ be the type of algebras such that $\cdot$ is a binary function symbol and e is a nullary function symbol, then the set of identities $E = \{x \cdot e \approx x,\ e \cdot x \approx x,\ x \cdot (y \cdot z) \approx (x \cdot y) \cdot z\}$ of type τ over $X \supseteq \{x, y, z\}$ defines the class of algebras of type τ that are monoids, i.e., an algebra $\mathbf{A} = (A, \{\cdot_A, e_A\})$ of type τ is a monoid if and only if $\mathbf{A} \models E$. □

A class of algebras K of the same type is an *equational class* if there exists a family of equations E such that for every algebra $\mathbf{A}$ of type τ we have that $\mathbf{A} \in K$ if and only if $\mathbf{A} \models E$.

Let K be a class of algebras of the same type. We say that K is *closed under subalgebras* if for every $\mathbf{A} \in K$ and every subalgebra $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in K$. We say that K is *closed under homomorphic images*, or *closed under quotients*, if for every $\mathbf{A} \in K$ and every homomorphic image $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in K$. We say that K is *closed under products* if for every set I and $\{\mathbf{A}_i\}_{i \in I} \subseteq K$ we have that $\prod_{i \in I} \mathbf{A}_i \in K$. Similarly, K is *closed under finite products* if it is closed under products for every finite index set I. Classes of algebras of the same type that are closed under homomorphic images, subalgebras and products are called *varieties of algebras*.

We have that equational classes are closed under homomorphic images, subalgebras and products. By celebrated Birkhoff's theorem, see, e.g., [18] and [27, Theorem II.11.9], we also have the converse. That is.

Theorem 9 (Birkhoff). *Let K be a class of algebras of the same type τ. Then K is an equational class if and only if K is a variety.*

Proof. (Sketch) If K is an equational class then K is a variety since equations are preserved by subalgebras, homomorphic images and products [27, Lemma II.11.3.].

Conversely, assume that K is a variety. Then for any infinite set X of variables let $\{T_\tau(X) \overset{e_i}{\twoheadrightarrow} A_i\}_{i\in I}$ be the set of all surjective homomorphisms, up to isomorphism, with codomain in K. Consider the product $\mathbf{A} = \prod_{i\in I} A_i$ and let $e : T_\tau(X) \to A$ be the unique homomorphism such that $\pi_i \circ e = e_i$, where $\pi_i : A \to A_i$ is the projection homomorphism, $i \in I$. Then we have that K is an equational class with $\ker(e)$ as a set of defining equations [27, Lemma II.11.8.]. □

Furthermore, each variety is defined by a unique set of equations if we restrict our attention to sets of equations that are equational theories, cf. [27, Definition II.14.6].

An *equational theory for a type* τ is a class of equations E such that for every set of variables X we have:

i) $(t, t) \in E$ for every $t \in T_\tau(X)$.

ii) $(t_1, t_2) \in E$ implies $(t_2, t_1) \in E$ for every $t_1, t_2 \in T_\tau(X)$.

iii) $(t_1, t_2), (t_2, t_3) \in E$ implies $(t_1, t_3) \in E$ for every $t_1, t_2, t_3 \in T_\tau(X)$.

iv) For every $f \in \tau$ of arity $n_f \geq 1$ and $(t_i, t'_i) \in E$, $1 \leq i \leq n_f$, we have that $(f(t_1, \ldots, t_{n_f}), f(t'_1, \ldots, t'_{n_f})) \in E$.

v) For every $(t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n)) \in E$ and every $r_i \in T_\tau(Y)$, $1 \leq i \leq n$, we have that $(t_1(r_1, \ldots, r_n), t_2(r_1, \ldots, r_n)) \in E$.

Properties i), ii) and iii) are the properties of an equivalence relation, i.e., reflexive, symmetric and transitive, respectively. Property iv) is the congruence property, and property v) is the substitution property.

According to this, by Birkhoff's theorem, we have a one–to–one correspondence between classes of algebras of the same type that are varieties and equational theories. A similar result can be obtained for the case of *pseudovarieties of algebras*, which are classes of finite algebras closed under homomorphic images, subalgebras and finite products. Pseudovarieties of algebras, also known as *varieties of finite algebras*, are exactly directed unions of equational classes of finite algebras [12, 14, 37]. Categorical versions of Birkhoff's theorem for varieties of algebras and pseudovarieties of algebras, together with their local versions, will be proved in Chapter 5.

1.3 Coalgebras for an endofunctor

This section is similar (dual) to Section 1.1. We define the general notion of coalgebras for an endofunctor, fix some notation and show some examples. See [76] for more details and examples on this subject. We start by defining the category of coalgebras for an endofunctor.

Definition 10. Let $\mathcal{C}$ be a category and $G : \mathcal{C} \to \mathcal{C}$ an endofunctor on $\mathcal{C}$. We define the category $\texttt{coalg}(G)$ of *G–coalgebras* as the category whose objects are pairs $\mathbf{X} = (X, \beta_X)$ such that $X \in \mathcal{C}$ and $\beta_X \in \mathcal{C}(X, G(X))$ and a morphism $\texttt{coalg}(G)(\mathbf{X}, \mathbf{Y})$, with $\mathbf{Y} = (Y, \beta_Y)$, is a morphism $f \in \mathcal{C}(X, Y)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Pairs (X, β_X) as above are called *G–coalgebras*.

By using the following lemma, we can equivalently define deterministic automata as coalgebras for an endofunctor.

Lemma 11. *Let X, Y and A be sets, $f \in \texttt{Set}(X, Y)$, $g_1 \in \texttt{Set}(X, X^A)$ and $g_2 \in \texttt{Set}(Y, Y^A)$ then the commutativity of the following two diagrams are equivalent:*

$$\begin{tikzcd} Y^A & X^A \arrow[l, "\texttt{Set}(A,f)"'] \\ Y \arrow[u, "g_2"] & X \arrow[l, "f"] \arrow[u, "g_1"'] \end{tikzcd} \qquad \begin{tikzcd} Y & X \arrow[l, "f"'] \\ A \times Y \arrow[u, "g_2'"] & A \times X \arrow[l, "id_A \times f"] \arrow[u, "g_1'"'] \end{tikzcd}$$

where $g_i'(a, s) = g_i(s)(a)$.

Proof. Let $x \in X$ and $a \in A$, then we have:

$$\begin{aligned} (\texttt{Set}(A, f) \circ g_1)(x)(a) = (g_2 \circ f)(x)(a) &\Leftrightarrow \texttt{Set}(A, f)(g_1(x))(a) = g_2(f(x))(a) \\ &\Leftrightarrow (f \circ g_1(x))(a) = g_2'(a, f(x)) \\ &\Leftrightarrow f(g_1(x)(a)) = g_2'((id_A \times f)(a, x)) \\ &\Leftrightarrow f(g_1'(a, x)) = g_2'((id_A \times f)(a, x)) \\ &\Leftrightarrow (f \circ g_1')(a, x) = (g_2' \circ (id_A \times f))(a, x) \end{aligned}$$

□

The previous lemma also follows from the general fact that the functor $A \times \_$ is left adjoint to the functor $\texttt{Set}(A, \_)$ (see Example 18).

Now we show some examples of coalgebras for an endofunctor.

Example 12 (Deterministic automata as coalgebras)**.** Let A be a set of symbols, i.e., an *alphabet*. Consider the functor $G : \texttt{Set} \to \texttt{Set}$ given by $G(\_) = \texttt{Set}(A, \_)$. By the previous lemma, we have that deterministic automata are the same as G–coalgebras. □

Example 13 (Transition systems with output)**.** Let A be a set of symbols and consider the functor $G : \mathtt{Set} \to \mathtt{Set}$ defined on objects as $G(X) = A \times X$ and $G(f) = id_A \times f$ on morphisms. A G–coalgebra is a *transition system with outputs on* A, i.e., a pair (X, β_X) such that $\beta_X \in \mathtt{Set}(X, A \times X)$. If $\beta_X(x_1) = (a, x_2)$ then a is considered as the *output* from the state x_1, and x_2 is the *next state*, this is usually depicted as $x_1 \xrightarrow{a} x_2$. If (X, β_X) and (Y, β_Y) are two transition systems with outputs on A then a morphism $f \in \mathtt{coalg}(G)((X, \beta_X), (Y, \beta_Y))$ is a function $f \in \mathtt{Set}(X, Y)$ such that for every $x_1, x_2 \in X$ and $a \in A$ with $x_1 \xrightarrow{a} x_2$, i.e., $\beta_X(x_1) = (a, x_2)$, we have that $f(x_1) \xrightarrow{a} f(x_2)$, i.e., $\beta_Y(f(x_1)) = (a, f(x_2))$. $\square$

If $\mathcal{C}$ is a category that has an initial object $Y \in \mathcal{C}$, then also $\mathtt{coalg}(F)$ has an initial object, namely $(Y, !_{GY})$. In the case of the previous example, the initial transition system with outputs on A is the transition system $(\emptyset, !_{A\times\emptyset})$.

Similarly to the previous section, we now define the dual concept of a free algebra, that is, a cofree coalgebra.

Definition 14. Let $\mathcal{C}$ be a category, $G : \mathcal{C} \to \mathcal{C}$ an endofunctor on $\mathcal{C}$ and $Y \in \mathcal{C}$. The *cofree* G*–coalgebra over* Y *generators* is the cofree U–object over Y (see, Definition 2), where $U : \mathtt{coalg}(G) \to \mathcal{C}$ is the *forgetful functor* defined on objects as $U(Y, \beta_Y) = Y$ for every $(Y, \beta_Y) \in \mathtt{coalg}(G)$ and $U(f) = f$ for every morphism f in $\mathtt{coalg}(F)$.

As examples of cofree coalgebras we have the following.

Example 15 (Example 12 continued)**.** Let A be a set of symbols, i.e., an *alphabet*. Consider the functor $G : \mathtt{Set} \to \mathtt{Set}$ given by $G(\_) = \mathtt{Set}(A, \_)$ and the set $2 = \{0, 1\}$. Then we have that the cofree G–coalgebra over 2 is given by $(2^{A^*}, \varsigma)$, where $\varsigma : 2^{A^*} \to \mathtt{Set}(A, 2^{A^*})$ is defined as $\varsigma(L)(a)(w) = L(aw)$. In this case the morphism $\epsilon_2 : 2^{A^*} \to 2$ is defined as $\epsilon_2(L) = L(\varepsilon)$.

Let $\mathbf{X} = (X, \beta_X)$ be a deterministic automaton on A, where $\beta_X : X \to X^A$, and let $c : X \to 2$ be a function, which we think of as a two–colouring of the states. Then, by cofreeness of $(2^{A^*}, \varsigma)$, the unique morphism $c^\flat \in \mathtt{coalg}(G)(\mathbf{X}, (2^{A^*}, \varsigma))$ such that $\epsilon_2 \circ c^\flat = c$ is defined as:

$$c^\flat(x)(w) = \begin{cases} c(x) & \text{if } w = \varepsilon \\ c^\flat(\beta_X(x)(a))(u) & \text{if } w = au. \end{cases}$$

In this case, $c^\flat(x)$ is exactly the language the automaton $\mathbf{X}$ accepts from the state x with the colouring c.

Note that the G–algebra $(2^{A^*}, \varsigma')$, where $\varsigma' : 2^{A^*} \to \mathtt{Set}(A, 2^{A^*})$ is defined as $\varsigma'(L)(a)(w) = L(wa)$, is also a cofree G–coalgebra over 2, which is isomorphic in $\mathtt{coalg}(G)$ to $(2^{A^*}, \varsigma)$. $\square$

Example 16 (Example 13 continued)**.** Let A be a set of symbols and consider the functor $G : \mathtt{Set} \to \mathtt{Set}$ defined on objects as $G(X) = A \times X$ and $G(f) = id_A \times f$ on morphisms. The cofree G–coalgebra on 1 generator is the G–coalgebra $(A^\omega, \langle head, tail\rangle)$ such that $\langle head, tail\rangle : A^\omega \to A \times A^\omega$ is defined as $\langle head, tail\rangle(f) =$

$(head(f), tail(f))$, where $head(f) = f(0)$ and $tail(f)(n) = f(n+1)$, $n \in \omega$. In this case the morphism ϵ_1 is the trivial function in $\mathtt{Set}(A^\omega, 1)$. Elements in A^w are called *streams on* A.

Let $\mathbf{X} = (X, \beta_X)$ be a G–coalgebra, where $\beta_X : X \to A \times X$, and let $c \in \mathtt{Set}(X, 1)$ be the unique morphism into 1. Then, by cofreeness of $(A^\omega, \langle head, tail\rangle)$, the unique morphism $c^\flat \in \mathtt{coalg}(G)(\mathbf{X}, (A^\omega, \langle head, tail\rangle))$ such that $\epsilon_1 \circ c^\flat = c$ is given by:

$$c^\flat(x)(0) = \pi_1(\beta_X(x)) \quad \text{and} \quad c^\flat(x)(n+1) = c^\flat(\pi_2(\beta_X(x)))(n).$$

That is, $c_\flat(x)$ is the stream on A we obtain from the state x.

In general, the cofree G–coalgebra on Y generators is the G–coalgebra $((Y \times A)^\omega, \langle h, t\rangle)$ such that $\langle h, t\rangle : (Y \times A)^\omega \to A \times (Y \times A)^\omega$ is given by $\langle h, t\rangle(f) = (h(f), t(f))$, where $h(f) = \pi_2(f(0))$ and $t(f)(n) = f(n+1)$, $n \in \omega$, and the morphism $\epsilon_Y \in \mathtt{Set}((Y \times A)^\omega, Y)$ is given by $\epsilon_Y(f) = \pi_1(f(0))$. Here $\pi_1 : Y \times A \to Y$ and $\pi_2 : Y \times A \to A$ are the projection maps. $\square$

Note that, in general, for any endofunctor $G : \mathtt{Set} \to \mathtt{Set}$, the cofree G–coalgebra over $X \in \mathtt{Set}$ is essentially the same as the final ${}_XG$–coalgebra, where ${}_XG$ is the functor ${}_XG(\_) := X \times G(\_)$. For instance, in the setting of the previous example, $(2^{A^*}, \langle \epsilon_2, \varsigma\rangle)$ is the final ${}_2G$–coalgebra, where $\langle \epsilon_2, \varsigma\rangle : 2^{A^*} \to 2 \times 2^{A^*}$ is obtained from ϵ_2 and ς from the universal property of the product $2 \times 2^{A^*}$.

Another observation regarding the previous examples is how the coinduction principle is related with cofree coalgebras (this notion comes from the observation that the induction principle is related with free algebras). In fact, the definition of the $c^\flat$ above is a coinductive one. The previous fact is represented by the slogan "coinduction=use of finality for coalgebras", see [55].

1.4 Adjunctions, contravariant adjunctions and duality

In this section, we define the concepts of an adjunction, a contravariant adjunction (also called a dual adjunction) and a duality between two categories. We start by defining the concept of an adjunction.

Definition 17. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that F is *left adjoint of* G or that G is *right adjoint* of F, denoted as $F \dashv G$, if for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there is a bijection $\Phi_{X,Y} : \mathcal{D}(F(X), Y) \to \mathcal{C}(X, G(Y))$ which is natural in both X and Y. The latter means that for every morphism $f \in \mathcal{C}(X, X')$ and $g \in \mathcal{D}(Y, Y')$ the following two diagrams commute:

$$\begin{tikzcd}
\mathcal{D}(F(X),Y) \arrow[r, "\Phi_{X,Y}"] & \mathcal{C}(X,G(Y)) & \mathcal{D}(F(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\mathcal{D}(F(X),g)"] & \mathcal{C}(X,G(Y)) \arrow[d, "\mathcal{C}(X,G(g))"'] \\
\mathcal{D}(F(X'),Y) \arrow[r, "\Phi_{X',Y}"'] \arrow[u, "\mathcal{D}(F(f),Y)"] & \mathcal{C}(X',G(Y)) \arrow[u, "\mathcal{C}(f,G(Y))"'] & \mathcal{D}(F(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathcal{C}(X,G(Y'))
\end{tikzcd}$$

Note that the functor F on the expression $\mathcal{D}(F(X),Y)$ is on the left and the functor G on the expression $\mathcal{C}(X,G(Y))$ is on the right, which helps to remember that F is left adjoint of G and that G is right adjoint of F.

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, the following properties are equivalent to the property of $F \dashv G$ (see, e.g., [4, 66]):

i) There exist natural transformations $\eta : Id_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow Id_{\mathcal{D}}$ such that $G\epsilon \circ \eta G = Id_G$ and $\epsilon F \circ F\eta = Id_F$. That is, for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have that $G(\epsilon_Y) \circ \eta_{G(Y)} = Id_{G(Y)}$ and $\epsilon_{F(X)} \circ F(\eta_X) = Id_{F(X)}$. The previous two equalities are known as the *triangle identities*.

ii) There exists a natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow GF$ that satisfies the following universal property (cf. Definition 1):

(UP) For every $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and every morphism $f \in \mathcal{C}(X, G(Y))$ there exists a unique morphism $f^\sharp \in \mathcal{D}(F(X), Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
GF(X) \arrow[r, "G(f^\sharp)"] & G(Y) \\
X \arrow[u, "\eta_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The morphism $f^\sharp$ is called the *extension* of f.

iii) There exists a natural transformation $\epsilon : FG \Rightarrow Id_{\mathcal{D}}$ that satisfies the following universal property (cf. Definition 2):

(UP) For every $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and every morphism $f \in \mathcal{D}(F(X), Y)$ there exists a unique morphism $f^\flat \in \mathcal{C}(X, G(Y))$ such that the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f^\flat)"] \arrow[dr, "f"'] & FG(Y) \arrow[d, "\epsilon_Y"] \\
& Y
\end{tikzcd}$$

The natural transformations η and ϵ above are called *unit* and *counit* of the adjunction, respectively. As examples of adjunctions we have the following.

Example 18. Let $\mathcal{C} = \mathcal{D} = \mathtt{Set}$, $A \in \mathtt{Set}$ and consider the functors $F : \mathtt{Set} \to \mathtt{Set}$ and $G : \mathtt{Set} \to \mathtt{Set}$ given by $F(X) = A \times X$ and $G(Y) = Y^A$. Then we have that $F \dashv G$ which can be verified by any of the following facts:

i) The bijective correspondence $\Phi_{X,Y} : \mathtt{Set}(A \times X, Y) \to \mathtt{Set}(X, Y^A)$ defined as $\Phi_{X,Y}(f) = \hat{f}$, where $\hat{f}(x)(a) := f(a, x)$, which is natural in both X and Y.

ii) The natural transformation $\eta : Id_{\mathtt{Set}} \Rightarrow GF$ such that every $\eta_X : X \to (A \times X)^A$ is defined as $\eta_X(x)(a) = (a, x)$, which satisfies the universal property above.

iii) The natural transformation $\epsilon : FG \Rightarrow Id_{\mathtt{Set}}$ such that every $\epsilon_Y : A \times Y^A \to Y$ is defined as $\epsilon_Y(a, f) = f(a)$, which satisfies the universal property above. □

Example 19. For a given type of algebras τ, let $F_\tau : \mathtt{Set} \to \mathtt{Set}$ be its corresponding polynomial functor (see Example 5). Let $\mathcal{C} = \mathtt{Set}$, $\mathcal{D} = \mathtt{alg}(F_\tau)$, $U : \mathtt{alg}(F_\tau) \to \mathtt{Set}$ be the forgetful functor and $T_\tau : \mathtt{Set} \to \mathtt{alg}(F_\tau)$ be the functor such that $T_\tau(X)$ is the τ–algebra of τ–terms on X. Then $T_\tau \dashv U$ which can be easily verified by considering the natural transformation $\eta : Id_{\mathtt{Set}} \Rightarrow UT_\tau$ such that every $\eta_X : X \Rightarrow UT_\tau(X)$ is the inclusion function, which satisfies the universal property above. □

Now we turn to define the concept of a contravariant adjunction which is the special case of an adjunction in which we replace the category $\mathcal{D}$ by $\mathcal{D}^{op}$. Given two contravariant functors $F\colon \mathcal{C} \rightarrowtail \mathcal{D}$ and $G\colon \mathcal{D} \rightarrowtail \mathcal{C}$, we say that they form a *contravariant adjunction*, also called a *dual adjunction*, between F and G, denoted by $F \dashv\vdash G$ or $G \dashv\vdash F$, if for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is a bijective correspondence $\Phi_{X,Y} : \mathcal{D}(Y, F(X)) \to \mathcal{C}(X, G(Y))$ which is natural in both X and Y, i.e., for every morphism $f \in \mathcal{C}(X, X')$ and $g \in \mathcal{D}(Y, Y')$ the following two diagrams commute:

$$
\begin{tikzcd}
\mathcal{D}(Y,F(X)) \arrow[r, "\Phi_{X,Y}"] & \mathcal{C}(X,G(Y)) \\
\mathcal{D}(Y,F(X')) \arrow[u, "\mathcal{D}(Y,F(f))"'] \arrow[r, "\Phi_{X',Y}"'] & \mathcal{C}(X',G(Y)) \arrow[u, "\mathcal{C}(f,G(Y))"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(Y,F(X)) \arrow[r, "\Phi_{X,Y}"] & \mathcal{C}(X,G(Y)) \\
\mathcal{D}(Y',F(X)) \arrow[u, "\mathcal{D}(g,F(X))"'] \arrow[r, "\Phi_{X,Y'}"'] & \mathcal{C}(X,G(Y')) \arrow[u, "\mathcal{C}(X,G(g))"]
\end{tikzcd}
$$

Observe that both F and G are on the right side, i.e., on the codomain of the hom-sets. Such a contravariant adjunction can be equivalently defined by two units $\eta^{FG}\colon Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF}\colon Id_{\mathcal{C}} \Rightarrow GF$ that satisfy the triangle identities $G\eta^{FG} \circ \eta^{GF}_G = Id_G$ and $F\eta^{GF} \circ \eta^{FG}_F = Id_F$.

The following are examples of contravariant adjunctions.

Example 20. Consider the case $\mathcal{D} = \mathcal{C} = \mathtt{Set}$ and the contravariant functors $F, G : \mathtt{Set} \rightarrowtail\!\!\!\!\!\times \mathtt{Set}$ such that $F(X) = G(X) = \mathtt{Set}(X, A)$, where A is a fixed set. Then we have that $F \dashv\vdash G$ since the bijective correspondence $\Phi_{X,Y} : \mathtt{Set}(Y, \mathtt{Set}(X, A)) \to \mathtt{Set}(X, \mathtt{Set}(Y, A))$ given by $\Phi_{X,Y}(f)(x)(y) := f(y)(x)$, $x \in X$ and $y \in Y$, is natural in both X and Y. □

Example 21. Consider the case $\mathcal{D} = \mathcal{C} = \mathtt{Vec}_{\mathbb{K}}$, where $\mathtt{Vec}_{\mathbb{K}}$ is the category of vector spaces over the field $\mathbb{K}$ and linear maps, and the contravariant functors $F, G : \mathtt{Vec}_{\mathbb{K}} \rightarrowtail\!\!\!\!\!\times \mathtt{Vec}_{\mathbb{K}}$ such that $F(X) = G(X) = \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K})$. That is, $F(X) = G(X)$ is the dual space of X. Then we have that $F \dashv\vdash G$ since the bijective correspondence $\Phi_{X,Y} : \mathtt{Vec}_{\mathbb{K}}(Y, \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K})) \to \mathtt{Vec}_{\mathbb{K}}(X, \mathtt{Vec}_{\mathbb{K}}(Y, \mathbb{K}))$ given by $\Phi_{X,Y}(f)(x)(y) := f(y)(x)$, $x \in X$ and $y \in Y$, is natural in both X and Y. □

Example 22. Consider the category $\mathcal{D} = \mathtt{BA}$ of Boolean algebras and Boolean algebra homomorphisms (see, e.g., [27]), $\mathcal{C} = \mathtt{Set}$ and the contravariant functors $F : \mathtt{Set} \rightarrowtail\!\!\!\!\!\times \mathtt{BA}$ and $G : \mathtt{BA} \rightarrowtail\!\!\!\!\!\times \mathtt{Set}$ such that $F(X) = \mathtt{Set}(X, 2)$ and $G(Y) = \mathtt{BA}(Y, 2)$. That is, $F(X)$ is the Boolean algebra of subsets of X with the usual set theoretic operations and $G(Y)$ is the set of ultrafilters of Y. Then we have that $F \dashv\vdash G$ since the bijective correspondence $\Phi_{X,Y} : \mathtt{BA}(Y, \mathtt{Set}(X, 2)) \to \mathtt{Set}(X, \mathtt{BA}(Y, 2))$ given by $\Phi_{X,Y}(f)(x)(y) := f(y)(x)$, $x \in X$ and $y \in Y$, is natural in both X and Y. We can also justify that $F \dashv\vdash G$ by defining the units $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ for every $X \in \mathtt{Set}$ and $Y \in \mathtt{BA}$ as the morphisms $\eta^{FG}_Y \in \mathtt{BA}(Y, \mathtt{Set}(\mathtt{BA}(Y, 2), 2))$ and $\eta^{GF}_X \in \mathtt{Set}(X, \mathtt{BA}(\mathtt{Set}(X, 2), 2))$ such that $\eta^{FG}_Y(y)(u) = u(b)$ and $\eta^{GF}_X(x)(f) = f(x)$. □

We close this section by studying the concept of a duality between two categories, which is a special case of a contravariant adjunction. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we say that they are *dual categories* if there exists contravariant functors $F : \mathcal{C} \rightarrowtail\!\!\!\!\!\times \mathcal{D}$ and $G : \mathcal{D} \rightarrowtail\!\!\!\!\!\times \mathcal{C}$ and natural isomorphisms $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ such that $G\eta^{FG} \circ \eta^{GF}_G = Id_G$ and $F\eta^{GF} \circ \eta^{FG}_F = Id_F$. We now illustrate the dualities we will mention in this thesis.

Example 23 (Duality between $\mathtt{Set}$ and $\mathtt{CABA}$)**.** We have that the category $\mathtt{Set}$ is dual to the category $\mathtt{CABA}$ of complete atomic Boolean algebras and complete Boolean algebra morphisms.

Recall that a Boolean algebra B is *complete* if any subset S of B has a join $\bigvee S$. An element $a \in B$ is an *atom* if $b < a$ implies $b = 0$ or, equivalently, if $a = b \vee c$ implies $a = b$ or $a = c$. We say that B is *atomic* if every element in B is the join of atoms (not necessarily a finite join). We denote the set of atoms of B as $\mathrm{At}(B)$. Note that $\mathrm{At}(B)$ is isomorphic, as a set, to $\mathtt{CABA}(B, 2)$. We have that every object $B \in \mathtt{CABA}$ is isomorphic to $\mathcal{P}(X)$ for $X = \mathrm{At}(B)$, where $\mathcal{P}(X)$ denotes the power set of X, i.e., the set of all subsets of X, which is an object in $\mathtt{CABA}$ with the operations of union, intersection and complement with respect to X. Note that the atoms of $\mathcal{P}(X)$ are the singleton sets $\{x\}$, $x \in X$.

We have that the category $\mathtt{Set}$ and $\mathtt{CABA}$ are dual categories. In fact, the pair of contravariant functors that define the duality between $\mathtt{Set}$ and $\mathtt{CABA}$ are given by $\mathtt{Set}(\_, 2) : \mathtt{Set} \rightarrowtail\!\!\!\!\!\times \mathtt{CABA}$ and $\mathtt{CABA}(\_, 2) : \mathtt{CABA} \rightarrowtail\!\!\!\!\!\times \mathtt{Set}$. Note that $\mathtt{Set}(X, 2)$ is isomorphic in $\mathtt{CABA}$ to $2^X \cong \mathcal{P}(X)$, where 2 is the two–element set, and $\mathtt{CABA}(B, 2)$ is isomorphic in $\mathtt{Set}$ to the set $\mathrm{At}(B)$ of atoms of B, where 2 is the two–element Boolean algebra.

Basic facts about Boolean algebras can be found in [27, IV.1] and facts for the duality between $\mathtt{Set}$ and $\mathtt{CABA}$ can be found in [33, 10.24 Theorem.]. □

Example 24 (Stone duality between $\mathtt{BA}$ and $\mathtt{St}$)**.** The category $\mathtt{BA}$ of Boolean algebras with Boolean algebra morphisms is dual to the category $\mathtt{St}$ of Stone spaces with continuous functions. A *Stone space*, also called a *Boolean space*, is a topological space which is Hausdorff, compact and has a basis of clopen sets.

The duality between $\mathtt{BA}$ and $\mathtt{St}$ is given by the contravariant functors $\mathtt{BA}(\_, 2) : \mathtt{BA} \leftarrowtail\!\!\!\rightarrow \mathtt{St}$ and $\mathtt{St}(\_, 2) : \mathtt{St} \leftarrowtail\!\!\!\rightarrow \mathtt{BA}$, where 2 represents the two–element Boolean algebra in $\mathtt{BA}$ and the two–element discrete space in $\mathtt{St}$, respectively. In this case, topology $\mathtt{BA}(A, 2)$, for a given $A \in \mathtt{BA}$, is the subspace topology of the product 2^X, where 2 has the discrete topology, and the Boolean algebra operations in $\mathtt{St}(X, 2)$, for a given $X \in \mathtt{St}$, are the usual set theoretic operations of union, intersection and complement. Note that $\mathtt{BA}(A, 2)$ is the set of ultrafilters of B and that $\mathtt{St}(X, 2)$ is the set of clopens of X. More details of this duality can be found in, e.g., [27, IV.4]. □

Example 25 (Duality between $\mathtt{Poset}$ and $\mathtt{AlgCDL}$)**.** We have that the category $\mathtt{Poset}$ of partially ordered sets with order preserving functions is dual to the category $\mathtt{AlgCDL}$ of algebraic completely distributive lattices with complete lattice homomorphisms.

Recall that, see [33], an element k in a complete lattice L is *compact* if $k \leq \bigvee S$ implies that $k \leq \bigvee T$ for some finite subset T of S. A complete lattice L is *algebraic* if every element in L is the join of compact elements. Hence, objects in $\mathtt{AlgCDL}$ are complete distributive lattices that are algebraic and satisfy the infinite distributive laws.

The duality between $\mathtt{Poset}$ and $\mathtt{AlgCDL}$ is given by the contravariant functors $\mathtt{Poset}(\_, \mathbf{2}_c) : \mathtt{Poset} \leftarrowtail\!\!\!\rightarrow \mathtt{AlgCDL}$ and $\mathtt{AlgCDL}(\_, \mathbf{2}_c) : \mathtt{AlgCDL} \leftarrowtail\!\!\!\rightarrow \mathtt{Poset}$, where $\mathbf{2}_c$ is the two–element chain (which is an object in $\mathtt{Poset}$ as well as an object in $\mathtt{AlgCDL}$). Note that for any $\mathbf{X} \in \mathtt{Poset}$ we have that $\mathtt{Poset}(\mathbf{X}, \mathbf{2}_c)$ is (isomorphic to) the object in $\mathtt{AlgCDL}$ that consists of all downsets of $\mathbf{X}$, with the usual operations of intersection and union, and that for any object L in $\mathtt{AlgCDL}$ we have that $\mathtt{AlgCDL}(L, \mathbf{2}_c)$ is (isomorphic to) the object in $\mathtt{Poset}$ that consists of all completely join–prime elements in L with the order inherited from L. An element $p \in L$ is *completely join–prime* if $p \leq \bigvee S$ implies $p \leq s$ for some $s \in S$.

Basic facts about partially ordered sets and (algebraic completely distributive) lattices can be found in [33]. Facts about the duality between $\mathtt{Poset}$ and $\mathtt{AlgCDL}$ are mentioned in [33, 10.29 Theorem.]. □

Example 26 (Duality between $\mathtt{JSL}$ and $\mathtt{StJSL}$)**.** The category $\mathtt{JSL}$ of join semilattices with 0 and join preserving functions that also preserve 0 is dual to the category of $\mathtt{StJSL}$ of join semilattices with 0 that have a Stone topology (also called Boolean topology in [27, IV Definition 4.1.]) and join preserving continuous functions that preserve 0. A topological space has a *Stone topology* if it is Hausdorff, compact and has a basis of clopen subsets.

The duality between $\mathtt{JSL}$ and $\mathtt{StJSL}$ is given by the contravariant functors $\mathtt{JSL}(\_, \mathbf{2}) : \mathtt{JSL} \to \mathtt{StJSL}$ and $\mathtt{StJSL}(\_, \mathbf{2}) : \mathtt{StJSL} \to \mathtt{JSL}$, where $\mathbf{2}$ is the two–element join semilattice (which is an element in $\mathtt{JSL}$ as well as an element in $\mathtt{StJSL}$). For any object L in $\mathtt{JSL}$, the object $\mathtt{JSL}(L, \mathbf{2})$ has its structure inherited by the product $\prod_{l \in L} \mathbf{2}$, i.e., the join operation is componentwise and the topology is the subspace topology of the product $\prod_{l \in L} \mathbf{2}$. Similarly, for every object T in $\mathtt{StJSL}$, the object $\mathtt{StJSL}(T, \mathbf{2})$ has the operation of join defined componetwise. Elements in $\mathtt{JSL}(L, \mathbf{2})$ and $\mathtt{StJSL}(T, \mathbf{2})$ are called in [53] *characters* of L and T, respectively.

Facts and general properties of this duality can be found in [53, 32]. □

Example 27 (Duality between $\mathtt{Vec}_{\mathbb{K}}$ and $\mathtt{StVec}_{\mathbb{K}}$)**.** For a finite field $\mathbb{K}$, denote by $\mathtt{Vec}_{\mathbb{K}}$ the category of vector spaces over $\mathbb{K}$ with linear maps and denote by $\mathtt{StVec}_{\mathbb{K}}$ the category of topological vector spaces over $\mathbb{K}$ with a Stone topology with linear maps that are continuous.

The duality between $\mathtt{Vec}_{\mathbb{K}}$ and $\mathtt{StVec}_{\mathbb{K}}$ is given by the contravariant functors $\mathtt{Vec}_{\mathbb{K}}(\_, \mathbb{K}) : \mathtt{Vec}_{\mathbb{K}} \to \mathtt{StVec}_{\mathbb{K}}$ and $\mathtt{StVec}_{\mathbb{K}}(\_, \mathbb{K}) : \mathtt{StVec}_{\mathbb{K}} \to \mathtt{Vec}_{\mathbb{K}}$, where $\mathbb{K}$ is the one dimentional space over $\mathbb{K}$ (which is an element in $\mathtt{Vec}_{\mathbb{K}}$ as well as an element in $\mathtt{StVec}_{\mathbb{K}}$). For any object S in $\mathtt{Vec}_{\mathbb{K}}$, the object $\mathtt{Vec}_{\mathbb{K}}(S, \mathbb{K})$ has its structure inherited by the product $\prod_{s \in S} \mathbb{K}$, i.e., the operations are componentwise and the topology is the subspace topology of the product $\prod_{s \in S} \mathbb{K}$. Similarly, for every object T in $\mathtt{StVec}_{\mathbb{K}}$, the object $\mathtt{StVec}_{\mathbb{K}}(T, \mathbb{K})$ has componentwise operations.

Facts and general properties of this duality can be found in [16, 32]. □

For more (general) dualities that will lead to more applications of the results presented in this thesis the reader is referred to, e.g., [32, 29].

Chapter 2

Equations and coequations for deterministic automata

The concept of a deterministic automaton has been broadly studied in computer science as a model or machine which allows us to classify finite sequences of symbols, called *words*, from a given set considered as an *alphabet* [64, 77, 35, 36]. A *deterministic automaton* on a set A, where A is called its *alphabet*, is a pair (X, α_X) such that X is a set, whose elements are called *states*, and $\alpha_X : A \times X \to X$ is a function, called its *transition function*. According to this, if the automaton is at the state $x \in X$ and we input the symbol $a \in A$, then the automaton makes a transition to the state $\alpha_X(a, x)$. The main purpose of studying deterministic automata is to classify languages (sets of words on A) that can be "recognized" by *finite deterministic automata*, i.e., deterministic automata with a finite set of states.

It is worth mentioning that in our definition of deterministic automata we do not consider accepting/final states nor an initial state, contrary to classical definitions such as in [64]. The main reason for this is to consider deterministic automata as algebras as well as coalgebras. Nevertheless, accepting states can be considering when cofree automata on 2 generators are studied and initial states can be considered when free automata on 1 generator are studied.

Deterministic automata can be studied by using categorical and algebraic approaches [35, 36, 13, 79], from which many of the classical concepts defined in automata theory have a mathematical and equivalent counterpart. This not only allows us to formalize and understand all the concepts from a different perspective, but also to study mathematical concepts for the case of deterministic automata such as equations and coequations.

The study of equations is a subject that has been studied in logic and more specifically in universal algebra [27, 18]. As every deterministic automaton on an alphabet A is an algebra of a certain type (namely, an algebra of type $\tau = A$ in which every function symbol $a \in A$ is a unary function symbol), we get a canonical definition of equations for deterministic automata. In this chapter, we mainly focus on equations with only one variable as in [13, 79].

Categorical approaches generalize the notion of equations, which is a concept that is studied for algebras for a functor and particularly it can be studied for deterministic automata. From this approach, a categorical dual notion of equations, which are called *coequations*, is naturally obtained for coalgebras for an endofunctor, which is another equivalent point of view for deterministic automata.

Intuitively, coequations allow us to classify coalgebras by restricting their behaviour, which in case of deterministic automata is given by the set of languages that can be accepted. This is an interesting notion, because it will allow us the study of classes of automata by restricting the languages they can accept. Which is a similar but different phenomenon as in the case of equations.

In this chapter, we focus on the study of equations and coequations for deterministic automata. Equations and coequations for deterministic automata have been studied in, e.g., [13, 79]. We will provide the necessary (categorical) definitions, illustrate both concepts with some examples and some basic facts. Then we show a correspondence between equations and coequations for deterministic automata and obtain the duality result between equations and coequations in [13], whose generalization will be shown in Chapter 4. This duality result follows from the duality between `Set` and `CABA` by suitable restrictions on each category, namely, by considering surjective monoid homomorphisms with free domain on the `Set` side. Such monoid homomorphisms come from surjective maps $A^* \to \texttt{trans}(X, \alpha_X)$ between the monoid A^* and the transition monoid $\texttt{trans}(X, \alpha_X)$ of a deterministic automaton (X, α_X) on A. Additionally, we will study classes of deterministic automata that can be represented by equations as well as coequations, which are the regular varieties of deterministic automata [79].

We defined deterministic automata on A, see Example 4, as F–algebras for the functor $F : \texttt{Set} \to \texttt{Set}$ defined as $F(X) = A \times X$ on objects and $F(f) = id_A \times f$ on morphisms. In the case that A and X are finite sets, we can draw the diagram of the automaton (X, α_X) which is the diagram with nodes in X and an arrow from a node x_1 to a node x_2 with label a, as in $x_1 \xrightarrow{a} x_2$, for every $a \in A$ and $x_1, x_2 \in X$ such that $\alpha_X(a, x_1) = x_2$. The following is an example of this notation.

Example 28. If $A = \{a, b\}$ and $X = \{x_1, x_2, x_3\}$, then the following diagram

$$\begin{tikzcd} x_1 \arrow[r, "{a,b}"] & x_2 \arrow[loop above, "a"] \arrow[r, bend left, "b"] & x_3 \arrow[loop above, "a"] \arrow[l, bend left, "b"] \end{tikzcd}$$

represents the automaton (X, α_X) such that its transition function is given by $\alpha_X(a, x_1) = \alpha_X(b, x_1) = x_2$, $\alpha_X(a, x_2) = x_2$, $\alpha_X(b, x_2) = x_3$, $\alpha_X(a, x_3) = x_3$ and $\alpha_X(b, x_3) = x_2$. □

We denote by A^* the set of all words with symbols in A. That is, every element $w \in A^*$ is of the form $w = a_1 \cdots a_n$, $n \in \mathbb{N}$, where each $a_i \in A$, $1 \le i \le n$. In the particular case that $n = 0$, we obtain the *empty word* which we denote by ε.

Notation. Given a deterministic automaton (X, α_X) on A, $w \in A^*$ and $x \in X$, we define the state $w(x) \in X$ by induction as follows:

$$w(x) = \begin{cases} x & \text{if } w = \epsilon, \\ \alpha_X(a, u(x)) & \text{if } w = au,\ u \in A^*,\ a \in A \end{cases}$$

thus $w(x)$ is the state we reach from x by processing the word w from right to left, cf. Example 7.

Example 29 (Example 28 continued)**.** If we consider the automaton (X, α_X) given in Example 28, then we have the following:

$$aab(x_1) = x_2, \quad aaaba(x_1) = x_3, \quad bab^8(x_3) = x_2, \quad \varepsilon(x_2) = x_2, \quad a^3ba^7b^3(x_1) = x_3.$$

An easy way to remember how to do the previous calculations is by introducing parenthesis for each symbol in A. For example:

$$aab(x_1) = aa(b(x_1)) = aa(x_2) = a(a(x_2)) = a(x_2) = x_2.$$

□

Remark. Using the previous notation, a homomorphism of automata $h : (X, \alpha_X) \to (Y, \alpha_Y)$ is a function $h : X \to Y$ such that for every $a \in A$ and $x \in X$ we have $h(a(x)) = a(h(x))$. This notation coincides with the notation used for algebras of type $\tau = A$ where each symbol in A is a unary function symbol (see Example 5). In fact, automata on A are exactly algebras of the type τ just described.

2.1 Equations for deterministic automata

Now we turn our attention to the study of equations for deterministic automata, as defined in [13], which, informally, are pairs of words $(u, v) \in A^* \times A^*$. This notion of equations as pairs of words in $A^* \times A^*$ correspond exactly to equations on one variable for algebras of the type $\tau = A$ in which every element in $\tau = A$ is a unary function symbol. That is, a pair $(u, v) \in A^* \times A^*$ corresponds to the equation $\forall x\ u(x) = v(x)$, also denoted as $u(x) \approx v(x)$, studied in universal algebra [27, Definition II.11.1]. An automaton satisfies the equation $(u, v) \in A^* \times A^*$ if the automaton cannot distinguish between processing the word u and processing the word v from any given state. This is defined as follows.

Definition 30 (cf. [13, Definition 1])**.** Let A be an alphabet, an *equation* on A is a pair $(u, v) \in A^* \times A^*$. We say that the automaton (X, α_X) on A *satisfies* the equation (u, v), denoted as $(X, \alpha_X) \models (u, v)$, if for every $x \in X$ we have $u(x) = v(x)$. We denote by $\mathtt{Eq}(X, \alpha_X)$ the set of equations that (X, α_X) satisfies, that is

$$\mathtt{Eq}(X, \alpha_X) = \{(u, v) \in A^* \times A^* \mid (X, \alpha_X) \models (u, v)\}.$$

As $(X, \alpha_X) \models (u, v)$ if and only if $(X, \alpha_X) \models (v, u)$, for $u, v \in A^*$, we can denote the equation (u, v) as $u = v$ and then we have:

$$(X, \alpha_X) \models u = v \quad \Leftrightarrow \quad \forall\, x \in X\ u(x) = v(x).$$

Given a set E of equations on A, we say that (X, α_X) satisfies E, denoted as $(X, \alpha_X) \models E$, if $(X, \alpha_X) \models u = v$ for every $u = v \in E$.

Remark. Note that an equation (u, v) is exactly the equation $\forall x \in X\ u(x) = v(x)$, also denoted as $u(x) \approx v(x)$ (cf. [27, Definition II.11.1]).

The next example illustrates some particular cases of equations satisfied by a deterministic automaton and it describes how to obtain a generator set for the set $\mathtt{Eq}(X, \alpha_X)$ in the case that X and A are finite sets. The general case for obtaining a generator set for $\mathtt{Eq}(X, \alpha_X)$, where X and A are finite sets, can be easily obtained from what it is described in the example.

Example 31. Let $A = \{a, b\}$, $X = \{x_1, x_2\}$ and consider the automaton (X, α_X) on A given by the following diagram:

$$\begin{tikzcd}
x_1 \arrow[loop above, "b"] \arrow[r, bend left, "a"] & x_2 \arrow[loop above, "a"] \arrow[l, bend left, "b"]
\end{tikzcd}$$

Then (X, α_X) satisfies equations such as $a = aaa$, $baa = bb$ and $bab = bb$, but it does not satisfy the equation $ab = ba$ nor the equation $\epsilon = b$, since $ab(x_1) = x_2 \neq x_1 = ba(x_1)$ and $\epsilon(x_2) = x_2 \neq x_1 = b(x_2)$.

Now, how can we find all the equations that (X, α_X) satisfies? This is in general a nontrivial task since there could be infinitely many of them. For example, since (X, α_X) satisfies $a = aaa$ then it also satisfies any equation of the form $a = a^{2n+1}$ for $n \geq 1$, but all of them are obtained from $a = aaa$ by replacing a by aaa, since $a = aaa$, as follows:

$$a = aaa = aaaaa = aaaaaaa = a^7 = a^9 = a^{11} = \cdots$$

In this sense, each equation $a = a^{2n+1}$, $n \geq 1$, is generated (i.e., can be deduced by using substitution and transitivity) by the single equation $a = aaa$. In the cases that A and X are finite sets we can find a finite set of equations that generates all the equations satisfied by (X, α_X). We will illustrate how to obtain a generator set of equations for the automaton given above, whose general algorithm can be easily described.

By Definition 30 above we have that $(X, \alpha_X) \models u = v$ iff $\forall\, x \in X\ u(x) = v(x)$, so we are going to consider all the states of (X, α_X) at the same time and make transitions for all the symbols in A to find when the state we reach with two different words is the same. That is, we put all the states of (X, α_X) in the tuple (x_1, x_2) and start to make transitions, according to (X, α_X), for each symbol in A.

In fact, if we make an a transition and a b transition from the tuple (x_1, x_2) we get the tuples (x_2, x_2) and (x_1, x_1), respectively, which is illustrated in the following picture:

$$\begin{tikzcd}
& (x_1, x_2) \arrow[dl, "a"'] \arrow[dr, "b"] & \\
(x_2, x_2) & & (x_1, x_1)
\end{tikzcd}$$

Until now, there are no different words, starting from (x_1, x_2), that will take us to the same tuple, hence we have not found any nontrivial equations yet. But, as we have new tuples, namely (x_2, x_2) and (x_1, x_1), we need to do the transitions from those states for every symbol in A. We can start with (x_2, x_2) by making all the transitions for every symbol in A to obtain the following:

$$\begin{tikzcd}
& (x_1, x_2) \arrow[dl, "a"'] \arrow[dr, "b"] & \\
(x_2, x_2) \arrow[loop left, "a"] \arrow[rr, bend right, "b"'] & & (x_1, x_1)
\end{tikzcd}$$

In each of those transitions, we found two different words (paths starting from (x_1, x_2)) that will take us to the same tuple. In fact,

i) The words a and aa take us from (x_1, x_2) to the same tuple (x_2, x_2), which means that the equation $a = aa$ is satisfied by (X, α_X).

ii) The words b and ba take us from (x_1, x_2) to the same tuple (x_1, x_1), which means that the equation $b = ba$ is satisfied by (X, α_X).

Note that in i) and ii) we always start from the tuple (x_1, x_2) which is the tuple that represents all the states of the automaton. This process terminates since there are at most $|X|^{|X|}$ tuples.

Also, the equations obtained in i) and ii) above are given by the two shortest paths that take us to the same tuple, which in some sense is the minimum information we want to capture in an equation. For example, the words $baaa$ and b will take us to the same tuple, i.e., the automaton satisfies the equation $baaa = b$, but $baaa = b$ can be deduced from $a = aa$ and $b = ba$. Therefore, the equations in i) and ii) above are enough to deduce every equation that comes from the previous diagram, i.e., to deduce by using reflexivity, symmetry, transitivity, substitution and concatenation.

Now, we still have to do the transitions from the tuple (x_1, x_1) to find new equations and/or new tuples. By making all the transitions from the tuple (x_1, x_1) we obtain the following:

$$
\begin{tikzcd}
 & (x_1, x_2) \arrow[dl, "a"'] \arrow[dr, "b"] & \\
(x_2, x_2) \arrow[loop left, "a"] \arrow[rr, bend right, "b"'] & & (x_1, x_1) \arrow[loop right, "b"] \arrow[ll, bend right, "a"']
\end{tikzcd}
$$

Again, in each of those transitions we found two different words that will take us to the same tuple. In fact,

iii) The words a and ab take us from (x_1, x_2) to the same tuple (x_2, x_2), which means that the equation $a = ab$ is satisfied (X, α_X).

iv) The words b and bb take us from (x_1, x_2) to the same tuple (x_2, x_2), which means that the equation $b = bb$ is satisfied (X, α_X).

Finally, as every tuple in the previous diagram has all the transitions for each symbol in A, we have finished the process for finding a generating set for the equations that the automaton (X, α_X) satisfies. Hence, a generating set for $\mathtt{Eq}(X, \alpha_X)$ is given by the equations:

$$a = aa, \quad ba = b, \quad a = ab, \quad b = bb$$

That is, every equation in $\mathtt{Eq}(X, \alpha_X)$ can be deduced from the four equations above by using reflexivity, symmetry, transitivity, substitution and concatenation. $\square$

We will now study some properties of equations for deterministic automata and show their relation to mathematical concepts such as (free) monoids, monoid congruence, and illustrate this with a categorical approach by using (commutative) diagrams.

For any given set A, let $A^* = (A^*, \cdot, \varepsilon)$ be the *free monoid on* A, i.e., the free U–object over A (see Definition 1) where $U : \mathtt{Mon} \to \mathtt{Set}$ is the forgetful functor from the category $\mathtt{Mon}$ of monoids and monoid homomorphisms into $\mathtt{Set}$. Here, A^* is the set of words with symbols on A and $\cdot$ is given by the concatenation of words. The function $\eta_A : A \to A^*$ is given by $\eta_A(a) = a$, $a \in A$. For any monoid $(M, \cdot, e)$ and any function $f : A \to M$ the unique monoid homomorphism $f^\sharp$ from $(A^*, \cdot, \varepsilon)$ to $(M, \cdot, e)$ such that $U(f^\sharp) \circ \eta_A = f$ is canonically defined for any $w = a_1 \cdots a_n$, $n \geq 1$, $a_i \in A$ as:

$$f^\sharp(w) = f^\sharp(a_1 \cdots a_n) = f(a_1) \cdot \cdots \cdot f(a_n).$$

This (universal) property says that in order to get a monoid homomorphism from $(A^*, \cdot, \varepsilon)$ to $(M, \cdot, e)$ it is enough to define a function $f : A \to M$. We now define the concept of a monoid congruence.

Definition 32. Let $M = (M, \cdot, e)$ be a monoid. A *monoid congruence* of M is an equivalence relation θ on M such that for every $(m, n), (x, y) \in \theta$ we have that $(m \cdot x, n \cdot y) \in \theta$. The previous condition is equivalent to the condition that

$(m,n) \in \theta$ and $x \in M$ implies $(m \cdot x, n \cdot x), (x \cdot m, x \cdot n) \in \theta$. The equivalence class of an element $m \in M$ with respect to θ is denoted by m/θ. Note that $(e,e) \in \theta$ for every monoid congruence θ since θ is a reflexive relation.

From the adjunction $(\_) \times X \dashv (\_)^X$ we have that there is a one–to–one correspondence between functions $f : A \times X \to Y$ and functions $\widehat{f} : A \to Y^X$ which is given by the equation $f(a,x) = \widehat{f}(a)(x)$. Hence, for an automaton (X, α_X) on A we have the function $\widehat{\alpha_X} : A \to X^X$ from which, by the universal property of the free monoid A^* on A, we get a unique monoid homomorphism $\widehat{\alpha_X}^\sharp$ from the free monoid $A^* = (A^*, \cdot, \varepsilon)$ to the monoid $X^X = (X^X, \circ, id_X)$ such that $U(\widehat{\alpha_X}^\sharp) \circ \eta_A = \widehat{\alpha_X}$. From this, we have the following result.

Lemma 33. *Let (X, α_X) be a deterministic automaton on A. Then for every $x \in X$ and $w \in A^*$ we have that $\widehat{\alpha_X}^\sharp(w)(x) = w(x)$.*

Proof. We proof this by induction on the length of w. In fact,

i) If $w = \varepsilon$, then $\widehat{\alpha_X}^\sharp(\varepsilon) = id_X$, which implies that $\widehat{\alpha_X}^\sharp(\varepsilon)(x) = id_X(x) = x = \varepsilon(x)$.

ii) If $w = au$ with $a \in A$ and $u \in A^*$, then we have:

$$\begin{aligned}\widehat{\alpha_X}^\sharp(au)(x) &= \left(\widehat{\alpha_X}^\sharp(a) \circ \widehat{\alpha_X}^\sharp(u)\right)(x) = \widehat{\alpha_X}^\sharp(a)\left(\widehat{\alpha_X}^\sharp(u)(x)\right) \\ &= \widehat{\alpha_X}(a)(u(x)) = \alpha_X(a, u(x)) = au(x).\end{aligned}$$

where the first equality follows from the fact that $\widehat{\alpha_X}^\sharp$ is a monoid homomorphism, the second one from definition of composition $\circ$, the third one from the induction hypothesis and the fact that $a \in A$, the fourth one from the definition of $\widehat{\alpha_X}$, and the last one is the notation we defined. □

From the previous lemma, we get the following corollary that describes the set $\mathtt{Eq}(X, \alpha_X)$ as a kernel of a monoid homomorphism.

Corollary 34. *Let (X, α_X) be a deterministic automaton on A. Then $\mathtt{Eq}(X, \alpha_X) = \ker(\widehat{\alpha_X}^\sharp)$. In particular, $\mathtt{Eq}(X, \alpha_X)$ is a congruence of the monoid A^*.* □

Another relevant concept is that of the transition monoid of a deterministic automaton which is defined as follows.

Definition 35. Let (X, α_X) be a deterministic automaton on A. The *transition monoid* $\mathtt{trans}(X, \alpha_X)$ of (X, α_X) is the monoid defined as:

$$\mathtt{trans}(X, \alpha_X) := (\mathrm{Im}(\widehat{\alpha_X}^\sharp), \circ, id_X).$$

Note that, by the first isomorphism theorem [27, Theorem II.6.12], we have that $\mathtt{trans}(X, \alpha_X)$ is isomorphic to $A^*/\,\mathtt{Eq}(X, \alpha_X)$ (cf. [13, Theorem 28]).

Example 36 (Example 31 continued). For $A = \{a, b\}$ and $X = \{x_1, x_2\}$ we considered the automaton (X, α_X) on A given by the diagram:

$$\begin{tikzcd}
x_1 \arrow[loop above, "b"] \arrow[r, bend left, "a"] & x_2 \arrow[loop above, "a"] \arrow[l, bend left, "b"]
\end{tikzcd}$$

In order to get a generating set for the equations of (X, α_X) we constructed the diagram:

$$\begin{tikzcd}
& (x_1, x_2) \arrow[dl, "a"'] \arrow[dr, "b"] & \\
(x_2, x_2) \arrow[loop left, "a"] \arrow[rr, bend right, "b"'] & & (x_1, x_1) \arrow[loop right, "b"] \arrow[ll, bend right, "a"']
\end{tikzcd}$$

From this diagram we can obtain the function $\widehat{\alpha_X}^\sharp(w)$ for each $w \in A^*$. In fact, the function $\widehat{\alpha_X}^\sharp(w)$ is the function that maps each element in the tuple (x_1, x_2), which is the tuple containing all the different elements in X, to its corresponding element in the tuple we reach from (x_1, x_2) by processing the word w from right to left in the diagram above. For instance, with the word $abbab$ we reach the tuple (x_2, x_2) from the tuple (x_1, x_2). This means that $\widehat{\alpha_X}^\sharp(abbab)$ maps each element in (x_1, x_2) to its corresponding element in (x_2, x_2), i.e., $\widehat{\alpha_X}^\sharp(abbab)(x_1) = \widehat{\alpha_X}^\sharp(abbab)(x_2) = x_2$. According to this, since $\mathtt{trans}(X, \alpha_X) := (\mathrm{Im}(\widehat{\alpha_X}^\sharp), \circ, id_X)$ and it is isomorphic to $A^*/\,\mathtt{Eq}(X, \alpha_X)$, we obtain the equations $\mathtt{Eq}(X, \alpha_X)$ of (X, α_X) by looking at the pairs of words (u, v) such that from the tuple (x_1, x_2) we get the same tuple by processing the word u and by processing the word v from right to left. This is what we did in Example 31 but we restricted our attention to the equations that generated the congruence $\mathtt{Eq}(X, \alpha_X)$. In this case, we have that $\mathtt{Eq}(X, \alpha_X) = \langle (a, aa), (ba, b), (a, ab), (b, bb) \rangle$, where the right–hand side denotes the least congruence on A^* that contains $\{(a, aa), (ba, b), (a, ab), (b, bb)\}$. □

From Corollary 34 we have that $\mathtt{Eq}(X, \alpha_X)$ is a congruence of the monoid A^* for every (X, α_X). On the other hand, for every congruence θ of A^* there exists an automaton (X, α_X) on A such that $\mathtt{Eq}(X, \alpha_X) = \theta$, namely, the automaton $(A^*/\theta, f_\theta)$ where $f_\theta(a, w/\theta) = aw/\theta$. The automaton $A^*/\,\mathtt{Eq}(X, \alpha_X)$ is what in [13, 79] is called $\mathbf{free}(X, \alpha_X)$. Additionally, if E is a set of equations on A then we have that $(X, \alpha_X) \models E$ if and only if $(X, \alpha_X) \models \langle E \rangle$, where $\langle E \rangle$ denotes the least congruence of A^* containing E.

We have the following categorical characterization of satisfaction of equations for deterministic automata.

Proposition 37. *Let (X, α_X) be a deterministic automaton on A and let E be a set of equations on A. Denote by $\langle E \rangle$ the least congruence on A^* containing E and let*

$\nu_E : A^* \to A^*/\langle E\rangle$ be the canonical homomorphism such that $\nu_E(w) = w/\langle E\rangle$. Then $(X, \alpha_X) \models E$ if and only if $\widehat{\alpha_X}^\sharp$ factors through ν_E, i.e., there exists $g : A^*/\langle E\rangle \to X^X$ such that the following diagram commutes:

$$\begin{tikzcd}
A^* \arrow[rr, "\widehat{\alpha_X}^\sharp"] \arrow[dr, "\nu_E"'] & & X^X \\
& A^*/\langle E\rangle \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Proof. We have that $\widehat{\alpha_X}^\sharp$ factors through ν_E if and only if $\langle E\rangle = \ker(\nu_E) \subseteq \ker(\widehat{\alpha_X}^\sharp) = \mathtt{Eq}(X, \alpha_X)$, which holds if and only if $(X, \alpha_X) \models \langle E\rangle$. $\square$

By the previous proposition, we have that every set E of equations induces the surjective monoid homomorphism[1] $\nu_E : A^* \to A^*/\langle E\rangle$. Conversely, every surjective monoid homomorphism $h : A^* \to M$ defines the set of equations (in fact, a congruence) $\ker(h)$. Therefore, we can abstractly regard equations as a special kind of arrows in a category, e.g., regular epimorphisms with free domain. This abstract concept of equations will be defined in Chapter 4 and studied in the subsequent chapters. A similar characterization for satisfaction of equations in terms of a commutative diagram, as the one in the previous proposition, is shown in [13, Section 4] by using free $(A \times \_)$–algebras on 1 generator.

In this section, we mainly focused on the study of equations for the case of one variable, in the sense of [27, Definition II.11.1]. That is, if we consider deterministic automata as algebras of type $\tau = A$, in which every element in $\tau = A$ is a unary function symbol, then we have two kinds of equations for each $u, v \in A^*$, namely

$$\forall x\ u(x) = v(x) \ \text{ and } \ \forall x \forall y\ u(x) = v(y),$$

also denoted as $u(x) \approx v(x)$ and $u(x) \approx v(y)$, respectively. These are the kind of equations studied in universal algebra. The equation $u(x) \approx v(x)$, in which the variable on the left–hand side is the same as the variable on the right–hand side, is known as a *regular equation* [71]. Regular equations for deterministic automata can be identified with pairs $(u, v) \in A^* \times A^*$ as in [13].

Our main purpose of focusing on regular equations is to relate our work with [13, 79] to show that classes of deterministic automata defined by regular equations can be equivalently defined by sets of coequations.

2.2 Coequations for deterministic automata

Coequations for deterministic automata have been studied in [13, 79]. In this section, we do a similar work as in the previous section, we define the concept of

[1] Surjective algebra homomorphisms are also known in universal algebra as algebra epimorphisms [27, Definition II.6.1]. In the category Mon of monoids and monoid homomorphisms, surjective monoid homomorphisms are exactly regular epimorphisms [4, 7.72. Examples].

coequations for deterministic automata, give some examples and basic facts. All of this will be generalized in Chapter 4 by using a general categorical approach. In this section, deterministic automata are regarded as coalgebras (X, β_X) for the functor $G(\_) = \mathtt{Set}(A, \_)$. We start by defining coequations for deterministic automata.

Definition 38. Let A be a set. A set of *coequations* on A is a subset S of 2^{A^*}, that is, a set of languages on A.

Given a deterministic automaton (X, β_X) on A and a function $c : X \to 2$, which we think of as a two–colouring of X, we define the the *observability map* $o_c : X \to 2^{A^*}$ as the function given by $o_c(x)(w) = c(w(x))$. That is, $o_c(x) : A^* \to 2$ is the language that (X, β_X) accepts from the state $x \in X$ according to the colouring c, cf. Example 15. We interpret a set of coequations S as the set of "behaviours" that we are interested in. In this sense, an automaton on A will satisfy the set of coequations S if all the languages it can accept for any two–colouring are contained in S. This is defined as follows.

Definition 39. Let A be a set, (X, β_X) an automaton on A and let S be a set of coequations on A. We say that (X, β_X) *satisfies* the set of coequations S, denoted as $(X, \beta_X) \models S$, if for every $c \in \mathtt{Set}(X, 2)$ we have that $\mathrm{Im}(o_c) \subseteq S$. That is, S contains all the languages the automaton (X, β_X) can accept.

The following examples illustrate this concept of satisfaction of coequations.

Example 40. Let $A = \{a, b\}$ and consider the following automaton (X, β_X) on A:

$$\begin{tikzcd} x_1 \arrow[r, bend left, "{a,b}"] & x_2 \arrow[l, bend left, "{a,b}"] \end{tikzcd}$$

Then, for every $S \subseteq 2^{A^*}$ we have $(X, \beta_X) \models S$ if and only if $\{\emptyset, L_{\text{odd}}, L_{\text{even}}, A^*\} \subseteq S$ where L_{odd} and L_{even} are the sets of words in A^* with an odd and even number of symbols, respectively. □

Example 41. Let $A = \{a\}$ and consider the following automaton (X, β_X) on A:

$$\begin{tikzcd} x_1 \arrow[r, bend left, "a"] & x_2 \arrow[l, bend left, "a"] & & x_3 \arrow[rr, "a"] & & x_4 \arrow[dl, "a"] \\ & & & & x_5 \arrow[ul, "a"] & \end{tikzcd}$$

Then, for every $S \subseteq 2^{A^*}$ we have:

$$(X, \beta_X) \models S \text{ iff } \{\emptyset, (aa)^*, a(aa)^*, (aaa)^*, a(aaa)^*, aa(aaa)^*, A^*\} \subseteq S.$$

□

As in the case of equations for deterministic automata, we can also give a characterization of $(X, \beta_X) \models\!\!\!\models S$ by means of a commutative diagram. In fact, if we consider the functor $G := \mathtt{Set}(A, \_) : \mathtt{Set} \to \mathtt{Set}$ and we let $U : \mathtt{coalg}(G) \to \mathtt{Set}$ be the forgetful functor, then we have that the cofree U–object over 2 is the object $(2^{A^*}, \varsigma) \in \mathtt{coalg}(G)$, where $\varsigma : 2^{A^*} \to \mathtt{Set}(A, 2^{A^*})$ is such that $\varsigma(L)(a)(w) = L(aw)$, and the morphism $\epsilon_2 \in \mathtt{Set}(2^{A^*}, 2)$ is given by $\epsilon_2(L) = L(\varepsilon)$. That is, for every $(X, \beta_X) \in \mathtt{coalg}(G)$ and $c \in \mathtt{Set}(X, 2)$ there exists a unique morphism $c^\flat$ in $\mathtt{coalg}(G)((X, \beta_X), (2^{A^*}, \varsigma))$, namely $c^\flat = o_c$, such that $c = \epsilon_2 \circ U(c^\flat)$, cf. Example 15. From this, we have the following.

Proposition 42. *Let A be a set, (X, β_X) an automaton on A and $S \subseteq 2^{A^*}$. Then, $(X, \beta_X) \models\!\!\!\models S$ if and only if for every $c \in \mathtt{Set}(X, 2)$ the morphism o_c factors through the inclusion $i_S \in \mathtt{Set}(S, 2^{A^*})$, i.e., for every $c \in \mathtt{Set}(X, 2)$ there exists g_c such that the following diagram commutes:*

$$\forall c \in \mathtt{Set}(X,2) \qquad \begin{tikzcd} X \arrow[rr, "o_c"] \arrow[dr, dashed, "g_c"'] & & 2^{A^*} \\ & S \arrow[ur, "i_S"'] & \end{tikzcd}$$

□

In the setting of the previous proposition, the set given by $\mathtt{Coeq}(X, \beta_X) := \bigcup_{c \in \mathtt{Set}(X,2)} \mathrm{Im}(o_c)$ is the minimum set of coequations that (X, β_X) satisfies, i.e., for any subset $S \subseteq 2^{A^*}$ we have $(X, \beta_X) \models\!\!\!\models S$ if and only if $\mathtt{Coeq}(X, \beta_X) \subseteq S$, [13, Proposition 6].

Note the similarity between the diagram in the previous proposition and the diagram in Proposition 37. The two diagrams are dual to each other in the sense that reversing the direction of the arrows of one of them will give us the same kind of diagram as the other one. In this case, surjective functions become injective functions and vice versa, after reversing the arrows. This is a dual relationship between the concepts of equations and coequations.

2.3 Duality between equations and coequations

In this section, we state a duality result between equations and coequations for deterministic automata, which is a result that is shown in [13]. We explain how this duality follows from the duality between the category Set and the category CABA of complete atomic Boolean algebras (see Example 23) if we ‘restrict’ it on Set to monoid congruences and automata homomorphisms. It is worth mentioning that we will work with special kinds of automata here. On one hand, for considering equations, we will consider quotients of the monoid A^* as automata whose transition structure is given by concatenation, on the other hand, for considering coequations, we will consider subobjects of the object 2^{A^*} in CABA as automata whose transition structure is given by derivatives, this also means that the set of states of such subobject is the carrier set of an object CABA. This will be formally stated through this section.

Given a monoid M, we have that there is a one–to–one correspondence between monoid congruences of M and surjective monoid homomorphisms with domain M. In fact, if θ is a monoid congruence on M then we have that the natural map $\nu_\theta : M \to M/\theta$ defined as $\nu_\theta(m) = m/\theta$ is a surjective monoid homomorphism, and for any surjective monoid homomorphism $e : M \to N$ we have that $\ker(e)$ is a monoid congruence of M. In this case we have that the monoid N is isomorphic to $M/\ker(e)$ and that $\ker(\nu_\theta) = \theta$.

Let $S \subseteq 2^{A^*}$. We say that S is *closed under left derivatives* if for every $L \in S$ and $u \in A^*$ we have that ${}_uL \in S$, where ${}_uL(w) := L(wu)$, $w \in A^*$. The element ${}_uL$ is called the *left derivative of L with respect to u*. Similarly, S is *closed under right derivatives* if for every $L \in S$ and $u \in A^*$ we have that $L_u \in S$, where $L_u(w) := L(uw)$, $w \in A^*$. The element L_u is called the *right derivative of L with respect to u*.

By using the duality between CABA and Set we obtain the objects in CABA that correspond to monoid congruences of A^*.

Proposition 43. *Let θ be an equivalence relation on A^* and let $\nu_\theta : A^* \to A^*/\theta$ be the canonical map such that $\nu_\theta(w) = w/\theta$. Then ν_θ is a monoid homomorphism if and only if the injective function $\mathtt{Set}(\nu_\theta, 2) : \mathtt{Set}(A^*/\theta, 2) \to \mathtt{Set}(A^*, 2)$ is such that the object $\mathrm{Im}(\mathtt{Set}(\nu_\theta, 2)) \in$ CABA, which is given by*

$$\mathrm{Im}(\mathtt{Set}(\nu_\theta, 2)) = \{f \circ \nu_\theta \mid f \in \mathtt{Set}(A^*/\theta, 2)\},$$

is closed under left and right derivatives.

Proof. For every $u \in A^*$ define the functions $l_u, r_u \in \mathtt{Set}(A^*, A^*)$ as $l_u(w) = wu$ and $r_u(w) = uw$. Then we have that $\nu_\theta : A^* \to A^*/\theta$ is a monoid homomorphism if and only if for every $u \in A^*$ there exists $g, g' \in \mathtt{Set}(A^*/\theta, A^*/\theta)$ such that the following diagrams commute:[2]

$$\begin{tikzcd} A^* \arrow[r, "l_u"] \arrow[d, "\nu_\theta"'] & A^* \arrow[d, "\nu_\theta"] \\ A^*/\theta \arrow[r, dashed, "g"'] & A^*/\theta \end{tikzcd} \qquad \begin{tikzcd} A^* \arrow[r, "r_u"] \arrow[d, "\nu_\theta"'] & A^* \arrow[d, "\nu_\theta"] \\ A^*/\theta \arrow[r, dashed, "g'"'] & A^*/\theta \end{tikzcd}$$

which by duality, i.e., by applying the contravariant functor $\mathtt{Set}(\_, 2)$, means that the object $\mathrm{Im}(\mathtt{Set}(\nu_\theta, 2)) \in$ CABA is closed under left and right derivatives. □

According to the previous proposition, subsets of 2^{A^*} that are objects in CABA and are closed under left and right derivatives, which in [13] are called *preformations of languages*, are in one–to–one correspondence with monoid congruences of

[2]Thanks to Henning Urbat who told me about characterizing the property of ν_θ being a monoid homomorphism by means of commutative diagrams.

A^*. Now, to obtain the duality between monoid congruences of A^* and preformation of languages given in [13] we need to restrict the morphisms on monoid congruences to surjective automata morphisms. That is, given a monoid congruence θ on A^* we can consider the set A^*/θ as an automaton whose transition function $f_\theta : A \times A^*/\theta \to A^*/\theta$ is given by $f_\theta(a, w/\theta) = (aw)/\theta$. Then, we define the category $\mathcal{D}$ whose objects are quotients of A^*, i.e., objects of the form A^*/θ, where θ is a monoid congruence and whose morphisms are surjective automata morphisms, i.e., a morphism from A^*/θ_1 to A^*/θ_2 is a surjective function $h \in \texttt{Set}(A^*/\theta_1, A^*/\theta_2)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \times A^*/\theta_1 \arrow[r, "id_A \times h"] \arrow[d, "f_{\theta_1}"'] & A \times A^*/\theta_2 \arrow[d, "f_{\theta_2}"] \\
A^*/\theta_1 \arrow[r, "h"'] & A^*/\theta_2
\end{tikzcd}$$

By applying the contravariant functor $\texttt{Set}(\_, 2)$ we have, by duality, that commutativity of the previous diagram is equivalent to commutativity of the diagram:

$$\begin{tikzcd}
\texttt{Set}(A \times A^*/\theta_1, 2) & \texttt{Set}(A \times A^*/\theta_2, 2) \arrow[l, "\texttt{Set}(id_A \times h, 2)"'] \\
\texttt{Set}(A^*/\theta_1, 2) \arrow[u, "\texttt{Set}(f_{\theta_1}, 2)"] & \texttt{Set}(A^*/\theta_2, 2) \arrow[l, "\texttt{Set}(h, 2)"] \arrow[u, "\texttt{Set}(f_{\theta_2}, 2)"']
\end{tikzcd}$$

Finally, commutativity of the previous diagram is equivalent, by Lemma 11, to commutativity of the following diagram:

$$\begin{tikzcd}
\texttt{Set}(A^*/\theta_1, 2) & & \texttt{Set}(A^*/\theta_2, 2) \arrow[ll, "\texttt{Set}(h, 2)"'] \\
\texttt{Set}(A^*/\theta_1, 2) \times A \arrow[u, "f'_{\theta_1}"] & (\dagger) & \texttt{Set}(A^*/\theta_2, 2) \times A \arrow[ll, "\texttt{Set}(h, 2) \times id_A"] \arrow[u, "f'_{\theta_2}"']
\end{tikzcd}$$

where f'_{θ_i} is defined as $f'_{\theta_i}(L, a)(w/\theta) = L((aw)/\theta_i)$.

Note that the diagram ($\dagger$) means that $\texttt{Set}(h, 2)$ is a homomorphism between the automaton $(\texttt{Set}(A^*/\theta_2, 2), f'_{\theta_2})$ and the automaton $(\texttt{Set}(A^*/\theta_1, 2), f'_{\theta_1})$, which is injective since h is surjective. All in all, we have that the category $\mathcal{D}$ is dual to the category $\mathcal{C}$ whose objects are preformations of languages and whose morphisms are injective automata homomorphism. This is exactly the duality shown in [13]. We will come back to this setting to establish a general duality between equations and coequations in Chapter 4. Also, note that $\mathrm{Im}(\texttt{Set}(\nu_\theta, 2))$ is the same as $\mathbf{cofree}(A^*/\theta)$ defined in [13] and that for a preformation of languages

S, that is, a subset S of 2^{A^*} in CABA which is closed under left and right derivatives, we have that $\mathtt{CABA}(S, 2)$ is the same as $\mathbf{free}(S)$ defined in [13]. Therefore, from the previous observation we get the identities $\mathbf{free} \circ \mathbf{cofree}(A^*/\theta) = A^*/\theta$ and $\mathbf{cofree} \circ \mathbf{free}(S) = S$, see [13, Theorem 22].

As an application of the previous facts we have the following:

Example 44. Given a family of languages $\mathcal{L} \subseteq 2^{A^*}$, we can construct an automaton (X, β_X) representing the family $\mathcal{L}$ in the following sense:

> For every $L \in \mathcal{L}$ there exists $x \in X$ and $c \in 2^X$ such that $o_c(x) = L$.

We can construct an automaton that represents $\mathcal{L}$ with the minimum number of states and moreover that satisfies the following stronger property:

> There exists $x \in X$ such that for every $L \in \mathcal{L}$ there exists $c \in 2^X$ such that $o_c(x) = L$.

The construction is a follows: let $P(\mathcal{L})$ be the least preformation of languages containing $\mathcal{L}$, that is, the least subobject in CABA of 2^{A^*} which is closed under left and right derivatives. Then the the dual $\mathtt{CABA}(\iota, 2)$ of the inclusion $\iota : P(\mathcal{L}) \to 2^{A^*}$ induces a surjective monoid homomorphism $\nu_\theta : A^* \to A^*/\theta$. Now, by duality, the automaton A^*/θ has the desired property. In fact, if $L \in P(\mathcal{L})$ then L is the join of atoms in $P(\mathcal{L})$ and the colouring $c_L \in \mathtt{Set}(A^*/\theta, 2)$ that colours with colour 1 every atom below L and 0 every other atom is such that $L = c_L \circ \nu_\theta$ which implies that $L = o_{c_L}(\varepsilon/\theta)$. □

The previous example gives us a way to construct a single program (automaton) for a specific set of behaviours (set of languages) in an efficient way (minimum number of states) with the property that the initial configuration (initial state) of the program is the same for every desired behaviour. Here is a small illustration of this fact.

Example 45. Let $A = \{a, b\}$ and consider the following family of languages on A^*

$$\mathcal{L} = \{(a \cup b)^+, L_{\text{odd}}, L_{\text{even}}\}$$

where L_{odd} and L_{even} are the sets of words in A^* with an odd and even number of symbols, respectively. We would like to construct an automaton (X, β_X) with the property that there exists $x_0 \in X$ such that for every $L \in \mathcal{L}$ there exists $c_L \in 2^X$ such that $o_{c_L}(x_0) = L$. According to the previous example, we only need to construct the least preformation of languages $P(\mathcal{L})$ containing $\mathcal{L}$. In this case, $P(\mathcal{L})$ is the preformation of languages (with 8 elements) whose atoms are

$$A_1 = \{\varepsilon\}, \quad A_2 = L_{\text{odd}}, \quad \text{and} \quad A_3 = L_{\text{even}} \setminus \{\varepsilon\}$$

Clearly $\mathcal{L} \subseteq P(\mathcal{L})$ since $(a \cup b)^+ = A_2 \cup A_3$, $L_{\text{odd}} = A_2$, and $L_{\text{even}} = A_1 \cup A_3$. Then the automaton with the desired property is given by the atoms of $P(\mathcal{L})$, which is the following:

$$
\begin{tikzcd}
A_1 \arrow[r, "{a,b}"] & A_2 \arrow[r, bend left, "{a,b}"] & A_3 \arrow[l, bend left, "{a,b}"]
\end{tikzcd}
$$

Whose initial state x_0 is the atom containing the empty word ε, i.e., $x_0 = A_1$. Clearly, for the colourings $c_1 = \{A_2, A_3\}$, $c_2 = \{A_2\}$, and $c_3 = \{A_1, A_3\}$ we have that

$$o_{c_1}(x_0) = (a \cup b)^+, \quad o_{c_2}(x_0) = L_{\text{odd}}, \quad \text{and} \quad o_{c_3}(x_0) = L_{\text{even}}.$$

Note that each colouring is obtained from the representation of the corresponding language as a join of atoms. For instance, the colouring c_1 for the language $(a \cup b)^+$ is obtained from the equation $(a \cup b)^+ = A_2 \cup A_3$. □

2.4 Varieties and covarieties of automata

We conclude this chapter by studying special classes of deterministic automata: varieties of automata and covarieties of automata. The concept of a variety and a covariety are key concepts which are studied in universal algebra and universal coalgebra, respectively, see, e.g., [27, 76]. Varieties of algebras are defined as classes of algebras of the same type that are closed under homomorphic images (also called quotients), subalgebras and products. Dually, varieties of coalgebras are defined as classes of coalgebras of the same type that are closed under homomorphic images, subcoalgebars and coproducts.

An important theorem that characterizes varieties of algebras is the Birkhoff theorem, which states that a class of algebras is a variety if and only if it can be defined by equations [18]. A similar theorem for the case of coalgebras was proved in [76]. In this chapter, we study these concepts for the specific case of deterministic automata, which can be seen as either algebras or coalgebras for an endofunctor (see Example 4 and Example 12).

Let (X, α_X) and (Y, α_Y) be deterministic automata on A. We say that (X, α_X) is a *subautomaton* of (Y, α_Y) if there exists an injective homomorphism of automata from (X, α_X) to (Y, α_Y). We say that (X, α_X) is a *quotient* or a *homomorphic image* of (Y, α_Y) if there exists a surjective homomorphism of automata from (Y, α_Y) onto (X, α_X). Given a family $\{(X_i, \alpha_i)\}_{i \in I}$ of automata on A, we define the *product* $\prod_{i \in I}(X_i, \alpha_i)$ as the deterministic automaton:

$$\prod_{i \in I}(X_i, \alpha_i) := \left(\prod_{i \in I} X_i, \alpha\right)$$

such that $\alpha : A \times \prod_{i \in I} X_i \to \prod_{i \in I} X_i$ is given by $\alpha(a, f)(i) = \alpha_i(a, f(i))$, $a \in A$, $i \in I$. We define the *coproduct* $\coprod_{i \in I}(X_i, \alpha_i)$ as the deterministic automaton:

$$\coprod_{i \in I}(X_i, \alpha_i) := \left(\coprod_{i \in I} X_i, \alpha'\right)$$

where $\coprod_{i\in I} X_i$ is the set $\coprod_{i\in I} X_i := \bigcup_{i\in I} X \times \{i\}$ and $\alpha' : A \times \coprod_{i\in I} X_i \to \coprod_{i\in I} X_i$ is given by $\alpha'(a,(x,i)) = (\alpha_i(a,x), i)$, $a \in A$, $i \in I$ and $x \in X_i$.

Let K be a class of automata on A. We say that K is *closed under subautomata* if for every $(Y, \alpha_Y) \in K$ and every subautomaton (X, α_X) of (Y, α_Y) we have that $(X, \alpha_X) \in K$. We say that K is *closed under quotients* if for every $(Y, \alpha_Y) \in K$ and every quotient (X, α_X) of (Y, α_Y) we have that $(X, \alpha_X) \in K$. We say that K is *closed under products* if for every family $\{(X_i, \alpha_i)\}_{i\in I} \subseteq K$ we have that $\prod_{i\in I}(X_i, \alpha_i) \in K$. We say that K is *closed under coproducts* if for every family $\{(X_i, \alpha_i)\}_{i\in I} \subseteq K$ we have that $\coprod_{i\in I}(X_i, \alpha_i) \in K$. From this, we define the concepts of a variety of automata and a covariety of automata as follows.

Definition 46. Let K be a class of automata on A. We say that K is a *variety of automata* on A if K is closed under subautomata, quotients and products. We say that K is a *covariety of automata* on A if K is closed under subautomata, quotients and coproducts.

By Birkhoff's theorem, we have that varieties of algebras are exactly classes that are defined by equations, see, e.g., [27, 18]. Examples of varieties of algebras include: deterministic automata, semigroups, monoids, groups, lattices and Boolean algebras. In all those cases, we can axiomatize the given kind of algebras by a set of equations, e.g., for the case of semigroups we only need the identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ of associativity. Dually, covarieties of coalgebras are exactly classes that are defined by coequations, see, e.g., [76]. Examples of covarieties of coalgebras include: deterministic automata, transition systems with output, dynamical systems and labeled transition systems with eventually constant behaviour. In all those cases, we can characterize the given class by restricting their behaviour, e.g., eventually constant streams for the case of labeled transition systems with eventually constant behaviour.

The next example shows a variety of automata that is not a covariety.

Example 47. Let $A = \{a, b\}$, and consider the variety V_1 generated by the automaton (X, α_X) on A given by

$$x \xrightarrow{a,b} y \circlearrowleft a,b$$

That is, V_1 is the least variety containing (X, α_X). Then, an automaton $(Y, \alpha_Y) \in V_1$ if and only if there exists $s \in Y$ such that for every $y \in Y$, $\alpha_Y(a, y) = \alpha_Y(b, y) = s$, that is, an automaton is in V_1 if and only if there is no difference between a and b transitions, and there is a state, called a *sink*, that is reachable from any state by inputting the letter a (or, equivalently, that is reachable from any state by inputting the letter b).

Note that V_1 is not a covariety since the coproduct $(X, \alpha_X) + (X, \alpha_X)$ has no sink. A set of defining equations for V_1 is $E = \{a(x) \approx b(y)\}$. Here, $a(x) \approx b(y)$ is the equation $\forall x \forall y\ (a(x) = b(y))$. $\square$

The next example shows a covariety of automata that is not a variety.

Example 48. Let $A = \{a\}$ and consider the following automaton (X, α_X) on A:

$$\begin{tikzcd} x_1 \arrow[r, bend left, "a"] & x_2 \arrow[l, bend left, "a"] \end{tikzcd} \qquad \begin{tikzcd} x_3 \arrow[rr, "a"] & & x_4 \arrow[dl, "a"] \\ & x_5 \arrow[ul, "a"] & \end{tikzcd}$$

Let C_1 be the covariety generated by (X, α_X). Then C_1 is defined by the set of coequations $S = \{\emptyset, (aa)^*, a(aa)^*, (aaa)^*, a(aaa)^*, aa(aaa)^*, A^*\}$. From this, we have that C_1 is not closed under products since $(X, \alpha_X) \times (X, \alpha_X)$ recognizes the language $(aaaaaa)^*$ which is not an element in S. $\square$

As closure properties of sum and products are both defined for deterministic automata, we can ask if we can characterize classes of deterministic automata that are varieties and also covarieties. In fact, we have the following.

Theorem 49. *Let K be a class of deterministic automata on A. The following are equivalent:*

i) K is variety and a covariety of deterministic automata. That is, K is closed under subautomata, quotients, products and coproducts.

ii) K is defined by a preformation of languages, that is, there exists $S \subseteq 2^{A^*}$ in CABA *that is closed under left and right derivatives such that $K = \mathrm{Mod}(S)$.*

iii) K is defined by a monoid congruence on A^*. That is, there exists a monoid congruence θ on A^* such that $K = \mathrm{Mod}(\theta)$.

Here $\mathrm{Mod}(S) := \{(X, \alpha_X) \mid (X, \alpha_X) \Vdash\!\!= S\}$ *and* $\mathrm{Mod}(\theta) := \{(X, \alpha_X) \mid (X, \alpha_X) \models \theta\}$.

Varieties of deterministic automata that are closed under coproducts are instances of *regular varieties* [44].

Proof. $i) \Rightarrow iii)$ By Birkhoff's theorem, if we consider automata on A as algebras of type $\tau = A$ in which each symbol in A is a unary function symbol, we have that $K = \mathrm{Mod}(E)$ for some set of equations E. Each equation in E is of the form $\forall x\ u(x) = v(x)$ or $\forall x, y\ u(x) = v(y)$. Consider the automata $\mathbf{2}_d := (2, d)$ such that for each $n \in 2$ and $a \in A$ we have $d(a, n) = n$, that is, $\mathbf{2}_d$ is the coproduct of two copies of the trivial automaton. Note that $\mathbf{2}_d \in K$ since it is the coproduct of two copies of the trivial automaton. Hence E cannot contain equations of the form $\forall x, y\ u(x) = v(y)$ since those kind of equations are not satisfied by $\mathbf{2}_d$. Therefore, all the equations are of the form $\forall x\ u(x) = v(x)$ which can be identified with elements $(u, v) \in A^*$ and by closing E under substitution and operations in τ we get the congruence θ of A^* such that $K = \mathrm{Mod}(E) = \mathrm{Mod}(\theta)$.

$iii) \Rightarrow i)$ It is trivial since $\mathrm{Mod}(\theta)$ is closed under subautomata, quotients, products and sums.

$ii) \Leftrightarrow iii)$ Let $S \subseteq 2^{A^*}$ in CABA closed under left and right derivatives. By duality, i.e., by applying the functor $\texttt{CABA}(\_, 2)$ to the inclusion $\iota : S \to 2^{A^*}$, we get a surjective monoid homomorphism $\nu_\theta : A^* \to A^*/\theta$ (see Proposition 43). We will now prove that $\mathrm{Mod}(S) = \mathrm{Mod}(\theta)$ by using the fact that A^* is the free U–object over 1, where $U : \texttt{alg}_\tau \to \texttt{Set}$ is the forgetful functor from the category $\texttt{alg}_\tau$ of algebras of type $\tau = A$ to Set, where each operation symbol in A is a unary function symbol. Let (X, α_X) be an automaton on A, $x \in X$ and $c : X \to 2$, then we have the following commutative diagram:

$$\begin{tikzcd}
A^* \arrow[r, "x^\sharp"] \arrow[rr, bend left, "o_c(x)"] & X \arrow[r, "c"] & 2 \\
1 \arrow[u, "\eta_1"] \arrow[ur, "x"'] & &
\end{tikzcd}$$

Assume that $(X, \alpha_X) \in \mathrm{Mod}(S)$, that is, that $o_c(x) \in S$. For a fixed $x \in X$, the previous fact defines a morphism $g_x : 2^X \to S$ in CABA which is given by $g_x(c) = o_c(x)$, which by duality gives us a function $h_x : A^*/\theta \to X$. Note that $x^\sharp = h_x \circ \nu_\theta$, which follows from duality since $\texttt{Set}(x^\sharp, 2) = \iota \circ g_x$. Now, for any $(u, v) \in \theta$ we have that $u(x) = x^\sharp(u) = (h_x \circ \nu_\theta)(u) = (h_x \circ \nu_\theta)(v) = x^\sharp(v) = v(x)$. That is, $(X, \alpha_X) \in \mathrm{Mod}(\theta)$.

Conversely, assume that $(X, \alpha_X) \in \mathrm{Mod}(\theta)$. Then, for every $x \in X$ the homomorphism $x^\sharp$ factors through ν_θ as $x^\sharp = h_x \circ \nu_\theta$. Then, by duality, we have $\texttt{Set}(x^\sharp, 2) = \iota \circ \texttt{Set}(h_x, 2)$ which means that for every colouring $c : X \to 2$ we have that $o_c(x) = c \circ x^\sharp \in S$. Therefore $(X, \alpha_X) \in \mathrm{Mod}(S)$. $\square$

As a corollary of the previous proof we have.

Corollary 50. *Let $S \subseteq 2^{A^*}$ be a preformation of languages and θ a congruence on A^*. Then for every automaton (X, α_X) on A we have:*

i) $(X, \alpha_X) \models \theta$ *if and only if* $(X, \alpha_X) \mathrel{|\!\!\models} \mathrm{Im}(\texttt{Set}(\nu_\theta, 2))$.

ii) $(X, \alpha_X) \mathrel{|\!\!\models} S$ *if and only if* $(X, \alpha_X) \models \texttt{CABA}(S, 2)$.

Here $\nu_\theta : A^* \to A^*/\theta$ is the canonical map such that $\nu_\theta(w) = w/\theta$.

Proof. The two statements follow from the previous proof as follows:

i) This follows from the fact that $\mathrm{Mod}(\theta) = \mathrm{Mod}(S)$ for $S = \mathrm{Im}(\texttt{Set}(\nu_\theta, 2))$.

ii) Consider the inclusion map $\iota : S \to 2^{A^*}$. Then, by using the duality between Set and CABA and Proposition 43, we have that $\texttt{CABA}(\iota, 2) : \texttt{CABA}(2^{A^*}, 2) \cong A^* \to \texttt{CABA}(S, 2)$ is a surjective monoid homomorphism. Therefore, we have that $\mathrm{Mod}(S) = \mathrm{Mod}(\theta)$, where θ is the congruence given by the partition $\texttt{CABA}(S, 2)$, that is, θ is the congruence on A^* such that $A^*/\theta \cong \texttt{CABA}(S, 2)$, which is given by $\theta = \ker(\texttt{CABA}(\iota, 2))$.

□

The following example shows how a regular variety can be described in the three equivalent ways shown in Theorem 49.

Example 51. Let $A = \{a, b\}$, and consider the regular variety V_2 generated by the automaton (X, α_X) on A given by

$$\begin{tikzcd} x \arrow[r, "{a,b}"] & y \arrow[loop right, "{a,b}"] \end{tikzcd}$$

Then, by Theorem 49, V_2 can be described in three different ways, namely:

i) As the closure under subautomata, quotients, products and coproducts of the set $\{(X, \alpha_X)\}$, which means that an automaton $(Y, \alpha_Y) \in V_2$ if and only if (Y, α_Y) is the sum of elements in V_1 (see Example 47).

ii) $V_2 = \mathrm{Mod}(\theta)$ where θ is the congruence generated by $\{a = b, aa = a\}$.

iii) $V_2 = \mathrm{Mod}(S)$ where S is the preformation of languages $S = \{\emptyset, \{\epsilon\}, A^+, A^*\}$ where $A^+ = A^* \smallsetminus \{\epsilon\}$. □

Now we discuss the kind of equations we have considered from the perspective of universal algebra. The equations for automata we considered were pairs (u, v) such that $u, v \in A^*$. This kind of equation corresponds to the equation $\forall x\ u(x) = v(x)$, also denoted as $u(x) \approx v(x)$, that are studied in universal algebra [27, Definition II.11.1]. Equations in which the set of variables used in each term are the same are called *regular equations*. They were first introduced by Płonka [71], which in the case of deterministic automata on A can be identified with pairs (u, v), $u, v \in A^*$. It is worth mentioning that a Birkhoff–type theorem for regular varieties was formulated by Taylor in [88, p. 4] in which the algebra $\mathbf{2}_d$ in the previous proof is generally defined, which is called the *sup–algebra* of type τ. The importance of the algebra $\mathbf{2}_d$ is that an equation holds in $\mathbf{2}_d$ if and only if it is regular [44, Lemma 2.1].

By using the characterization of regular varieties given by Taylor [88, p. 4], we have that a variety of automata is closed under coproducts if and only if it contains $\mathbf{2}_d$. This fact can be proved directly by noticing that $\mathbf{2}_d$ is the coproduct of two copies of the trivial (one element) algebra and, conversely, that the coproduct of a family $\{(X_i, \alpha_i)\}_{i \in I}$ can be obtained as a homomorphic image of the algebra $\prod_{i \in I}(X_i, \alpha_i) \times \prod_{i \in I} \mathbf{2}_d$. In fact, let $\phi : I \to \prod_{i \in I} 2$ be an injective function and $i_0 \in I$ a fixed element, then the function $h : \prod_{i \in I} X_i \times \prod_{i \in I} 2 \to \coprod_{i \in I} X_i$ defined by

$$h(f, p) = \begin{cases} (i_0, f(i_0)) & \text{if } p \notin \mathrm{Im}(\phi), \\ (i, f(i)) & \text{if } p \in \mathrm{Im}(\phi) \text{ and } \phi(i) = p, \end{cases}$$

is a surjective homomorphism onto $\coprod_{i \in I}(X_i, x_i)$. Notice that h is well-defined since ϕ is injective.

Additionally, note that the property of the class $K = \mathrm{Mod}(S)$ being closed under products can be proved directly from the fact that S is a preformation of languages as follows: Consider a family $\{(X_i, \alpha_i)\}_{i\in I} \subseteq \mathrm{Mod}(S)$ and let $(\prod_{i\in I} X_i, \alpha)$ be the product of that family. Fix a colouring $c : \prod_{i\in I} X_i \to 2$ and $f \in \prod_{i\in I} X_i$, we want to show that $o_c(f) \in S$, which follows the fact that S is a complete Boolean algebra and from the equality

$$o_c(f) = \bigvee_{y\in c^{-1}(1)} \left(\bigwedge_{i\in I} o_{\delta_{y(i)}}(x(i)) \right)$$

where $\delta_{y(i)} : X_i \to 2$ is such that $\delta_{y(i)}(s) = 1$ if and only if $s = y(i)$. In fact,

$$\begin{aligned} w \in o_c(f) &\Leftrightarrow \exists y \in c^{-1}(1) \;\; w(f) = y \\ &\Leftrightarrow \exists y \in c^{-1}(1) \forall i \in I \;\; w(f(i)) = y(i) \\ &\Leftrightarrow \exists y \in c^{-1}(1) \forall i \in I \;\; w \in o_{\delta_{y(i)}}(f(i)) \\ &\Leftrightarrow w \in \bigvee_{y\in c^{-1}(1)} \left(\bigwedge_{i\in I} o_{\delta_{y(i)}}(f(i)) \right). \end{aligned}$$

2.5 Discussion

The purpose of this chapter was to study equations and coequations for deterministic automata. Both concepts were already studied in [13], where a duality result between them was obtained. We studied the basic definitions of equations and coequations for deterministic automata with the purpose of introducing both notions and give the reader an intuitive idea abouth them. Especially for the concept of coequations which is less known.

We showed how the duality result between equations and coequations proved in [13] can be obtained from the duality between `Set` and `CABA` by considering the proper restrictions and using commutative diagrams, something which was not done in [13]. Additionally, we presented the notion of satisfaction of equations and coequations as commutative diagrams. It is worth mentioning that in [13] a variation of Proposition 37 is shown by using free $(A \times \_)$–algebras on 1 generator instead of monoids.

The study of regular varieties for automata and their relation with coequations is a new contribution which is based on the paper [79]. In fact, we showed in Theorem 49 that regular varieties of automata can be equivalently defined by coequations that are preformation of languages. The study of regular varieties and their corresponding defining equations has been previously done in [71, 88, 44]. It is worth mentioning that varieties of deterministic automata that are not regular cannot be defined by coequations, since they are not closed under sums. Therefore, the study of varieties of deterministic automata that are not regular was not considered for the purpose of this chapter.

One of the main motivations in the presentation of this chapter is how equations and coequations can be studied from a categorical point of view, i.e., by using arrows. In this case, we showed how sets of equations for deterministic automata can be represented as a surjective arrow from a free object and, dually, coequations can be represented as an injective arrow into a cofree object. This approach of capturing equations as a special kind of arrows is generally considered for proving categorical versions of Birkhoff's theorem such as [15] and will be studied in more detail in Chapter 4.

We showed in Corollary 34 that the set of equations that a deterministic automaton satisfies is a monoid congruence on A^*. Also, in Theorem 49 we used a monoid congruence on A^* to describe regular varieties of automata. This notion of a monoid congruence on A^* is essentially the same as a *fully invariant congruence on the set* $T_\tau(1)$ of terms of type $\tau = A$ over 1, which is the same as an *equational theory of type* τ *over* 1 [27, II.14]. Hence, for any class K of automata closed under subautomata, quotients, products and coproducts, the monoid congruence on A^* such that $K = \mathrm{Mod}(\theta)$ is unique. We will study equational theories from a categorical point of view in Chapter 5.

Chapter 3

Equations and coequations for weighted automata

Weighted automata are a generalization of non-deterministic automata introduced by Schützenberger [83]. Every transition is associated with an input letter from an alphabet A and a weight expressing the cost (or probability, time, resources needed) of its execution. This weight is typically an element of a semiring. The multiplication of the semiring is used to accumulate the weight of a path by multiplying the weights of each transition in the path, while the addition of the semiring computes the weight of a string w by summing up the weights of the paths labeled with w [34]. In this way, the behaviour of weighted automata is given in terms of formal power series, i.e., functions assigning a weight to each finite string w over A.

Weighted automata may have a non-deterministic behaviour because different transitions from the same state may be labeled by the same input letter, with possibly different weights. However, they can be determinized by assigning a linear structure to the state-space using a generalization of the powerset construction for non-deterministic automata [23]. Moreover, determinized weighted automata are typically infinite-state, but determinization allows us to study weighted automata both from an algebraic and a coalgebraic perspective. From the algebraic perspective, a (determinized) weighted automaton is just an algebra with a unary operation for each input symbol, whereas coalgebraically, a weighted automaton is a deterministic transition system with output weights associated to each state.

In this chapter, we study equations and coequations for weighted automata. Similarly to the previous chapter, we will define the notion of equations and coequations for the case of weighted automata and then show their relationship by showing a duality result between them. A more general case is also considered by introducing the concept of linear equations. First, we start with some preliminaries and concepts that are needed to define weighted automata.

3.1 Preliminaries

In this section, we define some of the concepts we need in order to define the concept of a weighted automaton. We will first introduce the concept of a semiring and then define the concept of a semimodule over a given semiring. From this, the category $\mathtt{Smod}_S$ of semimodules over a semiring S with linear maps as morphisms will be defined. We start by defining the concept of a semiring.

Definition 52. A *semiring* S is an algebra $S = (S, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are binary operations and 0 and 1 are nullary operations, such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, $\cdot$ distributes over $+$ on the left and on the right, and $0 \cdot s = s \cdot 0 = 0$ for every $s \in S$.

As examples of a semiring we have the following.

Example 53. The following are some examples of a semiring:

i) The semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers with the usual addition and multiplication. Similarly, the integers, (non–negative) rational numbers, (non–negative) real numbers and complex numbers with the usual operations are also semirings.

ii) The set of ideals of a given ring under addition and multiplication of ideals.

iii) The Boolean semiring $\mathbb{B} = (2, +, \cdot, 0, 1)$, where $2 = \{0, 1\}$, addition is defined as the 'or' operation $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$ and the multiplication as usual.

iv) The *tropical semiring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$ where $x \oplus y = \mathtt{max}\{x, y\}$ and $x \odot y = x + y$. This semiring is also known as the *max–plus* semiring.

v) The $n \times n$ matrices with entries on any given semiring with addition and multiplication of matrices.

vi) Every ring is a semiring if we forget the operation of additive inverse. □

Now, we define the concept of a semimodule over a semiring.

Definition 54. Let S be a semiring, a *semimodule over* S, or *S–semimodule*, is a commutative monoid $R = (R, +, 0)$ together with an S-left-action $\cdot : S \times R \to R$ such that

$$\begin{array}{lll} (s + s') \cdot r = s \cdot r + s' \cdot r & 0 \cdot r = 0 & 1 \cdot r = r \\ s \cdot (r + r') = s \cdot r + s \cdot r' & s \cdot 0 = 0 & s \cdot (s' \cdot r) = (s \cdot s') \cdot r \end{array}$$

for any $s, s' \in S$ and $r, r' \in R$. We will often write sr instead of $s \cdot r$.

A *linear map* between S–semimodules R_1 and R_2 is a function $h : R_1 \to R_2$ such that for any $x, y \in R_1$ and $c, d \in S$, $h(cx + dy) = ch(x) + dh(y)$. We denote the category of S–semimodules with linear maps as $\mathtt{Smod}_S$.

Let $U : \mathtt{Smod}_{\mathrm{S}} \to \mathtt{Set}$ be the forgetful functor, then the free U–object over a set X, which is called the *free S–semimodule on the generators X* and which we denote by $V(X)$, is the S–semimodule whose underlying set is given by $V(X) = \{\phi \in S^X \mid \mathrm{supp}(\phi)$ is finite$\}$, where $\mathrm{supp}(\phi)$, the *support of* ϕ, is defined as $\mathrm{supp}(\phi) = \{x \in X \mid \phi(x) \neq 0\}$. Addition in $V(X)$ is componentwise, $0 \in V(X)$ is the constant function with 0 as its value, and the action of S over $V(X)$ is multiplication of a constant by a function.

For $\phi \in V(X)$ we have the correspondence $\phi \Leftrightarrow s_1x_1 + \cdots + s_nx_n$, where $\mathrm{supp}(\phi) = \{x_1, \ldots, x_n\}$ and $\phi(x_i) = s_i$, $i = 1, \ldots, n$. Note that we are using formal sums here. According to this correspondence there is a copy of X in $V(X)$, namely $x \mapsto 1x$; in this case, $1x$ will be simply denoted as x. According to this, the function $\eta_X : X \to V(X)$ for the free U–object over X is given by $\eta_X(x) = 1x = x$ and, for a given S–semimodule R, the extension $f^\sharp \in \mathtt{Smod}_{\mathrm{S}}(V(X), R)$ of a function $f \in \mathtt{Set}(X, R)$ is the linear map given by:

$$f^\sharp(c_1x_1 + \cdots + c_nx_n) = c_1f(x_1) + \cdots + c_nf(x_n).$$

If A and X are sets, then the set $V(X)^A$ is the carrier set of an S–semimodule, called the *A power of the S–semimodule $V(X)$*, whose S–semimodule structure is defined componentwise. That is, if $f, g \in V(X)^A$, $a \in A$ and $s \in S$, then we have:

$$(f + g)(a) = f(a) + g(a) \text{ and } (s \cdot f)(a) = s \cdot f(a).$$

Now we define the notion of a weighted automaton.

Definition 55. Let A be a set and S a semiring. A *weighted automaton with input alphabet A and weights over a semiring S* is a pair (X, β_X), where X is a set (not necessarily finite) and $\beta_X : X \to V(X)^A$ is a function.

As in the case of deterministic automata, in cases that the set A and X are finite sets, we can represent the weighted automaton (X, β_X) by drawing an arrow from the state x to the state x_i with label $a : s_i$, as in $x \xrightarrow{a:s_i} x_i$, for every $a \in A$ and $x, x_i \in X$ such that $\beta_X(x)(a) = s_1x_1 + \cdots s_nx_n$, $1 \leq i \leq n$, $s_i \in S$. The following example illustrates this notation.

Example 56. Let $A = \{a, b\}$, $X = \{x_1, x_2\}$ and let R be the semiring (field) $R = \mathbb{Z}_3$ of integers modulo 3, then the diagram:

$$\begin{tikzcd}
x_1 \arrow[loop left, "{a:1, b:2}"] \arrow[rr, bend left, "b:1"] & & x_2 \arrow[ll, bend left, "b:2"] \arrow[loop right, "{a:1, b:1}"]
\end{tikzcd}$$

represents the weighted automaton (X, β_X) such that $\beta_X(x_1)(a) = x_1$, $\beta_X(x_1)(b) = 2x_1 + x_2$, $\beta_X(x_2)(a) = x_2$ and $\beta_X(x_2)(b) = 2x_1 + x_2$. $\square$

3.2 Equations for weighted automata

In this section, we study equations for weighted automata. We defined weighted automata with input alphabet A and weights over a semiring S as pairs (X, β_X) such that $\beta_X : X \to V(X)^A$. Now as $V(X)$ is the free U–object over X, where $U : \mathtt{Smod}_{\mathrm{S}} \to \mathtt{Set}$ is the forgetful functor, then we have the extension $\beta_X^\sharp : V(X) \to V(X)^A$ of β_X, i.e., we have the deterministic automaton $(V(X), \beta_X^\sharp)$ on A and hence we can study and define equations for (X, β_X) as equations for $(V(X), \beta_X^\sharp)$.

Definition 57. Let A be a set. An *equation on A* is a pair $(u, v) \in A^* \times A^*$. Given a semiring S and a weighted automaton (X, β_X) with input alphabet A and weights over S, we say that (X, β_X) satisfies the equation (u, v), denoted as $(X, \beta_X) \models (u, v)$, if the deterministic automaton $(V(x), \beta_X^\sharp)$ satisfies (u, v), i.e., if for every $\phi \in V(X)$ we have that $u(\phi) = v(\phi)$. Note that since $\beta_X^\sharp$ is the linear extension of β_X, $(X, \beta_X) \models (u, v)$ is equivalent to the property $u(x) = v(x)$ for every $x \in X$.

For cases that A and X are finite sets, we can find a generating set for the equations that a given weighted automaton satisfies. We do this in a similar way as in the case of deterministic automata, cf. Example 31.

Example 58. Let $A = \{a, b\}$, $X = \{x_1, x_2\}$ and let R be the semiring (field) $R = \mathbb{Z}_3$ of integers modulo 3 and consider the weighted automaton (X, β_X) with diagram:

$$
\begin{tikzcd}
x_1 \arrow[loop left, "{a:1, b:2}"] \arrow[rr, bend left, "b:1"] & & x_2 \arrow[loop right, "{a:1, b:1}"] \arrow[ll, bend left, "b:2"]
\end{tikzcd}
$$

Then, to find a generating set for the equations (X, β_X) satisfies we start from the tuple (x_1, x_2) that contains all the states in X and we make all the possible transitions to get deterministic automaton generated by (x_1, x_2) which is the following one:

$$
\begin{tikzcd}
(x_1, x_2) \arrow[loop above, "a"] \arrow[r, "b"] & (2x_1 + x_2, 2x_1 + x_2) \arrow[loop above, "a"] \arrow[r, "b"] & (0, 0) \arrow[loop above, "{a, b}"]
\end{tikzcd}
$$

Therefore, a generating set for the equations (X, β_X) satisfies is $\{a = \varepsilon, bb = bbb\}$. $\square$

3.3 Coequations for weighted automata

In this section, we study coequations for weighted automata. For an alphabet A and a semiring S, elements in S^{A^*} are called *power series*. If we colour the states of a weighted automaton (X, β_X), with input alphabet A and weights over S, by elements in S then we can associate to every element ϕ in $V(X)$ a power series, the *behaviour* of ϕ with respect to the given colouring. In fact, given an S–colouring $f : X \to S$ of the states of X we get the S–colouring $f^\sharp : V(X) \to S$ of $V(X)$ and hence, for every $\phi \in V(X)$, we define the behaviour $o_f(\phi) \in S^{A^*}$ as the power series defined by $o_f(\phi)(w) = f^\sharp(w(\phi))$, $w \in A^*$. The map $o_f : V(X) \to S^{A^*}$ just defined is the *observability map* of (X, β_X) associated to f. The following example shows how the observability map is defined for the weighted automaton given in Example 58.

Example 59. Consider the weighted automaton given in Example 58 and the $\mathbb{Z}_3$–colouring $f : X \to \mathbb{Z}_3$ such that $f(x_1) = 2$ and $f(x_2) = 0$. Then we have that for any $s_1, s_2 \in \mathbb{Z}_3$:

$$o_f(s_1x_1 + s_2x_2)(w) = \begin{cases} 2s_1 & \text{if } n_b(w) = 0 \\ s_1 + s_2 & \text{if } n_b(w) = 1 \\ 0 & \text{if } n_b(w) \geq 2 \end{cases}$$

where $n_b(w)$ denotes the number of b's in w. Note that the previous calculation easily follows from the deterministic automaton we drew in Example 58 to get the equations of (X, β_X). In general, for the same weighted automaton, we have that for any S–colouring $g : X \to S$ the function o_g is such that:

$$o_g(s_1x_1 + s_2x_2)(w) = \begin{cases} s_1g(x_1) + s_2g(x_2) & \text{if } n_b(w) = 0 \\ s_1\left(2g(x_1) + g(x_2)\right) + s_2\left(2g(x_1) + g(x_2)\right) & \text{if } n_b(w) = 1 \\ 0 & \text{if } n_b(w) \geq 2 \end{cases}$$

□

We now define coequations for weighted automata as follows.

Definition 60. Let A be an alphabet and S be a semiring. A set of *coequations* for A over S is a subset $Q \subseteq S^{A^*}$, that is, a set of power series. We say that a weighted automaton (X, β_X) *satisfies* Q, denoted as $(X, \beta_X) \models Q$, if for every $f \in \mathtt{Set}(X, S)$ we have $\mathrm{Im}(o_f) \subseteq Q$.

From the previous definition and the previous example we have the following.

Example 61. Consider the weighted automaton (X, β_X) given in Example 58. Then $(X, \beta_X) \models Q$ if and only if Q contains the set

$$\{o_g(s_1x_1 + s_2x_2) \mid g \in \mathtt{Set}(X, S), s_1, s_2 \in S\},$$

where $o_g(s_1x_1 + s_2x_2)$ is defined as in the previous example. □

3.4 Duality between equations and coequations

In this section, we provide a duality for equations and coequations for weighted automata. For this purpose, similar to the case of deterministic automata, we will make use of the duality between the category $\mathtt{Set}$ and a category $\mathcal{C}$ whose objects are isomorphic to objects of the form S^X, $X \in \mathtt{Set}$, where S is a set with at least two elements, and whose morphisms are isomorphic to morphisms of the form $\mathtt{Set}(f, S) : S^Y \to S^X$ for $f \in \mathtt{Set}(X, Y)$. We will avoid the formal definition of the category $\mathcal{C}$ but we will use the description just given and the fact that the contravariant functor $\mathtt{Set}(\_, S) : \mathtt{Set} \rtimes\!\!\longrightarrow \mathcal{C}$ is part of the duality.

As in the case of deterministic automata, we define the category $\mathcal{D}$ whose objects are quotients of the deterministic automaton A^*, i.e., objects of the form A^*/θ, where θ is a monoid congruence and its transition function $f_\theta : A \times A^*/\theta \to A^*/\theta$ is given by $f_\theta(a, w/\theta) = (aw)/\theta$, and whose morphisms are surjective automata morphisms, i.e., a morphism from A^*/θ_1 to A^*/θ_2 is a surjective function $h \in \mathtt{Set}(A^*/\theta_1, A^*/\theta_2)$ such that $f_{\theta_2} \circ (id_A \times h) = h \circ f_{\theta_1}$.

Now, every quotient $\nu_\theta : A^* \to A^*/\theta$ induces the injective map $\mathtt{Set}(\nu_\theta, S)$ in $\mathcal{C}$ which we identify with the subset $\mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ of S^{A^*}. In order to establish the duality, we need to characterize the properties of $\mathrm{Im}(\mathtt{Set}(\nu_\theta, S)) = \{f \circ \nu_\theta \mid f \in \mathtt{Set}(A^*/\theta, S)\}$.

Lemma 62. *Let A be a set, θ a monoid congruence on A^* and $\nu_\theta : A^* \to A^*/\theta$ the canonical map given by $\nu_\theta(w) = w/\theta$. Then the subset $Q = \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ of S^{A^*} satisfies the following properties:*

i) Q is closed under left and right derivatives. That is, if $g \in Q$ and $u \in A^*$ then ${}_ug, g_u \in Q$, where ${}_ug(v) = g(vu)$ and $g_u(v) = g(uv)$, $v \in A^*$.

ii) $\mathbf{B}(Q) \overset{def}{=} \{\mathrm{supp}(g) \mid g \in Q\} \subseteq \mathtt{Set}(A^*, 2)$ is an object in $\mathtt{CABA}$ with the usual set–theoretic operations.

iii) The set Q is determined by the atoms $\mathtt{CABA}(\mathbf{B}(Q), 2)$ of $\mathbf{B}(Q)$ in the sense that

$$Q = \left\{ g \in S^{A^*} \;\middle|\; g = \bigvee_{k \in \mathtt{CABA}(\mathbf{B}(Q),2)} s_k k,\ s_k \in S \right\},$$

where in the expression above, $g(w) = s_k$ if and only if $w \in k$.

Conversely, every subset Q of S^{A^*} satisfying the three properties above defines the congruence quotient of A^* given by $\mathtt{CABA}(\mathbf{B}(Q), 2)$. Additionally, the previous correspondence between congruences of A^* and subsets Q of S^{A^*} satisfying the three properties above is bijective.

Proof. For every $u \in A^*$ define the functions $l_u, r_u \in \mathtt{Set}(A^*, A^*)$ as $l_u(w) = wu$ and $r_u(w) = uw$. Then we have that $\nu_\theta : A^* \to A^*/\theta$ is a monoid homomorphism if and only if for every $u \in A^*$ there exists $g, g' \in \mathtt{Set}(A^*/\theta, A^*/\theta)$ such that the following diagrams commute:

$$
\begin{tikzcd}
A^* \arrow[r, "l_u"] \arrow[d, "\nu_\theta"'] & A^* \arrow[d, "\nu_\theta"] \\
A^*/\theta \arrow[r, dashed, "g"'] & A^*/\theta
\end{tikzcd}
\qquad
\begin{tikzcd}
A^* \arrow[r, "r_u"] \arrow[d, "\nu_\theta"'] & A^* \arrow[d, "\nu_\theta"] \\
A^*/\theta \arrow[r, dashed, "g'"'] & A^*/\theta
\end{tikzcd}
$$

which by duality, i.e., by applying the contravariant functor $\mathtt{Set}(\_, S)$, means that the object $Q = \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ is closed under left and right derivatives, which proves i).

To prove ii), note that $\{\mathrm{supp}(g) \mid g \in Q\} = \{f \circ \nu_\theta \mid f \in \mathtt{Set}(A^*/\theta, 2)\} = \mathrm{Im}(\mathtt{Set}(\nu_\theta, 2)) \in \mathtt{CABA}$.

To prove iii), let $g \in Q = \mathrm{Im}(\mathtt{Set}(\nu_\theta, S)) = \{f \circ \nu_\theta \mid f \in \mathtt{Set}(A^*/\theta, S)\}$, i.e., put $g = f_g \circ \nu_\theta$. As $\mathrm{supp}(g) \in \mathbf{B}(Q) \in \mathtt{CABA}$ then it is the join of atoms, i.e., elements in $\mathtt{CABA}(\mathbf{B}(Q), 2)$, and for each such atom $k \in \mathtt{CABA}(\mathbf{B}(Q), 2)$ define s_k as $s_k = (f_g \circ \nu_\theta)(w)$ for some $w \in k$. Conversely, each $\bigvee_{k \in \mathtt{CABA}(\mathbf{B}(Q),2)} s_k k$ is equal to $f \circ v_\theta \in Q$, where $f \in \mathtt{Set}(A^*/\theta, S)$ is defined as $f(w/\theta) = s_k$ and k is the unique atom that contains w.

Note that such subsets Q of S^{A^*} satisfying the three properties above are completely determined by $\mathbf{B}(Q)$ and they are in one–to–one correspondence. Hence, the lemma follows from what was proved in Proposition 43. $\square$

Subsets Q of S^{A^*} that satisfy the three properties above are called *closed subsystems* of S^{A^*}, see [81, Section 4]. Since closed subsystems are closed under right derivatives, we can consider every closed subsystem Q as a deterministic automaton (Q, f_Q) where $f_Q : Q \to Q^A$ is defined as $f_Q(g)(a) = g_a$. It is worth mentioning that $\mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ is the same as $\mathbf{cofree}(A^*/\theta)$ defined in [81] and that $\mathtt{CABA}(\mathbf{B}(Q), 2)$ is the same as $\mathbf{free}(Q)$ defined in [81]. Therefore, from the previous lemma and duality we get the identities $\mathbf{free} \circ \mathbf{cofree}(A^*/\theta) = A^*/\theta$ and $\mathbf{cofree} \circ \mathbf{free}(Q) = Q$, see [81, Theorem 5 and Corollary 8].

The next two examples illustrate the previous relationship between closed subsystems and monoid congruences.

Example 63. Let $A = \{a, b\}$ and let S be a semiring. Let $Q \subseteq S^{A^*}$ be the set

$$Q = \left\{ g \in S^{A^*} \;\middle|\; \forall w \in A^* g(w) = g\left(b^{n_b(w)}\right) \right\},$$

where $n_b(w)$ is the number of b's in the word w. Then one can verify that Q is a closed subsystem, i.e., it satisfies the three properties of the lemma above. Note that the set $\mathtt{CABA}(\mathbf{B}(Q), 2)$ is given by $\{a^*, a^*ba^*, a^*(ba^*)^2, a^*(ba^*)^3, \ldots\}$ and that $Q = \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$, if we take for θ the monoid congruence generated by $\{a = \varepsilon\}$. $\square$

Example 64. Let $A = \{a, b\}$ and let θ be the monoid congruence on A^* generated by $\{a = \varepsilon, bb = bbb\}$. Then we have that $A^*/\theta = \{\varepsilon/\theta, b/\theta, bb/\theta\}$. If $S = \mathbb{Z}_3$, then by the previous lemma we know that

$$\mathrm{Im}(\mathtt{Set}(\nu_\theta, S)) = \{s_1(\varepsilon/\theta) + s_2(b/\theta) + s_3(bb/\theta) \mid s_i \in \mathbb{Z}_3\},$$

that is, $\mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ has 27 elements. For instance, we can verify that the set $\mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ is closed under left derivatives by noticing that:

$$\begin{aligned}
{}_a\left(s_1(\varepsilon/\theta) + s_2(b/\theta) + s_3(bb/\theta)\right) &= s_1(\varepsilon/\theta) + s_2(b/\theta) + s_3(bb/\theta) \\
{}_b\left(s_1(\varepsilon/\theta) + s_2(b/\theta) + s_3(bb/\theta)\right) &= s_2(\epsilon/\theta) + s_3(b/\theta + bb/\theta).
\end{aligned}$$

□

We now define the categories of monoid congruences on A^* and of closed subsystems of S^{A^*} which will be dual categories. In fact, we define the categories $\mathcal{D}$ and $\mathcal{K}$ as follows:

$$\begin{aligned}
\mathrm{objects}(\mathcal{D}) &= \{(A^*/\theta, f_\theta) \mid \theta \text{ is a monoid congruence on } A^*\} \\
\mathrm{arrows}(\mathcal{D}) &= \{e : A^*/\theta_1 \to A^*/\theta_2 \mid e \text{ is a surjective automata homomorphism}\} \\
\mathrm{objects}(\mathcal{K}) &= \{(Q, f_Q) \mid Q \text{ is a closed subsystem of } S^{A^*}\} \\
\mathrm{arrows}(\mathcal{K}) &= \{m : Q \to Q' \mid m \text{ is an injective automata homomorphism}\}
\end{aligned}$$

Using a similar argument as for the case of deterministic automata, we have that the categories $\mathcal{D}$ and $\mathcal{K}$ are dual, [81, Theorem 10]. We illustrate this duality with the following example.

Example 65. Let $A = \{a, b\}$ and let S be a semiring. Let θ_1 be the monoid congruence on A^* generated by $\{a = \varepsilon, bb = bbb\}$ and let θ_2 be the monoid congruence on A^* generated by $\{a = \varepsilon, b = bb\}$. Clearly we have that $\theta_1 \subseteq \theta_2$ and hence, we have the surjective automata homomorphism $e : A^*/\theta_1 \to A^*/\theta_2$ given by $e(w/\theta_1) = w/\theta_2$. By the duality just described, we have the following situation:

$$\begin{tikzcd}
\varepsilon/\theta_1 \arrow[loop above, "a"] \arrow[r, "b"] & b/\theta_1 \arrow[loop above, "a"] \arrow[r, "b"] \arrow[ddd, "e"'] & bb/\theta_1 \arrow[loop above, "{a,b}"] & & \{s_1(\varepsilon/\theta_1) + s_2(b/\theta_1) + s_3(bb/\theta_1) \mid s_i \in S\} \\
& & {} \arrow[r, mapsto, "\mathtt{Set}(_, S)"] & {} & \\
& & {} & {} \arrow[l, mapsto, "\mathtt{CABA}(\mathbf{B}(\ ), 2)"'] & \\
\varepsilon/\theta_2 \arrow[loop above, "a"] \arrow[r, "b"] & b/\theta_2 \arrow[loop above, "{a,b}"] & & & \{s_1(\varepsilon/\theta_2) + s_2(b/\theta_2) \mid s_i \in S\} \arrow[uuu, "m"]
\end{tikzcd}$$

where the homomorphism $m : \mathtt{Set}(A^*/\theta_2, S) \to \mathtt{Set}(A^*/\theta_1, S)$ is given by

$$m(s_1(\varepsilon/\theta_2) + s_2(b/\theta_2)) = s_1(\varepsilon/\theta_1) + s_2(b/\theta_1) + s_2(bb/\theta_1).$$

□

Similar to the case of deterministic automata, given a set of equations E on A, we can define the class of weighted automata on A that satisfy E. We can also do the same for the case of coequations, i.e., define the class of weighted automata that satisfy a given set of coequations on A. Now, we have that classes of weighted automata defined by monoid congruences are the same as classes of weighted automata defined by closed subsystems.

Theorem 66. *Let A be a set and let S be a semiring. Let $Q \subseteq S^{A^*}$ be a closed subsystem, θ a monoid congruence on A^*, and (X, β_X) a weighted automaton with input symbols in A and weights in S. Then*

i) $(X, \beta_X) \models \theta$ *if and only if* $(X, \beta_X) \Vdash \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$.

ii) $(X, \beta_X) \Vdash Q$ *if and only if* $(X, \beta_X) \models \mathtt{CABA}(\mathbf{B}(Q), 2)$.

Proof. $i)$ Let $\iota : \mathrm{Im}(\mathtt{Set}(\nu_\theta, S)) \to S^{A^*}$ be the inclusion homomorphism, i.e., $\iota \cong \mathtt{Set}(\nu_\theta, S)$ and let $P := \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$. For every $\phi \in V(X)$ and $f \in \mathtt{Set}(X, S)$ we have the following commutative diagram:

$$\begin{tikzcd}
A^* \arrow[r, "\phi^\sharp"] \arrow[rr, bend left, "o_f(\phi)"] & V(X) \arrow[r, "f^\sharp"] & S \\
1 \arrow[u, "\eta_1"] \arrow[ur, "\phi"'] & X \arrow[u, "\eta_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Assume that $(X, \beta_X) \models \theta$. Then, for every $\phi \in V(X)$ the homomorphism $\phi^\sharp$ factors through ν_θ as $\phi^\sharp = h_s \circ \nu_\theta$. Then, by duality, we have $\mathtt{Set}(\phi^\sharp, S) = \iota \circ \mathtt{Set}(h_s, S)$ which means that for every colouring $f : X \to S$ we have that $o_f(\phi) = f^\sharp \circ \phi^\sharp \in P$. Therefore $(X, \beta_X) \Vdash P$.

Conversely, assume that $(X, \beta_X) \Vdash P$, that is, that $o_f(\phi) \in P$. Fix $(u, v) \in \theta$, if $(X, \beta_X) \not\models (u, v)$ then there exists $x \in X$ such that $u(x) \neq v(x)$, since $u(x), v(x) \in V(X)$ are formal sums that are different then there exists $f : X \to S$ such that $f^\sharp(u(x)) \neq f^\sharp(v(x))$, i.e., $o_f(x)(u) \neq o_f(x)(v)$. Now, as $o_f(x) \in P = \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ then there exists $c_f : A^*/\theta \to S$ such that $o_f(x) = c_f \circ \nu_\theta$ but then

$$o_f(x)(u) = (c_f \circ \nu_\theta)(u) = c_f(u/\theta) = c_f(v/\theta) = (c_f \circ \nu_\theta)(v) = o_f(x)(v)$$

which contradicts $o_f(x)(u) \neq o_f(x)(v)$. Therefore $(X, \beta_X) \models (u, v)$ and hence $(X, \beta_X) \models \theta$.

$ii)$ Follows from $i)$ since monoid congruences on A^* are in one–to–one correspondence with closed subsystems by Lemma 62. $\square$

Remark. In the previous proof, to prove that $(X, \beta_X) \Vdash P$ implies $(X, \beta_X) \models \theta$, we cannot proceed as in the proof of Theorem 49. That is, for a fixed $\phi \in X$, and assuming that $(X, \beta_X) \Vdash P$, we have the function $g_\phi : S^X \to P$ given by $g_\phi(f) = o_f(\phi)$, but then we do not know if by duality we will get a function $h_\phi : A^*/\theta \to X$

such that $\mathtt{Set}(h_\phi, S) \cong g_\phi$ (see diagram in the previous proof). Also, switching to $S^{V(X)}$ instead of S^X does not help since there are elements in $\gamma \in S^{V(X)}$ that are not of the form $\gamma = f^\sharp$ for some $f \in S^X$.

As an illustration of the previous theorem we have the following.

Example 67. Let $A = \{a, b\}$ and let S be a semiring. Let θ be the monoid congruence on A^* generated by $\{a = \varepsilon\}$. Then we have that $P := \mathrm{Im}(\mathtt{Set}(\nu_\theta, S))$ is the closed subsystem $P = \left\{g \in S^{A^*} \;\middle|\; \forall w \in A^* g(w) = g\left(n_b(w)\right)\right\}$. In this case, we have, for a weighted automaton (X, β_X) with inputs on A and weights on S, that

$$(X, \beta_X) \models \theta \;\Leftrightarrow\; (X, \beta_X) \mid\models P \;\Leftrightarrow\; \forall x \in X \;\beta_X(a, x) = x.$$

□

3.5 Linear equations and coequations

In this section, we work with a more general kind of equations for weighted automata than the ones introduced in Section 3.2. We defined equations to be pairs $(u, v) \in A^* \times A^*$ and we said that a weighted automaton (X, β_X) with inputs in A and weights on S satisfies (u, v) if for any $x \in X$, $u(x) = v(x)$. Now, since $u(x) \in V(X)$ (the free S-semimodule generated by X) for any $u \in A^*$, it makes sense to define expressions like

$$(s_1 w_1 + \cdots + s_n w_n)(x) := s_1 w_1(x) + \cdots + s_n w_n(x)$$

for elements $s_i \in S$ and $w_i \in A^*$, that is, for any $\varphi = \sum_{i=1}^n s_i w_i \in V(A^*)$ and $x \in X$ we get an element $\varphi(x) \in V(X)$. In this case we can ask whether $\varphi(x) = \psi(x)$ holds or not for some given $\varphi, \psi \in V(A^*)$ and $x \in X$. This motivates the following definition.

Definition 68. Let A be a set. A pair $(\varphi, \psi) \in V(A^*)$ will be called a *linear equation* on A. Note that this is now a more general kind of equation since $A^* \subseteq V(A^*)$. We write $(X, \beta_X) \models (\varphi, \psi)$, and say: (X, β_X) satisfies the equation (φ, ψ), if for every $x \in X$, $\varphi(x) = \psi(x)$.

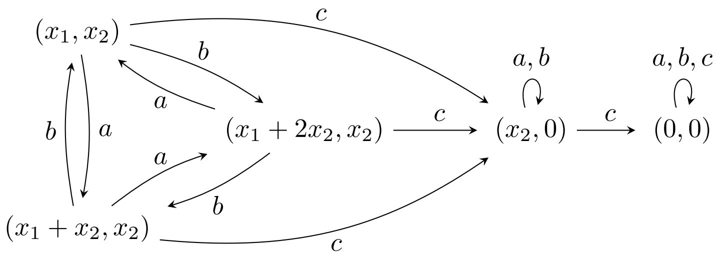
The following example shows some particular cases of satisfaction of linear equations and compares it with the concept of an equation.

Example 69. Let (X, β_X) be the weighted automaton with inputs in $A = \{a, b, c\}$ and weights on $\mathbb{Z}_3$ given by the following diagram:

$$\begin{tikzcd} x_1 \arrow[loop left, "{a:1, b:1}"] \arrow[rr, "{a:1, b:2, c:1}"] & & x_2 \arrow[loop right, "{a:1, b:1}"] \end{tikzcd}$$

Then one easily verifies that the following linear equations are satisfied by (X, β_X): $ac = c$, $a + c = b$, and $2c + b = b^2$.

Now, if we construct the deterministic automaton on A to get the equations in $A^* \times A^*$ that (X, β_X) satisfies, we get the automaton:



Hence, the monoid congruence $\mathtt{Eq}(X, \beta_X)$ of equations that (X, β_X) satisfies is generated by:

$$\{c^2 = c^3, b^3 = \epsilon, a^3 = \epsilon, ab = \epsilon, ba = \epsilon, ac = c, bc = c, ca = c, cb = c\}.$$

Observe that none of the linear equations $a + c = b$ or $2c + b = b^2$ can be deduced from $\mathtt{Eq}(X, \beta_X)$, even though (X, β_X) satisfies both of them. □

As the previous example shows, equations of the form $(\varphi, \psi) \in V(A^*) \times V(A^*)$ that are satisfied by (X, β_X) are interesting. We now turn to define the notion of linear automata.

Definition 70. Let $\mathbb{K}$ be a field and A be a set. A $\mathbb{K}$*–linear automaton,* or $\mathtt{Vec}_{\mathbb{K}}$ *automaton,* on an alphabet A is a pair (X, β_X) such that $X \in \mathtt{Vec}_{\mathbb{K}}$ and $\beta_X \in \mathtt{Vec}_{\mathbb{K}}(X, X^A)$. Note that every weighted automaton over a field $\mathbb{K}$ yields a $\mathtt{Vec}_{\mathbb{K}}$ automaton and conversely. Note that we have the converse since every $X \in \mathtt{Vec}_{\mathbb{K}}$ is free, i.e., it has a basis, which is not the case in $\mathtt{Smod}_{\mathbb{S}}$.

Let $X \in \mathtt{Vec}_{\mathbb{K}}$ and let A be a set. We denote by $A \cdot X$ the coproduct $\coprod_{a \in A} X$ in $\mathtt{Vec}_{\mathbb{K}}$, whose underlying set is given by the set of all functions in X^A with finite support, i.e., an element $f \in A \cdot X$ is an element $f \in X^A$ such that the set $\operatorname{supp}(f) = \{a \in A \mid f(a) \neq 0\}$ is finite. Operations in $A \cdot X$ are componentwise, that is, if $f, g \in A \cdot X \subseteq X^A$, $a \in A$ and $k \in \mathbb{K}$, then:

$$(f + g)(a) = f(a) + g(a) \ \text{ and } \ (k \cdot f)(a) = k \cdot f(a).$$

Similarly to the case of deterministic automata, we can see a $\mathbb{K}$–linear automaton as an algebra as well as a coalgebra. In fact, every $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) on A corresponds to a pair $(X, \widetilde{\beta_X})$ where $\widetilde{\beta_X} : A \cdot X \to X$. The previous observation follows from the following commutative diagram in $\mathtt{Vec}_{\mathbb{K}}$:

$$\begin{tikzcd} X \arrow[r, "\iota_{a'}"] \arrow[d, "\beta_X"'] \arrow[dr, "\pi_{a'} \circ \beta_X"] & A \cdot X = \coprod_{a \in A} X \arrow[d, "\widetilde{\beta_X}"] \\ \prod_{a \in A} X = X^A \arrow[r, "\pi_{a'}"'] & X \end{tikzcd}$$

where $\pi_{a'}$ is the a'–th projection and $\iota_{a'}$ is the a'–th inclusion obtained from the universal property of the product and coproduct, respectively. The previous algebraic and coalgebraic perspective allows us to study equations and coequations for $\mathtt{Vec}_{\mathbb{K}}$ automata, respectively.

We will study equations for $\mathtt{Vec}_{\mathbb{K}}$ automata by taking into account the situation we just described. In fact, for a fixed set A, consider the endofunctor $H : \mathtt{Vec}_{\mathbb{K}} \to \mathtt{Vec}_{\mathbb{K}}$ defined as $H(X) = A \cdot X = \coprod_{a\in A} X$. Then we have that a $\mathtt{Vec}_{\mathbb{K}}$ automaton is exactly an H–algebra. Consider the forgetful functor $U : \mathtt{alg}(H) \to \mathtt{Vec}_{\mathbb{K}}$ from the category of H–algebras to $\mathtt{Vec}_{\mathbb{K}}$, then the free U–object over $\mathbb{K} \in \mathtt{Vec}_{\mathbb{K}}$ is the object $V(A^*) = (V(A^*), \varpi) \in \mathtt{alg}(H)$ where the morphism $\varpi : A\cdot V(A^*) \to V(A^*)$ in $\mathtt{Vec}_{\mathbb{K}}$ is such that $(\varpi \circ \iota_a)(s_1w_1 + \cdots + s_nw_n) = s_1aw_1 + \cdots + s_naw_n$ and the morphism $\eta_{\mathbb{K}} : \mathbb{K} \to V(A^*)$ is such that $\eta_{\mathbb{K}}(1) = \varepsilon$. From this, we define equations for $\mathtt{Vec}_{\mathbb{K}}$ automata as follows.

Definition 71. An *equation* for $\mathtt{Vec}_{\mathbb{K}}$ automata is a pair $(\varphi, \psi) \in V(A^*)\times V(A^*)$. We say that a $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) *satisfies* (φ, ψ), denoted as $(X, \beta_X) \models (\varphi, \psi)$, if for every $x \in X$ we have that $(\varphi, \psi) \in \ker(\overline{x}^\sharp)$, where $\overline{x} \in \mathtt{Vec}_{\mathbb{K}}(\mathbb{K}, X)$ is the linear map such that $\overline{x}(1) = x$ and $\overline{x}^\sharp \in \mathtt{alg}(H)((V(A^*), \gamma), (X, \beta_X))$ is the extension of the linear map $\overline{x}$. Or, equivalently, $(X, \beta_X) \models (\varphi, \psi)$, if for every $t \in \mathtt{Vec}_{\mathbb{K}}(\mathbb{K}, X)$ we have that $(\varphi, \psi) \in \ker(t^\sharp)$.

Similar to the case of deterministic automata, we have the following kind of congruences for $\mathtt{Vec}_{\mathbb{K}}$ automata.

An equivalence relation θ on $V(A^*)$ is a *linear congruence* on $V(A^*)$ if:

i) $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \theta$ imply $(\varphi_1 + \varphi_2, \psi_1 + \psi_2) \in \theta$.

ii) $(\varphi, \psi) \in \theta$, $k \in \mathbb{K}$, and $a \in A$ imply $(k\varphi, k\psi), (a\varphi, a\psi), (\varphi a, \psi a) \in \theta$. Here, for $\phi = \sum_{i=1}^n k_iw_i \in V(A^*)$, the elements $k\phi$, $a\phi$, and ϕa in $V(A^*)$ are defined as

$$k\phi := \sum_{i=1}^{n} kk_iw_i, \quad a\phi = \sum_{i=1}^{n} k_iaw_i, \quad \phi a = \sum_{i=1}^{n} k_iw_ia.$$

If θ is a linear congruence on $V(A^*)$ then the set $V(A^*)/\theta$ has the structure of a $\mathtt{Vec}_{\mathbb{K}}$ automaton which is the structure map f_θ in $\mathtt{Vec}_{\mathbb{K}}(A \cdot V(A^*)/\theta, V(A^*)/\theta)$ that makes the linear morphism $\nu_\theta : V(A^*) \to V(A^*)/\theta$ an H–algebra morphism in $\mathtt{alg}(H)((V(A^*), \varpi), (V(A^*)/\theta, f_\theta))$, where $\nu_\theta(\phi) = \phi/\theta$. We denote by $\mathtt{Eq}(X, \beta_X)$ the set of equations that the $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) satisfies, which can be easily shown to be a linear congruence on $V(A^*)$. Note that $\mathtt{Eq}(X, \beta_X)$ is the intersection $\bigcap_{x\in X} \ker(\overline{x}^\sharp) = \bigcap_{t\in \mathtt{Vec}_{\mathbb{K}}(\mathbb{K},X)} \ker(t^\sharp)$.

To study coequations for $\mathtt{Vec}_{\mathbb{K}}$ automata, we use a dual argument by seeing $\mathtt{Vec}_{\mathbb{K}}$ automata as coalgebras for a functor. In fact, for a fixed set A, consider the endofunctor $G : \mathtt{Vec}_{\mathbb{K}} \to \mathtt{Vec}_{\mathbb{K}}$ defined as $G(X) = X^A = \prod_{a\in A} X$. Then we have that a $\mathtt{Vec}_{\mathbb{K}}$ automaton is exactly a G–coalgebra. Consider the forgetful functor $U : \mathtt{coalg}(G) \to \mathtt{Vec}_{\mathbb{K}}$ from the category of G–coalgebras to $\mathtt{Vec}_{\mathbb{K}}$. Then we have that the cofree U–object over $\mathbb{K}$ is the object $\mathbb{K}^{A^*} = (\mathbb{K}^{A^*}, \sigma)$ where the morphism

$\sigma : \mathbb{K}^{A^*} \to (\mathbb{K}^{A^*})^A$ in $\mathtt{Vec}_{\mathbb{K}}$ is such that $\sigma(g)(a) = g_a$, where $g_a(w) = g(aw)$, and the morphism $\epsilon_{\mathbb{K}} : \mathbb{K}^{A^*} \to \mathbb{K}$ is defined as $\epsilon_{\mathbb{K}}(g) = g(\varepsilon)$. We define coequations for $\mathtt{Vec}_{\mathbb{K}}$ automata as follows.

Definition 72. A set of *coequations* is a subspace S of $\mathbb{K}^{A^*}$. We say that a $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) *satisfies* S, denoted as $(X, \beta_X) \Vvdash S$, if for every $d \in \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K})$ we have that $\mathrm{Im}(d^\flat) \subseteq S$, where the morphism $d^\flat : X \to \mathbb{K}^{A^*}$ is obtained from the universal property of the cofree U–object $\mathbb{K}^{A^*}$. The morphism $d^\flat$ is also called the *observability map with respect to* d, which is also denoted as o_d. For every $x \in X$ the element $d^\flat(x) = o_d(x)$ is the behaviour of the state x with respect to the colouring d.

The previous two situations for equations and coequations for $\mathtt{Vec}_{\mathbb{K}}$ automata are captured by the following commutative diagram in $\mathtt{Vec}_{\mathbb{K}}$:

$$\begin{tikzcd}
V(A^*) \arrow[rr, "\overline{x}^\sharp"] & & X \arrow[rr, "d^\flat"] \arrow[drr, "d"'] & & \mathbb{K}^{A^*} \arrow[d, "\epsilon_{\mathbb{K}}"] \\
\mathbb{K} \arrow[u, "\eta_{\mathbb{K}}"] \arrow[urr, "\overline{x}"'] & & & & \mathbb{K}
\end{tikzcd}$$

where $x \in X$. Note that $d^\flat(x)(w) = (d \circ \overline{x}^\sharp)(w)$. Also, the minimum set of coequations that a given $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) satisfies, denoted as $\mathtt{Coeq}(X, \beta_X)$, is given by the subspace of $\mathbb{K}^{A^*}$ generated by the union $\bigcup_{d \in \mathtt{Vec}_{\mathbb{K}}(X,\mathbb{K})} \mathrm{Im}(d^\flat)$. We have that $\mathtt{Eq}(X, \beta_X)$ is a linear congruence, from which we get that $V(A^*)/\mathtt{Eq}(X, \beta_X)$ has the structure of a $\mathtt{Vec}_{\mathbb{K}}$ automaton which is the one induced by the surjective morphism $\nu_{\mathtt{Eq}(X,\beta_X)}$. In a similar way, the space $\mathtt{Coeq}(X, \beta_X)$ also has the structure of a $\mathtt{Vec}_{\mathbb{K}}$ automaton, which is the one induced by the inclusion map $\iota_{\mathtt{Coeq}(X,\beta_X)} \in \mathtt{Vec}_{\mathbb{K}}(\mathtt{Coeq}(X, \beta_X), \mathbb{K}^{A^*})$, that is, $\mathtt{Coeq}(X, \beta_X)$ is closed under right derivatives. In fact, for every $d \in \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K})$ and $x \in X$ we have that $d^\flat(x) = d \circ \overline{x}^\sharp$, from which we get that $d^\flat(x)_a = (d \circ \pi_a \circ \beta_X)^\flat(x)$, since for every $w \in A^*$ we have:

$$\begin{aligned}
d^\flat(x)_a(w) &= d^\flat(x)(aw) = (d \circ \overline{x}^\sharp)(aw) = (d \circ \overline{x}^\sharp \circ \varpi)(a, w) = (d \circ \widetilde{\beta_X})(a, \overline{x}^\sharp(w)) \\
&= (d \circ \widetilde{\beta_X} \circ \iota_a)(\overline{x}^\sharp(w)) = (d \circ \pi_a \circ \beta_X)(\overline{x}^\sharp(w)) = (d \circ \pi_a \circ \beta_X \circ \overline{x}^\sharp)(w) \\
&= (d \circ \pi_a \circ \beta_X)^\flat(x)(w)
\end{aligned}$$

where the identity $(d \circ \overline{x}^\sharp \circ \varpi)(a, w) = (d \circ \widetilde{\beta_X})(a, \overline{x}^\sharp(w))$ follows from the fact that $\overline{x}^\sharp$ is a morphism in $\mathtt{alg}(H)$ and the equality $(d \circ \widetilde{\beta_X} \circ \iota_a)(\overline{x}^\sharp(w)) = (d \circ \pi_a \circ \beta_X)(\overline{x}^\sharp(w))$ follows from the commutative square that relates β_X and $\widetilde{\beta_X}$. Hence, the result follows from the fact that elements in $\mathtt{Coeq}(X, \beta_X)$ are linear combinations of the elements above (clearly, derivatives distribute over linear combinations).

For a $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) denote by $\mathbf{free}(X, \beta_X)$ the $\mathtt{Vec}_{\mathbb{K}}$ automaton $V(A^*)/\mathtt{Eq}(X, \beta_X)$ and by $\mathbf{cofree}(X, \beta_X)$ the $\mathtt{Vec}_{\mathbb{K}}$ automaton $\mathtt{Coeq}(X, \beta_X)$. Then we have the following.

Theorem 73. *Let A be a set and let θ be a linear congruence on $V(A^*)$. Then we have that* $(\mathbf{free} \circ \mathbf{cofree})(V(A^*)/\theta) = V(A^*)/\theta$.

Proof. We will make use of the following facts:

a) For every $d \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K})$, $\phi \in V(A^*)$ and $w \in A^*$ we have $d^\flat(\phi/\theta)(w) = d(w\phi/\theta)$.

b) For every $g \in \mathbb{K}^{A^*}$ we have that $\overline{g}^\sharp \in \mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K}^{A^*})$ is the linear map such that for every $u, w \in A^*$, $\overline{g}^\sharp(u)(w) = g_u(w) = g(uw)$. That is, if $\phi = \sum_{i=1}^n s_i w_i$, then $\overline{g}^\sharp(\phi)(w) = \sum_{i=1}^n s_i g(w_i w)$.

We have to show that $\mathtt{Eq}(\mathbf{cofree}(V(A^*)/\theta)) = \theta$.

($\subseteq$) Consider any pair $(\phi, \psi) \in \mathtt{Eq}(\mathbf{cofree}(V(A^*)/\theta))$ and assume that $\phi/\theta \neq 0/\theta$ in $V(A^*)/\theta$. As $V(A^*)/\theta$ is a vector space and $\phi/\theta \neq 0/\theta$, there exists a basis $\mathcal{B}$ of $V(A^*)/\theta$ containing ϕ/θ, now define the linear map $d_\phi \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K})$ such that $d_\phi(\phi/\theta) = 1$ and $d_\phi(\varphi/\theta) = 0$ for any $\varphi/\theta \in \mathcal{B} \setminus \{\phi/\theta\}$. Consider the element $d^\flat_\phi(\varepsilon/\theta) \in \mathbf{cofree}(V(A^*)/\theta)$. Then, as $(\phi, \psi) \in \mathtt{Eq}(\mathbf{cofree}(V(A^*)/\theta))$, we have that $\phi(d^\flat_\phi(\varepsilon/\theta)) = \psi(d^\flat_\phi(\varepsilon/\theta))$, in particular

$$(\star) \qquad \phi(d^\flat_\phi(\varepsilon/\theta))(\varepsilon) = \psi(d^\flat_\phi(\varepsilon/\theta))(\varepsilon).$$

Now, if we put $\phi = \sum_{i=1}^n s_i w_i$, then form a) and b) above we have that :

$$(\star\star) \qquad \begin{aligned} \phi(d^\flat_\phi(\varepsilon/\theta))(\varepsilon) &= \overline{d^\flat_\phi(\varepsilon/\theta)}^\sharp(\phi)(\varepsilon) = \sum_{i=1}^n s_i \overline{d^\flat_\phi(\varepsilon/\theta)}^\sharp(w_i)(\varepsilon) \\ &= \sum_{i=1}^n s_i d^\flat_\phi(\varepsilon/\theta)(w_i) = \sum_{i=1}^n s_i d_\phi(w_i/\theta) \\ &= d_\phi(\phi/\theta) = 1 \end{aligned}$$

A similar argument shows that $\psi(d^\flat_\phi(\varepsilon/\theta))(\varepsilon) = d_\phi(\psi/\theta)$ which implies, from $(\star)$ above, that $d_\phi(\psi/\theta) = 1$. From this equality and the definition of d_ϕ we get that

$$\psi/\theta = \phi/\theta + \sum_{j=1}^m t_j \varphi_j/\theta \qquad (\dagger)$$

for some $\varphi_j/\theta \in \mathcal{B} \setminus \{\phi/\theta\}$ and $t_j \in \mathbb{K}$.

Now, for every $j = 1, \ldots, m$, define the map $d_{\varphi_j} \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K})$ such that $d_{\varphi_j}(\varphi_j/\theta) = 1$ and $d_{\varphi_j}(\varphi/\theta) = 0$ for any $\varphi/\theta \in \mathcal{B} \setminus \{\varphi_j/\theta\}$, and consider the element $d^\flat_{\varphi_j}(\varepsilon/\theta) \in \mathbf{cofree}(V(A^*)/\theta)$. Then, as $(\phi, \psi) \in \mathtt{Eq}(\mathbf{cofree}(V(A^*)/\theta))$, we have that $\phi(d^\flat_{\varphi_j}(\varepsilon/\theta)) = \psi(d^\flat_{\varphi_j}(\varepsilon/\theta))$, in particular

$$\phi(d^\flat_{\varphi_j}(\varepsilon/\theta))(\varepsilon) = \psi(d^\flat_{\varphi_j}(\varepsilon/\theta))(\varepsilon)$$

Similar to the reasoning above, we have that:

$$0 = d_{\phi_j}(\varphi) = \phi(d^\flat_{\varphi_j}(\varepsilon/\theta))(\varepsilon) = \psi(d^\flat_{\varphi_j}(\varepsilon/\theta))(\varepsilon) = d_{\varphi_j}(\psi) = t_j$$

which from equation (†) we get that $\psi/\theta = \phi/\theta$, i.e. $(\psi, \phi) \in \theta$.

($\supseteq$) Let $(\phi, \psi) \in \theta$, $d \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K})$, $\varphi \in V(A^*)$ and put $x = d^\flat(\varphi/\theta)$. We need to prove that $\phi(x) = \psi(x)$ in $\mathbb{K}^{A^*}$. In fact, for any $w \in A^*$, by using a) and b) above and a similar reasoning as in ($\star\star$) above we get that

$$\phi(x)(w) = d(\phi w \varphi/\theta) = d(\psi w \varphi/\theta) = \psi(x)(w)$$

which is true since $(\phi, \psi) \in \theta$ implies $(\phi w \varphi, \psi w \varphi) \in \theta$ because θ is a linear congruence. □

Note that the inclusion from right to left above does not hold in general when considering semimodules (over a semiring) instead of vector spaces. In fact, let $\mathbb{N}$ be the semiring of natural numbers with the usual sum and product, $A = \{a, b\}$, and let θ be the linear congruence on $V(A^*)$ associated to the partition $\{\{0\}, V(A^*) \smallsetminus \{0\}\}$ of $V(A^*)$. Then $V(A^*)/\theta \cong \mathbb{B}$, the Boolean semiring, where the action of $\mathbb{N}$ on $\mathbb{B}$ is given by $n \cdot x = 0$ if and only if $n = 0$ or $x = 0$. However, one can verify that $\mathbf{cofree}(V(A^*)/\theta)$ has only one element and therefore it satisfies any identity. It follows that $V(A^*)/\theta$ cannot be a subset of $(\mathbf{free} \circ \mathbf{cofree})(V(A^*)/\theta) = 1$.

Similarly to the case of weighted automata we can show that for a $\mathtt{Vec}_{\mathbb{K}}$ automaton (X, β_X) and a linear congruence θ we have that

$$(X, \beta_X) \models \theta \text{ if and only if } (X, \beta_X) \Vdash \mathbf{cofree}(V(A^*)/\theta).$$

Example 74. Let $A = \{x, y\}$, and let $\theta = \langle xy = yx \rangle$ be the linear congruence generated by the equation $xy = yx$. Then the $\mathtt{Vec}_{\mathbb{K}}$ automaton $V(A^*)/\theta$ is isomorphic to $\mathbb{K}[x, y]$, the ring of polynomials on indeterminates x and y with coefficients in $\mathbb{K}$. Here the transition function on $\mathbb{K}[x, y]$ is (right) multiplication, that is, for a polynomial $p(x, y) \in \mathbb{K}[x, y]$ we have that $x(p(x, y)) = p(x, y)x$, and $y(p(x, y)) = p(x, y)y$. Then, $\mathbf{cofree}\,(V(A^*)/\theta)$ is the set

$$\left\{ L \in \mathbb{K}^{A^*} \;\middle|\; \forall w_1, w_2 \Big(n_x(w_1) = n_x(w_2) \wedge n_y(w_1) = n_y(w_2) \Rightarrow L(w_1) = L(w_2) \Big) \right\},$$

where $n_x(w)$ is the number of x's in the word $w \in A^*$. Notice that $\mathbf{cofree}\,(V(A^*)/\theta)$ is isomorphic to $\mathbb{K}^{\mathcal{M}(\mathbb{K}[x,y])}$ where $\mathcal{M}(\mathbb{K}[x, y])$ are the monic monomials in $\mathbb{K}[x, y]$. □

Example 75. Let $A = \{x, y\}$, and, for a fixed $k \in \mathbb{K} \smallsetminus \{0\}$, let $\theta = \langle xy = yx, y - k = 0 \rangle$ be the linear congruence generated by the equations $xy = yx$ and $y - k = 0$. Then the $\mathtt{Vec}_{\mathbb{K}}$ automaton $V(A^*)/\theta$ is isomorphic to $\mathbb{K}[x, y]/\langle y - k \rangle \cong \mathbb{K}[x]$, where $\langle y - k \rangle$ is the ideal generated by $y - k$. A similar calculation as in the previous example shows that $\mathbf{cofree}\,(V(A^*)/\theta) \cong \mathbb{K}^{\{x\}^*} \cong \mathbb{K}^{\mathbb{N}}$. □

Example 76. Let A be a finite alphabet, and θ a linear congruence on $V(A^*)$ such that for any $x, y \in A$, $(xy, yx) \in \theta$. We claim that $\theta = \langle C \rangle$ for some finite $C \subseteq \theta$. In fact, if $A = \{x_1, \ldots, x_n\}$ then $V(A^*)/\theta \cong \mathbb{K}[x_1, \ldots, x_n]/I$ for the ideal I of $\mathbb{K}[x_1, \ldots, x_n]$ given by $I = \{\phi - \psi \mid (\phi, \psi) \in \theta\}$, which is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$ since θ is a linear congruence (here $\phi - \psi$ is calculated as in $\mathbb{K}[x_1, \ldots, x_n]$). Then, by Hilbert basis theorem, we have that I is finitely generated, say $I = \langle \varphi_1, \ldots, \varphi_m \rangle$. It follows that

$$\theta = \langle \{x_i x_j = x_j x_i \mid 1 \leq i < j \leq n\} \cup \{\varphi_l = 0 \mid 1 \leq l \leq m\} \rangle.$$

□

Now, as A^* is a basis for the space $V(A^*)$, i.e., $V(A^*)$ is the free vector space over the set A^*, then we have that $\mathbb{K}^{A^*} = \mathtt{Set}(A^*, \mathbb{K})$ and $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ are isomorphic in $\mathtt{Vec}_{\mathbb{K}}$. In order to complete the duality, we work with $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ instead of $\mathbb{K}^{A^*}$ and use the duality between $\mathtt{Vec}_{\mathbb{K}}$ and $\mathtt{StVec}_{\mathbb{K}}$, where $\mathbb{K}$ is a finite field. We complete the duality with the following result.

Proposition 77. *Let $\mathbb{K}$ be a finite field. Let S be a subobject of $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ in $\mathtt{StVec}_{\mathbb{K}}$ which is closed under left right derivatives, then $S = (\mathbf{cofree} \circ \mathbf{free})(S)$.*

Proof. Let $i : S \hookrightarrow \mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ be the inclusion morphism in $\mathtt{StVec}_{\mathbb{K}}$. Then, by applying the functor $\mathtt{StVec}_{\mathbb{K}}(\_, \mathbb{K})$ we get the surjective morphism $\mathtt{StVec}_{\mathbb{K}}(i, \mathbb{K}) =: e : \mathtt{StVec}_{\mathbb{K}}(\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K}), \mathbb{K}) \twoheadrightarrow \mathtt{StVec}_{\mathbb{K}}(S, \mathbb{K})$ in $\mathtt{Vec}_{\mathbb{K}}$. By duality, the object $\mathtt{StVec}_{\mathbb{K}}(\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K}), \mathbb{K})$ is isomorphic to $V(A^*)$ in which each element $\phi \in V(A^*)$ corresponds to the evaluation morphism $ev_\phi \in \mathtt{StVec}_{\mathbb{K}}(\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K}), \mathbb{K})$ such that $ev_\phi(f) = f(\phi)$. Also, since e is onto then $\mathtt{StVec}_{\mathbb{K}}(S, \mathbb{K})$ is isomorphic in $\mathtt{Vec}_{\mathbb{K}}$ to $V(A^*)/\theta$ where $\theta = \ker(e)$.

Claim: $\theta = \mathtt{Eq}(S)$.

In fact, let $\phi, \psi \in V(A^*)$ then:

$$\begin{aligned}(\phi, \psi) \in \theta &\Leftrightarrow (\phi, \psi) \in \ker(e) \Leftrightarrow ev_\phi \circ i = ev_\psi \circ i \\ &\Leftrightarrow \forall s \in S \ \ (ev_\phi \circ i)(s) = (ev_\psi \circ i)(s) \Leftrightarrow \forall s \in S \ \ s(\phi) = s(\psi) \\ &\Leftrightarrow (\phi, \psi) \in \mathtt{Eq}(S)\end{aligned}$$

where the last equivalence follows from the fact that S is closed under derivatives.

Finally, by the claim we have that $\mathbf{free}(S) = V(A^*)/\theta$ and from that equality it follows that $(\mathbf{cofree} \circ \mathbf{free})(S) = \mathbf{cofree}(V(A^*)/\theta) = S$. In fact, we have that

$$\begin{aligned}\mathbf{cofree}(V(A^*)/\theta) &= \left\{ d \circ \overline{\phi/\theta}^{\sharp} \mid d \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K}), \phi/\theta \in V(A^*)/\theta \right\} \\ &= \{d \mid d \in \mathtt{Vec}_{\mathbb{K}}(V(A^*)/\theta, \mathbb{K})\} \cong S\end{aligned}$$

where the isomorphism follows by definition of $V(A^*)/\theta$ and duality and the last equality follows from the fact that S is closed under derivatives (since the morphisms $\overline{\phi/\theta}^{\sharp}$ are just translations and therefore can be omitted by closure under derivatives). □

3.6 Discussion

The contents and results of this chapter were based on the paper [81], which can be basically seen as a natural next step of what we presented for the case of deterministic automata in the previous chapter. Even though some of the contents are similar to the case of deterministic automata, their proofs had to be modified and in some cases completely changed.

The first step in developing this chapter was to observe that every weighted automaton induces a deterministic automaton in which the states are elements of a free semimodule. Hence, we studied equations and coequations for weighted automata as equations and coequations for the deterministic automata they induce. Later, we switched to the category $\mathtt{Vec}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ to work with $\mathbb{K}$–linear automata and did a similar work in which the notion of equations is a more general one since they need to induce an object in $\mathtt{Vec}_{\mathbb{K}}$.

Chapter 4

Equations and coequations, categorically

Equations play a fundamental role in (universal) algebra and in axiomatizations of mathematical structures such as, e.g., semigroups, monoids, groups, lattices, rings, modules over a ring, vector spaces, Boolean algebras and Heyting algebras. Algebraic structures of the same type that are equationally defined are also characterized as classes of algebras closed under homomorphic images, subalgebras and products, which is the statement of Birkhoff's theorem [18]. There are categorical versions of Birkhoff's theorem in the literature such as [15], for which the role and generalization of equations from a categorical point of view is made by using regular epimorphisms with regular–projective domain. The previous notion is equivalent, in the classical case, to quotients of a term algebra, whose elements in its kernel are exactly classical equations. Other categorical generalizations for equations were also considered in [52, 65, 50, 54, 38, 25]. A common aspect in most of those approaches is the one of representing equations by means of an epimorphism with projective domain. For the purpose of this chapter, we will consider equations as epimorphisms with a free object as its domain (a surjective homomorphism from a term algebra in the classical case), but any other kind of epimorphisms can also be considered to get a notion of equations.

The categorical dual of equations is the notion of *coequations*. Coequations were studied extensively in the search for a dual of Birkhoff's theorem and the specification of classes of coalgebras (see, e.g., [61, 62, 3, 2, 11, 75, 31, 46, 49, 84, 76]). In our case, coequations will be represented as monomorphisms with cofree codomain. Again, any other kind of monomorphisms can be considered to get a notion of coequations.

After defining the abstract concept of equations and coequations, we will define the natural notion of a morphism between equations and of a morphism between coequations in order to define their corresponding categories. We will show how the category of equations and the category of coequations relate to each other under the setting of a contravariant adjunction. For this purpose, we state a lifting theorem for contravariant adjunctions from [51] and, in the case of a duality, we will add one more layer to get a duality result between equations and coequations.

A similar work will be done for the case of Eilenberg–Moore categories, i.e., for algebras over a monad and coalgebras over a comonad. In this case, we present a result for lifting contravariant adjunctions to Eilenberg–Moore categories, which is based on the paper [80]. We give necessary and sufficient conditions to lift a contravariant adjunction to Eilenberg–Moore categories. Additionally, we will show how to define a monad from a comonad in the case of a contravariant adjunction and how to define a comonad from a monad in the case of a duality. As an application, we derive dualities between equations and coequations for the case of dynamical systems and deterministic automata.

We will start by defining the notions of equations and coequations from a categorical point of view and illustrate these notions with some examples.

4.1 Basic definitions

Let L be an endofunctor on a category $\mathcal{D}$ and S be an object in $\mathcal{D}$. Assume that the free U–object $\mathfrak{F}(S) \in \mathtt{alg}(L)$ over S exists, where $U : \mathtt{alg}(L) \to \mathcal{D}$ is the forgetful functor. We define *equations* for L on S generators (or variables in S) as epimorphisms with domain $\mathfrak{F}(S)$, i.e., elements $e_{Q_S} \in \mathtt{alg}(L)(\mathfrak{F}(S), \mathbf{Q_S})$ that are epimorphisms in $\mathcal{D}$ for some $\mathbf{Q_S} \in \mathtt{alg}(L)$[1]. Observe that if L is a polynomial functor on $\mathtt{Set}$ (see Example 19 and, e.g., [76, Section 10]) then equations can be identified with L-congruences θ of $\mathfrak{F}(S)$, since $\mathfrak{F}(S)/\theta \cong \mathbf{Q_S}$ in $\mathtt{alg}(L)$ for $\theta = \ker(e_{Q_S})$, and elements in θ are pairs of terms with variables on the set S. This corresponds to the classical definition of equations in universal algebra [27, Definition II.11.1]. We say that an algebra $\mathbf{X} = (X, \alpha_X) \in \mathtt{alg}(L)$ satisfies the equation e_{Q_S}, denoted as $(X, \alpha_X) \models e_{Q_S}$, if for any morphism $f \in \mathcal{D}(S, X)$ the morphism $f^\sharp$, which is given by freeness of $\mathfrak{F}(S)$, factors through e_{Q_S}, i.e., there exists $g_f \in \mathtt{alg}(L)(\mathbf{Q_S}, \mathbf{X})$ such that the following diagram in $\mathtt{alg}(L)$ commutes:

$$\begin{tikzcd} & Q_S \arrow[dr, dashed, "g_f"] & \\ \mathfrak{F}(S) \arrow[ur, "e_{Q_S}"] \arrow[rr, "f^\sharp"'] & & X \end{tikzcd} \qquad \forall f \in \mathcal{D}(S, X)$$

We have a bijective correpondence between morphisms $f \in \mathcal{D}(S, X)$ and morphisms $h \in \mathtt{alg}(L)(\mathfrak{F}(S), \mathbf{X})$, which is given by freeness of $\mathfrak{F}(S)$ under the assignment $f \mapsto f^\sharp$ and $h \mapsto h \circ \eta_S$. Therefore, the property $(X, \alpha_X) \models e_{Q_S}$ is equivalent

[1] We treat the case of epimorphisms in the base category $\mathcal{D}$, but other kind of epimorphisms can be considered in order to define a notion of equations (e.g., regular epimorphisms, extremal epimorphisms, split epimorphisms. Note that the forgetful functor $U : \mathtt{alg}(L) \to \mathcal{D}$ reflects epimorphisms since it is faithful, i.e., if e is a morphism in $\mathtt{alg}(L)$ such that $U(e)$ is an epimorphism, then e is also an epimorphism.)

to the property that every morphism $h \in \mathtt{alg}(L)(\mathfrak{F}(S), \mathbf{X})$ factors through e_{Q_S} in $\mathtt{alg}(L)$, i.e., there exists $g_h \in \mathtt{alg}(L)(\mathbf{Q_S}, \mathbf{X})$ such that the following diagram in $\mathtt{alg}(L)$ commutes:

$$\begin{tikzcd} & Q_S \arrow[dr, dashed, "g_h"] & \\ \mathfrak{F}(S) \arrow[ur, "e_{Q_S}"] \arrow[rr, "\forall h"'] & & X \end{tikzcd}$$

Now, assuming that the free U–object on S generators $\mathfrak{F}(S) \in \mathtt{alg}(L)$ exists, we define the category $\mathtt{eq}(L, S)$ of equations for L on S generators as follows:

Objects of $\mathtt{eq}(L, S)$: morphisms $e_{Q_S} \in \mathtt{alg}(L)(\mathfrak{F}(S), \mathbf{Q_S})$, some $\mathbf{Q_S} \in \mathtt{alg}(L)$, such that $U(e_{Q_S})$ is an epimorphism.

Arrows of $\mathtt{eq}(L, S)$: for $e_{Q_S}, e_{Q'_S} \in \mathtt{eq}(L, S)$, a morphism $f \in \mathtt{eq}(L, S)(e_{Q_S}, e_{Q'_S})$ is a morphism $f \in \mathtt{alg}(L)(\mathbf{Q_S}, \mathbf{Q'_S})$ such that the following diagram in $\mathtt{alg}(L)$ commutes:

$$\begin{tikzcd} \mathfrak{F}(S) \arrow[r, two heads, "e_{Q_S}"] \arrow[dr, two heads, "e_{Q'_S}"'] & Q_S \arrow[d, "f"] \\ & Q'_S \end{tikzcd}$$

Notice that morphisms in $\mathtt{eq}(L, S)$ are necessarily epimorphisms and that $\mathtt{eq}(L, S)$ is isomorphic to the co-slice category $\mathfrak{F}(S) \downarrow \mathtt{alg}(L)$ with the restriction that the objects are exactly epimorphisms in $\mathcal{D}$ with domain $\mathfrak{F}(S)$.

The next example illustrates this notion of equations for the case of deterministic automata.

Example 78. Consider the $\mathtt{Set}$ endofunctor L given by $L(X) = A \times X$, where A is a fixed set.

The free U–object on $S = 1$ is given by $A^* = (A^*, \varrho)$ where $\varrho\colon A \times A^* \to A^*$ is defined by $\varrho(a, w) = aw$ and the unit $\eta_1 \colon 1 \to A^*$ maps the unique element in 1 to the empty word $\varepsilon \in A^*$, i.e., $\eta_1 = \varepsilon$.

As we saw in Examples 4 and 5, elements in $\mathtt{alg}(L)$ are deterministic automata on A or, equivalently, algebras of type $\tau = A$, where each $a \in A = \tau$ is a unary function symbol. In this case, congruences on A^* (which are also called *left congruences*) are equivalence relations $\theta \subseteq A^* \times A^*$ such that for any $a \in A$ and $(u, v) \in \theta$ we have that $(au, av) \in \theta$. Congruences θ on A^* correspond to equations as defined above, by letting $A^*/\theta = (A^*/\theta, f_\theta) \in \mathtt{alg}(L)$ where f_θ is given by $f_\theta(a, w/\theta) = aw/\theta$ and the epimorphism (equation) ν_θ associated to θ is the morphism $\nu_\theta \in \mathtt{alg}(L)(A^*, A^*/\theta)$ that maps every word to its equivalence class with respect to θ.

An L-algebra (X, α_X) satisfies the equation ν_θ, also denoted as $(X, \alpha_X) \models \theta$, if and only if for every $(u, v) \in \theta$ and any $x \in X$, we have $\overline{x}^\sharp(u) = \overline{x}^\sharp(v)$, where $\overline{x} \in \mathtt{Set}(1, X)$ is the function such that $\overline{x}(0) = x$.

Notice that the function $\varrho' \colon A \times A^* \to A^*$ defined as $\varrho'(a, w) = wa$ is such that the algebra (A^*, ϱ') is also a free U–object on 1, which gives us the notion of *right congruence* as a corresponding notion of equation. In this case, satisfaction of equations coincides with satisfaction of equations as defined in [13] and in Chapter 2 of this thesis. □

We dualize the definition of equations to obtain the definition of coequations, e.g., [76, 61, 62]. Let B be an endofunctor on a category $\mathcal{C}$ and R be an object in $\mathcal{C}$. Assume that the cofree V–object $\mathfrak{C}(R) \in \mathtt{coalg}(B)$ over R exists, where $V : \mathtt{coalg}(B) \to \mathcal{C}$ is the forgetful functor. We define *coequations* for B on R colours as monomorphisms with codomain $\mathfrak{C}(R)$, i.e., elements $m_{S_R} \in \mathtt{coalg}(B)(\mathbf{S_R}, \mathfrak{C}(R))$ that are monomorphisms in $\mathcal{C}$ for some $\mathbf{S_R} \in \mathtt{coalg}(B)$[2]. We say that a B–coalgebra $\mathbf{Y} = (Y, \beta_Y)$ satisfies the coequation m_{S_R}, denoted as $(Y, \beta_Y) \mathrel{|\!\!\models} m_{S_R}$ (notice the difference between the symbols: $\models$ for equations and $\mathrel{|\!\!\models}$ for coequations), if for every morphism (R–colouring of Y) $f \in \mathcal{C}(Y, R)$ the morphism $f^\flat$, which is given by cofreeness of $\mathfrak{C}(R)$, factors through m_Q. Equivalently, we have that $(Y, \beta_Y) \mathrel{|\!\!\models} m_{S_R}$ if every $h \in \mathtt{coalg}(B)(\mathbf{Y}, \mathfrak{C}(R))$ factors through m_{S_R}, i.e., there exists $g_h \in \mathtt{coalg}(B)(\mathbf{Y}, \mathbf{S_R})$ such that the following diagram commutes:

$$\begin{tikzcd} & S_R \arrow[dr, hook, "m_{S_R}"] & \\ Y \arrow[ur, dashed, "g_h"] \arrow[rr, "\forall h"'] & & \mathfrak{C}(R) \end{tikzcd}$$

Assuming that the cofree V–object on R colours $\mathfrak{C}(R) \in \mathtt{coalg}(B)$ exists, we define the category $\mathtt{coeq}(B, R)$ of coequations for B on R colours whose objects are morphisms $m_{S_R} \in \mathtt{coalg}(B)(\mathbf{S_R}, \mathfrak{C}(R))$ for some $\mathbf{S_R} \in \mathtt{coalg}(B)$ such that $V(m_{S_R})$ is a monomorphism, and, a morphism from m_{S_R} to $m_{S'_R}$ in $\mathtt{coeq}(B, R)$ is a morphism $g \in \mathtt{coalg}(B)(\mathbf{S_R}, \mathbf{S'_R})$ such that $m_{S'_R} \circ g = m_{S_R}$. Notice that morphisms in $\mathtt{coeq}(B, R)$ are necessarily monomorphisms and that $\mathtt{coeq}(B, R)$ is isomorphic to the slice category $\mathtt{coalg}(B) \downarrow \mathfrak{C}(R)$ with the restriction that the objects are exactly monomorphisms in $\mathcal{C}$ with codomain $\mathfrak{C}(R)$.

The next example illustrates this definition of coequations for the case of deterministic automata.

Example 79. For a given set A, consider the $\mathtt{Set}$ endofunctor B defined by $B(X) = X^A$, and consider the two–element set $R = 2$ of colours. Then the cofree B-coalgebra on 2 colours is given by $2^{A^*} = (2^{A^*}, \varsigma)$ where $\varsigma \colon 2^{A^*} \to (2^{A^*})^A$ is given

[2] In a similar way as the case of equations, we choose monomorphisms in the base category $\mathcal{C}$, but other kind of monomorphisms can be considered in order to define a notion of coequations (e.g., regular monomorphisms, extremal monomorphisms, split monomorphisms.) Note that the forgetful functor $V : \mathtt{coalg}(B) \to \mathcal{C}$ reflects monomorphisms since it is faithful, i.e., if m is a morphism in $\mathtt{coalg}(B)$ such that $V(m)$ is a monomorphism, then m is also a monomorphism.

by right derivative, i.e., $\varsigma(L)(a) = L_a$, where $L_a(w) = L(aw)$, $w \in A^*$ and the counit $\epsilon_2 \colon 2^{A^*} \to 2$ is given by $\epsilon_2(L) = L(\varepsilon)$. Given a B–coalgebra $\mathbf{Y} = (Y, \beta_Y)$ and a two–colouring $c : Y \to 2$ of Y, the map $c^\flat \in \mathtt{coalg}(B)(\mathbf{Y}, 2^{A^*})$ maps every state $y \in Y$ to the language $c^\flat(y)$ it accepts according to the colouring c, i.e., $c^\flat(y)(\varepsilon) = c(y)$ and $c^\flat(y)(aw) = c^\flat(\beta_Y(y)(a))(w)$.

Coequations for B on R correspond to subsets of 2^{A^*} that are closed under right derivatives, i.e., subcoalgebras of 2^{A^*}. Given any monomorphism (coequation) m_S with codomain 2^{A^*} and a B-coalgebra (Y, β_Y), we have that $(Y, \beta_Y) \mathrel{\|}\!\!\models m_S$ if and only if for every two–colouring $c \in \mathtt{Set}(Y, 2)$ of Y, the set of those languages accepted by the states of the automaton (Y, β_Y) with respect to c is contained in $\mathrm{Im}(m_S)$. This coincides with satisfaction of coequations as defined in [13].

Similarly to the previous example, the function $\varsigma' \colon 2^{A^*} \to (2^{A^*})^A$ given by left derivative $\varsigma'(L)(a) = {}_aL$, where ${}_aL(w) = L(wa)$, $w \in A^*$, is such that $(2^{A^*}, \varsigma')$ is also a cofree B-coalgebra for which the corresponding notion of coequations are subsets of 2^{A^*} closed under left derivatives. $\square$

4.2 Lifting contravariant adjunctions

In this section, we study the notion of a contravariant adjunction and how to lift contravariant adjunctions to categories of algebras and coalgebras, according to [51, 56]. We instantiate this abstract approach in examples of constructions on various kinds of automata.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and contravariant functors $F : \mathcal{C} \rtimes\!\!\longrightarrow \mathcal{D}$ and $G : \mathcal{D} \rtimes\!\!\longrightarrow \mathcal{C}$ that form a contravariant adjunction, i.e., $F \dashv\vdash G$ (see Section 1.4). Let B be an endofunctor on $\mathcal{C}$ and L be an endofunctor on $\mathcal{D}$. That is, we have the situation depicted in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[loop left, "B"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\dashv\vdash"] & & \mathcal{D} \arrow[ll, bend left, "G"] \arrow[loop right, "L"] \end{tikzcd}$$

In this setting, we are interested in lifting the adjunction to a contravariant adjunction between lifted functors $\widehat{F} \colon \mathtt{coalg}(B) \to \mathtt{alg}(L)$ and $\widehat{G} \colon \mathtt{alg}(L) \to \mathtt{coalg}(B)$ of F and G, respectively, meaning that $U\widehat{F} = FV$ and $GU = V\widehat{G}$, as in the following picture:

$$
\begin{tikzcd}
\texttt{coalg}(B) \arrow[rr, bend left, "\widehat{F}"] \arrow[rr, phantom, "\dashv\vdash"] \arrow[dd, "V"'] & & \texttt{alg}(L) \arrow[ll, bend left, "\widehat{G}"] \arrow[dd, "U"] \\
& & \\
\mathcal{C} \arrow[loop left, "B"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\dashv\vdash"] & & \mathcal{D} \arrow[loop right, "L"] \arrow[ll, bend left, "G"]
\end{tikzcd}
\tag{4.1}
$$

where the vertical arrows U and V are forgetful functors. An important consequence of such a lifting is that, if $\texttt{alg}(L)$ has an initial object, then it is mapped by $\widehat{G}$ to a final object in $\texttt{coalg}(B)$.

In [51, 2.14. Theorem] it is shown that a sufficient condition for such a lifting is the existence of a natural isomorphism $\gamma\colon GL \Rightarrow BG$. This is summarized by the theorem below.

Theorem 80. *Let $F\colon \mathcal{C} \looparrowright \mathcal{D}$ and $G\colon \mathcal{D} \looparrowright \mathcal{C}$ be contravariant functors that form a contravariant adjunction. Let B be an endofunctor on $\mathcal{C}$ and L be an endofunctor on $\mathcal{D}$. If there is a natural isomorphism $\gamma\colon GL \Rightarrow BG$, then*

1. *The adjunction $F \dashv\vdash G$ lifts to an adjunction as in Diagram (4.1), i.e., to a contravariant adjunction between functors $\widehat{F}\colon \texttt{coalg}(B) \looparrowright \texttt{alg}(L)$ and $\widehat{G}\colon \texttt{alg}(L) \looparrowright \texttt{coalg}(B)$ such that $U\widehat{F} = FV$ and $GU = V\widehat{G}$, where the functors $U\colon \texttt{alg}(L) \to \mathcal{D}$ and $V\colon \texttt{coalg}(B) \to \mathcal{C}$ are the forgetful functors.*

2. *If F, G form a duality then $\widehat{F}, \widehat{G}$ form a duality as well.*

We will show a proof of the previous theorem to make the thesis self–contained. To prove the theorem we will use the following.

Lemma 81. *Let $F : \mathcal{C} \looparrowright \mathcal{D}$ and $G : \mathcal{D} \looparrowright \mathcal{C}$ be contravariant functors that form a contravariant adjunction with units $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$. Let B be an endofunctor on $\mathcal{C}$ and L be an endofunctor on $\mathcal{D}$. Let $\gamma : GL \Rightarrow BG$ be a natural isomorphism. Define the natural transformation $\rho : LF \Rightarrow FB$ as:*

$$\rho = FB\eta^{GF} \circ F\gamma_F^{-1} \circ \eta_{LF}^{FG}.$$

Then, the following two diagrams commute:

$$
\begin{tikzcd}
B \arrow[r, "B\eta^{GF}"] \arrow[d, "\eta_B^{GF}"'] \arrow[dr, phantom, "(1)"] & BGF \\
GFB \arrow[r, "G\rho"'] & GLF \arrow[u, "\gamma_F"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "L\eta^{FG}"] \arrow[d, "\eta_L^{FG}"'] \arrow[dr, phantom, "(2)"] & LFG \arrow[d, "\rho_G"] \\
FGL & FBG \arrow[l, "F\gamma"]
\end{tikzcd}
$$

Proof. Commutativity of (1) follows from the commutative diagram:

$$
\begin{tikzcd}
B \arrow[r, "B\eta^{GF}"] \arrow[d, "\eta_B^{GF}"'] \arrow[dr, phantom, "\text{a.}"] & BGF \arrow[r, "\gamma_F^{-1}"] \arrow[d, "\eta_{BGF}^{GF}"'] \arrow[dr, phantom, "\text{a.}"] & GLF \arrow[d, "\eta_{GLF}^{GF}"'] \arrow[dr, "Id_{GLF}"] \arrow[dr, phantom, "\text{c.}"'] & \\
GFB \arrow[r, "GFB\eta^{GF}"'] \arrow[rrr, dashed, bend right, "G\rho"'] & GFBGF \arrow[r, "GF\gamma_F^{-1}"'] & GLFGF \arrow[r, "G\eta_{LF}^{FG}"'] & GLF
\end{tikzcd}
$$

And, commutativity of (2) follows from the commutative diagram:

$$
\begin{tikzcd}
L \arrow[r, "\eta_L^{FG}"] \arrow[d, "L\eta^{FG}"'] \arrow[dr, phantom, "\text{a.}"] & FGL \arrow[r, "F\gamma^{-1}"] \arrow[d, "FGL\eta^{FG}"'] \arrow[dr, phantom, "\text{b.}"] & FBG \arrow[d, "FBG\eta^{FG}"'] \arrow[dr, "Id_{FBG}"] \arrow[dr, phantom, "\text{c.}"'] & \\
LFG \arrow[r, "\eta_{LFG}^{FG}"'] \arrow[rrr, dashed, bend right, "\rho_G"'] & FGLFG \arrow[r, "F\gamma_{FG}^{-1}"'] & FBGFG \arrow[r, "FB\eta_G^{GF}"'] & FBG
\end{tikzcd}
$$

where the following facts were used: a. Naturality of η^{FG}, b. Naturality of γ^{-1} and c. Triangle identity $G\eta^{FG} \circ \eta_G^{GF} = Id_G$. $\square$

Now we can prove Theorem 80.

Proof. Let $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ be the units of the contravariant adjunction. Define the natural transformation $\rho : LF \Rightarrow FB$ as

$$\rho = FB\eta^{GF} \circ F\gamma^{-1}F \circ \eta^{FG}LF.$$

Define the functors $\widehat{F} : \mathtt{coalg}(B) \rightarrowtail\!\!\!\!\longleftarrow \mathtt{alg}(L)$ and $\widehat{G} : \mathtt{alg}(L) \rightarrowtail\!\!\!\!\longleftarrow \mathtt{coalg}(B)$ on objects as:

$$(Y \xrightarrow{\beta_Y} BY) \xmapsto{\widehat{F}} (LFY \xrightarrow{\rho_Y} FBY \xrightarrow{F(\beta_Y)} FY)$$
$$(LX \xrightarrow{\alpha_X} X) \xmapsto{\widehat{G}} (GX \xrightarrow{G(\alpha_X)} GLX \xrightarrow{\gamma_X} BGX)$$

and $\widehat{F} = F$ and $\widehat{G} = G$ on morphisms.

Clearly $\widehat{F}$ and $\widehat{G}$ are liftings of F and G. Now to show that they form a contravariant adjunction we are going to show that the units η^{FG} and η^{GF} are the units for the contravariant adjunction between $\widehat{F}$ and $\widehat{G}$. That is, we have to show that if $\mathbf{Y} = (Y, \beta_Y) \in \mathtt{coalg}(B)$, then $\eta_Y^{GF} \in \mathtt{coalg}(B)\left(\mathbf{Y}, \widehat{G}\widehat{F}(\mathbf{Y})\right)$ and that if $\mathbf{X} = (X, \alpha_X) \in \mathtt{alg}(L)$ then $\eta_X^{FG} \in \mathtt{alg}(L)\left(\mathbf{X}, \widehat{F}\widehat{G}(\mathbf{X})\right)$. In fact,

i) The property of $\eta_Y^{GF} \in \mathtt{coalg}(B)\Big(\mathbf{Y}, \widehat{G}\widehat{F}(\mathbf{Y})\Big)$ is the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[rrr, "\beta_Y"] \arrow[d, "\eta_Y^{GF}"'] \arrow[dr, phantom, "\text{nat. of } \eta^{GF}"] & & & B(Y) \arrow[d, "B\eta_Y^{GF}"] \arrow[dll, "\eta^{GF}_{B(Y)}"'] \arrow[dl, phantom, "(1)"] \\
GF(Y) \arrow[r, "GF(\beta_Y)"'] & GFB(Y) \arrow[r, "G\rho_Y"'] & GLF(Y) \arrow[r, "\gamma_{F(Y)}"'] & BGF(Y)
\end{tikzcd}$$

ii) The property of $\eta_X^{FG} \in \mathtt{alg}(L)\Big(\mathbf{X}, \widehat{F}\widehat{G}(\mathbf{X})\Big)$ is the following commutative diagram:

$$\begin{tikzcd}
L(X) \arrow[rrr, "\alpha_X"] \arrow[d, "L\eta_X^{FG}"'] \arrow[drr, "\eta^{FG}_{L(X)}"] \arrow[dr, phantom, "(2)"] & & & X \arrow[d, "\eta_X^{FG}"] \arrow[dl, phantom, "\text{nat. of } \eta^{FG}"] \\
LFG(X) \arrow[r, "\rho_{G(X)}"'] & FBG(X) \arrow[r, "F\gamma_X"'] & FGL(X) \arrow[r, "FG(\alpha_X)"'] & FG(X)
\end{tikzcd}$$

Finally, in case that F and G form a duality, then the units η^{FG} and η^{GF} are natural isomorphisms which means that the lifting is also a duality. $\square$

The functors $\widehat{F}\colon \mathtt{coalg}(B) \times\!\!\longrightarrow \mathtt{alg}(L)$ and $\widehat{G}\colon \mathtt{alg}(L) \times\!\!\longrightarrow \mathtt{coalg}(B)$ in the previous theorem are defined on objects as:

$$(Y \xrightarrow{\beta_Y} BY) \overset{\widehat{F}}{\longmapsto} (LFY \xrightarrow{\rho_Y} FBY \xrightarrow{F(\beta_Y)} FY)$$
$$(LX \xrightarrow{\alpha_X} X) \overset{\widehat{G}}{\longmapsto} (GX \xrightarrow{G(\alpha_X)} GLX \xrightarrow{\gamma_X} BGX)$$

and on morphisms as $\widehat{F} = F$ and $\widehat{G} = G$. The natural transformation $\rho\colon LF \Rightarrow FB$ in the definition of $\widehat{F}$ is defined as the *mate* of the inverse $\gamma^{-1}\colon BG \Rightarrow GL$:

$$\rho \overset{\text{def.}}{=} FB\eta^{GF} \circ F\gamma^{-1}F \circ \eta^{FG}LF,$$

by using the units η^{GF} and η^{FG} of the adjunction. Natural transformations of the form $\rho\colon LF \Rightarrow FB$ and the definition of $\widehat{F}$ form the heart of the approach to *coalgebraic modal logic* based on contravariant adjunctions/dualities (see, e.g., [59, 57, 68, 24]). There is a one-to-one correspondence between such natural transformations and those of the form $BG \Rightarrow GL$, using the above construction (note that we use a natural transformation $BG \Rightarrow GL$ and not the inverse of a natural transformation as γ above). We are only interested in the case where the natural transformation $BG \Rightarrow GL$ is an isomorphism, to lift adjunctions, as in [60]. For notational convenience, the direction in $\gamma\colon GL \Rightarrow BG$ is reversed here.

In the rest of this section we provide examples and applications of Theorem 80 and the setting in Diagram (4.1).

Example 82 (From [60, Example 4]). For a fixed set A consider the following situation:

$$\begin{tikzcd} B \arrow[loop left] & \mathtt{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\dashv"] & \mathtt{Set} \arrow[l, bend left, "G"] & L \arrow[loop right] \end{tikzcd} \qquad \begin{gathered} F(X) = G(X) = 2^X \\ L(X) = (A \times X) + 1 \\ B(Y) = 2 \times Y^A \\ \gamma_X : 2^{A \times X + 1} \to 2 \times (2^X)^A \end{gathered}$$

Here L-algebras are *pointed deterministic automata* on A (cf. Example 78) and B-coalgebras are *two–coloured deterministic automata* on A (cf. Example 79). The contravariant functors F and G form a contravariant adjunction with unit $\eta^{FG} = \eta^{GF} : Id_{\mathtt{Set}} \Rightarrow GF$ such that $\eta^{FG}_X(x)(f) = f(x)$, $x \in X \in \mathtt{Set}$ and $f \in 2^X$, which, by Theorem 80, can be lifted to an adjunction between $\widehat{F}$ and $\widehat{G}$. The isomorphism $\gamma \colon GL \Rightarrow BG$ is defined for any X as the function $\gamma_X : 2^{A \times X+1} \to 2 \times (2^X)^A$ such that $\gamma_X(f) = (f(\cdot), \lambda a.\ \lambda x.f(a, x))$. Note that the natural transformation ρ is given by the function $\rho_Y : A \times 2^Y + 1 \to 2^{2 \times Y^A}$ such that $\rho_Y(\cdot)(j, f) = j$ and $\rho_Y(a, L)(j, f) = L(f(a))$.

Given a B-coalgebra $(Y, \langle c, \beta_Y \rangle)$, with $c \in \mathtt{Set}(Y, 2)$ and $\beta_Y \in \mathtt{Set}(Y, Y^A)$, we have that $\widehat{F}(Y, \langle c, \beta_Y \rangle) = (2^Y, [\alpha, i])$ where $\alpha \colon A \times 2^Y \to 2^Y$ and $i \colon 1 \to 2^Y$ are functions defined as follows:

$$\begin{aligned} i(\cdot) &= c^{-1}(\{1\}) = \text{accepting states of } (Y, \langle c, \beta_Y \rangle), \\ \alpha(a, Z) &= \{y \in Y \mid \beta_Y(y)(a) \in Z\}. \end{aligned}$$

Given an L-algebra $(X, [\alpha_X, i])$, with $i \in \mathtt{Set}(1, X)$ and $\alpha_X \in \mathtt{Set}(A \times X, X)$, we have that $\widehat{G}(X, [\alpha_X, i]) = (2^X, \langle c, \beta \rangle)$ where the functions $c \colon 2^X \to 2$ and $\beta \colon 2^X \to (2^X)^A$ are defined as:

$$\begin{gathered} c(Z) = 1 \ \text{ iff } \ i(\cdot) \in Z \\ \beta(Z)(a) = \{x \in X \mid \alpha_X(a, x) \in Z\} \end{gathered}$$

Recall from Example 78 that the initial L-algebra is given by $(A^*, [\eta_1, \varrho])$, where A^* is the free monoid with generators A and identity element ε, $\eta_1 \colon 1 \to A^*$ is the empty word ε and $\varrho \colon A \times A^* \to A^*$ is the concatenation function given by $\varrho(a, w) = aw$. Because of the contravariant adjunction, the initial L-algebra is sent by $\widehat{G}$ to the final B-coalgebra, given by $\widehat{G}(A^*, [\eta_1, \varrho]) = (2^{A^*}, \langle \epsilon_2, \varsigma \rangle)$ where $\epsilon_2(L) = L(\varepsilon)$ and $\varsigma(L)(a)(w) = L(aw)$. Note that the final B-coalgebra is not sent by $\widehat{F}$ to the initial L-algebra. $\square$

Example 83. For a fixed set A consider the following situation:

$$\begin{tikzcd} B \arrow[loop left] & \mathtt{CABA} \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathtt{Set} \arrow[l, bend left, "G"] & L \arrow[loop right] \end{tikzcd} \qquad \begin{gathered} F(Y) = \mathtt{CABA}(Y, 2) \\ G(X) = 2^X \\ L(X) = (A \times X) + 1 \\ B(Y) = 2 \times Y^A \\ \gamma_X : 2^{A \times X + 1} \to 2 \times (2^X)^A \end{gathered}$$

The contravariant functors F and G form a contravariant adjunction, in fact a duality, whose units $\eta^{FG} : Id_{\mathtt{Set}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathtt{CABA}} \Rightarrow GF$, which are natural isomorphisms, are given by $\eta^{FG}_X(x)(f) = f(x)$ and $\eta^{GF}_Y(y)(h) = h(y)$. By Theorem 80, this duality can be lifted to a duality between $\widehat{F}$ and $\widehat{G}$ if we consider the canonical natural isomorphism $\gamma\colon GL \Rightarrow BG$ defined for every X as the morphism $\gamma_X\colon 2^{A\times X+1} \to 2 \times (2^X)^A$ such that $\gamma_X(f) = (f(\cdot), \lambda a.\ \lambda x. f(a,x))$. Note that the natural transformation ρ is given by the function $\rho_Y : A\times \mathtt{CABA}(Y,2)+1 \to \mathtt{CABA}(2\times Y^A, 2)$ defined as $\rho_Y(\cdot)(i,f) = i$ and $\rho_Y(a,h)(i,f) = h(f(a))$. That is, $\rho_Y(\cdot) = \pi_1$, where $\pi_1 \in \mathtt{CABA}(2\times Y^A, 2)$ is such that $\pi_1(i,f) = i$, and $\rho_Y(a,h) = h\circ\pi_a$, where $\pi_a \in \mathtt{CABA}(2\times Y^A, 2)$ is such that $\pi_a(i,f) = f(a)$.

Given a B-coalgebra $(Y, \langle c, \beta_Y\rangle)$, we have that $\widehat{F}(Y,\langle c,\beta_Y\rangle) = (\mathtt{CABA}(Y,2), [\alpha, i])$ where the functions $\alpha\colon A\times \mathtt{CABA}(Y,2) \to \mathtt{CABA}(Y,2)$ and $i\colon 1 \to \mathtt{CABA}(Y,2)$ are defined as follows:

$$i(\cdot) = c \ \text{ and } \ \alpha(a,h) = h\circ\pi_a\circ\beta_Y$$

In particular, if $P \subseteq 2^{A^*}$ is a *preformation of languages* [13, Definition 11], i.e., $P \in \mathtt{CABA}$ and it is closed under left and right derivatives [3], then $(P, \langle\epsilon_2, \varsigma'\rangle) \in \mathtt{coalg}(B)$ where $\epsilon_2(L) = L(\varepsilon)$ and $\varsigma'(L)(a) = {}_aL$. In this case, $\widehat{F}(P,\langle\epsilon_2,\varsigma'\rangle) = \mathbf{free}(P)$ which is the quotient A^*/θ where θ is the set, in fact congruence, of all equations satisfied by the automaton (P,ς), where $\varsigma(L)(a) = L_a$ (see [13]).

Given an L-algebra $(X, [\alpha_X, i])$, we have that $\widehat{G}(X,[\alpha_X,i]) = (2^X, \langle c,\beta\rangle)$ where the $\mathtt{CABA}$ morphisms $c\colon 2^X \to 2$ and $\beta\colon 2^X \to (2^X)^A$ are defined as

$$c(Z) = 1 \ \text{ iff } \ i(\cdot) \in Z$$
$$\beta(Z)(a) = \{x \in X \mid \alpha_X(a,x) \in Z\}.$$

In particular, if θ is a congruence of the monoid A^* then $(A^*/\theta, [f_\theta, \varepsilon/\theta]) \in \mathtt{alg}(L)$ where $f_\theta(a, w/\theta) = aw/\theta$. In this case, $\widehat{G}(A^*/\theta, [f_\theta,\varepsilon/\theta]) \cong \mathbf{cofree}(A^*/\theta)$ which is the minimum set of coequations that the automaton $(A^*/\theta, f'_\theta)$ satisfies, where $f'_\theta(a, w/\theta) = wa/\theta$ (see [13]).

Similarly to the previous example, the initial L-algebra $A^* = (A^*, [\eta_1, \varrho])$, where $\eta_1 = \varepsilon$ and $\varrho(a,w) = aw$, is sent by $\widehat{G}$ to the final B-coalgebra $2^{A^*} = (2^{A^*}, \langle\epsilon_2,\varsigma\rangle)$, where $\epsilon_2(L) = L(\varepsilon)$ and $\varsigma(L)(a)(w) = L(aw)$. Also, because the contravariant adjunction is a duality, the final B-coalgebra 2^{A^*} is sent by $\widehat{F}$ to the initial L-algebra A^*. We will explore this case further in Section 4.3 to get dualities between sets of equations and sets of coequations. □

We can also show that in the setting of the previous example we can get the semantics of an alternating automaton. In fact, consider the adjunction $\mathcal{PP} \dashv U$ where $\mathcal{PP} : \mathtt{Set} \to \mathtt{CABA}$ is the composition of the contravariant powerset functor with itself and $U : \mathtt{CABA} \to \mathtt{Set}$ is the forgetful functor. Observe that, for any function $f : X \to Y$, $\mathcal{PP}(f)$ is defined as $\mathcal{PP}(f)(\mathcal{S}) = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{S}\}$. The

[3] $P \subseteq 2^{A^*}$ is closed under right (left) derivatives if for every $L\in P$ and $a \in A$, $L_a \in P$ (${}_aL \in P$). Here $L_a(w) = L(aw)$, and ${}_aL(w) = L(wa)$, $w \in A^*$.

unit of the adjunction $\eta : Id_{\texttt{Set}} \Rightarrow U\mathcal{P}\mathcal{P}$ is given by:

$$\begin{aligned} \eta_X : & X \to \mathcal{P}\mathcal{P}(X) \\ & x \mapsto \{S \subseteq X \mid x \in S\} \end{aligned}$$

i.e., $\eta_X(x)$ is the principal filter $\uparrow\{x\}$. The universal property for η that characterizes the adjunction can be verified as follows: for any $C \in \texttt{Set}$, $D \in \texttt{CABA}$, and $f \in \texttt{Set}(C, UD)$, the unique morphism $f^\sharp \in \texttt{CABA}(\mathcal{P}\mathcal{P}(C), D)$ such that $U(f^\sharp) \circ \eta_C = f$ is given by:

$$f^\sharp(\mathcal{S}) = \bigvee_{R \in \mathcal{S}} \bigwedge_{q \in R} f(q)$$

whose existence and uniqueness follows by applying $f^\sharp$ to the identity:

$$\mathcal{S} = \bigcup_{R \in \mathcal{S}} \bigcap_{q \in R} \eta_C(q).$$

Given an alternating automaton $\langle c, \beta \rangle : X \to 2 \times (\mathcal{P}\mathcal{P}(X))^A$ we can define the semantics of $\langle c, \beta \rangle$ by using the adjunction $\mathcal{P}\mathcal{P} \dashv U$ and the setting given in the previous example to define the language $L(x) \in 2^{A^*}$ accepted by $x \in X$ as $L(x) := (\tilde{c}^\flat \circ \eta_X)(x)$ by using the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\langle c, \beta \rangle"'] & \mathcal{P}\mathcal{P}(X) \arrow[r, "\tilde{c}^\flat"] \arrow[d, "\langle c, \beta \rangle^\sharp =: \langle \tilde{c}, \tilde{\beta} \rangle"] & 2^{A^*} \arrow[d, "\langle \epsilon_2, \varsigma \rangle"] \\ & 2 \times (\mathcal{P}\mathcal{P}(X))^A \arrow[r, "2 \times (\tilde{c}^\flat)^A"'] & 2 \times (2^{A^*})^A \end{tikzcd}$$

from the definition of $\langle \tilde{c}, \tilde{\beta} \rangle := \langle c, \beta \rangle^\sharp$ we have that:

$$\langle \tilde{c}, \tilde{\beta} \rangle(\mathcal{S}) = \bigvee_{R \in \mathcal{S}} \bigwedge_{q \in R} \langle c, \beta \rangle(q)$$

this means that

$$\tilde{c}(\mathcal{S}) = 1 \;\Leftrightarrow\; c \in \mathcal{S}$$

and that, for any $K \subseteq X$ we have that

$$K \in \tilde{\beta}(\mathcal{S})(a) \;\Leftrightarrow\; \exists R \in \mathcal{S} \;\; \forall q \in R \;\; K \in \tilde{\beta}(\eta_X(q))(a).$$

From this, we have that the language $L(x)$ is defined as:

$$\begin{aligned} \varepsilon \in L(x) &\Leftrightarrow c(x) = 1 \\ aw \in L(x) &\Leftrightarrow \exists R \in \beta(x)(a) \;\; \forall q \in R \;\; w \in L(q) \end{aligned}$$

which is the expected semantics for an alternating automaton.

Example 84. Let A be a fixed set and consider the following situation:

$$
\begin{tikzcd}
\texttt{Vec}_{\mathbb{K}} \arrow[loop left, "B"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \texttt{Vec}_{\mathbb{K}} \arrow[loop right, "L"] \arrow[l, bend left, "G"]
\end{tikzcd}
\qquad
\begin{aligned}
F(X) = G(X) &= \texttt{Vec}_{\mathbb{K}}(X, \mathbb{K}) \\
L(X) &= \mathbb{K} + (A \cdot X) \\
B(Y) &= \mathbb{K} \times Y^A \\
\gamma_X \colon (\mathbb{K} + A \cdot X)^\partial &\to \mathbb{K} \times (X^\partial)^A
\end{aligned}
$$

Here $X^\partial := \texttt{Vec}_{\mathbb{K}}(X, \mathbb{K})$, the *dual space* of X, and $A \cdot X := \coprod_{a \in A} X$. We have that the contravariant functors F and G form a contravariant adjunction with unit $\eta^{GF}_X = \eta^{FG}_X : X \longrightarrow \texttt{Vec}_{\mathbb{K}}(\texttt{Vec}_{\mathbb{K}}(X, \mathbb{K}), \mathbb{K})$ given by $\eta^{FG}_X(x)(\varphi) = \varphi(x)$. By Theorem 80, this contravariant adjunction can be lifted to a contravariant adjunction between $\widehat{F}$ and $\widehat{G}$ if we consider the canonical natural isomorphism $\gamma \colon GL \Rightarrow BG$ defined for every X as the map $\gamma_X \colon (\mathbb{K} + A \cdot X)^\partial \to \mathbb{K} \times (X^\partial)^A$ such that $\gamma_X(\varphi) = (\varphi(1), \lambda a.\ \lambda x.\varphi(a, x))$. Note that the natural transformation ρ is given by the linear transformation $\rho_Y : \mathbb{K} + A \cdot Y^\partial \to (\mathbb{K} \times Y^A)^\partial$ which is defined on the canonical basis as $\rho_Y(1)(k, f) = k$ and $\rho_Y(a, \varphi)(k, f) = (\varphi \circ f)(a)$.

Given a B-coalgebra $(Y, \langle c, \beta_Y \rangle)$, we have that $\widehat{F}(Y, \langle c, \beta_Y \rangle) = (Y^\partial, [i, \alpha])$ where $i \colon \mathbb{K} \to Y^\partial$ and $\alpha \colon A \cdot Y^\partial \to Y^\partial$ are linear maps which are defined on the canonical basis as:

$$
\begin{aligned}
i(1)(y) &= c(y) \\
\alpha(a, \varphi)(y) &= (\varphi \circ \beta_Y(y))(a) = \varphi(\beta_Y(y)(a))
\end{aligned}
$$

In particular, if $S \subseteq \mathbb{K}^{A^*}$ is a subspace such that for every $f \in S$ and $a \in A$, $f_a, {}_af \in S$, where $f_a(w) = f(aw)$ and ${}_af(w) = f(wa)$, $w \in A^*$, then we have that $(S, \langle \epsilon_2, \varsigma' \rangle) \in \texttt{coalg}(B)$ where $\epsilon_2(f) = f(\varepsilon)$ and $\varsigma'(f)(a) = {}_af$. In this case, $\widehat{F}(S, \langle \epsilon_2, \varsigma' \rangle) \cong \mathbf{free}(S)$ which is the quotient $V(A^*)/\theta$ where θ is the set, in fact linear congruence, of all linear equations satisfied by the automaton (S, ς). Here $V(A^*) = \{\phi \colon A^* \to \mathbb{K} \mid \mathrm{supp}(\phi) \text{ is finite}\}$, where $\mathrm{supp}(\phi) = \{w \in A^* \mid \phi(w) \neq 0\}$ is the support of ϕ, and the function ς is defined as $\varsigma(f)(a) = f_a$ (see [81]).

Given an L-algebra $(X, [i, \alpha_X])$, we have that $\widehat{G}(X, [i, \alpha_X]) = (X^\partial, \langle c, \beta \rangle)$ where the linear maps $c \colon X^\partial \to \mathbb{K}$ and $\beta \colon X^\partial \to (X^\partial)^A$ are defined as

$$
c(\varphi) = \varphi(i(1)) \ \text{ and } \ \beta(\varphi)(a)(x) = \varphi(\alpha_X(a, x)).
$$

In particular, if $\theta \subseteq V(A^*) \times V(A^*)$ is a linear congruence on $V(A^*)$, then we have that $(V(A^*)/\theta, [f_\theta, \varepsilon/\theta]) \in \texttt{alg}(L)$, where $f_\theta(a, \phi/\theta) = a\phi/\theta$, and we have $\widehat{G}(V(A^*)/\theta, [f_\theta, \varepsilon/\theta]) \cong \mathbf{cofree}(V(A^*)/\theta)$ which is the minimum set of coequations (power series) satisfied by the automaton $(V(A^*)/\theta, f'_\theta)$, where $f'_\theta(a, \phi/\theta) = (\phi a)/\theta$ (see [81]).

Notice that the contravariant adjunction is not a duality, but if we restrict to vector spaces of finite dimension then we get a duality. In the latter case there is no initial L-algebra or, equivalently, there is no final B-coalgebra. □

4.3 Duality between equations and coequations

We defined equations as epimorphisms from a free U–object and coequations as monomorphisms into a cofree V–object. In the previous section, we have seen how to relate initial algebras and final coalgebras by lifting contravariant adjunctions and dualities. In this section, we describe how to apply these liftings to obtain a correspondence between equations and coequations.

If we lift a contravariant adjunction between contravariant functors F and G to a contravariant adjunction $\widehat{F}\colon \mathtt{coalg}(B) \times\!\!\!\longrightarrow \mathtt{alg}(L)$ and $\widehat{G}\colon \mathtt{alg}(L) \times\!\!\!\longrightarrow \mathtt{coalg}(B)$ as in the previous section, then the functor $\widehat{G}$ sends the initial L-algebra to the final B-coalgebra, and $\widehat{G}$ sends epimorphisms to monomorphisms. As a consequence, equations are sent by $\widehat{G}$ to coequations. However, $\widehat{F}$ does not map coequations to equations, in general.

In order to obtain a full correspondence between equations and coequations, suppose that the contravariant adjunction between F and G is a duality (and that there is a natural isomorphism $\gamma\colon GL \Rightarrow BG$). Then, by Theorem 80, the duality between F and G lifts to a duality between $\widehat{F}$ and $\widehat{G}$. In this case, we can add another level to the picture in (4.1), yielding a duality between equations and coequations:

$$
\begin{tikzcd}
\mathtt{coeq}(B,G(S)) \arrow[rr, bend left, "\widehat{F}"] \arrow[rr, phantom, "\cong"] \arrow[d, "V'"'] & & \mathtt{eq}(L,S) \arrow[ll, bend left, "\widehat{G}"] \arrow[d, "U'"] \\
\mathtt{coalg}(B) \arrow[rr, bend left, "\widehat{F}"] \arrow[rr, phantom, "\cong"] \arrow[d, "V"'] & & \mathtt{alg}(L) \arrow[ll, bend left, "\widehat{G}"] \arrow[d, "U"] \\
\mathcal{C} \arrow[loop left, "B"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\cong"] & & \mathcal{D} \arrow[loop right, "L"] \arrow[ll, bend left, "G"]
\end{tikzcd}
\tag{4.2}
$$

where $\mathtt{eq}(L,S)$ and $\mathtt{coeq}(B,G(S))$ are the categories of equations for L on S generators and coequations for B on $G(S)$ colours, respectively, U and V are forgetful functors, and U' and V' are the canonical functors defined as $U'(e_X) = \mathbf{X}$ and $V'(m_Y) = \mathbf{Y}$ on objects and $U'(f) = f$ and $V'(g) = g$ on morphisms. From this, we have the following result.

Theorem 85. *Let $F\colon \mathcal{C} \times\!\!\!\longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \times\!\!\!\longrightarrow \mathcal{C}$ be contravariant functors that form a duality. Let B be an endofunctor on $\mathcal{C}$, L be an endofunctor on $\mathcal{D}$ with an object S in $\mathcal{D}$ such that the free U–object $\mathfrak{F}(S) \in \mathtt{alg}(L)$ over $S \in \mathcal{D}$ exists, where $U : \mathtt{alg}(L) \to \mathcal{D}$ is the forgetful functor. If there is a natural isomorphism $\gamma\colon GL \Rightarrow BG$ then:*

1. *The duality between F and G lifts to a duality between contravariant functors $\widehat{F}\colon$ $\texttt{coeq}(B, G(S)) \rightarrowtail\!\!\!\!\!\longrightarrow \texttt{eq}(L, S)$ and $\widehat{G}\colon$ $\texttt{eq}(L, S) \rightarrowtail\!\!\!\!\!\longrightarrow \texttt{coeq}(B, G(S))$, as in Diagram* (4.2)*, where $\widehat{F}(Y, \beta_Y) = (F(Y), F(\beta_Y) \circ \rho_Y)$ and $\widehat{G}(X, \alpha_X) = (G(X), \gamma_X \circ G(\alpha_X))$, for $(Y, \beta_Y) \in \texttt{coalg}(B)$ and $(X, \alpha_X) \in \texttt{alg}(L)$, and $\widehat{F}(f) = F(f)$ and $\widehat{G}(g) = G(g)$ for morphisms f in $\texttt{coalg}(B)$ and g in $\texttt{alg}(L)$, and the natural transformation $\rho : LF \Rightarrow FB$ is defined as:*

$$\rho = FB\eta^{GF} \circ F\gamma^{-1}F \circ \eta^{FG}LF.$$

where $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ and $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ are the units of the contravariant adjunction.

2. *Given $e_P \in \texttt{eq}(L, S)$, $m_Q \in \texttt{coeq}(B, G(S))$, $(X, \alpha_X) \in \texttt{alg}(L)$, and $(Y, \beta_Y) \in \texttt{coalg}(B)$ we have:*

   i) $(X, \alpha_X) \models e_P$ *if and only if* $\widehat{G}(X, \alpha_X) \mathrel{|\!\!\models} \widehat{G}(e_P)$.

   ii) $\widehat{F}(Y, \beta_Y) \models \widehat{F}(m_Q)$ *if and only if* $(Y, \beta_Y) \mathrel{|\!\!\models} m_Q$. □

As an application of the previous theorem we have the following.

Example 86. (cf. Example 83) For a fixed set A consider the following situation:

$$\begin{tikzcd} B \arrow[loop left] & \texttt{CABA} \arrow[r, bend left, tail, "F"] \arrow[r, phantom, "\cong"] & \texttt{Set} \arrow[l, bend left, tail, "G"] \arrow[loop right] & L \end{tikzcd} \qquad \begin{aligned} F(Y) &= \texttt{CABA}(Y, 2) \\ G(X) &= 2^X \\ L(X) &= A \times X \\ B(Y) &= Y^A \\ \gamma_X\colon 2^{A\times X} &\to (2^X)^A \end{aligned}$$

If we put $S = 1$, then we get a duality between $\texttt{eq}(L, 1)$, whose objects can be identified with left congruences of A^*, and $\texttt{coeq}(B, 2)$, whose objects can be identified with subalgebras $Q \subseteq 2^{A^*}$ in $\texttt{CABA}$ that are closed under right derivatives. The previous fact was obtained by considering the free U–object on S $A^* = (A^*, \varrho)$ where $\varrho(a, w) = aw$. Note that we can also get a duality between right congruences and subalgebras of 2^{A^*} in $\texttt{CABA}$ that are closed under left derivatives by considering (A^*, ϱ') such that $\varrho'(a, w) = wa$ instead of (A^*, ϱ).

Additionally, from this setting, if we consider congruences of A^* and subalgebras of 2^{A^*} that are closed both under left and right derivatives, we can derive the duality between equations and coequations that was shown in [13, Theorem 22] and in Section 2.3. We will come back to this situation in a more general setting in Section 4.5.1 and also in a slightly different setting in Section 4.5.2. □

Example 87. (cf. Section 3.5) For a fixed set A and a finite field $\mathbb{K}$ consider the following situation:

$$
\begin{tikzcd}
\mathtt{StVec}_{\mathbb{K}} \arrow[loop left, "B"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathtt{Vec}_{\mathbb{K}} \arrow[l, bend left, "G"] \arrow[loop right, "L"]
\end{tikzcd}
\qquad
\begin{gathered}
F(Y) = \mathtt{StVec}_{\mathbb{K}}(Y, \mathbb{K}) \\
G(X) = \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K}) \\
L(X) = A \cdot X \\
B(Y) = Y^A \\
\gamma_X \colon \mathtt{Vec}_{\mathbb{K}}(A \cdot X, \mathbb{K}) \to \mathtt{Vec}_{\mathbb{K}}(X, \mathbb{K})^A
\end{gathered}
$$

where the isomorphism γ is obtained from the universal property of the coproduct $A \cdot X = \coprod_{a \in A} X$. Now, if we put $S = 1$, then we get a duality between $\mathtt{eq}(L, 1)$, whose objects can be identified with left linear congruences of $V(A^*)$, and $\mathtt{coeq}(B, 2)$, whose objects can be identified with subobjects Q of $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ in $\mathtt{StVec}_{\mathbb{K}}$ that are closed under right derivatives. The previous fact was obtained by considering the free U–object on S $V(A^*) = (V(A^*), \varpi)$ where $\varpi : A \cdot V(A^*) \to V(A^*)$ in $\mathtt{Vec}_{\mathbb{K}}$ is such that $(\varpi \circ \iota_a)(s_1 w_1 + \cdots + s_n w_n) = s_1 a w_1 + \cdots + s_n a w_n$, where $\iota_a : V(A^*) \to A \cdot V(A^*)$ is the a–th inclusion. Note that we can also get a duality between right linear congruences and subobjects of $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ in $\mathtt{StVec}_{\mathbb{K}}$ that are closed under left derivatives by considering $(V(A^*), \varpi')$ such that $(\varpi' \circ \iota_a)(s_1 w_1 + \cdots + s_n w_n) = s_1 w_1 a + \cdots + s_n w_n a$ instead of $(V(A^*), \varpi)$.

Additionally, from this setting, if we consider linear congruences of $V(A^*)$ and subobjects of $\mathtt{Vec}_{\mathbb{K}}(V(A^*), \mathbb{K})$ in $\mathtt{StVec}_{\mathbb{K}}$ that are closed both under left and right derivatives we get the duality proved in Section 3.5. $\square$

Example 88. In this example, we explicitly show that if the contravariant adjunction is not a duality then sets of coequations are not always sent to sets of equations. Let A be the set $A = \{a, b\}$ and consider the situation:

$$
\begin{tikzcd}
\mathtt{Set} \arrow[loop left, "B"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\underset{\top}{\perp}"] & \mathtt{Set} \arrow[l, bend left, "G"] \arrow[loop right, "L"]
\end{tikzcd}
\qquad
\begin{gathered}
F(X) = G(X) = 2^X \\
L(X) = A \times X \\
B(Y) = Y^A \\
\gamma_X \colon 2^{A \times X} \to (2^X)^A
\end{gathered}
$$

In this case, consider the set $S = 1$ of generators. The free U–object in $\mathtt{alg}(L)$ over S is given by $A^* = (A^*, \varrho)$, where $\varrho(a, w) = aw$, and unit $\eta_1 = \varepsilon$, where $U : \mathtt{alg}(L) \to \mathtt{Set}$ is the forgetful functor. The cofree V–object on $G(S) = 2$ colours is the coalgebra $2^{A^*} = (2^{A^*}, \varsigma) = \widehat{G}(A^*)$, where $\varsigma(L)(a)(w) = L(aw)$, and counit $\epsilon_2(L) = L(\varepsilon)$, where $V : \mathtt{coalg}(B) \to \mathtt{Set}$ is the forgetful functor. Now, consider the element $m_Q \in \mathtt{coeq}(B, 2)$ where Q is the B–coalgebra $Q = (\{\emptyset, A^*\}, \beta)$ such that for every $a \in A$, $\beta(\emptyset)(a) = \emptyset$ and $\beta(A^*)(a) = A^*$, and m_Q is the inclusion map $m_Q \colon Q \to 2^{A^*}$ (note that $m_Q \in \mathtt{coalg}(B)(Q, 2^{A^*})$ since $\varsigma \circ m_Q = (m_Q)^A \circ \beta$). Now, the codomain of $\widehat{F}(m_Q)$ is $(2^Q, \alpha)$ where $\alpha(a, f) = \alpha(b, f) = f$ for all $f \in 2^Q$ (this definition of α follows from Example 82).

We have that $2^Q = (2^Q, \alpha)$ cannot be a homomorphic image of A^*. In fact, if there exists an epimorphism $e \in \mathtt{alg}(L)(A^*, 2^Q)$ then there is a right congruence θ of A^* such that $(A^*/\theta, f_\theta) \cong (2^Q, \alpha)$ which means that A^*/θ has four equivalence classes and for each equivalence class $w/\theta \in A^*/\theta$ we have that $w/\theta = aw/\theta =$

bw/θ, which is a contradiction since the last equalities imply that there is only one equivalence class. □

4.3.1 Equations for coalgebras

In this section, we define equations for coalgebras by using liftings of contravariant adjunctions. The concepts presented here can be dualized to define coequations for algebras.

Assume we have a contravariant adjunction between functors $F\colon \mathcal{C} \leftarrow\!\!\!\!\times\!\!\!\rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \leftarrow\!\!\!\!\times\!\!\!\rightarrow \mathcal{C}$ which is lifted to a contravariant adjunction between contravariant functors $\widehat{F}\colon \mathtt{coalg}(B) \leftarrow\!\!\!\!\times\!\!\!\rightarrow \mathtt{alg}(L)$ and $\widehat{G}\colon \mathtt{alg}(L) \leftarrow\!\!\!\!\times\!\!\!\rightarrow \mathtt{coalg}(B)$ for an endofunctor B on $\mathcal{C}$ and an endofunctor L on $\mathcal{D}$. Given an equation $e_P \in \mathtt{eq}(L, S)$ for some S in $\mathcal{D}$, we define, for a given coalgebra (Y, β_Y) in $\mathtt{coalg}(B)$, $(Y, \beta_Y) \models e_P$, and say that the coalgebra (Y, β_Y) satisfies the equation e_P, as:

$$(Y, \beta_Y) \models e_P \quad \overset{def}{\iff} \quad \widehat{F}(Y, \beta_Y) \models e_P.$$

Notice that if $\widehat{F}$ and $\widehat{G}$ form a duality then the property $\widehat{F}(Y, \beta_Y) \models e_P$ is equivalent to $(Y, \beta_Y) \mid\!\models \widehat{G}(e_P)$. One could be tempted to use $(Y, \beta_Y) \mid\!\models \widehat{G}(e_P)$ as a definition for $(Y, \beta_Y) \models e_P$ since $\widehat{G}(e_P) \in \mathtt{coeq}(B, G(S))$ but we prefer to avoid this since the dual argument is not true in general, i.e., given $m_Q \in \mathtt{coeq}(B, G(S))$, $\widehat{F}(m_Q)$ is not always in $\mathtt{eq}(L, S)$, as it was shown in Example 88.

Now we illustrate the previous definition of satisfaction of equations for coalgebras in the following example.

Example 89. Consider the situation given in Example 82 and let $S = \emptyset$. Then we have that for a B-coalgebra $Y = (Y, \langle c, \beta_Y \rangle)$, $c \in \mathtt{Set}(Y, 2)$ and $\beta_Y \in \mathtt{Set}(Y, Y^A)$, and a left congruence θ on A^*:

$$\begin{aligned}(Y, \langle c, \beta_Y \rangle) \models \theta \quad &\Leftrightarrow \quad \forall (u, v) \in \theta \;\; u(i(\cdot)) = v(i(\cdot)) \text{ in } \widehat{F}(Y, \langle c, \beta_Y \rangle) \\ &\Leftrightarrow \quad \forall (u, v) \in \theta \;\; \{y \in Y \mid c(u(y)) = 1\} = \{y \in Y \mid c(v(y)) = 1\}.\end{aligned}$$

In words, a left congruence θ on A^* is satisfied by $(Y, \langle c, \beta_Y \rangle)$ if for every pair $(u, v) \in \theta$ the set of states that accept u coincides with the set of states that accept v.

In Example 78 we also defined satisfaction of left congruences for deterministic automata, as the canonical notion that arises by viewing (the transition structure of) automata as algebras. According to this, if we consider (Y, β_Y) as an $A \times Id_{\mathtt{Set}}$–algebra, we have a direct definition for $(Y, \beta_Y) \models \theta$. We conclude this example by showing the relation between $(Y, \langle c, \beta_Y \rangle) \models \theta$ and $(Y, \beta_Y) \models \theta$.

Consider the coloured automaton $(Y, \langle c, \beta_Y \rangle)$ on $A = \{a\}$ given by:

$$\begin{tikzcd} y_1 \arrow[r, "a"] & y_2 \arrow[r, "a"] & y_3 \arrow[loop right, "a"] \end{tikzcd}$$

If we denote by $\langle u = v\rangle$ the least left congruence containing the pair $(u, v) \in A^* \times A^*$, then we have that $(Y, \langle c, \beta_Y\rangle) \models \langle a = aa\rangle$ since

$$\{y \in Y \mid c(a(y)) = 1\} = \{y_1, y_2, y_3\} = \{y \in Y \mid c(aa(y)) = 1\}$$

but $(Y, \beta_Y) \not\models \langle a = aa\rangle$ since $a(y_1) = y_2 \neq y_3 = aa(y_1)$. We now prove that $(Y, \beta_Y) \models \langle u = v\rangle$ implies $(Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle$ and that the converse holds if $(Y, \langle c, \beta_Y\rangle)$ is minimal.

Proposition 90. *Let $(Y, \langle c, \beta_Y\rangle)$ be a two–coloured automaton on A, and let $(u, v) \in A^* \times A^*$. Then $(Y, \beta_Y) \models \langle u = v\rangle$ implies $(Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle$.*

Proof. If $(Y, \beta_Y) \models \langle u = v\rangle$ then, for any $y \in Y$ we have that $u(y) = v(y)$, thus

$$\{y \in Y \mid c(u(y)) = 1\} = \{y \in Y \mid c(v(y)) = 1\}$$

which means that $(Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle$. □

A coalgebra $\mathbf{Y} = (Y, \langle c, \beta_Y\rangle) \in \mathtt{coalg}(B)$ is *minimal* if the unique observability B–coalgebra morphism $o_c \in \mathtt{coalg}(B)(\mathbf{Y}, 2^{A^*})$ is injective.

Proposition 91. *Let $u, v \in A^*$ and let $(Y, \langle c, \beta_Y\rangle)$ be a minimal B-coalgebra. Then*

$$(Y, \beta_Y) \models \langle u = v\rangle \textit{ if and only if } (Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle.$$

Proof. The direction (⇒) was proved in Proposition 90. To prove the other direction, assume that $(Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle$ and let us prove that $(Y, \beta_Y) \models \langle u = v\rangle$. In fact, assume by contradiction that there exists $y' \in Y$ such that $u(y') \neq v(y')$. As $(Y, \langle c, \beta_Y\rangle)$ is minimal there exists $w \in A^*$ such that $c(wu(y')) \neq c(wv(y'))$ implying that

$$\{y \in Y \mid c(wu(y)) = 1\} \neq \{y \in Y \mid c(wv(y)) = 1\}$$

which contradicts the fact that $(Y, \langle c, \beta_Y\rangle) \models \langle u = v\rangle$ since $(wu, wv) \in \langle u = v\rangle$ because $\langle u = v\rangle$ is a left congruence. □

This completes our example of equations for coalgebras.

4.4 Lifting contravariant adjunctions to Eilenberg–Moore categories

In this section, we extend the results from the previous sections, on lifting adjunctions and dualities, to the case that the endofunctor L is a monad and the endofunctor B is a comonad. We state the main theorem for lifting contravariant adjunctions to Eilenberg–Moore categories (Theorem 92), and obtain a theorem for dualities between equations and coequations as a consequence. Further, given either a monad or a comonad, we show how to derive a corresponding canonical comonad or monad, respectively. We start by recalling the definition of Eilenberg–Moore categories, see, e.g., [9].

Given a category $\mathcal{D}$ and a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, we denote the category of Eilenberg–Moore T–algebras, also called T–algebras, and their homomorphisms by $\mathtt{Alg}(\mathsf{T})$. Note the calligraphic difference between $\mathtt{alg}(T)$ and $\mathtt{Alg}(\mathsf{T})$. Objects in $\mathtt{Alg}(\mathsf{T})$ are pairs $\mathbf{X} = (X, \alpha_X)$ where X is an object in $\mathcal{D}$ and $\alpha_X \in \mathcal{D}(TX, X)$ is a morphism $\alpha_X : TX \to X$ in $\mathcal{D}$ that satisfies the identities $\alpha_X \circ \eta_X = id_X$ and $\alpha_X \circ T(\alpha_X) = \alpha_X \circ \mu_X$. Note that every object in $\mathtt{Alg}(\mathsf{T})$ is an object in $\mathtt{alg}(T)$ but the converse is does not hold in general. A homomorphism from a T–algebra $\mathbf{X_1} = (X_1, \alpha_1)$ to a T–algebra $\mathbf{X_2} = (X_2, \alpha_2)$ is a morphism $h \in \mathcal{D}(X_1, X_2)$ such that $h \circ \alpha_1 = \alpha_2 \circ T(h)$.

Dually, given a category $\mathcal{C}$ and a comonad $\mathsf{B} = (B, \epsilon, \delta)$ on $\mathcal{C}$, $\mathtt{Coalg}(\mathsf{B})$ denotes the category of Eilenberg–Moore B–coalgebras, also called B–coalgebras. Note the calligraphic difference between $\mathtt{coalg}(B)$ and $\mathtt{Coalg}(\mathsf{B})$. Objects in $\mathtt{Coalg}(\mathsf{B})$ are pairs $\mathbf{Y} = (Y, \beta_Y)$ where Y is an object in $\mathcal{C}$ and $\beta_Y \in \mathcal{C}(Y, BY)$ satisfies the identities $\epsilon_Y \circ \beta_Y = id_Y$ and $B(\alpha_Y) \circ \alpha_Y = \delta_Y \circ \alpha_Y$. Note that every object in $\mathtt{Coalg}(\mathsf{B})$ is an object in $\mathtt{coalg}(B)$ but the converse is does not hold in general. A homomorphism from a B–coalgebra $\mathbf{Y_1} = (Y_1, \beta_1)$ to a B–coalgebra $\mathbf{Y_2} = (Y_2, \beta_2)$ is a morphism $h \in \mathcal{C}(Y_1, Y_2)$ such that $\beta_2 \circ h = B(h) \circ \beta_1$.

Assume a contravariant adjunction between $F \colon \mathcal{C} \rightarrowtail\!\!\!\to \mathcal{D}$ and $G \colon \mathcal{D} \rightarrowtail\!\!\!\to \mathcal{C}$, a monad $\mathsf{L} = (L, \eta, \mu)$ on $\mathcal{D}$, and a comonad $\mathsf{B} = (B, \epsilon, \delta)$ on $\mathcal{C}$, as summarized in the following picture:

$$\begin{tikzcd}
\mathsf{B} = (B, \epsilon, \delta) \arrow[r, phantom, "\circlearrowright"] & \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{D} \arrow[ll, bend left, "G"] \arrow[r, phantom, "\circlearrowleft"] & \mathsf{L} = (L, \eta, \mu)
\end{tikzcd}$$

Then we can ask under what conditions the contravariant adjunction can be lifted to functors $\widehat{F} \colon \mathtt{Coalg}(\mathsf{B}) \rightarrowtail\!\!\!\to \mathtt{Alg}(\mathsf{L})$ and $\widehat{G} \colon \mathtt{Alg}(\mathsf{L}) \rightarrowtail\!\!\!\to \mathtt{Coalg}(\mathsf{B})$ on the Eilenberg–Moore categories. Similar to the approach in Section 4.2, we require a natural isomorphism $\gamma \colon GL \Rightarrow BG$, but for the current case we also require γ to satisfy certain conditions that relate the monad L and the comonad B.

Theorem 92. *Let $F \colon \mathcal{C} \rightarrowtail\!\!\!\to \mathcal{D}$ and $G \colon \mathcal{D} \rightarrowtail\!\!\!\to \mathcal{C}$ be contravariant functors that form a contravariant adjunction. Let $\mathsf{L} = (L, \eta, \mu)$ be a monad on $\mathcal{D}$, and $\mathsf{B} = (B, \epsilon, \delta)$ a comonad on $\mathcal{C}$. If there is a natural isomorphism $\gamma \colon GL \Rightarrow BG$ such that the following two diagrams commute:*

$$\begin{tikzcd}
G & BG \arrow[l, "\epsilon_G"'] \\
& GL \arrow[u, "\gamma"'] \arrow[ul, "G\eta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
BBG & & BG \arrow[ll, "\delta_G"'] \\
BGL \arrow[u, "B\gamma"] & GLL \arrow[l, "\gamma_L"] & GL \arrow[u, "\gamma"'] \arrow[l, "G\mu"]
\end{tikzcd}
\tag{4.3}$$

then F and G lift to functors $\widehat{F}\colon$ $\mathtt{Coalg}(\mathsf{B}) \times\!\!\!\longrightarrow \mathtt{Alg}(\mathsf{L})$ and $\widehat{G}\colon$ $\mathtt{Alg}(\mathsf{L}) \times\!\!\!\longrightarrow \mathtt{Coalg}(\mathsf{B})$, respectively, such that $\widehat{F}$ and $\widehat{G}$ form a contravariant adjunction. Additionally, if F and G form a duality then $\widehat{F}$ and $\widehat{G}$ form a duality.

Proof. Follows from [58] (see the contravariant adjunction as an adjunction between $\mathcal{D}$ and $\mathcal{C}^{op}$ with $G \dashv F$. In this case, having a comonad on $\mathcal{C}$ is the same as having a monad in $\mathcal{C}^{op}$). By [58, Lemma 1], using the fact that γ^{-1} is a natural transformation from BG to GL, there exists a functor $\widehat{G}\colon$ $\mathtt{Alg}(\mathsf{L}) \times\!\!\!\longrightarrow \mathtt{Coalg}(\mathsf{B})$ that is a lifting. By [58, Theorem 4], using the fact that γ is a natural isomorphism, there exists $\widehat{F}\colon$ $\mathtt{Coalg}(\mathsf{B}) \times\!\!\!\longrightarrow \mathtt{Alg}(\mathsf{L})$ such that $\widehat{F} \dashv\vdash \widehat{G}$, which is necessarily a lifting (up to isomorphism). $\square$

Remark. We mention the explicit definition of the functors $\widehat{G}\colon$ $\mathtt{Alg}(\mathsf{L}) \times\!\!\!\longrightarrow \mathtt{Coalg}(\mathsf{B})$ and $\widehat{F}\colon$ $\mathtt{Coalg}(\mathsf{B}) \times\!\!\!\longrightarrow \mathtt{Alg}(\mathsf{L})$ in the theorem above. In fact, define $\rho : LF \Rightarrow FB$ as $\rho = FB\eta^{GF} \circ F\gamma_F^{-1} \circ \eta_{LF}^{FG}$, where η^{GF} and η^{FG} are the units of the contravariant adjunction. Then, the definition of $\widehat{F}$ and $\widehat{G}$ are given by $\widehat{F}(Y, \beta_Y) = (F(Y), F(\beta_Y) \circ \rho_Y)$ and $\widehat{G}(X, \alpha_X) = (G(X), \gamma_X \circ G(\alpha_X))$ on objects and $\widehat{F} = F$ and $\widehat{G} = G$ on morphisms.

From the previous theorem, we derive dualities between equations and coequations in Eilenberg–Moore categories, which follows in a similar way as in Section 4.3. In this case, we do not need to explicitly assume the existence of free algebras since the algebra $\mathbf{LS} = (L(S), \mu_S) \in \mathtt{Alg}(\mathsf{L})$ is the free U–object over S, where $U : \mathtt{Alg}(\mathsf{L}) \to \mathcal{D}$ is the forgetful functor. In fact, we have the adjunction $\overline{L} \dashv U$, where $\overline{L} : \mathcal{D} \to \mathtt{Alg}(\mathsf{L})$ is defined as $\overline{L}(X) = \mathbf{LX}$ on objects, $\overline{L}(f) = L(f)$ on morphisms. Note that the unit for the adjunction $\overline{L} \dashv U$ is the unit η of the monad L and the extension $f^\sharp \in \mathtt{Alg}(\mathsf{L})(\mathbf{LS}, \mathbf{A})$ of a morphism $f \in \mathcal{D}(S, A)$ is given by $f^\sharp = \alpha_A \circ Lf$. Dually, the coalgebra $\mathbf{BR} = (B(R), \delta_R) \in \mathtt{Coalg}(\mathsf{B})$ is the cofree V–object on R, where $V : \mathtt{Coalg}(\mathsf{B}) \to \mathcal{C}$ is the forgetful functor. In fact, we have that the functor $\overline{B} : \mathcal{C} \to \mathtt{Coalg}(\mathsf{B})$ defined as $\overline{B}(Y) = \mathbf{BY}$ on objects and $\overline{B}(g) = B(g)$ on morphisms is such that $V \dashv \overline{B}$. Note that the counit for the adjunction $V \dashv \overline{B}$ is the counit ϵ of the comonad B and the map $f^\flat \in \mathtt{Alg}(\mathsf{L})(\mathbf{A}, \mathbf{BR})$ associated to a morphism $f \in \mathcal{C}(A, R)$ is given by $f^\flat = Bf \circ \beta_A$. In this case, for an object $S \in \mathcal{D}$, we define the category $\mathtt{Eq}(\mathsf{L}, S)$ as follows:

Objects of $\mathtt{Eq}(\mathsf{L}, S)$: epimorphisms $e_X \in \mathtt{Alg}(\mathsf{L})(\mathbf{LS}, \mathbf{X})$, for some $\mathbf{X} \in \mathtt{Alg}(\mathsf{L})$.

Arrows of $\mathtt{Eq}(\mathsf{L}, S)$: for $e_{X_1}, e_{X_2} \in \mathtt{Eq}(\mathsf{L}, S)$, a morphism $f \in \mathtt{Eq}(\mathsf{L}, S)(e_{X_1}, e_{X_2})$ is a morphism $f \in \mathtt{Alg}(\mathsf{L})(\mathbf{X_1}, \mathbf{X_2})$ such that the following diagram commutes:

$$\begin{tikzcd}
LS \arrow[r, two heads, "e_{X_1}"] \arrow[dr, "e_{X_2}"'] & X_1 \arrow[d, "f"] \\
& X_2
\end{tikzcd}$$

Dually, for an object $R \in \mathcal{C}$, we define the category $\mathtt{Coeq}(\mathsf{B}, R)$ whose objects are monomorphisms $m_Y \in \mathtt{Coalg}(\mathsf{B})(\mathbf{Y}, \mathbf{BR})$, for some $\mathbf{Y} = (Y, \beta_Y) \in \mathtt{Coalg}(\mathsf{B})$, and a morphism $g \in \mathtt{Coeq}(\mathsf{B}, R)(m_{Y_1}, m_{Y_2})$ is a morphism $g \in \mathtt{Coalg}(\mathsf{B})(\mathbf{Y_1}, \mathbf{Y_2})$ such that $m_{Y_2} \circ g = m_{Y_1}$. With these definitions we have the following.

Theorem 93. *Let $F\colon \mathcal{C} \leftarrow\!\!\!\times\!\!\rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \leftarrow\!\!\!\times\!\!\rightarrow \mathcal{C}$ be contravariant functors that form a duality. Let $\mathsf{L} = (L, \eta, \mu)$ be a monad on $\mathcal{D}$, and $\mathsf{B} = (B, \epsilon, \delta)$ a comonad on $\mathcal{C}$. If there is a natural isomorphism $\gamma\colon GL \Rightarrow BG$ making the diagrams* (4.3) *commute, then:*

1. *The duality between F and G lifts to a duality between contravariant functors $\widehat{F}\colon \mathtt{Coeq}(\mathsf{B}, G(S)) \leftarrow\!\!\!\times\!\!\rightarrow \mathtt{Eq}(\mathsf{L}, S)$ and $\widehat{G}\colon \mathtt{Eq}(\mathsf{L}, S) \leftarrow\!\!\!\times\!\!\rightarrow \mathtt{Coeq}(\mathsf{B}, G(S))$, where $\widehat{F}(Y, \beta_Y) = (F(Y), F(\beta_Y) \circ \rho_Y)$ and $\widehat{G}(X, \alpha_X) = (G(X), \gamma_X \circ G(\alpha_X))$, for $(Y, \beta_Y) \in \mathtt{Coalg}(\mathsf{B})$ and $(X, \alpha_X) \in \mathtt{Alg}(\mathsf{L})$, and $\widehat{F}(f) = F(f)$ and $\widehat{G}(g) = G(g)$ for morphisms f in $\mathtt{Coalg}(\mathsf{B})$ and g in $\mathtt{Alg}(\mathsf{L})$, and the natural transformation $\rho : LF \Rightarrow FB$ is defined as:*

$$\rho = FB\eta^{GF} \circ F\gamma^{-1}F \circ \eta^{FG}LF.$$

   where $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ and $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ are the units of the contravariant adjunction.

2. *Given $e_P \in \mathtt{Eq}(\mathsf{L}, S)$, $m_Q \in \mathtt{Coeq}(\mathsf{B}, G(S))$, $(X, \alpha_X) \in \mathtt{Alg}(\mathsf{L})$, and $(Y, \beta_Y) \in \mathtt{Coalg}(\mathsf{B})$ we have that:*

   i) $(X, \alpha_X) \models e_P$ if and only if $\widehat{G}(X, \alpha_X) \mathrel{|\!\!\models} \widehat{G}(e_P)$.

   ii) $\widehat{F}(Y, \beta_Y) \models \widehat{F}(m_Q)$ if and only if $(Y, \beta_Y) \mathrel{|\!\!\models} m_Q$. □

Now, it is worth mentioning that the converse of Theorem 92 also holds.

Theorem 94. *Let $F\colon \mathcal{C} \leftarrow\!\!\!\times\!\!\rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \leftarrow\!\!\!\times\!\!\rightarrow \mathcal{C}$ be contravariant functors that form a contravariant adjunction. Let $\mathsf{L} = (L, \eta, \mu)$ be a monad on $\mathcal{D}$, and $\mathsf{B} = (B, \epsilon, \delta)$ a comonad on $\mathcal{C}$. Assume that F lifts to a functor $\widehat{F}\colon \mathtt{Coalg}(\mathsf{B}) \leftarrow\!\!\!\times\!\!\rightarrow \mathtt{Alg}(\mathsf{L})$ and G lifts to a functor $\widehat{G}\colon \mathtt{Alg}(\mathsf{L}) \leftarrow\!\!\!\times\!\!\rightarrow \mathtt{Coalg}(\mathsf{B})$, such that $\widehat{F}$ and $\widehat{G}$ form a contravariant adjunction. Then, there is a natural isomorphism $\gamma\colon GL \Rightarrow BG$ such that the following two diagrams commute:*

$$\begin{tikzcd} G & BG \arrow[l, "\epsilon_G"'] \arrow[dl, phantom, "\star"] \\ & GL \arrow[u, "\gamma"'] \arrow[ul, "G\eta"] \end{tikzcd} \qquad \begin{tikzcd} BBG & & BG \arrow[ll, "\delta_G"'] \\ BGL \arrow[u, "B\gamma"] & GLL \arrow[l, "\gamma_L"] \arrow[u, phantom, "\star\star"] & GL \arrow[l, "G\mu"] \arrow[u, "\gamma"'] \end{tikzcd}$$

Proof. Let $U : \mathtt{Alg}(\mathsf{L}) \to \mathcal{D}$ and $V : \mathtt{Coalg}(\mathsf{B}) \to \mathcal{C}$ be the forgetful functors. Define the functors $\overline{L} : \mathcal{D} \to \mathtt{Alg}(\mathsf{L})$ and $\overline{B} : \mathcal{C} \to \mathtt{Coalg}(\mathsf{B})$ on objects as $\overline{L}(X) = \mathbf{LX} = (L(X), \mu_X)$ and $\overline{B}(Y) = \mathbf{BY} = (B(Y), \delta_Y)$, and on morphisms as $\overline{L}(f) = L(f)$

and $\overline{B}(g) = B(g)$. Note that $\overline{L}$ and $\overline{B}$ are well–defined on objects since L is a monad and B is a comonad, respectively, and they are well–defined on morphisms by naturality of μ and naturality of δ, respectively. Now, by using the fact that L is a monad and B is a comonad, resectively, we have that $\overline{L} \dashv U$ and $V \dashv \overline{B}$. Note that the unit of the adjunction $\overline{L} \dashv U$ is η and the counit of the adjunction $V \dashv \overline{B}$ is ϵ. Hence we have the following (not necessarily commutative) diagram:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathtt{Coalg}(\mathsf{B}) \arrow[r, bend left, "\widehat{F}"] \arrow[r, phantom, "\dashv\vdash"] \arrow[d, bend left, "V"] \arrow[d, phantom, "\vdash"] & \mathtt{Alg}(\mathsf{L}) \arrow[l, bend left, "\widehat{G}"] \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
\mathsf{B} = (B, \epsilon, \delta) \circlearrowright \mathcal{C} \arrow[u, bend left, "\overline{B}"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\dashv\vdash"] & \mathcal{D} \circlearrowleft \mathsf{L} = (L, \eta, \mu) \arrow[l, bend left, "G"] \arrow[u, bend left, "\overline{L}"]
\end{tikzcd}
$$

Let $\eta^{FG} : Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} : Id_{\mathcal{C}} \Rightarrow GF$ be the units of the contravariant adjunction $F \dashv\vdash G$ and let $\eta^{\widehat{G}\widehat{F}} : Id_{\mathtt{Coalg}(\mathsf{B})} \Rightarrow \widehat{G}\widehat{F}$ and $\eta^{\widehat{F}\widehat{G}} : Id_{\mathtt{Alg}(\mathsf{L})} \Rightarrow \widehat{F}\widehat{G}$ be the units of the contravariant adjunction $\widehat{F} \dashv\vdash \widehat{G}$. Now, as the contravariant adjunction $F \dashv\vdash G$ lifts to $\widehat{F} \dashv\vdash \widehat{G}$, the we have that:

$$GU = V\widehat{G}, \;\; FV = U\widehat{F}, \;\; U\eta^{\widehat{F}\widehat{G}} = \eta^{FG}U, \text{ and } V\eta^{\widehat{G}\widehat{F}} = \eta^{GF}V$$

By composing the adjunctions, we have that $\widehat{G}\overline{L} \dashv\vdash U\widehat{F}$ and $\overline{B}G \dashv\vdash FV$ which, by using the fact that $U\widehat{F} = FV$, implies that $\widehat{G}\overline{L} \cong \overline{B}G$, by uniqueness of adjunctions. Let $\alpha : \widehat{G}\overline{L} \Rightarrow \overline{B}G$ be a natural isomorphism. By construction of α, we have the following commutative diagrams:

$$
\begin{tikzcd}[column sep=huge]
\overline{B}V \arrow[d, "\overline{B}\eta^{GF}V"'] & Id_{\mathtt{Coalg}(\mathsf{B})} \arrow[l, "\eta^{\overline{B}V}"'] \arrow[r, "\eta^{\widehat{G}\widehat{F}}"] \arrow[d, phantom, "\dagger"] & \widehat{G}\widehat{F} \arrow[d, "\widehat{G}\epsilon^{\overline{L}U}\widehat{F}"] \\
\overline{B}GFV & & \widehat{G}\overline{L}U\widehat{F} \arrow[ll, "\alpha U\widehat{F}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
FG \arrow[d, "F\epsilon G"'] & Id_{\mathcal{C}} \arrow[l, "\eta^{FG}"] \arrow[r, "\eta"'] \arrow[d, phantom, "\dagger\dagger"] & L \arrow[d, "U\eta^{\widehat{F}\widehat{G}}\overline{L}"] \\
FBG \arrow[rr, "FV\alpha"'] & & U\widehat{F}\widehat{G}\overline{L}
\end{tikzcd}
$$

where $\eta^{\overline{B}V} : Id_{\mathtt{Coalg}(\mathsf{B})} \Rightarrow \overline{B}V$ is the unit of the adjunction $V \dashv \overline{B}$ and $\epsilon^{\overline{L}U} : \overline{L}U \Rightarrow Id_{\mathtt{Alg}(\mathsf{L})}$ is the counit of $\overline{L} \dashv U$.

Now, we have that $V\alpha$ is a natural isomorphism from $V\widehat{G}\overline{L} = GU\overline{L} = GL$ to $V\overline{B}G = BG$. We prove that $\gamma := V\alpha : GL \Rightarrow BG$ makes the diagrams $\star$ and $\star\star$ above commute. In fact, commutativity of $\star$ follows from the diagram:

$$
\begin{tikzcd}
G \arrow[d, "\eta^{GF}G"'] \arrow[dd, bend right=60, "Id_G"'] & BG \arrow[l, "\epsilon G"'] \arrow[d, "\eta^{GF}BG"] \arrow[l, phantom, "\text{nat. } \eta^{GF}", shift right=4ex] & GL \arrow[l, "V\alpha"'] \arrow[d, "V\eta^{\widehat{G}\widehat{F}}\widehat{G}\overline{L}"] \arrow[l, phantom, "\text{nat. } \eta^{GF}", shift right=4ex] \arrow[dd, bend left=60, "Id_{GL}"] \\
GFG \arrow[d, "G\eta^{FG}"'] & GFBG \arrow[l, "GF\epsilon G"'] \arrow[d, phantom, "\dagger\dagger"] & GU\widehat{F}\widehat{G}\overline{L} \arrow[l, "GFV\alpha"'] \arrow[d, "GU\eta^{\widehat{F}\widehat{G}}\overline{L}"] \\
G & & GL \arrow[ll, "G\eta"]
\end{tikzcd}
$$

and commutativity of $\star\star$ follows from using the identities $\mu = U\epsilon^{\overline{L}U}\overline{L}$, $\delta = V\eta^{\overline{B}V}\overline{B}$ and the commutative diagram:

$$
\begin{tikzcd}
BBG & & BG \arrow[ll, "V\eta^{\overline{B}V}\overline{B}G"'] \\
BGL \arrow[u, "BV\alpha"] \arrow[rr, phantom, "\text{nat. } \eta^{\overline{B}V}", shift left=4ex] & & GL \arrow[u, "V\alpha"'] \arrow[ll, "V\eta^{\overline{B}V}\widehat{G}\overline{L}"] \arrow[dl, "GU\epsilon^{\overline{L}U}\overline{L}"] \\
& GLL \arrow[ul, "V\alpha L"] \arrow[u, phantom, "\bullet"] &
\end{tikzcd}
$$

where $\bullet$ follows from the commutative diagram:

$$
\begin{tikzcd}
\overline{B}V\widehat{G} \arrow[d, "\overline{B}V\eta^{\widehat{G}\widehat{F}}\widehat{G}"'] & & \widehat{G} \arrow[ll, "\eta^{\overline{B}V}\widehat{G}"'] \arrow[d, "\eta^{\widehat{G}\widehat{F}}\widehat{G}"] \arrow[ll, phantom, "\dagger", shift right=4ex] \\
\overline{B}V\widehat{G}\widehat{F}\widehat{G} \arrow[d, "\overline{B}V\widehat{G}\eta^{\widehat{F}\widehat{G}}"'] & \widehat{G}\overline{L}U\widehat{F}\widehat{G} \arrow[l, "\alpha U\widehat{F}\widehat{G}"'] \arrow[d, "\widehat{G}\overline{L}U\eta^{\widehat{F}\widehat{G}}"'] \arrow[dl, phantom, "\text{nat. } \alpha"] & \widehat{G}\widehat{F}\widehat{G} \arrow[l, "\widehat{G}\epsilon^{\overline{L}U}\widehat{F}\widehat{G}"'] \arrow[d, "\widehat{G}\eta^{\widehat{F}\widehat{G}}"] \arrow[dl, phantom, "\text{nat. } \epsilon^{\overline{L}U}"] \\
\overline{B}V\widehat{G} \arrow[uu, bend left=60, "Id_{\overline{B}V\widehat{G}}"] & \widehat{G}\overline{L}U \arrow[l, "\alpha U"] & \widehat{G} \arrow[l, "\widehat{G}\epsilon^{\overline{L}U}"] \arrow[uu, bend right=60, "Id_{\widehat{G}}"']
\end{tikzcd}
$$

□

Remark. Note that in the previous proof the fact that L is a monad and the fact that B is a comonad were essential in order to define the functors $\overline{L}$ and $\overline{B}$ and to guarantee that $\overline{L} \dashv U$ and $V \dashv \overline{B}$. The later two adjunctions were essential in order to obtain the isomorphism γ.

We now proceed with special cases of our setting where, given the contravariant adjunction and a comonad B on $\mathcal{C}$, we can canonically define a monad L on $\mathcal{D}$ such that the contravariant adjunction lifts, Section 4.4.1. We can also do it in the opposite way, i.e., define a comonad from a given monad, but in this case additional assumptions are required, Section 4.4.2.

4.4.1 Defining a monad from a comonad

In this part, we start with a contravariant adjunction between contravariant functors $F\colon \mathcal{C} \rtimes\!\!\longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \rtimes\!\!\longrightarrow \mathcal{C}$ and a comonad $\mathsf{B} = (B, \epsilon, \delta)$ on $\mathcal{C}$. That is, we have the following setting:

$$\begin{tikzcd} \mathsf{B} = (B,\epsilon,\delta) \arrow[r, phantom, "\circlearrowright"] & \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{D} \arrow[ll, bend left, "G"] \end{tikzcd}$$

The purpose is to find a canonical monad $\mathsf{L} = (L, \eta, \mu)$ on $\mathcal{D}$ and a lift of the contravariant adjunction between F and G to a contravariant adjunction between $\widehat{F}\colon \mathtt{Coalg}(\mathsf{B}) \rtimes\!\!\longrightarrow \mathtt{Alg}(\mathsf{L})$ and $\widehat{G}\colon \mathtt{Alg}(\mathsf{L}) \rtimes\!\!\longrightarrow \mathtt{Coalg}(\mathsf{B})$. We choose $L = FBG$, and define $\eta\colon Id_{\mathcal{D}} \Rightarrow L$ and $\mu\colon LL \Rightarrow L$ by:

$$\begin{aligned} \eta &= (Id_{\mathcal{D}} \xRightarrow{\eta^{FG}} FG \xRightarrow{F\epsilon_G} FBG) \\ \mu &= (FBGFBG \xRightarrow{FB\eta^{GF}_{BG}} FBBG \xRightarrow{F\delta_G} FBG) \end{aligned} \tag{4.4}$$

where η^{FG} and η^{GF} are the units of the contravariant adjunction. With this choice of (L, η, μ) we have the following result.

Proposition 95. *Let $F\colon \mathcal{C} \rtimes\!\!\longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \rtimes\!\!\longrightarrow \mathcal{C}$ be contravariant functors that form a contravariant adjunction. Let $\mathsf{B} = (B, \epsilon, \delta)$ be a comonad on $\mathcal{C}$. Then (L, η, μ) with $L = FBG$ and η, μ defined as in* (4.4) *is a monad on $\mathcal{D}$.*

Additionally, if η^{GF} is a natural isomorphism, then the contravariant adjunction between the functors F and G lifts to a contravariant adjunction between contravariant functors $\widehat{F}\colon \mathtt{Coalg}(\mathsf{B}) \rtimes\!\!\longrightarrow \mathtt{Alg}(\mathsf{L})$ and $\widehat{G}\colon \mathtt{Alg}(\mathsf{L}) \rtimes\!\!\longrightarrow \mathtt{Coalg}(\mathsf{B})$. In this case, if F and G form a duality then the lifting $\widehat{F}$ and $\widehat{G}$ is also a duality.

Proof. We have that $\mathsf{L} = (L, \eta, \mu)$ is the monad induced by the composite of the following two adjunctions:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, bend left, "G"] \arrow[rr, phantom, "\perp"] & & \mathcal{C}^{op} \arrow[ll, bend left, "F"] \arrow[rr, bend left, "H"] \arrow[rr, phantom, "\perp"] & & \mathtt{Alg}(\mathsf{B}^{op}) \arrow[ll, bend left, "J"] \end{tikzcd}$$

(see, e.g., [4, Chapter V]), where B^{op} is the monad on $\mathcal{C}^{op}$ that is dual to the monad B on $\mathcal{C}$, and the functors H and J are defined as $H(X) = (BX, \delta_X)$ and $J(Y, \alpha_Y) = Y$ on objects and $H(g) = B(g)$ and $J(f) = f$ on morphisms.

The last part of the proposition will be a consequence of Theorem 92 for which $\gamma = (\eta^{GF}_{BG})^{-1} : GL \Rightarrow BG$. In fact, we only need to prove that the following two diagrams commute:

$$
\begin{tikzcd}
G & BG \arrow[l, "\epsilon_G"'] \arrow[dl, phantom, "\text{i.}"] \\
 & GL \arrow[ul, "G\eta"] \arrow[u, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
BBG & & BG \arrow[ll, "\delta_G"'] \\
BGL \arrow[u, "B\gamma"] & GLL \arrow[l, "\gamma_L"] \arrow[u, phantom, "\text{ii.}"] & GL \arrow[l, "G\mu"] \arrow[u, "\gamma"']
\end{tikzcd}
$$

Commutativity of i. follows from the commutative diagram:

$$
\begin{tikzcd}
G & BG \arrow[l, "\epsilon_G"'] \arrow[dl, phantom, "\text{nat. of } \eta^{GF}"] \\
GFG \arrow[u, "G\eta^{FG} = (\eta^{GF}_G)^{-1}"] & GL \arrow[l, "GF\epsilon_G"] \arrow[u, "\gamma = (\eta^{GF}_{BG})^{-1}"']
\end{tikzcd}
$$

And commutativity of ii. follows from the commutative diagram:

$$
\begin{tikzcd}
BBG & & & BG \arrow[lll, "\delta_G"'] \\
BGL \arrow[u, "B(\eta^{GF}_{BG})^{-1}"] & GLL \arrow[l, "(\eta^{GF}_{BGL})^{-1}"] \arrow[ul, phantom, "\text{nat. of } \eta^{GF}"] & GFBBG \arrow[l, "GFB\eta^{GF}_{BG}"] \arrow[ull, "(\eta^{GF}_{BBG})^{-1}"'] \arrow[ur, phantom, "\text{nat. of } \eta^{GF}"] & GL \arrow[l, "GF\delta_G"] \arrow[u, "\gamma = (\eta^{GF}_{BG})^{-1}"']
\end{tikzcd}
$$

□

4.4.2 Defining a comonad from a monad

We now prove a similar result as Proposition 95 in order to define a comonad on $\mathcal{C}$ if we have a monad on $\mathcal{D}$. In order to do this we will assume that the contravariant adjunction is a duality so we can use the fact that the units of the contravariant adjunction are isomorphisms.

Assume that we have a contravariant adjunction between contravariant functors $F \colon \mathcal{C} \looparrowright \mathcal{D}$ and $G \colon \mathcal{D} \looparrowright \mathcal{C}$, and let $\mathsf{L} = (L, \eta, \mu)$ be a monad on $\mathcal{D}$. Define the endofunctor B on $\mathcal{C}$ as $B = GLF$. Now, if we assume that the contravariant adjunction is a duality with units $\eta^{FG} \colon Id_{\mathcal{D}} \Rightarrow FG$ and $\eta^{GF} \colon Id_{\mathcal{C}} \Rightarrow GF$ that are natural isomorphisms. Then we can define natural transformations $\epsilon\colon B \Rightarrow Id_{\mathcal{C}}$ and $\delta\colon B \Rightarrow BB$ as:

$$\begin{aligned} \epsilon &= (GLF \xRightarrow{G\eta_F} GF \xRightarrow{(\eta^{GF})^{-1}} Id_{\mathcal{C}}) \\ \delta &= (GLF \xRightarrow{G\mu_F} GLLF \xRightarrow{GL(\eta^{FG})^{-1}_{LF}} GLFGLF) \end{aligned} \tag{4.5}$$

Under the previous assumptions and choice of (B, ϵ, δ) we get:

Proposition 96. *Let $F\colon \mathcal{C} \rightarrowtail \mathcal{D}$ and $G\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be contravariant functors that form a duality. Let $\mathsf{L} = (L, \eta, \mu)$ be a monad on $\mathcal{D}$. Then (B, ϵ, δ), where $B = GLF$ and ϵ, δ are defined as in* (4.5)*, is a comonad on $\mathcal{C}$. Further, the duality between F and G lifts to a duality between $\widehat{F}\colon$ $\mathtt{Coalg}(\mathsf{B}) \rightarrowtail \mathtt{Alg}(\mathsf{L})$ and $\widehat{G}\colon$ $\mathtt{Alg}(\mathsf{L}) \rightarrowtail \mathtt{Coalg}(\mathsf{B})$.*

Proof. To prove that $\mathsf{B} = (B, \epsilon, \delta)$ is a comonad we have to prove that the following diagrams commute:

$$\begin{tikzcd} B & BB \arrow[l, "B\epsilon"'] \arrow[r, "\epsilon_B"] & B \\ & B \arrow[ul, "Id_B"] \arrow[u, "\delta"'] \arrow[ur, "Id_B"'] \arrow[ul, phantom, "1." near end, shift left=4] \arrow[ur, phantom, "2." near end, shift right=4] \end{tikzcd} \qquad \begin{tikzcd} BBB & & BB \arrow[ll, "B\delta"'] \\ BB \arrow[u, "\delta_B"] & & B \arrow[ll, "\delta"] \arrow[u, "\delta"'] \arrow[ull, phantom, "3."] \end{tikzcd}$$

Commutativity of the previous diagrams follow from proof of Proposition 95 by a dual argument. That is, by replacing η, L, μ, F, G and B by ϵ, B, δ, G, F and L, respectively, in the proof of Proposition 95 and by reversing the arrows. In this case, arrows including η^{GF} or η^{FG} will be replaced by $(\eta^{FG})^{-1}$ and $(\eta^{GF})^{-1}$, respectively.

The last part of the proposition will be a consequence of Theorem 92 by considering $\gamma = GL(\eta^{FG})^{-1} : GL \Rightarrow BG$. In fact, we only need to prove that the following two diagrams commute:

$$\begin{tikzcd} G & BG \arrow[l, "\epsilon_G"'] \\ & GL \arrow[ul, "G\eta"] \arrow[u, "\gamma"'] \arrow[ul, phantom, "\mathrm{i.}", shift left=4] \end{tikzcd} \qquad \begin{tikzcd} BBG & & BG \arrow[ll, "\delta_G"'] \\ BGL \arrow[u, "B\gamma"] & GLL \arrow[l, "\gamma_L"] \arrow[u, phantom, "\mathrm{ii.}"] & GL \arrow[l, "G\mu"] \arrow[u, "\gamma"'] \end{tikzcd}$$

Commutativity of i. follows from the commutative diagram:

$$\begin{tikzcd} G & GFG \arrow[l, "(\eta^{GF}_G)^{-1} = G\eta^{FG}"'] & BG \arrow[l, "G\eta_{FG}"'] \arrow[dl, phantom, "\text{nat. of } \eta"] \\ & & GL \arrow[ull, "G\eta"] \arrow[u, "\gamma = (GL\eta^{FG})^{-1}"'] \end{tikzcd}$$

And commutativity of ii. follows from the commutative diagram:

$$\begin{tikzcd}
BBG & GLLFG \arrow[l, "GL(\eta^{FG}_{LFG})^{-1}"'] & BG \arrow[l, "G\mu_{FG}"'] \\
BGL \arrow[u, "BGL(\eta^{FG})^{-1}"] \arrow[ur, phantom, "\text{nat. of } \eta^{FG}"] & GLL \arrow[l, "GL(\eta^{FG}_{L})^{-1}"] \arrow[u, "GLL(\eta^{FG})^{-1}"'] \arrow[ur, phantom, "\text{nat. of } \mu"] & GL \arrow[l, "G\mu"] \arrow[u, "\gamma = GL(\eta^{FG})^{-1}"']
\end{tikzcd}$$

□

4.5 Applications

In this section, we will apply results from this chapter to study equations and coequations for dynamical systems and deterministic automata. We illustrate both cases by using Eilenberg–Moore categories.

4.5.1 Equations and coequations for dynamical systems

Let $M = (M, \cdot, e)$ be a monoid, let $\mathsf{L} = (L, \eta, \mu)$ be the monad on $\mathtt{Set}$ defined as:

$$L(X) = X \times M \qquad \begin{aligned} \eta_X \colon X &\to X \times M \\ x &\mapsto (x, e) \end{aligned} \qquad \begin{aligned} \mu_X \colon (X \times M) \times M &\to X \times M \\ (x, m, n) &\mapsto (x, m \cdot n) \end{aligned}$$

and let $\mathsf{B} = (B, \epsilon, \delta)$ be the comonad on $\mathtt{CABA}$ defined as:

$$B(Y) = Y^M \qquad \begin{aligned} \epsilon_Y \colon Y^M &\to Y \\ f &\mapsto f(e) \end{aligned} \qquad \begin{aligned} \delta_Y \colon Y^M &\to (Y^M)^M \\ f &\mapsto \lambda_m.\lambda_n f(n \cdot m) \end{aligned}$$

where the $\mathtt{CABA}$ structure on $Z^M = B(Z)$, for a given $Z \in \mathtt{CABA}$, is componentwise. Consider the duality between $\mathtt{CABA}$ and $\mathtt{Set}$ given by the contravariant functors $F\colon \mathtt{CABA} \leftarrow\!\!\!\!\!\times\!\!\!\longrightarrow \mathtt{Set}$ and $G\colon \mathtt{Set} \leftarrow\!\!\!\!\!\times\!\!\!\longrightarrow \mathtt{CABA}$ defined as $F(\_) = \mathtt{CABA}(\_, 2)$ and $G(\_) = \mathtt{Set}(\_, 2)$, if we consider the natural isomorphism $\gamma\colon GL \Rightarrow BG$ given by the canonical isomorphism $\gamma_X \colon 2^{X \times M} \to (2^X)^M$ then we can easily verify the hypothesis of Theorem 92 to lift the duality between F and G from the following setting:

$$\begin{tikzcd}
\mathsf{B} = (B, \epsilon, \delta) \arrow[r, bend left, phantom, ""] & \mathtt{CABA} \arrow[loop left] \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathtt{Set} \arrow[l, bend left, "G"] \arrow[loop right] & \mathsf{L} = (L, \eta, \mu)
\end{tikzcd}
\qquad
\begin{aligned}
F(Y) &= \mathtt{CABA}(Y, 2) \\
G(X) &= \mathtt{Set}(X, 2) \\
L(X) &= X \times M \\
B(Y) &= Y^M \\
\gamma_X \colon 2^{X \times M} &\to (2^X)^M
\end{aligned}$$

Observe that elements $(X, \alpha_X) \in \texttt{Alg}(\mathsf{L})$ are *dynamical systems*, also called *monoid actions*, on $\texttt{Set}$ for the monoid M, that is, a set X together with a map $\alpha_X \colon X \times M \to X$ that satisfies the properties $\alpha_X(x, e) = x$ and $\alpha_X(\alpha_X(x, m), n) = \alpha_X(x, m\cdot n)$. Further, an object $(Y, \beta_Y) \in \texttt{Coalg}(\mathsf{B})$ is an object $Y \in \texttt{CABA}$ with a map $\beta_Y \in \texttt{CABA}(Y, Y^M)$ such that $\beta_Y(y)(e) = y$ and $\beta_Y(\beta_Y(y)(m))(n) = \beta_Y(y)(n \cdot m)$.

We are going to consider equations and coequations for dynamical systems for the particular case that the set of generators is $S = 1$. We have that the free U–object $\mathfrak{F}(1)$ in $\texttt{Alg}(\mathsf{L})$ over $S = 1$ is $\mathfrak{F}(1) = (M, \varrho)$ where $\varrho\colon M \times M \to M$ is given by $\varrho(m, n) = m \cdot n$ and the unit $\eta_1 \colon 1 \to M$ is given by $\eta_1 = e$, the identity element in M. On the other hand, the cofree V–object $\mathfrak{C}(G(1)) = \mathfrak{C}(2)$ in $\texttt{Coalg}(\mathsf{B})$ over 2 colours is $\mathfrak{C}(2) = (2^M, \varsigma')$, where $\varsigma' \colon 2^M \to (2^M)^M$ is given by $\varsigma'(f)(n)(m) = f(m \cdot n)$ and the counit $\epsilon_2 \colon 2^M \to 2$ is given by $\epsilon_2(f) = f(e)$.

According to this, equations in $\texttt{Eq}(\mathsf{L}, 1)$ correspond to quotients $M/\theta = (M/\theta, f_\theta)$ where $\theta \subseteq M \times M$ is a *right congruence* on M, i.e. an equivalence relation such that for any $p \in M$, $(m, n) \in \theta$ implies $(m \cdot p, n \cdot p) \in \theta$, and the function $f_\theta \colon M/\theta \times M \to M/\theta$ is given by $f_\theta(m/\theta, n) = m \cdot n/\theta$. On the other hand coequations in $\texttt{Coeq}(\mathsf{L}, 2)$ correspond to *left-closed-subsystems* $Q = (Q, \varsigma')$, i.e. subalgebras Q of the complete atomic Boolean algebra 2^M such that for any $f \in Q$ and $m \in M$, $\varsigma'(f)(m) \in Q$.

Now, by using Theorem 93, we have as a consequence a correspondence between right congruences and left-closed-subsystems for dynamical systems.

Proposition 97. *There is a duality between* $\texttt{Eq}(\mathsf{L}, 1)$ *and* $\texttt{Coeq}(\mathsf{B}, 2)$ *given by* $\widehat{F}$ *and* $\widehat{G}$ *that induces a duality between right congruences on* M *and left-closed-subsystems of* 2^M.

Proof. Duality between $\texttt{Eq}(\mathsf{L}, 1)$ and $\texttt{Coeq}(\mathsf{B}, 2)$ follows Theorem 93 by taking $S = 1$. Now, given a right congruence θ on M its corresponding left-closed-subsystem is $\mathrm{Im}(\widehat{G}(\nu_\theta))$ where $\nu_\theta \in \texttt{Alg}(\mathsf{L})(M, M/\theta)$ is the canonical morphism defined as $\nu_\theta(m) = m/\theta$ and, given a left-closed-subsystem $\mathbf{Q} = (Q, \varsigma')$ its corresponding right congruence is $\ker(\widehat{F}(i_Q))$ where $i_Q \in \texttt{Coalg}(\mathsf{B})(\mathbf{Q}, 2^M)$ is the inclusion morphism. This correspondence between right congruences and left-closed-subsystems is a duality induced by $\widehat{F}$ and $\widehat{G}$. $\square$

Using this duality one can prove that right congruences on M and left-closed subsystems of 2^M characterize the same classes of dynamical systems.

Proposition 98. *Let* (X, α_X) *be a dynamical system on* M*, i.e.,* $(X, \alpha_X) \in \texttt{Alg}(\mathsf{L})$. *For a right congruence* θ *on* M *let* $\nu_\theta \in \texttt{Alg}(\mathsf{L})(M, M/\theta)$ *be the canonical epimorphism (equation) defined as* $\nu_\theta(m) = m/\theta$. *The following are equivalent:*

i) $(X, \alpha_X) \models \nu_\theta$.

ii) For every colouring $c \colon X \to 2$ *and any* $x \in X$ *we have that*

$$\{m \in M \mid c(\alpha_X(x)(m)) = 1\} \in \mathrm{Im}(\widehat{G}(\nu_\theta)).$$

Proof. ($\Rightarrow$): Assume that $(X, \alpha_X) \models \nu_\theta$, i.e., for every $z \in \mathtt{Set}(1, X)$ we have that $z^\sharp$ factors in $\mathtt{Alg}(\mathsf{L})$ through the epimorphism ν_θ as $z^\sharp = g_z \circ \nu_\theta$. That is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\nu_\theta"] \arrow[rr, bend right, "z^\sharp"'] & M/\theta \arrow[r, dashed, "g_z"] & X \\ 1 \arrow[u, "\eta_1"] \arrow[urr, bend right, "z"'] & & \end{tikzcd}$$

Therefore, by applying the functor G, we have that for every $d \in \mathtt{CABA}(2^X, 2)$ the morphism $d^\flat$, given by cofreeness of $\mathfrak{C}(2)$, factors in $\mathtt{Coalg}(\mathsf{B})$ through 2^{ν_θ} as $d^\flat = 2^{\nu_\theta} \circ h_d$, which means that $\mathrm{Im}(d^\flat) \subseteq \mathrm{Im}(2^{\nu_\theta}) = \mathrm{Im}(\widehat{G}(\nu_\theta))$. This is shown in the following commutative diagram:

$$\begin{tikzcd} 2^X \arrow[r, dashed, "h_d"] \arrow[rr, bend right, "d^\flat"'] \arrow[drr, bend right, "d"'] & 2^{M/\theta} \arrow[r, "2^{\nu_\theta}"] & 2^M \arrow[d, "\epsilon_2"] \\ & & 2 \end{tikzcd}$$

In particular, for a fixed $x \in X$, if we define the morphism $d_x \in \mathtt{CABA}(2^X, 2)$ as $d_x(f) = f(x)$, then we have that $\mathrm{Im}({d_x}^\flat) \subseteq \mathrm{Im}(\widehat{G}(\nu_\theta))$. Now, given a function $c : X \to 2$, put $\{x_i\}_{i \in I} = c^{-1}(1)$ and define $f_i \in 2^X$ as $f_i(x') = 1$ iff $x' = x_i$, $i \in I$. Then we have that ${d_x}^\flat(f_i) \in \mathrm{Im}(\widehat{G}(\nu_\theta))$ for every $i \in I$, but

$${d_x}^\flat(f_i) = \{m \in M \mid \alpha_X(x)(m) = x_i\}$$

Hence, $\{m \in M \mid \alpha_X(x)(m) = x_i\} \in \mathrm{Im}(\widehat{G}(\nu_\theta))$, and since $\mathrm{Im}(\widehat{G}(\nu_\theta)) \in \mathtt{CABA}$ we have that

$$\{m \in M \mid c(\alpha_X(x)(m)) = 1\} = \bigvee_{i \in I} \{m \in M \mid \alpha_X(x)(m) = x_i\} \in \mathrm{Im}(\widehat{G}(\nu_\theta)).$$

($\Leftarrow$) Assume that for any $c \in \mathtt{Set}(X, 2)$ and for any $x \in X$ we have that $\{m \in M \mid c(\alpha_X(x)(m)) = 1\} \in \mathrm{Im}(\widehat{G}(\nu_\theta))$. For any $x_0 \in X$ define $c_{x_0} \in \mathtt{Set}(X, 2)$ as $c_{x_0}(x') = 1$ iff $x' = x_0$. Hence, for any $x_0 \in X$ and any $x \in X$ we have $\{m \in M \mid c_{x_0}(\alpha_X(x)(m)) = 1\} \in \mathrm{Im}(\widehat{G}(\nu_\theta))$, but

$$\{m \in M \mid c_{x_0}(\alpha_X(x)(m)) = 1\} = \{m \in M \mid \alpha_X(x)(m) = x_0\} = {d_x}^\flat(c_{x_0})$$

if we denote by d_x the element in $\mathtt{CABA}(2^X, 2)$ given by $d_x(f) = f(x)$. So, for a fixed $x \in X$ and every $x_0 \in X$ we have that ${d_x}^\flat(c_{x_0}) \in \mathrm{Im}(\widehat{G}(\nu_\theta))$, and, as $\mathrm{Im}(\widehat{G}(\nu_\theta)) \in \mathtt{CABA}$, then $\mathrm{Im}({d_x}^\flat) \subseteq \mathrm{Im}(\widehat{G}(\nu_\theta))$, which means that ${d_x}^\flat$ factors through the inclusion map $i : \mathrm{Im}(\widehat{G}(\nu_\theta)) \to 2^M$ as ${d_x}^\flat = i \circ h_x$ in $\mathtt{Coalg}(\mathsf{B})$. That is, the following diagram commutes:

$$\begin{tikzcd}
2^X \arrow[r, dashed, "h_x"] \arrow[rr, bend right=10, "{d_x}^\flat"'] \arrow[rrd, bend right=15, "d_x"'] & \mathrm{Im}(\widehat{G}(\nu_\theta)) \arrow[r, "i"] & 2^M \arrow[d, "\epsilon_2"] \\
& & 2
\end{tikzcd}$$

Therefore, by applying the functor F, we have that for $x \in X$, $x^\sharp$ factors through $\widehat{F}(i) = \mathtt{CABA}(i, 2)$ as $x^\sharp = g_s \circ \mathtt{CABA}(i, 2)$. That is, the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "{\mathtt{CABA}(i,2)}"] \arrow[rr, bend right=10, "x^\sharp"'] & \mathtt{CABA}(\mathrm{Im}(\widehat{G}(\nu_C)), 2) \arrow[r, dashed, "g_x"] & X \\
1 \arrow[u, "\eta_1"] \arrow[rru, bend right=10, "x"'] & &
\end{tikzcd}$$

which means that $\ker(x^\sharp) \supseteq \ker(\mathtt{CABA}(i, 2)) = \ker(\widehat{F}(i)) = \theta$. Now, as $x \in X$ was arbitrary, then we have that $(X, \alpha_X) \models \nu_\theta$. □

If M is the free monoid on A generators then we get [79, Corollary 14]. In this case, property ii) in the previous proposition is the definition for satisfaction of coequations given in [13] where the set of coequations considered is $\mathrm{Im}(\widehat{G}(e_C))$.

4.5.2 Equations and coequations for automata

In this part, we study the case of equations and coequations for deterministic automata. Consider the following setting:

$$\begin{tikzcd}
\mathtt{CABA} \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathtt{Set} \arrow[l, bend left, "G"] \arrow[loop right, "\mathsf{L} = (L, \eta, \mu)"]
\end{tikzcd}$$

where $F(Y) = \mathtt{CABA}(Y, 2)$, $G(X) = \mathtt{Set}(X, 2)$, and L is the monad given by:

$$L(X) = X^* = \coprod_{i \in \mathbb{N}} X^i \qquad \begin{array}{rl} \eta_X \colon & X \to X^* \\ & x \mapsto x \end{array} \qquad \begin{array}{rl} \mu_X \colon & (X^*)^* \to X^* \\ & w_1 \cdots w_n \mapsto w_1 \cdots w_n \end{array}$$

According to Proposition 96, as F and G form a duality, we get a comonad $\mathsf{B} = (B, \epsilon, \delta)$ on $\mathtt{CABA}$ and a duality between $\mathtt{Coalg}(\mathsf{B})$ and $\mathtt{Alg}(\mathsf{L})$. Observe that $\mathtt{Alg}(\mathsf{L})$ is isomorphic to the category of monoids.

For any set A, $L(A) = A^*$ is the free monoid on A generators, with unit morphism η_A and multiplication μ_A. That is, (A^*, μ_A) is the free U–object over A, where $U : \mathtt{Alg}(\mathsf{L}) \to \mathtt{Set}$ is the forgetful functor. Now we will fix the set A and show how the notion of satisfaction of equations given in [13] for a deterministic automaton on A can be equivalently defined in this setting. In fact, given a deterministic automaton $(X, \alpha_X \colon X \times A \to A)$ on A we can use the correspondence:

$$\frac{\alpha_X \colon X \times A \to X}{\widehat{\alpha_X} \colon A \to X^X}$$

to work with the monoid $X^X = (X^X, \beta) \in \mathtt{Alg}(\mathsf{L})$ with composition of functions as multiplication β. We have that homomorphic images of A^*, i.e., elements in $\mathtt{Eq}(\mathsf{L}, A)$, correspond to congruences of the monoid A^*. Given any congruence θ of A^* we have that $(X, \alpha_X) \models \theta$, if the unique extension $\widehat{\alpha_X}^\sharp \in \mathtt{Alg}(\mathsf{L})(A^*, X^X)$ of $\widehat{\alpha_X}$ factors in $\mathtt{Alg}(\mathsf{L})$ through the canonical morphism $\nu_\theta \colon A^* \to A^*/\theta$. That is, we have that $(X, \alpha_X) \models \theta$ if there exists $g \in \mathtt{Alg}(\mathsf{L})(A^*/\theta, X^X)$ such that the following diagram commutes:

$$\begin{tikzcd}
A^* \arrow[r, "\nu_\theta"] \arrow[rr, bend right, "\widehat{\alpha_X}^\sharp"'] & A^*/\theta \arrow[r, dashed, "g"] & X^X \\
A \arrow[u, "\eta_A"] \arrow[urr, bend right, "\widehat{\alpha_X}"'] & &
\end{tikzcd}$$

this means that for any $(u, v) \in \theta$ the transition functions $f_u, f_v \in X^X$, where $f_w(x) = w(x)$, $w \in A^*$, are the same. This is the notion of satisfaction of equations we previously defined in Example 78, and which appears in [13].

We apply G to the previous diagram to get the following diagram:

$$\begin{tikzcd}
2^{(A^*)} \arrow[d, "2^{\eta_A}"'] & 2^{(A^*/\theta)} \arrow[l, "2^{\nu_\theta}"'] & 2^{(X^X)} \arrow[l, dashed, "2^g"'] \arrow[ll, bend left, "2^{\widehat{\alpha_X}^\sharp}"] \arrow[dll, bend left, "2^{\widehat{\alpha_X}}"] \\
2^A & &
\end{tikzcd}$$

Now, the equality $2^{\widehat{\alpha_X}^\sharp} = 2^{\nu_\theta} \circ 2^g$ in the diagram above implies that:

$$\mathrm{Im}(2^{\widehat{\alpha_X}^\sharp}) \subseteq \mathrm{Im}(2^{\nu_\theta}) = \{L \in 2^{A^*} \mid \forall\, (u, v) \in \theta, L(u) = L(v)\},$$

which is an object in $\mathtt{CABA}$ and it is closed under left and right derivatives since $\mathrm{Im}(2^{\widehat{\alpha_X}^\sharp}) \cong 2^{(A^*/\theta)}$ and $2^{\widehat{\alpha_X}^\sharp}$ is a B–coalgebra morphism, where the isomorphism follows from the fact that 2^{ν_θ} is injective, since ν_θ is surjective.

By Theorem 93, we get a duality between $\mathtt{Eq}(\mathsf{L}, A)$ and $\mathtt{Coeq}(\mathsf{B}, G(A))$ which is the duality between equations and coequations given in [13, Theorem 22]. Additionally, using the previous commutative diagrams, one can prove the equivalence between i) and ii) given in Proposition 98 for the case that $M = A^*$, the congruences θ are congruences of A^*, and the coequations $\mathrm{Im}(\widehat{G}(\nu_\theta))$ are subalgebras of 2^M that are closed under left and right derivatives, cf. [79, Theorem 17].

4.6 Discussion

In this chapter, we developed the general theory about equations and coequations that we use in this thesis, from a categorical point of view. We already studied particular cases in Chapter 2 and Chapter 3, where equations and coequations for deterministic automata and weighted automata were introduced, respectively. In both cases, we showed a duality result which is also obtained from the general theory of the present chapter. The main idea in this chapter was to consider equations as special arrows in a category, namely, epimorphisms with free domain. Dually, we considered coequations as monomorphisms with cofree codomain. Epimorphisms and monomorphisms are the categorical generalization of surjective and injective functions, respectively.

The main idea of considering equations from a categorical point of view was initially made in [15] where a categorical version of Birkhoff's theorem is proved. Later, other categorical approaches for equations and coequations were also studied in, e.g., [52, 65, 50, 54, 38, 25, 76]. In [15], a categorical version of Birkhoff's theorem was formulated by considering equations as regular epimorphisms with regular–projective domain. From [15], dual versions can be easily obtained and hence the idea of defining coequations as a special kind of monomorphisms. In [10], coequations are defined as regular subobjects of a cofree coalgebra, i.e., a special kind of monomorphism. In [61], coequations are called modal rules or modal formulas, and they are represented by morphisms in M, usually a monomorphism, for a given $(\mathscr{E}, M)$–category [61, Definition 2.4.1]. In [25], equations are presented as pairs of arrows, left–hand side and right–hand side, and the definition of satisfaction of an equation is in terms of coequalizing those two arrows, this property can be presented in terms of their coequalizer, when it exists, and hence equations are a special kind of epimorphism. A similar idea of defining equations with left–hand side and right–hand side is explored in [38]. In [76], the role of the coequations that define a covariety is achieved by considering subsystems of a cofree coalgebra, i.e., a monomorphism. In this chapter, we presented a general approach by considering equations as epimorphisms with free domain and coequations as monomorphisms with cofree codomain. Special kinds of epimorphisms and monomorphisms can also be considered to obtain similar results.

Liftings of adjunctions to categories of algebras were studied in [51]. We included this theorem in this chapter and added another level by considering categories of equations and coequations in the case of a duality. In this case, we showed that the duality between the base categories can be lifted to a duality between equations and coequations. It is worth mentioning that we cannot obtain a similar result if we consider a contravariant adjunction that is not a duality, since coequations are not always mapped to equations, see Example 88.

We presented a similar work of lifting contravariant adjunctions for the case of Eilenberg–Moore categories. That is, we included monads and comonads into the picture. This work on Eilenberg–Moore categories was based on the paper [80]. We gave necessary and sufficient conditions to lift contravariant adjunctions to Eilenberg–Moore categories, and from that we also obtained liftings to categories of equations and coequations in the case that the contravariant adjunction is a duality. Additionally, we obtained results on how to obtain a monad from a given comonad and a contravariant adjunction and how to obtain a comonad from a given monad and a duality. Contravariant adjunctions also allowed us to define a notion of satisfaction of equations for coalgebras and satisfaction of coequations for algebras.

Our setting was general enough to obtain the duality of equations and coequations shown in [13] and generalize it to the case of dynamical systems. Which also gives us a more clear picture on how equations and coequations correspond to each other if we are in the setting of a contravariant adjunction that is a duality. Finally, the contents of this chapter will help us to get a basic understanding of the notion of equations and coequations that will be used in the next two chapters to obtain a categorical version of Birkhoff's theorem and abstract categorical versions of Eilenberg–type correspondences.

Chapter 5

Birkhoff's Theorem

In this chapter, we state a categorical version of Birkhoff's theorem for varieties of algebras over a monad. We will show that varieties of algebras are characterized as equational classes and that there is a one–to–one correspondence between varieties of algebras and equational theories. The definition of an equational theory will be given, which is a new concept and it will be used in the next chapter with the purpose of obtaining Eilenberg–type correspondences. The main purpose of the present chapter, apart from making the thesis self–contained, is to show a version of Birkhoff's theorem for algebras over a monad and to obtain a one–to–one correspondence between varieties of algebras and equational theories, contrary to some known categorical versions such as [10, 15] in which the existence of a defining family of equations for a variety is shown but no uniqueness. We will do a similar work for the case of pseudovarieties of algebras, local varieties of algebras and local pseudovarieties of algebras.

The main setting will be of a category with a factorization system together with a monad on the category. The factorization system will allow us to define the notion of homomorphic images, subalgebras and also to restrict the morphisms where equations are considered. The definition of a variety of algebras will be the standard one. On the other hand, to define the notion of an equational theory we based our approach on [27, Definition II.14.16] and also on early ideas in [78] for an Eilenberg–type correspondence for algebras over a monad. The main purpose of our definition of an equational theory is to derive Eilenberg–type correspondences in the next chapters. We will show the relation between equational theories and monad morphisms, the latter concept is used in a categorical characterization of HSP subcategories of Eilenberg–Moore algebras shown in [17].

We start by showing a categorical version for varieties of algebras over a monad, Birkhoff's theorem, whose proof is given by following the same ideas for standard proofs of Birkhoff's theorem, see, e.g., [27], but in our case we also take into account our notion of an equational theory to obtain a one–to–one correspondence between varieties of algebras and equational theories. Then we proceed to do a similar work for the finite case, i.e., pseudovarieties of algebras over a monad.

This is also known in some cases as Reiterman's theorem [73]. In our case, we based our proof on the fact that pseudovarieties of algebras are exactly directed unions of classes of finite algebras that are equational [12, 14, 37]. It is worth mentioning that the proof presented here avoids the use of topology, profinite techniques and implicit operations, which are the usual techniques to prove this theorem [73, 14, 28]. In this sense, the proof presented for characterizing pseudovarieties of algebras is more intuitive and easier to understand without needing the knowledge of topology or profinite techniques, even though those concepts can be easily brought to the scene by considering limits. Then we finish the chapter by doing a similar work to obtain local versions, in the sense that all the algebras considered are quotients of a given one.

In order to define the concept of a variety of algebras, we need to define the notion of a subalgebra, homomorphic image and product. The concept of a product is the standard one [66]. In order to define homomorphic images and subalgebras we will need the following concept of a factorization system.

Definition 99. Let $\mathcal{D}$ be a category and let $\mathscr{E}$ and $\mathscr{M}$ be classes of morphisms in $\mathcal{D}$. The pair $(\mathscr{E}, \mathscr{M})$ is called a *factorization system* on $\mathcal{D}$ if:

i) Each of $\mathscr{E}$ and $\mathscr{M}$ is closed under composition with isomorphisms,

ii) Every morphism f in $\mathcal{D}$ has a factorization $f = m \circ e$, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

iii) Given any commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, two heads, "e"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, hook, "m"'] & \cdot \end{tikzcd}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique *diagonal fill–in,* i.e., a unique morphism d such that the following diagram commutes:

$$\begin{tikzcd} \cdot \arrow[r, two heads, "e"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \arrow[dl, "d"'] \\ \cdot \arrow[r, hook, "m"'] & \cdot \end{tikzcd}$$

We will use the following fact about factorization systems [4].

Lemma 100. *Let $\mathcal{D}$ be a category, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. If T preserves the morphisms in $\mathscr{E}$ then* $\mathtt{Alg}(\mathsf{T})$ *inherits the same $(\mathscr{E}, \mathscr{M})$ factorization system.*

Proof. (*Sketch*) Given a morphism $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{A},\mathbf{B})$, factor f as $f = m \circ e$ in $\mathcal{D}$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$. Let $C \in \mathcal{D}$ be the codomain of e. Since $f = m \circ e$ is a morphism in $\mathtt{Alg}(\mathsf{T})$ then $m \circ e \circ \alpha_A = f \circ \alpha_A = \alpha_B \circ T(f) = \alpha_B \circ T(m) \circ T(e)$. From this equality, by using the diagonal fill–in property and the fact that $T(e) \in \mathscr{E}$, there exists a unique $\alpha_C \in \mathcal{C}(T(C), C)$ such that $e \circ \alpha_A = \alpha_C \circ T(e)$ and $m \circ \alpha_C = \alpha_B \circ T(m)$, that is, e and m are morphisms in $\mathtt{alg}(T)$.

Now, to prove that $\mathbf{C} = (C, \alpha_C) \in \mathtt{Alg}(\mathsf{T})$ we use the diagonal fill–in. The identity $id_C = \alpha_C \circ \eta_C$ is shown by proving that id_C and $\alpha_C \circ \eta_C$ are the diagonal fill–in of the square $m \circ e = m \circ e$. The identity $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ is shown by prooving that $\alpha_C \circ T\alpha_C$ and $\alpha_C \circ \mu_C$ are the diagonal–fill in of the square $(\alpha_B \circ T(\alpha_B) \circ TT(m)) \circ TT(e) = m \circ (e \circ \alpha_A \circ T(\alpha_A))$ (note that $TT(e) \in \mathscr{E}$ by the assumption that T preserves $\mathscr{E}$).

Uniqueness of factorizations and diagonal fill–in property in $\mathtt{Alg}(\mathsf{T})$ follow from uniqueness of factorizations and diagonal fill–in property in $\mathcal{D}$, respectively. □

5.1 Varieties of algebras

Varieties of algebras have been studied in universal algebra and equational logic. In particular, Birkhoff's variety theorem (see Theorem 9 and, e.g., [18, 27]) states that a class of algebras of the same type is a variety, i.e., it is closed under homomorphic images, subalgebras and (not necessarily finite) products, if and only if it is definable by equations. As a consequence, for a fixed type of algebras, we get a one–to–one correspondence between varieties of algebras and equational theories. Birkhoff's theorem has been generalized to a categorical level, see, e.g., [4, 10, 15, 17], to characterize subcategories of a given category that are, in some sense, equationally defined. In this section, we provide, under mild assumptions, a Birkhoff's theorem for varieties of T–algebras, Theorem 105, where T is a monad. In order to derive Eilenberg–type correspondences in the subsequent chapters, we will also prove that there is a one–to–one correspondence between varieties of T–algebras and equational T–theories. A categorical definition of an equational T–theory will be given.

The definition of variety of T–algebras, which depends on the concept of homomorphic images and subalgebras, will be defined by using a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$. In order to define the concept of equational T–theories, we base our approach on [27, Definition II.14.16]. After providing the assumptions and basic definitions needed to state Theorem 105, its proof will easily follow from the assumptions needed by following the same ideas for standard proofs of Birkhoff's theorem, see, e.g., [27]. We start by fixing the setting for the theorem and by listing the assumptions we need.

We fix a complete category $\mathcal{D}$, a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$ and a full subcategory $\mathcal{D}_0$ of $\mathcal{D}$. We will use the following assumptions:

(B1) The factorization system $(\mathscr{E}, \mathscr{M})$ is such that every map in $\mathscr{E}$ is an epimorphism[1].

(B2) For every $X \in \mathcal{D}_0$, the free T–algebra $\mathbf{TX} = (T(X), \mu_X)$ is *projective with respect to $\mathscr{E}$ in* $\mathtt{Alg}(\mathsf{T})$. That is, for every $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{B})$ with $X \in \mathcal{D}_0$ and $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$ there exists $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ such that the following diagram commutes:

$$\begin{tikzcd} & T(X) \arrow[dl, "h"'] \arrow[d, dashed, "g"] \\ B & A \arrow[l, two heads, "e"] \end{tikzcd}$$

(B3) For every $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ there exists $X_A \in \mathcal{D}_0$ and $s_A \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_A}, \mathbf{A}) \cap \mathscr{E}$.

(B4) T preserves morphisms in $\mathscr{E}$.

(B5) For every $X \in \mathcal{D}_0$, there is, up to isomorphism, only a set of T–algebra morphisms in $\mathscr{E}$ with domain $\mathbf{TX}$.

The notion of a variety of T–algebras, which depends on the concept of homomorphic images and subalgebras, will be defined by using the factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$, which is lifted to $\mathtt{Alg}(\mathsf{T})$ using (B4), Lemma 100. The role of $\mathcal{D}_0$ is that the objects from which "variables" for the equations are considered are objects in $\mathcal{D}_0$. Assumption (B2) of $\mathbf{TX}$ being projective with respect to $\mathscr{E}$, $X \in \mathcal{D}_0$, will play a fundamental role in relating varieties of algebras with equational theories. Assumption (B3) guarantees that every algebra in $\mathtt{Alg}(\mathsf{T})$ is the homomorphic image of a free T–algebra with object of generators from $\mathcal{D}_0$. Condition (B5) will allow us to define the equational theory for a given variety of algebras. Condition (B1) will provide the condition that different equational T–theories (Definition 102) define different varieties of T–algebras, Proposition 107.

For Birkhoff's classical variety theorem [18], we can take $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections, and T to be the term monad for a given type of algebras τ, i.e., $T(X) = T_\tau(X)$, the set of terms of type τ on the set of variables X (see Example 103 and Example 110). Another important example will be given by $\mathcal{D} = \mathtt{Poset}$, with $\mathcal{D}_0 =$ discrete posets (i.e., we do not want the "variables" to be ordered) to obtain a Birkhoff's theorem for ordered algebras [20].

We now give the necessary definitions to formulate our Birkhoff's theorem for T–algebras. We start by defining varieties of T–algebras.

Definition 101. Let $\mathcal{D}$ be a complete category, T a monad on $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Let K be a class of algebras in $\mathtt{Alg}(\mathsf{T})$. We say that K *is closed under $\mathscr{E}$–quotients* if $\mathbf{B} \in K$ for every $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$ with $\mathbf{A} \in K$. We say that K *is closed under $\mathscr{M}$–subalgebras* if $\mathbf{B} \in K$ for every $m \in \mathtt{Alg}(\mathsf{T})(\mathbf{B}, \mathbf{A}) \cap \mathscr{M}$

[1]Note that this condition implies that $\mathscr{M}$ contains the extremal monomorphisms of $\mathcal{D}$, [4, 14.10 Proposition]. Condition (B1) will only be used to prove that the correspondence between varieties of algebras and equational theories is bijective, see Proposition 107.

with $\mathbf{A} \in K$. We say that K *is closed under products* if $\prod_{i\in I} \mathbf{A}_i \in K$ for every set I such that $\mathbf{A}_i \in K$, $i \in I$. A class V of algebras in $\mathtt{Alg}(\mathsf{T})$ is called a *variety of* T*–algebras* if it is closed under $\mathscr{E}$–quotients, $\mathscr{M}$–subalgebras and products.

Now, we define one of the main concepts of this thesis that will play a role in the obtainment of Eilenberg–type correspondences in the subsequent chapters, namely, the concept of an equational T–theory.

Definition 102. Let $\mathcal{D}$ be a category, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. An *equational* T*–theory on* $\mathcal{D}_0$ is a family of T–algebra morphisms (equations) $\mathtt{E} = \{T(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X\in\mathcal{D}_0}$ in $\mathscr{E}$ such that for any $X, Y \in \mathcal{D}_0$ and any $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TY})$ there exists $g' \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q_X}, \mathbf{Q_Y})$ such that the following diagram commutes:

$$\begin{tikzcd} T(X) \arrow[r, "\forall g"] \arrow[d, two heads, "e_X"'] & T(Y) \arrow[d, two heads, "e_Y"] \\ Q_X \arrow[r, dashed, "g'"'] & Q_Y \end{tikzcd}$$

Intuitively, in the setting $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections, and T to be the term monad for a given type of algebras τ, we have that for every object $X \in \mathcal{D}_0$ (i.e., a set of variables) the morphism e_X, which we asssume to be a surjection, represents the set of equations $\ker(e_X)$, which is a congruence on $\mathbf{TX}$, i.e., it is an equivalence relation on $T(X)$ which is closed under the componentwise algebraic operations. In this case, the algebra $\mathbf{Q_X}$ is the free algebra on X generators in the variety associated to the equational theory. Commutativity of the diagram above means that the family of all equations $\{\ker(e_X)\}_{X\in\mathcal{D}_0}$ is closed under any substitution $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TY})$. The previous definition generalizes the definition of an equational theory to a categorical level, cf. [27, Definition II.14.16].

We have the following examples of equational theories.

Example 103. Consider the case $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. For a given type of algebras τ, consider the monad $\mathsf{T}_\tau = (T_\tau, \eta, \mu)$ such that $T_\tau(X)$ is the set of terms for τ on variables X, see [27, Definition II.10.1]. The unit $\eta_X : X \to T_\tau(X)$ is the inclusion function and multiplication $\mu_X : T_\tau(T_\tau(X)) \to T_\tau(X)$ is the identity map. Now, $\mathtt{Alg}(\mathsf{T}_\tau)$ is the category of algebras $\mathbf{A} = (A, \alpha_A)$ of type τ, where $\alpha_A : T_\tau(A) \to A$ is the evaluation $\alpha_A(t)$ in A of each term $t \in T_\tau(A)$. An equational T_τ–theory on $\mathcal{D}_0 = \mathtt{Set}$ is a family of surjective homomorphisms $\mathtt{E} = \{T_\tau(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X\in\mathtt{Set}}$ in $\mathtt{Alg}(\mathsf{T}_\tau)$ such that every $\ker(e_x)$ is a congruence on $T_\tau(X)$ and the family $\{\ker(e_X)\}_{X\in\mathcal{D}_0}$ is closed under substitution, i.e., for $(p(x_1, \dots, x_n), q(x_1, \dots, x_n)) \in \ker(e_X)$ and $r_x \in T_\tau(Y)$, $x \in X$, we have that $(p(r_{x_1}, \dots, r_{x_n}), q(r_{x_1}, \dots, r_{x_n})) \in \ker(e_Y)$, where $t(r_{x_1}, \dots, r_{x_n})$ is the term in $T_\tau(Y)$ obtained from $t(x_1, \dots, x_n) \in T_\tau(X)$ by replacing each variable x_i by r_{x_i}, $i = 1, \dots, n$. $\square$

Example 104. Consider the case $\mathcal{D} = \texttt{Poset}$, $\mathcal{D}_0 =$ the full subcategory of discrete posets, $\mathscr{E}$ = surjections and $\mathscr{M}$ = embeddings. Let τ be a type of algebras. An *ordered algebra of type* τ is a triple $A = (A, \leq_A, \{f_A : A^{n_f} \to A\}_{f\in\tau})$ such that $(A, \leq_A) \in \texttt{Poset}$ and all the functions $f_A : A^{n_f} \to A$ are order preserving, where the order in A^{n_f} is componentwise, $f \in \tau$. We can define the monad $\mathsf{T}_\tau = (T_\tau, \eta, \mu)$ where $T_\tau(X, \leq_X)$ is the poset $(T_\tau(X), \leq_{\mathsf{T}_\tau(X)})$ defined as: $x \leq_{\mathsf{T}_\tau(X)} y$ for every $x, y \in X$ such that $x \leq_X y$, and $f(t_1, \ldots, t_{n_f}) \leq_{\mathsf{T}_\tau(X)} f(q_1, \ldots, q_{n_f})$ for every $f \in \tau$ and terms $t_i, q_i \in T_\tau(X)$ such that $t_i \leq_{\mathsf{T}_\tau(X)} q_i$, $i = 1, \ldots, n_f$. Algebras in $\texttt{Alg}(\mathsf{T}_\tau)$ are ordered algebras of type τ.

An equational T_τ–theory is a family $\texttt{E} = \{T_\tau(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X\in\mathcal{D}_0}$ of surjective homomorphisms, which are trivially order preserving since $T_\tau(X)$ is discrete for any $X \in \mathcal{D}_0$, such that $\overrightarrow{\ker}(e_X)$ is an admissible preorder on $T_\tau(X)$ [2], where $\overrightarrow{\ker}(e_X) := \{(u, v) \mid e_X(u) \leq e_X(v)\}$, and the family $\{\overrightarrow{\ker}(e_x)\}_{X\in\mathcal{D}_0}$ is closed under substitution as in the previous example. In this case, $\overrightarrow{\ker}(e_X)$ represents the equations and inequations of terms with variables in X in the equational T_τ–theory. Note that if we take $\mathcal{D}_0 = \texttt{Poset}$ then condition (B2) does not hold. $\square$

Given a set of equations $\texttt{E} = \{T(X_i) \overset{e_i}{\twoheadrightarrow} Q_{X_i}\}_{i\in I}$ and an algebra $\mathbf{A} \in \texttt{Alg}(\mathsf{T})$, we say that $\mathbf{A}$ *satisfies* E, denoted as $\mathbf{A} \models \texttt{E}$, if $\mathbf{A}$ is E–*injective*, that is, if for every $i \in I$ and every $f \in \texttt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{A})$ there exists a T–algebra morphism g_f such that $f = g_f \circ e_i$.

Intuitively, $\mathbf{A} \models \texttt{E}$ if for every $i \in I$ and every assignment $f \in \texttt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{A})$ of the variables X_i to elements of the algebra $\mathbf{A}$, all the equations represented by $e_i \colon T(X_i) \twoheadrightarrow Q_{X_i}$ hold in $\mathbf{A}$. Given a set of equations E we denote the *models of* E by $\mathrm{Mod}(\texttt{E})$, that is:

$$\mathrm{Mod}(\texttt{E}) := \{\mathbf{A} \in \texttt{Alg}(\mathsf{T}) \mid \mathbf{A} \models \texttt{E}\}$$

A class K of T–algebras is *defined* by E if $K = \mathrm{Mod}(\texttt{E})$.

Now we state our categorical version of Birkhoff's theorem as follows.

Theorem 105 (Birkhoff's Theorem for T–algebras)**.** *Let $\mathcal{D}$ be a complete category, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B2) to (B5). Then a class K of* T*–algebras is a variety of* T*–algebras if and only if it is defined by a set of equations in which the domain of each each equation is $\mathbf{TX}$ for some $X \in \mathcal{D}_0$. Additionally, by assuming condition (B1), varieties of* T*–algebras are in one–to–one correspondence with equational* T*–theories on $\mathcal{D}_0$.*

5.1.1 Proof of Birkhoff's theorem for T–algebras

The proof for Birkhoff's theorem for T–algebras can be given by following the same ideas for standard proofs of Birkhoff's theorem, see, e.g., [27]. In our case, we will

[2] A preorder $\sqsubseteq$ on an ordered algebra $(A, \leq_A, \{f_A : A^{n_f} \to A\}_{f\in\tau})$ of type τ is *compatible* if for every $f \in \tau$ and $a_i, b_i \in A$ with $a_i \sqsubseteq b_i$, $i = 1, \ldots, n_f$, we have that $f_A(a_1, \ldots, a_{n_f}) \sqsubseteq f_A(b_1, \ldots, b_{n_f})$. A preorder $\sqsubseteq$ is *admissible* if it is compatible and $a \sqsubseteq b$ whenever $a \leq_A b$. The congruence $\theta_\sqsubseteq$ on A induced by the compatible preorder $\sqsubseteq$ is the relation $\theta_\sqsubseteq$ on A defined as $\theta_\sqsubseteq := \sqsubseteq \cap \sqsubseteq^{-1}$. Then $(A/\theta_\sqsubseteq, \leq_\sqsubseteq)$ is an ordered algebra with the order given by $[x] \leq_\sqsubseteq [y]$ iff $x \sqsubseteq y$. See [20].

also deal with equational T–theories and we have a fixed subcategory $\mathcal{D}_0$ whose objects represent "variables", which is the main difference with respect to some categorical versions such as [10, 15, 17]. Our proof follows the same ideas for standard proofs of (categorical) Birkhoff's theorem (see, e.g., [4, 15]), we include a proof of Theorem 105 in this thesis for the sake of completeness. We derive Theorem 105 from the following facts.

First, we have that the codomain of each arrow in an equational theory $\mathtt{E}$ belongs to $\mathrm{Mod}(\mathtt{E})$. That is, $\mathrm{Mod}(\mathtt{E})$ contains every free algebra on X for each choice of "variables" $X \in \mathcal{D}_0$.

Lemma 106. *Let $\mathcal{D}$ be a category, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ and $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$. Let $\mathtt{E} = \{T(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X \in \mathcal{D}_0}$ be an equational T–theory on $\mathcal{D}_0$. Assume (B2). Then $\mathbf{Q_X} \in \mathrm{Mod}(\mathtt{E})$ for every $X \in \mathcal{D}_0$.*

Proof. Let $Y \in \mathcal{D}_0$ and let $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{Q_X})$. Then we have the following commutative diagram:

$$\begin{tikzcd} T(Y) \arrow[r, two heads, "e_Y"] \arrow[rr, bend right, "f"'] \arrow[rrr, bend left=40, dashed, "g"] & Q_Y \arrow[r, dashed, "g'"] & Q_X & T(X) \arrow[l, two heads, "e_X"'] \end{tikzcd}$$

where $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{TX})$ is obtained from f and e_X using assumption (B2) and g' is obtained from the fact that $\mathtt{E}$ is an equational T–theory. Therefore, f factors through e_Y as $f = g' \circ e_Y$ and hence $\mathbf{Q_X} \in \mathrm{Mod}(\mathtt{E})$. $\square$

Next, we show that different equational theories will give us different class of algebras.

Proposition 107. *Let $\mathcal{D}$ be a category, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ and $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$. Assume (B1) and (B2). For $i = 1, 2$, let $\mathtt{E}_i = \{T(X) \overset{(e_i)_X}{\twoheadrightarrow} (Q_i)_X\}_{X \in \mathcal{D}_0}$ be an equational T–theory on $\mathcal{D}_0$. If $\mathtt{E}_1 \neq \mathtt{E}_2$ then $\mathrm{Mod}(\mathtt{E}_1) \neq \mathrm{Mod}(\mathtt{E}_2)$.*

Proof. As $\mathtt{E}_1 \neq \mathtt{E}_2$, there exists $X \in \mathcal{D}_0$ such that $(e_1)_X \neq (e_2)_X$, i.e., there is no isomorphism $\phi \in \mathtt{Alg}(\mathsf{T})((\mathbf{Q_1})_\mathbf{X}, (\mathbf{Q_2})_\mathbf{X})$ such that $\phi \circ (e_1)_X = (e_2)_X$. We have that $(\mathbf{Q_1})_\mathbf{X} \notin \mathrm{Mod}(\mathtt{E}_2)$ or $(\mathbf{Q_2})_\mathbf{X} \notin \mathrm{Mod}(\mathtt{E}_1)$. In fact, assume by contradiction that $(\mathbf{Q_1})_\mathbf{X} \in \mathrm{Mod}(\mathtt{E}_2)$ and $(\mathbf{Q_2})_\mathbf{X} \in \mathrm{Mod}(\mathtt{E}_1)$, then, from the fact that $(\mathbf{Q_1})_\mathbf{X} \in \mathrm{Mod}(\mathtt{E}_2)$, we get the commutative diagram:

$$\begin{tikzcd} T(X) \arrow[r, two heads, "(e_2)_X"] \arrow[rr, bend left, "(e_1)_X"] & (Q_2)_X \arrow[r, dashed, "g_{21}"] & (Q_1)_X \end{tikzcd}$$

i.e., there exists $g_{21} \in \mathtt{Alg}(\mathsf{T})((\mathbf{Q_2})_\mathbf{X}, (\mathbf{Q_1})_\mathbf{X})$ such that $g_{21} \circ (e_2)_X = (e_1)_X$. Similarly, from the fact that $(\mathbf{Q_2})_\mathbf{X} \in \mathrm{Mod}(\mathtt{E}_1)$, we get that there exists $g_{12} \in \mathtt{Alg}(\mathsf{T})((\mathbf{Q_1})_\mathbf{X}, (\mathbf{Q_2})_\mathbf{X})$ such that $g_{12} \circ (e_1)_X = (e_2)_X$. Hence we have that:

$$(e_2)_X = g_{12} \circ (e_1)_X = g_{12} \circ g_{21} \circ (e_2)_X$$

which implies that $g_{12} \circ g_{21} = id_{(Q_2)_X}$ since $(e_2)_X$ is epi by (B1). Similarly, $g_{21} \circ g_{12} = id_{(Q_1)_X}$, which implies that g_{12} is an isomorphism such that $g_{12} \circ (e_1)_X = (e_2)_X$ which is a contradiction. Hence $(\mathbf{Q_1})_\mathbf{X} \notin \mathrm{Mod}(\mathtt{E}_2)$ or $(\mathbf{Q_2})_\mathbf{X} \notin \mathrm{Mod}(\mathtt{E}_1)$ and, by the previous lemma, $(\mathbf{Q_i})_\mathbf{X} \in \mathrm{Mod}(\mathtt{E}_i)$, which implies that $\mathrm{Mod}(\mathtt{E}_1) \neq \mathrm{Mod}(\mathtt{E}_2)$. $\square$

The next proposition shows that, under conditions (B2) and (B4), every class defined by a set of equations, in which the domain of each equation is $\mathbf{TX}$ for some $X \in \mathcal{D}_0$, is a variety of T–algebras.

Proposition 108. *Let $\mathcal{D}$ be a complete category, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$. Assume (B2) and (B4). Let $\mathtt{E}$ be a set of equations in which the domain of each equation is $\mathbf{TX}$ for some $X \in \mathcal{D}_0$. Then $\mathrm{Mod}(\mathtt{E})$ is a variety of T–algebras.*

Proof. $\mathrm{Mod}(\mathtt{E})$ is nonempty since the algebra $\mathbf{1} = (1, !_{T(1)})$, where 1 is the terminal object in $\mathcal{D}$, is in $\mathrm{Mod}(\mathtt{E})$. Put $\mathtt{E} = \{T(X_i) \overset{e_i}{\twoheadrightarrow} Q_{X_i}\}_{i \in I}$, then:

i) $\mathrm{Mod}(\mathtt{E})$ is closed under $\mathscr{E}$–quotients: Let $\mathbf{A}, \mathbf{B} \in \mathtt{Alg}(\mathsf{T})$ with $\mathbf{A} \in \mathrm{Mod}(\mathtt{E})$ and let $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$. Let $i \in I$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{B})$, then we have the following commutative diagram:

$$\begin{tikzcd} T(X_i) \arrow[r, two heads, "e_i"] \arrow[rr, bend left, "k"] \arrow[rrr, bend left=40, "f"] & Q_{X_i} \arrow[r, dashed, "g_k"] & A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

where $k \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{A})$ was obtained from f using (B2) and the morphism $g_k \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q_{X_i}}, \mathbf{A})$ from the fact that $\mathbf{A} \in \mathrm{Mod}(\mathtt{E})$. Therefore f factors through e_i, i.e., $\mathbf{B} \in \mathrm{Mod}(\mathtt{E})$.

ii) $\mathrm{Mod}(\mathtt{E})$ is closed under $\mathscr{M}$–subalgebras: Let $\mathbf{A}, \mathbf{B} \in \mathtt{Alg}(\mathsf{T})$ with $\mathbf{A} \in \mathrm{Mod}(\mathtt{E})$ and let $m \in \mathtt{Alg}(\mathsf{T})(\mathbf{B}, \mathbf{A}) \cap \mathscr{M}$. Let $i \in I$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{B})$, then we have the following commutative diagram:

$$\begin{tikzcd} & Q_{X_i} \arrow[dr, dashed, "k"] \arrow[drr, dashed, bend left, "g_{m \circ f}"] & & \\ T(X_i) \arrow[ur, two heads, "e_i"] \arrow[rr, bend right, "f"'] & & B \arrow[r, hook, "m"] & A \end{tikzcd}$$

where $g_{m\circ f} \in \texttt{Alg}(\mathsf{T})(\mathbf{Q}_{\mathbf{X_i}}, \mathbf{A})$ was obtained from the fact that $\mathbf{A} \in \mathrm{Mod}(\texttt{E})$, and $k \in \texttt{Alg}(\mathsf{T})(\mathbf{Q}_{\mathbf{X_i}}, \mathbf{B})$ was obtained by the diagonal fill–in property of the factorization system $(\mathscr{E}, \mathscr{M})$ restricted to $\texttt{Alg}(\mathsf{T})$, by using (B4) in Lemma 100. Therefore, $\mathbf{B} \in \mathrm{Mod}(\texttt{E})$.

iii) $\mathrm{Mod}(\texttt{E})$ is closed under products: Let $\mathbf{A}_j \in \mathrm{Mod}(\texttt{E})$, $j \in J$, and let $\mathbf{A} = \prod_{j\in J} \mathbf{A}_j$ be their product in $\texttt{Alg}(\mathsf{T})$ with projections $\pi_j : A \to A_j$. Let $i \in I$ and $f \in \texttt{Alg}(\mathsf{T})(\mathbf{TX_i}, \mathbf{A})$, then we have the following commutative diagram:

$$\begin{tikzcd} & Q_{X_i} \arrow[dr, dashed, "g"] \arrow[drr, dashed, bend left, "g_{\pi_j\circ f}"] & & \\ T(X_i) \arrow[ur, "e_i"] \arrow[rr, bend right, "f"'] & & A \arrow[r, "\pi_j"] & A_j \end{tikzcd}$$

where $g_{\pi_j\circ f} \in \texttt{Alg}(\mathsf{T})(\mathbf{Q}_{\mathbf{X_i}}, \mathbf{A}_j)$ was obtained from the fact that $\mathbf{A}_j \in \mathrm{Mod}(\texttt{E})$, and $g \in \texttt{Alg}(\mathsf{T})(\mathbf{Q}_{\mathbf{X_i}}, \mathbf{A})$ was obtained by the universal property of the product. Finally, we have that $g \circ e_i = f$ since $\pi_j \circ g \circ e_i = \pi_j \circ f$ for every $j \in J$.

$\square$

The previous proposition shows that, under the assumptions of Theorem 105, every class defined by a set of equations, in which the domain of each equation is $\mathbf{TX}$ for some $X \in \mathcal{D}_0$, is a variety of T–algebras.

Now we prove the converse, that is, that every variety of T–algebras is defined by a set of equations (in fact, an equational T–theory).

Proposition 109. *Let $\mathcal{D}$ be a complete category, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ and $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$. Assume (B3), (B4) and (B5). Let V be a variety of T–algebras. Then $V = \mathrm{Mod}(\texttt{E})$ for some equational T–theory $\texttt{E}$ on $\mathcal{D}_0$.*

Notice that, if we assume (B2), the equational T–theory $\texttt{E}$ is unique by Proposition 107.

Proof. We prove the proposition in two steps: i) the construction of $\texttt{E}$, and ii) to show that $V = \mathrm{Mod}(\texttt{E})$. In fact:

i) For any $X \in \mathcal{D}_0$, let $H_X = \{T(X) \overset{(e_X)_i}{\twoheadrightarrow} (P_X)_i\}_{i\in I_X}$ be the collection of all T–algebra morphisms, up to isomorphism, in $\mathscr{E}$ with domain $\mathbf{TX}$ and codomain in the variety V. By (B5), H_X is a set. Put $\mathbf{P_X} = \prod_{i\in I_X} (\mathbf{P_X})_i$ and let $(\pi_X)_i \in \texttt{Alg}(\mathsf{T})(\mathbf{P_X}, (\mathbf{P_X})_i)$ be the ith–projection. Then we have the following commutative diagram in $\texttt{Alg}(\mathsf{T})$:

$$(\star) \qquad \begin{tikzcd} TX \arrow[rr, dashed, "k_X"] \arrow[dr, dashed, two heads, "e_X"'] \arrow[rrr, bend left, "(e_X)_i"] & & P_X \arrow[r, "(\pi_X)_i"] & (P_X)_i \\ & Q_X \arrow[ur, dashed, hook, "m_X"'] & & \end{tikzcd}$$

where $k_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{P_X})$ is obtained from the universal property of the product $\mathbf{P_X}$ and $k_X = m_X \circ e_X$ is the factorization of k_X in $\mathtt{Alg}(\mathsf{T})$ (use (B2) and apply Lemma 100). Note that $\mathbf{Q_X} \in V$ since it is an $\mathscr{M}$–subalgebra of a product of elements in V.

Claim: $\mathtt{E} = \{T(X) \overset{(e_X)_i}{\twoheadrightarrow} Q_X\}_{X \in \mathcal{D}_0}$ is an equational T–theory on $\mathcal{D}_0$.

Let $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TY})$ such that $X, Y \in \mathcal{D}_0$. We have to prove that there exists $g' \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q_X}, \mathbf{Q_Y})$ such that $g' \circ e_X = e_Y \circ g$. In fact, we have the following commutative diagram:

$$
\begin{tikzcd}
T(X) \arrow[rrrr, "g"] \arrow[d, two heads, "e_X"'] \arrow[drr, two heads, "e_{e_Y \circ g} = (e_X)_j"] \arrow[dr, phantom, "(\star)"] & & & & TY \arrow[d, two heads, "e_Y"] \\
Q_X \arrow[r, hook, "m_X"'] & P_X \arrow[r, "(\pi_X)_j"'] & S = (P_X)_j \arrow[rr, hook, "m_{e_Y \circ g}"'] & & Q_Y
\end{tikzcd}
$$

where $e_Y \circ g = m_{e_Y \circ g} \circ e_{e_Y \circ g}$ is the factorization of $e_Y \circ g$ and $\mathbf{S}$ is the codomain of $e_{e_Y \circ g}$. From that we have that $\mathbf{S}$ is an $\mathscr{M}$–subalgebra of $\mathbf{Q_Y} \in V$. Hence $\mathbf{S} \in V$ and therefore $\mathbf{S} = (\mathbf{P_X})_j$ and $e_{e_Y \circ g} = (e_X)_j$ for some $j \in I_X$. Therefore, $\mathtt{E}$ is an equational T–theory on $\mathcal{D}_0$.

ii) Let us prove that $V = \mathrm{Mod}(\mathtt{E})$.
($\supseteq$): Let $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ such that $\mathbf{A} \in \mathrm{Mod}(E)$. By assumption (B3), there exists $s_A \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_A}, \mathbf{A}) \cap \mathscr{E}$ with $X_A \in \mathcal{D}_0$. As $\mathbf{A} \in \mathrm{Mod}(E)$, the morphism s_A factors through e_{X_A} as $s_A = g_{s_A} \circ e_{X_A}$. Since $g_{s_A} \circ e_{X_A} = s_A \in \mathscr{E}$ and $e_{X_A} \in \mathscr{E}$ then $g_{s_A} \in \mathscr{E}$ [4, dual of 14.9 Proposition (1)]. Therefore, $\mathbf{A} \in V$ since $\mathbf{Q_{X_A}} \in V$.

($\subseteq$): Let $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ such that $\mathbf{A} \in V$. Let $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ such that $X \in \mathcal{D}_0$. Then we have the following commutative diagram in $\mathtt{Alg}(\mathsf{T})$:

$$
\begin{tikzcd}
A & & Z = (P_X)_i \arrow[ll, hook', "m_f"'] \\
& & P_X \arrow[u, "(\pi_X)_i"'] \\
TX \arrow[uu, "f"] \arrow[uurr, two heads, "e_f = (e_X)_i"] \arrow[rr, two heads, "e_X"'] & & Q_X \arrow[u, hook, "m_X"']
\end{tikzcd}
$$

where $f = m_f \circ e_f$ is the factorization of f with $m_f \in \mathscr{M}$ and $e_f \in \mathscr{E}$, which implies that $\mathbf{Z} \in V$ since $\mathbf{A} \in V$. Therefore, $\mathbf{Z} = (\mathbf{P_X})_i$ and $e_f = (e_X)_i$ for some $i \in I_X$. Hence the factorization of f through e_X follows from the definition of e_X (see i) above) which implies that $\mathbf{A} \in \mathrm{Mod}(E)$.

$\square$

Now, first part of Theorem 105 follows from Proposition 108 (by using (B2) and (B4)) and Proposition 109 (by using (B3), (B4) and (B5)). Uniqueness of the defining equational T–theory follows from Proposition 107 (by using (B1) and (B2)). $\square$

From Birkhoff's theorem for T–algebras we have the following.

Example 110. By considering the monad and the categories given in Example 103 we obtain the classical Birkhoff variety theorem [18]. $\square$

Example 111. By considering the monad and the categories given in Example 104 we obtain the Birkhoff variety theorem for ordered algebras [20]. $\square$

We now describe how we can derive an Eilenberg–type correspondence from our Birkhoff's theorem (this will be explained in full detail in subsequent chapters).

Example 112 (cf. [13, Theorem 39]). Consider the case $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, T the monad given by $TX = X^*$, where X^* is the free monoid on X, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. We have that conditions (B1) to (B5) are fullfilled. Therefore we have a one–to–one correspondence between varieties of monoids and equational T–theories. Now, consider the category $\mathcal{C} = \mathcal{C}_0 = \mathtt{CABA}$ which is dual to $\mathtt{Set}$ and let B be the comonad on $\mathtt{CABA}$ that is dual to the monad T on $\mathtt{Set}$, i.e, B is defined, up to isomorphism, as $B(2^X) = 2^{X^*}$. Then, by duality, we have a one–to–one correspondence between equational T–theories $\mathtt{E}$ and its dual $\mathtt{E}^\partial$, i.e., families of monomorphisms $\{S_X \xhookrightarrow{m_X} B(X)\}_{X \in \mathcal{C}_0 = \mathcal{C}}$ in $\mathtt{Coalg}(\mathsf{B})$ such that for any $X, Y \in \mathcal{C}_0$ and any $g \in \mathtt{Coalg}(\mathsf{B})(\mathbf{BX}, \mathbf{BY})$ there exists $g' \in \mathtt{Coalg}(\mathsf{B})(\mathbf{S_X}, \mathbf{S_Y})$ such that $m_Y \circ g = g' \circ m_X$.
This notion of the dual of an equational T–theory is equivalent with the –more complicated– notion of a variety of languages in [13, Definition 35] (see Example 144 for more details). In this setting, we get a one–to–one correspondence between varieties of monoids and duals of equational T–theories, i.e., varieties of languages. This is exactly the Eilenberg–type theorem [13, Theorem 39] and we will come back to this kind of example and provide more details in the next chapter. $\square$

It is worth mentioning that even though the Birkhoff's theorem we stated seems to be restricted for the case of T–algebras, there is no real limitation in this fact since we can always consider the identity monad on $\mathcal{D}$ and in this case $\mathtt{Alg}(\mathsf{T}) = \mathcal{D}$. Besides our interest of the study of algebras for a monad, which is the most common generalization for algebraic structures, we chose this approach for the following reasons:

i) We can study different algebraic structures by considering the same base category and the same factorization system on the base category.

ii) Equations are usually defined as certain kinds of epimorphisms with a projective domain, which in our case projective objects are usualy among the free objects $\mathbf{TX} = (TX, \mu_X) \in \mathtt{Alg}(\mathsf{T})$.

iii) Our main purpose is to derive Eilenberg–type correspondences in the subsequent chapters for which we make use of a dual category of $\mathtt{Alg}(\mathsf{T})$, which we already know how to construct from a dual category of the base category $\mathcal{D}$ by using Theorem 92 in the previous chapter.

Before we proceed with the case of finite algebras, we study the relationship of equational theories with the natural notion of monad morphisms. The following subsection will not be used in the rest of this thesis.

5.1.2 On equational T–theories

In this subsection, we prove some properties about equational T–theories and its relation with monad morphisms. We study this for the case $\mathcal{D}_0 = \mathcal{D}$. Thus, throughout this section we work with equational T–theories on $\mathcal{D}$ which are of the form $\mathtt{E} = \{T(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X \in \mathcal{D}}$, i.e., indexed by $\mathcal{D} = \mathcal{D}_0$. The first property we show is that the assignment $X \mapsto Q_X$ can be turned into a functor on $\mathcal{D}$ such that every e_X is the component of a natural transformation as follows.

Proposition 113. *Let $\mathcal{D}$ be a category, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $\mathtt{E} = \{T(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X \in \mathcal{D}}$ an equational T–theory on $\mathcal{D}$. Assume (B1). Define the operator S on objects in $\mathcal{D}$ as $S(X) = Q_X$ and for any morphism $f \in \mathcal{D}(X, Y)$ as $S(f) = (T(f))'$ where $(T(f))'$ is the unique morphism in $\mathtt{Alg}(\mathsf{T})$ such that $e_Y \circ T(f) = (T(f))' \circ e_X$, which is given by Definition 102. Then $S : \mathcal{D} \to \mathcal{D}$ is a functor and $e : T \Rightarrow S$ is a natural transformation.*

Proof. It follows from the definitions and using the fact that every e_X is epi by (B1). □

By using the previous proposition we prove now that the property for a T–algebra $\mathbf{A} = (A, \alpha_A)$ of satisfying $\mathtt{E}$ depends only on the fact that α_A factors through e_A.

Proposition 114. *Let $\mathcal{D}$ be a category, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $\mathtt{E} = \{T(X) \overset{e_X}{\twoheadrightarrow} Q_X\}_{X \in \mathcal{D}}$ an equational T–theory on $\mathcal{D}$. Assume (B1) and let $S : \mathcal{D} \to \mathcal{D}$ and $e : T \Rightarrow S$ as in the previous proposition. Let $\mathbf{A} = (A, \alpha_A) \in \mathtt{Alg}(\mathsf{T})$, then $\mathbf{A} \models \mathtt{E}$ if and only if α_A factors through e_A.*

Proof. The fact that α_A factors through e_A follows from the definition of $\mathbf{A} \models \mathtt{E}$ by considering the T–algebra morphism $\alpha_A : TA \to A$ in $\mathtt{Alg}(\mathsf{T})(\mathbf{TA}, \mathbf{A})$. Conversely, assume that α_A factors through e_A as $\alpha_A = g \circ e_A$ and let $X \in \mathcal{D}$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$, we have to show that f factors through e_X. In fact, this follows from the following commutative diagram:

$$\begin{tikzcd}
 & T(X) \arrow[drr, bend left, "f"] & & \\
T(X) \arrow[ur, "id_X"] \arrow[r, "T(\eta_X)"] \arrow[d, two heads, "e_X"'] \arrow[dr, phantom, "\text{nat. } e"] & TT(X) \arrow[u, "\mu_X"] \arrow[r, "T(f)"] \arrow[d, two heads, "e_{T(X)}"'] \arrow[dr, phantom, "\text{nat. } e"] & T(A) \arrow[r, "\alpha_A"] \arrow[d, two heads, "e_A"'] & A \\
S(X) \arrow[r, "S(\eta_X)"'] & ST(X) \arrow[r, "S(f)"'] & S(A) \arrow[ur, "g"'] &
\end{tikzcd}$$

□

We now recall the definition of a monad morphism.

Definition 115. Let $\mathcal{D}$ be a category and let $\mathsf{T} = (T, \eta, \mu)$ and $\mathsf{S} = (S, \iota, \nu)$ be monads on $\mathcal{D}$. A *monad morphism* ϱ from T to S, denoted as $\varrho : \mathsf{T} \to \mathsf{S}$ is a natural transformation $\varrho : T \Rightarrow S$ such that the following diagrams commute:

$$\begin{tikzcd}
Id_{\mathcal{D}} \arrow[r, "\eta"] \arrow[dr, "\iota"'] \arrow[dr, phantom, shift left=3, "\star"] & T \arrow[d, "\varrho"] \\
 & S
\end{tikzcd}
\qquad
\begin{tikzcd}
TT \arrow[r, "T\varrho"] \arrow[d, "\mu"'] & TS \arrow[r, "\varrho S"] \arrow[d, phantom, "\star\star"] & SS \arrow[d, "\nu"] \\
T \arrow[rr, "\varrho"'] & & S
\end{tikzcd}$$

In [17], it is shown, under mild assumptions, that there is a one–to–one correspondence between monad morphisms $\mathsf{T} \to \mathsf{S}$ for which the induced components are in $\mathscr{E}$ and full subcategories of $\mathtt{Alg}(\mathsf{T})$ that are closed under U–split epimorphisms, products and $\mathscr{M}$–subalgebras. We study the relationship between such monad morphisms $\mathsf{T} \to \mathsf{S}$ and equational T–theories.

Proposition 116. *Let $\mathcal{D}$ be a category and let $\mathsf{T} = (T, \eta, \mu)$ and $\mathsf{S} = (S, \iota, \nu)$ be monads on $\mathcal{D}$. Let $\varrho : \mathsf{T} \to \mathsf{S}$ be a monad morphism. Then:*

i) For every $X \in \mathcal{D}$ we have that $\mathbf{SX}_\varrho := (S(X), \nu_X \circ \varrho_{S(X)}) \in \mathtt{Alg}(\mathsf{T})$ and ϱ_X is a morphism in $\mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{SX}_\varrho)$.

ii) Let $(\mathscr{E}, \mathscr{M})$ be a factorization system on $\mathcal{D}$. Assume (B1) and (B4). If every component of ϱ is in $\mathscr{E}$ then, for every $X \in \mathcal{D}$, there is a unique morphism $h \in \mathcal{D}(TS(X), S(X))$ such that $\varrho_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, (S(X), h))$, namely $h = \nu_X \circ \varrho_{S(X)}$. Furthermore, $\mathtt{E}_\varrho := \{T(X) \overset{\varrho_X}{\twoheadrightarrow} S(X)\}_{X \in \mathcal{D}}$ is an equational T–theory on $\mathcal{D}$, where $\varrho_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{SX}_\varrho) \cap \mathscr{E}$.

Proof.

i) The property of $\mathbf{SX}_\varrho := (S(X), \nu_X \circ \varrho_{S(X)}) \in \mathtt{Alg}(\mathsf{T})$ follows from the commutative diagrams

$$
\begin{tikzcd}
S(X) \arrow[r, "\eta_{S(X)}"] \arrow[dr, "\iota_{S(X)}\ \star"] \arrow[ddr, "id_{S(X)}"'] & TS(X) \arrow[d, "\varrho_{S(X)}"] \\
 & SS(X) \arrow[d, "\nu_X"] \\
 & S(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
TTS(X) \arrow[rr, "\mu_{S(X)}"] \arrow[d, "T(\varrho_{S(X)})"'] \arrow[drr, phantom, "\star\star"] & & TS(X) \arrow[d, "\varrho_{S(X)}"] \\
TSS(X) \arrow[r, "\varrho_{SS(X)}"'] \arrow[d, "T(\nu_X)"'] \arrow[dr, phantom, "\text{nat. } \varrho"] & SSS(X) \arrow[r, "\nu_{S(X)}"'] \arrow[d, "S(\nu_X)"'] & SS(X) \arrow[d, "\nu_X"] \\
TS(X) \arrow[r, "\varrho_{S(X)}"'] & SS(X) \arrow[r, "\nu_X"'] & S(X)
\end{tikzcd}
$$

The property that $\varrho_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{SX}_\varrho)$ follows from $\star\star$.

ii) If $\varrho_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, (S(X), h))$ then $h \circ T(\varrho_X) = \varrho_X \circ \mu_X$. By $\star\star$ we have $\varrho_X \circ \mu_X = \nu_X \circ \varrho_{S(X)} \circ T(\varrho_X)$. Hence $h \circ T(\varrho_X) = \nu_X \circ \varrho_{S(X)} \circ T(\varrho_X)$ which implies, by (B1) and (B4), that $h = \nu_X \circ \varrho_{S(X)}$. We now prove that $\mathtt{E}_\varrho$ is an equational T–theory on $\mathcal{D}$. In fact, let $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TY})$, then we have that $g = \mu_Y \circ T(g) \circ T(\eta_X)$, which follows from the following commutative diagram:

$$
\begin{tikzcd}
TT(Y) \arrow[d, "\mu_Y"'] & TT(X) \arrow[l, "T(g)"'] \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, "Id_{T(X)}"] \\
T(Y) & T(X) \arrow[l, "g"]
\end{tikzcd}
$$

Now, the unique T–algebra morphism g' such that $g' \circ \varrho_X = \varrho_Y \circ g$ is given by $g' = \nu_Y \circ S(\varrho_Y) \circ S(g) \circ S(\eta_X)$, as the following commutative diagram shows:

$$
\begin{tikzcd}
T(X) \arrow[r, "T(\eta_X)"] \arrow[d, "\varrho_X"'] \arrow[rrrrr, bend left, "g"] \arrow[dr, phantom, "\text{nat. } \varrho"] & TT(X) \arrow[rr, "T(g)"] \arrow[d, "\varrho_{T(X)}"'] \arrow[dr, phantom, "\text{nat. } \varrho"] & & TT(Y) \arrow[rr, "\mu_Y"] \arrow[dl, "\varrho_{T(Y)}"'] \arrow[dr, "T(\varrho_Y)"] & & T(Y) \arrow[dd, "\varrho_Y"] \\
S(X) \arrow[r, "S(\eta_X)"'] \arrow[ddrrrrr, bend right, "g'"'] & ST(X) \arrow[r, "S(g)"'] & ST(Y) \arrow[rr, phantom, "\text{nat. } \varrho"] \arrow[dr, "S(\varrho_Y)"'] & & TS(Y) \arrow[dl, "\varrho_{S(Y)}"] \arrow[r, phantom, "\star\star"] & \\
 & & & SS(Y) \arrow[rr, "\nu_Y"'] & & S(Y)
\end{tikzcd}
$$

□

Now, we prove that, under mild assumptions, for any monad morphism $\varrho : \mathsf{T} \to \mathsf{S}$ such that each component is in $\mathscr{E}$, that the categories $\mathrm{Mod}(\mathtt{E}_\varrho)$ and $\mathtt{Alg}(\mathsf{S})$ are isomorphic.

Proposition 117. *Let $\mathcal{D}$ be a category, $\mathsf{T} = (T, \eta, \mu)$ and $\mathsf{S} = (S, \iota, \nu)$ monads on $\mathcal{D}$, $\varrho : \mathsf{T} \to \mathsf{S}$ a monad morphism and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B1) and (B4). If every component of ϱ is in $\mathscr{E}$ then the categories $\mathrm{Mod}(\mathtt{E}_\varrho)$ and $\mathtt{Alg}(\mathsf{S})$ are isomorphic.*

Proof. Define the functor $F : \mathrm{Mod}(\mathtt{E}_\varrho) \to \mathtt{Alg}(\mathsf{S})$ as $F(X, \alpha_X) = (X, \alpha'_X)$, where the morphism $\alpha'_X \in \mathtt{Alg}(\mathsf{T})(\mathbf{SX}_\varrho, (X, \alpha_X))$ is the unique morphism such that $\alpha'_X \circ \varrho_X = \alpha_X$ for $(X, \alpha_X) \in \mathrm{Mod}(\mathtt{E}_\varrho)$ and define F on morphisms as $F(f) = f$. Define the functor $G : \mathtt{Alg}(\mathsf{S}) \to \mathrm{Mod}(\mathtt{E}_\varrho)$ as $G(X, \alpha'_X) = (X, \alpha'_X \circ \varrho_X)$ and on morphism as $G(f) = f$. The fact that F and G are functors defining an isomorphism of categories is easily verified by using (B1), (B4), Proposition 114, Proposition 116 and the results given in [17, Section 3]. □

Until now, we proved, under mild assumptions, that every monad morphism $\varrho : \mathsf{T} \to \mathsf{S}$ such that every component is in $\mathscr{E}$ induces a variety of T–algebras $\mathrm{Mod}(\mathtt{E}_\varrho)$ which is isomorphic to $\mathtt{Alg}(\mathsf{S})$. Now, by the main theorem in [17], we have, under mild assumptions, that every variety K of T–algebras induces a monad S_K on $\mathcal{D}$ and a monad morphism $\varrho : \mathsf{T} \to \mathsf{S}_K$ with every component in $\mathscr{E}$ such that K is equivalent to $\mathtt{Alg}(\mathsf{S}_K)$.

Proposition 118. *Let $\mathcal{D}$ be a complete category, $\mathsf{T} = (T, \eta, \mu)$ a monad on $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B1), (B4) and that for any $\mathbf{X} \in \mathtt{Alg}(\mathsf{T})$, there is, up to isomorphism, only a set of T–algebra morphisms in $\mathscr{E}$ whose domain is $\mathbf{X}$. Let K be a variety of T–algebras and assume that $\mathscr{E}$ contains all the U–split epis, where $U : \mathtt{Alg}(\mathsf{T}) \to \mathcal{D}$ is the forgetful functor. Then there exists a monad S_K on $\mathcal{D}$ and a monad morphism $\alpha : \mathsf{T} \to \mathsf{S}_K$ with every component in $\mathscr{E}$ such that K is equivalent to $\mathtt{Alg}(\mathsf{S}_K)$.*

Proof. Follows from [17, 4.1. Theorem]. □

5.2 Varieties of finite algebras

This section is similar to the previous one with the restriction that all the algebras considered are finite. That is, we assume that the category $\mathcal{D}$ is a concrete category, with forgetful functor $U : \mathcal{D} \to \mathtt{Set}$, and we say that an algebra $\mathbf{A} = (A, \alpha) \in \mathtt{Alg}(\mathsf{T})$ is *finite* if $U(A)$ is a finite set. We state a categorical version of a Birkhoff–type theorem for finite T–algebras, some versions of this theorem are known as Reiterman's theorem [73]. We use the prefix 'pseudo' to indicate that all the algebras considered are finite. That is, a pseudovariety of T–algebras is a variety of finite T–algebras, which is a class of finite T–algebras closed under homomorphic images, subalgebras and finite producs. The Birkhoff–type theorem for varieties of finite algebras states that a class of finite algebras of the same type is a pseudovariety if and only if it is defined by "extended equations" [73, 14]. An "extended equation" is a concept that generalizes the concept of an equation and can be defined by using topological techniques or, alternatively, by implicit operations [73, 14]. Reiterman's proof for the Birkhoff–type theorem for varieties of finite algebras involves topological methods in which the set of n–ary implicit operations is the completion of the set of n–ary terms [73]. A topological approach was also explored by Banaschewski by using uniformities [14]. Recently, in [28], profinite techniques were used to define the concept of profinite equations which are the kind of equations that define pseudovarieties of T–algebras.

We provide a categorical version for a Birkhoff–type theorem for varieties of finite algebras, Theorem 122, which, under mild assumptions, establishes a one–to–one correspondence between pseudovarieties of T–algebras and pseudoequational T–theories. Different versions of this theorem such as [73, 14, 28] use topological approaches and/or profinite techniques. In the present chapter, topological approaches and profinite techniques are not used, thus avoiding constructions of certain limits and profinite completions, which gives us a better and basic understanding on how pseudovarieties are characterized. The main strategy we follow to state and prove our theorem is that pseudovarieties of algebras are exactly directed unions of equational classes of finite algebras, which is a fact that was proved in [12, 14, 37]. The definition of pseudoequational T–theories is based on the previous observation and the categorical dual of "varieties of languages" that was used by the author to derive an Eilenberg–type correspondence for T–algebras [78].

Throughout this section, we fix a complete concrete category $\mathcal{D}$ such that its forgetful functor preserves epis, monos and products, a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, a full subcategory $\mathcal{D}_0$ of $\mathcal{D}$ and a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$. We make the following assumptions:

(B_f1) The factorization system $(\mathscr{E}, \mathscr{M})$ is *proper*, that is, every morphism in $\mathscr{E}$ is an epimorphism and every morphism in $\mathscr{M}$ is a monomorphism.

(B_f2) For every $X \in \mathcal{D}_0$, the free T–algebra $\mathbf{TX} = (T(X), \mu_X)$ is *projective with respect to $\mathscr{E}$ in* $\mathtt{Alg}(\mathsf{T})$. That is, for every $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{B})$ with $X \in \mathcal{D}_0$ and $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$ there exists $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ such that $e \circ g = h$.

(B_f3) For every finite $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ there exists $X_A \in \mathcal{D}_0$ and $s_A \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_A}, \mathbf{A}) \cap \mathscr{E}$.

(B_f4) T preserves morphisms in $\mathscr{E}$.

In order to talk about finite algebras, we assume that the category $\mathcal{D}$ is a concrete category. That is, if $U : \mathcal{D} \to \mathtt{Set}$ is the forgetful functor for the concrete category $\mathcal{D}$, then an object $X \in \mathcal{D}$ is *finite* if $U(X)$ is a finite set. An algebra $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ is *finite* if its carrier object $A \in \mathcal{D}$ is finite. The algebras of interest will be the objects $\mathtt{Alg}_f(\mathsf{T})$ of finite algebras in $\mathtt{Alg}(\mathsf{T})$. The factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$, which is lifted to $\mathtt{Alg}(\mathsf{T})$ by using (B_f4) in Lemma 100, allows us to define the concept of homomorphic image and subalgebra. In this case, by using condition (B_f1) and the requirement that the forgetful functor U preserves epis, monos and products, we get the property that $\mathscr{M}$–subalgebras, $\mathscr{E}$–quotients and finite products of finite algebras are also finite. For instance, for the case of $\mathscr{E}$–quotients, if $\mathbf{A}, \mathbf{B} \in \mathtt{Alg}(\mathsf{T})$ are algebras and $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$ with $U(A)$ finite (i.e., $\mathbf{A}$ is finite), then $U(B)$ is also finite since $U(e) : U(A) \to U(B)$ is a surjective map by the assumption that U preserves epis. A similar argument shows that $\mathscr{M}$–subalgebras and finite products of finite algebras are also finite, by using the fact that U preserves monos and products, respectively. The purpose of the subcategory $\mathcal{D}_0$ is that the objects from which "variables" for the equations are considered are objects in $\mathcal{D}_0$. Assumption (B_f3) guarantees that every algebra is the homomorphic image of a free one with object of generators in $\mathcal{D}_0$.

To obtain a classical Birkhoff–type theorem for varieties of finite algebras we can consider $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections, and T to be the term monad for a given type of algebras τ, i.e., $T(X) = T_\tau(X)$, the set of terms of type τ on the set of variables X (see Example 103). Another important example will be given by $\mathcal{D} = \mathtt{Poset}$ and $\mathcal{D}_0$ to be the full subcategory of finite discrete posets (as before, we do not want the "variables" to be ordered).

Now, we will define the main concepts needed to state our categorical Birkhoff–type theorem for varieties of finite T–algebras. We start by defining the concept of a pseudoequational T–theory.

Definition 119. Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1) and (B_f4). A *pseudoequational* T*–theory on* $\mathcal{D}_0$ is an operator $\mathtt{P}$ on $\mathcal{D}_0$ such that for every $X \in \mathcal{D}_0$, $\mathtt{P}(X)$ is a nonempty collection of T–algebra morphisms in $\mathscr{E}$ with domain $\mathbf{TX}$ and finite codomain such that:

i) For every finite set I and $f_i \in \mathtt{P}(X)$, $i \in I$, there exists $f \in \mathtt{P}(X)$ such that every f_i factors through f, $i \in I$.

ii) For every $e \in \mathtt{P}(X)$ with codomain $\mathbf{A}$ and every T–algebra morphism $e' \in \mathscr{E}$ with domain $\mathbf{A}$ we have that $e' \circ e \in \mathtt{P}(X)$.

iii) For every $Y \in \mathcal{D}_0$, $f \in \mathtt{P}(X)$ and $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{TX})$ we have that $e_{f\circ h} \in \mathtt{P}(Y)$ where $f \circ h = m_{f\circ h} \circ e_{f\circ h}$ is the factorization of $f \circ h$ in $\mathtt{Alg}(\mathsf{T})$ by using the factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$, which is lifted to $\mathtt{Alg}(\mathsf{T})$ by using (B_f4) and Lemma 100.

Pseudovarieties of algebras are exactly directed unions of equational classes of finite algebras [12, 14, 37]. With this in mind, we can give an intuition of the previous definition. In fact, for each object $X \in \mathcal{D}_0$ of variables every morphism $f \in \mathtt{P}(X)$ represents a set of equations on X, namely $\ker(f)$, which can be equivalently given by a T–algebra morphism in $\mathscr{E}$ with domain $\mathbf{TX}$. Condition i) says that the set of all the equations on a fixed X is a directed set, i.e., for every set of equations $f_i \in \mathtt{P}(X)$, $i \in I$, with I finite, there is an upper bound $f \in \mathtt{P}(X)$. Here f is an upper bound of $\{f_i \mid i \in I\}$ if every f_i factors through f. Condition iii) says that all the equations considered are preserved under any substitution

$h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{TX})$ of variables in Y by terms in $T(X)$, this condition is related to the commutativity of the diagram given in Definition 102. Condition ii) is needed for uniqueness of the pseudoequational theory defining a given pseudovariety of algebras. In fact, two directed unions of equational classes of finite algebras can give us the same pseudovariety, but if we put the requirement of being downward closed, which is the requirement in condition ii), then we get uniqueness.

Given an algebra $\mathbf{A} \in \mathtt{Alg}_f(\mathsf{T})$, we say that $\mathbf{A}$ *satisfies* $\mathtt{P}$, denoted as $\mathbf{A} \models \mathtt{P}$, if for every $X \in \mathcal{D}_0$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ we have that f factors through some morphism in $\mathtt{P}(X)$. We denote by $\mathrm{Mod}_f(\mathtt{P})$ the *finite models* of $\mathtt{P}$, that is:

$$\mathrm{Mod}_f(\mathtt{P}) := \{\mathbf{A} \in \mathtt{Alg}_f(\mathsf{T}) \mid \mathbf{A} \models \mathtt{P}\}$$

A class K of finite T–algebras is *defined* by $\mathtt{P}$ if $K = \mathrm{Mod}_f(\mathtt{P})$.

Let K be a class of algebras in $\mathtt{Alg}_f(\mathsf{T})$. We say that K *is closed under finite products* if $\prod_{i \in I} \mathbf{A}_i \in K$ for every finite set I such that $\mathbf{A}_i \in K$, $i \in I$. We now define the concept of a pseudovariety of T–algebras.

Definition 120. Let $\mathcal{D}$ be a complete concrete category, T a monad on $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. A class K of finite algebras in $\mathtt{Alg}(\mathsf{T})$ is called a *pseudovariety of* T*–algebras* if it is closed under $\mathscr{E}$–quotients, $\mathscr{M}$–subalgebras and finite products.

The following example shows some known pseudovarieties.

Example 121. Consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections, and T to be the term monad for a given type of algebras τ. Then we have that equational classes of finite algebras are examples of pseudovarieties of T–algebras. For example, finite semigroups, finite monoids, finite groups, finite vector spaces, finite Boolean algebras, finite lattices, and so on. In [14], some non–equational examples of pseudovarieties are shown such as:

(1) the finite commutative monoids satisfying some identity $x^n = x^{n+1}$, $n = 1, 2, \ldots$,

(2) the finite cancellation monoids,

(3) the finite abelian p–groups, for a given prime number p, and

(4) the finite products of finite fields of a given prime characteristic.

In fact, every equation satisfied in the given pseudovariety is also satisfied in the larger pseudovariety, i.e., the pseudovariety of all commutative monoids for (1), the pseudovariety of all monoids for (2), the pseudovariety of all abelian groups for (3), and the pseudovariety of all commutative rings with unit of a given prime characteristic for (4). □

Now we can formulate our categorical Birkhoff–type theorem for varieties of finite T–algebras as follows.

Theorem 122 (Birkhoff–type theorem for varieties of finite T–algebras)**.** *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, monos and products, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1) to (B_f4). Then a class K of finite T–algebras is a pseudovariety of T–algebras if and only if is defined by a pseudoequational T–theory on $\mathcal{D}_0$. Additionally, pseudovarieties of T–algebras are in one–to–one correspondence with pseudoequational T–theories on $\mathcal{D}_0$.*

5.2.1 Proof of Birkhoff–type theorem for varieties of finite T–algebras

In this subsection, we prove Theorem 122. We start by proving that models of pseudoequational T–theories are pseudovarieties of T–algebras.

Proposition 123. *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, monos and products, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1), (B_f2) and (B_f4). Let $\mathtt{P}$ be a pseudoequational T–theory on $\mathcal{D}_0$. Then $\mathrm{Mod}_f(\mathtt{P})$ is a pseudovariety of T–algebras.*

Proof. Clearly $\mathrm{Mod}_f(\mathtt{P})$ is non empty since $\mathbf{1} = (1, !_{T(1)} \colon T(1) \to 1) \in \mathrm{Mod}_f(\mathtt{P})$, where 1 is the terminal object in $\mathcal{D}$, which is finite since the forgetful functor from $\mathcal{D}$ to $\mathtt{Set}$ preserves products. The proof of $\mathrm{Mod}_f(\mathtt{P})$ being closed under $\mathscr{E}$–quotients and $\mathscr{M}$–subalgebras is done in a similar way as in Proposition 108. Note that $\mathscr{E}$–quotients and $\mathscr{M}$–subalgebras of finite algebras are also finite since the forgetful functor from $\mathcal{D}$ to $\mathtt{Set}$ preserves epis and monos, respectively.

Now, let us prove that $\mathrm{Mod}_f(\mathtt{P})$ is closed under finite products. In fact, let $\mathbf{A}_i \in \mathrm{Mod}_f(\mathtt{P})$, $i \in I$ with I finite, and let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ be their product in $\mathtt{Alg}(\mathsf{T})$ with projections $\pi_i : A \to A_i$. We have that A is finite since the forgetful functor from $\mathcal{D}$ to $\mathtt{Set}$ preserves products, I is finite, and each A_i is finite. Let $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$, $X \in \mathcal{D}_0$. As $\mathbf{A}_i \in \mathrm{Mod}_f(\mathtt{P})$ then $\pi_i \circ f$ factors through some $e_i \in \mathtt{P}(X)$ as $\pi_i \circ f = g_i \circ e_i$. Since $\mathtt{P}$ is a pseudoequational T–theory there exists $e \in \mathtt{P}(X)$ such that every e_i factors through e as $h_i \circ e = e_i$, $i \in I$. Let $\mathbf{Q}$ be the codomain of e. Now, by definition of $\mathbf{A}$ there exists $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q}, \mathbf{A})$ such that $\pi_i \circ h = g_i \circ h_i$. As $\pi_i \circ f = \pi_i \circ h \circ e$ for every $i \in I$, then $f = h \circ e$, $e \in \mathtt{P}(X)$, which means that $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$. $\square$

Given a class K of algebras in $\mathtt{Alg}_f(\mathsf{T})$ define the operator $\mathtt{P}_K$ on $\mathcal{D}_0$ as follows:

$$\mathtt{P}_K(X) = \mathsf{T}\text{–algebra morphisms in } \mathscr{E} \text{ with domain } \mathbf{TX} \text{ and codomain in } K.$$

Now we show, under mild assumptions, that if K is a pseudovariety then $\mathtt{P}_K$ is a pseudoequational theory.

Proposition 124. *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, monos and products T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1) and (B_f4). Let K be a pseudovariety of T–algebras. Then $\mathtt{P}_K$ is a pseudoequational T–theory on $\mathcal{D}_0$.*

Proof. We have to prove properties i), ii), and iii) of Definition 119. In fact:

i) Let $X \in \mathcal{D}_0$, I a finite set and $f_i \in \mathtt{P}_K(X)$, $i \in I$. Let $\mathbf{A}_i \in K$ be the codomain of f_i. Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ with projections $\pi_i \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{A}_i)$. We have $\mathbf{A} \in K$. Now, by definition of $\mathbf{A}$, there exists $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ such that $\pi_i \circ f = f_i$. Let $f = m_f \circ e_f$ be the factorization of f with $e_f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{Q}) \cap \mathscr{E}$ and $m_f \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q}, \mathbf{A}) \cap \mathscr{M}$. We have that $\mathbf{Q} \in K$. Then e_f is a morphism in $\mathtt{P}_K(X)$ such that every f_i factors through e_f.

ii) Let $X \in \mathcal{D}_0$, $e \in \mathtt{P}_K(X)$ with codomain $\mathbf{A} \in K$, and $e' \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{B}) \cap \mathscr{E}$. We have that $\mathbf{B}$ is finite and that $\mathbf{B} \in K$. Therefore $e' \circ e \in \mathtt{P}_K(X)$.

iii) Let $X, Y \in \mathcal{D}_0$, $f \in \mathtt{P}_K(X)$ with codomain $\mathbf{A} \in K$, and $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{TX})$. Let $f \circ h = m_{f \circ h} \circ e_{f \circ h}$ be the factorization of $f \circ h$ with $e_{f \circ h} \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{Q}) \cap \mathscr{E}$ and $m_{f \circ h} \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q}, \mathbf{A}) \cap \mathscr{M}$. Then $\mathbf{Q} \in K$, which implies $e_{f \circ h} \in \mathtt{P}_K(Y)$.

$\square$

The next lemma shows that the codomain of every arrow in a pseudoequational theory satisfies the pseudoequational theory.

Lemma 125. *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis,monos and products, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B$_f$1), (B$_f$2) and (B$_f$4). Let $\mathtt{P}$ be a pseudoequational T–theory on $\mathcal{D}_0$. Let $X \in \mathcal{D}_0$ and $e \in \mathtt{P}(X)$ with codomain $\mathbf{A} \in \mathtt{Alg}_f(\mathsf{T})$, then $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$.*

Proof. Let $Y \in \mathcal{D}_0$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TY}, \mathbf{A})$. We have to show that f factors through some element in $\mathtt{P}(Y)$. We have the following commutative diagram:

$$\begin{tikzcd}
T(Y) \arrow[rr, "f"] \arrow[dr, "k"'] \arrow[ddr, "e_{e\circ k}"'] & & A \\
& T(X) \arrow[ur, "e"'] & \\
& Q \arrow[uur, hook, "m_{e\circ k}"'] &
\end{tikzcd}$$

where the morphism k is obtained from f and e by using (B$_f$2) and $e \circ k = m_{e \circ k} \circ e_{e \circ k}$ is the factorization of $e \circ k$. Then, from the previous diagram we have that $e_{e \circ k} \in \mathtt{P}(Y)$, since $e \in \mathtt{P}(X)$ and $\mathtt{P}$ is a pseudoequational T–theory. Therefore f factors through $e_{e \circ k} \in \mathtt{P}(Y)$, which implies that $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$. $\square$

To finish the proof of Theorem 122 we establish the following one–to–one correspondence between pseudoequational T–theories and pseudovarieties of T–algebras.

Proposition 126. *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, monos and products, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1), (B_f2) and (B_f4). Let $\mathtt{P}$ be a pseudoequational T–theory on $\mathcal{D}_0$ and let K be a pseudovariety of T–algebras. Then:*

i) $\mathtt{P}_{\mathrm{Mod}_f(\mathtt{P})} = \mathtt{P}$.

ii) Assume (B_f3), then $\mathrm{Mod}_f(\mathtt{P}_K) = K$.

Proof.

i) Let $X \in \mathcal{D}_0$, we have to prove that $\mathtt{P}_{\mathrm{Mod}_f(\mathtt{P})}(X) = \mathtt{P}(X)$.
($\subseteq$): Let $e \in \mathtt{P}_{\mathrm{Mod}_f(\mathtt{P})}(X)$ with codomain $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$. As $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$, there exists $e' \in \mathtt{P}(X)$ such that e factors through e' as $g \circ e' = e$. By (B$_f$1) and (B$_f$4) we have that g is a T–algebra morphism. As $g \circ e' = e \in \mathscr{E}$, then $g \in \mathscr{E}$, and, as $\mathtt{P}$ is a pseudoequational T–theory, then $g \circ e' = e \in \mathtt{P}(X)$.
($\supseteq$): Let $e \in \mathtt{P}(X)$ with codomain $\mathbf{A}$. By Lemma 125, $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P})$, i.e., $e \in \mathtt{P}_{\mathrm{Mod}_f(\mathtt{P})}(X)$.

ii) Let $\mathbf{A}$ be an object in $\mathtt{Alg}_f(\mathsf{T})$.
($\supseteq$): Assume that $\mathbf{A} \in K$. We have to show that $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P}_K)$. In fact, let $X \in \mathcal{D}_0$ and $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$. Let $f = m_f \circ e_f$ be the factorization of f with $e_f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{Q}) \cap \mathscr{E}$ and $m_f \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q}, \mathbf{A}) \cap \mathscr{M}$. Then $\mathbf{Q} \in K$, which implies that $e_f \in \mathtt{P}_K(X)$, i.e., $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P}_K)$.
($\subseteq$): Assume that $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P}_K)$. By (B$_f$3) there exists an object $X_A \in \mathcal{D}_0$ and $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX_A}, \mathbf{A}) \cap \mathscr{E}$. As $\mathbf{A} \in \mathrm{Mod}_f(\mathtt{P}_K)$, e factors through some $e' \in \mathtt{P}_K(X_A)$ as $e = g \circ e'$. Let $\mathbf{Q} \in K$ be the codomain of e'. As $g \circ e' = e \in \mathscr{E}$, then $g \in \mathscr{E}$ and $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q}, \mathbf{A})$ which implies that $\mathbf{A} \in K$ since $\mathbf{Q} \in K$.

□

The previous proposition finishes the proof of our Birkhoff–type theorem for varieties of finite T–algebras. □
Now, we derive a Birkhoff–type theorem for pseudovarieties of (ordered) algebras for a given type, then we show an example of a particular pseudovariety of algebras with its defining pseudoequational T–theory and finish this subsection by deriving Eilenberg's theorem [36, Theorem 34] to show a one–to–one correspondence between pseudovarieties of monoids and pseudovarieties of languages.

Example 127. Consider the case $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections and, for a given type of algebras τ, let T_τ be the term monad for τ. Then, by Theorem 122, a class of algebras of type τ is a pseudovariety if and only if it is defined by a pseudoequational T_τ–theory. □

Example 128. Consider the case $\mathcal{D} = \mathtt{Poset}$, $\mathcal{D}_0 =$ finite discrete posets, $\mathscr{E} =$ surjections, $\mathscr{M} =$ embeddings and, for a given type of algebras τ, let T_τ be the monad on $\mathtt{Poset}$ defined in Example 104. Then, by Theorem 122, a class of ordered algebras of type τ is a pseudovariety if and only if it is defined by a pseudoequational T_τ–theory. □

Example 129. Consider the case $\mathcal{D} = \texttt{Set}$, $\mathcal{D}_0 =$ finite sets, T the monad given by $T(X) = X^*$, where X^* is the free monoid on X, $\mathscr{E} =$ surjections, and $\mathscr{M} =$ injections. We have that conditions (B_f1) to (B_f4) are fullfilled. In this case, we have that $\texttt{Alg}(\mathsf{T})$ is the category of monoids. To describe the pseudovariety of all commutative monoids satisfying some identity $x^n = x^{n+1}$, $n = 1, 2, \ldots$, we define $\texttt{P}$ on $\mathcal{D}_0$ as follows:

- For every $X \in \mathcal{D}_0$, and $n = 1, 2 \ldots$, we define the surjective homomorphism of monoids $e_n : X^* \twoheadrightarrow \mathfrak{F}_n(X)$, where $\mathfrak{F}_n(X)$ is the free commutative monoid on X that satisfies the identity $x^n = x^{n+1}$. That is, $\mathfrak{F}_n(X) = (\texttt{Set}(X, \{0, 1, \ldots n\}), \cdot, 0)$ where $0 \in \texttt{Set}(X, \{0, 1, \ldots n\})$ is the zero function, i.e., $0(x) = 0$ for every $x \in X$, and $\cdot$ is defined on $\texttt{Set}(X, \{0, 1, \ldots n\})$ as $(f \cdot g)(x) = \min\{n, f(x) + g(x)\}$. Then define e_n on the set of generators X as $e_n(x) = \chi_x$, where $\chi_x(x) = 1$ and $\chi_x(y) = 0$ for $x \neq y$. Define $\texttt{P}(X)$ as:

$$\texttt{P}(X) = \{e' \circ e_n \mid n \in \mathbb{N}^+ \text{ and } e' \text{ is a T–alg. morphism in } \mathscr{E} \text{ with domain } \mathfrak{F}_n(X)\}$$

We have then that $\texttt{P}$ is a pseudoequational T–theory and $\mathrm{Mod}_f(\texttt{P})$ is the pseudovariety of all finite commutative monoids that satisfy some identity $x^n = x^{n+1}$, $n = 1, 2, \ldots$. □

In the next example, we derive Eilenberg's variety theorem [36, Theorem 3.4.]. Given a finite set Σ, i.e., an *alphabet*, a *language* L on Σ is a subset L of Σ^*, i.e., a collection of words with letters in Σ. We identify a language L on Σ by its characteristic function $L : \Sigma^* \to 2$. A language L on Σ is *recognizable* if there exists a finite monoid $\mathbf{A}$, a homomorphism of monoids $h : \Sigma^* \to A$ and a function $L' : A \to 2$ such that $L' \circ h = L$. We denote by $\mathrm{Rec}(\Sigma)$ the Boolean algebra of all recognizable languages on Σ. A *pseudovariety of languages* is an operator $\mathscr{L}$ such that for every finite set Σ we have:

i) $\mathscr{L}(\Sigma)$ is a subalgebra of the Boolean algebra $\mathrm{Rec}(\Sigma)$,

ii) $\mathscr{L}(\Sigma)$ is closed under left and right derivatives. That is, ${}_aL, L_a \in \mathscr{L}(\Sigma)$ for every $L \in \mathscr{L}(\Sigma)$ and $a \in \Sigma$, and

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every alphabet Γ, homomorphism of monoids $h : \Gamma^* \to \Sigma^*$ and $L \in \mathscr{L}(\Sigma)$, we have that $L \circ h \in \mathscr{L}(\Gamma)$.

Eilenberg's variety theorem [36, Theorem 34] says that there is a one–to–one correspondence between pseudovarieties of monoids and pseudovarieties of languages. This theorem is derived from Theorem 122 as follows.

Example 130 (Eilenberg's variety theorem)**.** Consider the setting as in the previous example, i.e., $\mathcal{D} = \texttt{Set}$, $\mathcal{D}_0 =$ finite sets, T the monad given by $T(X) = X^*$, where X^* is the free monoid on X, $\mathscr{E} =$ surjections, and $\mathscr{M} =$ injections. Then, we have a one–to–one correspondence between pseudovarieties of monoids, i.e., pseudovarieties of T–algebras, and pseudoequational T–theories on $\mathcal{D}_0$. Now, we have that pseudoequational T–theories on $\mathcal{D}_0$ are in one–to–one correspondence with pseudovarieties of languages. In fact, every pseudoequational T–theory P on $\mathcal{D}_0$ defines the pseudovariety of languages $\mathscr{L}_{\mathtt{P}}$ defined as $\mathscr{L}_{\mathtt{P}}(X) := \bigcup_{e \in \mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e, 2))$, and every pseudovariety of languages $\mathscr{L}$ defines the pseudoequational T–theory $\mathtt{P}_{\mathscr{L}}$ on $\mathcal{D}_0$ such that $\mathtt{P}_{\mathscr{L}}(X)$ is the collection of all T–algebra morphisms $e \in \mathscr{E}$ with domain $\mathbf{TX}$ and finite codomain such that $\mathrm{Im}(\mathtt{Set}(e, 2)) \subseteq \mathscr{L}(X)$, $X \in \mathcal{D}_0$. Furthermore, this correspondence is bijective, that is, for every pseudoequational T–theory P on $\mathcal{D}_0$ and every pseudovariety of languages $\mathscr{L}$ we have that $\mathtt{P} = \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}$ and $\mathscr{L} = \mathscr{L}_{\mathtt{P}_{\mathscr{L}}}$ (see Example 176 for more details). □

Birkhoff–type theorems for varieties of finite algebras, some of them also known as Reiterman's theorem [73], have been proved and generalized in [14, 28]. The approach in [73] was to consider implicit operations, which generalize the notion of terms. Equations given by implicit operations are the kind of equations that define pseudovarieties. The proof given in [73] involves the use of topology in which the set of n–ary implicit operations is the completion of the set of n–ary terms. In [14], a topological approach is also considered by using uniformities, and it is also shown that pseudovarieties are exactly directed unions of equational classes of finite algebras [14, Proposition 4]. In [28], a categorical approach is considered to prove a Birkhoff–type theorem for varieties of finite T–algebras, this is done by using profinite techniques to define the notion of profinite equation which are the kind of equations that allow to define and characterize pseudovarieties. In [28], for a given monad T on $\mathcal{D}$ they define the profinite monad $\widehat{\mathsf{T}}$ on the profinite completion $\widehat{\mathcal{D}}$ of the category $\mathcal{D}_f$ of finite objects in $\mathcal{D}$ which is done by using limits (in fact, right Kan extensions). The approach in the present work do not use topological nor profinite techniques, and it is based in the fact that pseudovarieties are exactly directed unions of equational classes of finite algebras, see [14, Proposition 4] and [12, 37]. Nevertheless, profinite and topological techniques can be easily brought to the scene in the present work if we identify the family of morphisms $\mathtt{P}(X)$ by its limit, where P is a pseudoequational T–theory. This would have led us to deal with profinite completions and topological spaces, in particular, profinite monoids, Stone spaces and Stone duality. We prefer to avoid this approach for the following reasons:

a) Make the present work more accessible to some readers.

b) To present a different approach without using topology and profinite techniques.

c) In the following chapters we derive Eilenberg–type correspondences, which deal with pseudocoequational theories rather than its dual, i.e., pseudoequational theories.

5.3 Local versions

In this section, we provide abstract versions of a Birkhoff–type theorem for local varieties of T–algebras and local pseudovarieties of T–algebras. Local pseudovarieties of algebras have been studied in [1, 43] in order to obtain local versions of Eilenberg–type correspondences, which will be derived in the subsequent chapters. The main idea for these local versions is to work with a fixed set of variables (or alphabet in the sense of [1, 43]), which in our notation reduces to consider the case in which the category $\mathcal{D}_0$ has only one object, say X. In order to do this, the kind of algebras considered in this local version are algebras that are generated by the object X in the following sense.

Definition 131. Let $\mathcal{D}$ be a category, T a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $X \in \mathcal{D}$. An algebra $\mathbf{A} \in \mathtt{Alg}(\mathsf{T})$ is X–generated if $\mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A}) \cap \mathscr{E}$ is nonempty.

The following example illustrates this concept.

Example 132. Consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections and T the free group monad. Then, a group is 1–generated if and only if it is a quotient of $\mathbf{T1} = \mathbb{Z}$, i.e., if and only if it is isomorphic to the group $\mathbb{Z}$ or to to the group $\mathbb{Z}_n$ of integers modulo n for some $n \in \mathbb{N}^+$. □

We have that $\mathscr{E}$–quotients of X–generated T–algebras are X–generated, but this property does not hold in general for $\mathscr{M}$–subalgebras and products. Thus, we will restrict our attention to X–generated $\mathscr{M}$–subalgebras, i.e., $\mathscr{M}$–subalgebras that are X–generated, and subdirect products. The latter are defined as follows.

Definition 133. Let $\mathcal{D}$ be a complete category, T a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ such that T preserves the morphisms in $\mathscr{E}$. Let $X \in \mathcal{D}$ and let $\mathbf{A}_i$ be an X–generated T–algebra with $e_i \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A}_i) \cap \mathscr{E}$, $i \in I$. We define the *subdirect product* of the family $\{(\mathbf{A}_i, e_i)\}_{i\in I}$ as the X–generated $\mathscr{M}$–subalgebra $\mathbf{S}$ of $\prod_{i\in I} \mathbf{A}_i$ described in the following commutative diagram:

$$
\begin{tikzcd}
 & T(X) \arrow[dl, "e_e"'] \arrow[d, "e"] \arrow[dr, "e_j"] & \\
S \arrow[r, hook, "m_e"'] & \prod_{i\in I} A_i \arrow[r, "\pi_j"'] & A_j
\end{tikzcd}
$$

where e is obtained from the morphisms e_j, $j \in I$, and the universal property of the product $\prod_{i\in I} \mathbf{A}_i$ and $e = m_e \circ e_e$ is the factorization of e. We say that the subdirect product $\mathbf{S}$ defined above is *finite* if I is a finite set.

To obtain local versions of a Birkhoff–type theorem, the concept of a local variety of T–algebras used is: classes of X–generated T–algebras closed under $\mathscr{E}$–quotients, X–generated $\mathscr{M}$–subalgebras and subdirect products. We state two local versions in the rest of this section, one for local varieties of algebras and one for local pseudovarieties of algebras. We start with a Birkhoff–type theorem for local varieties of T–algebras.

We fix a complete category $\mathcal{D}$, a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $X \in \mathcal{D}$. We will use the following assumptions:

(b1) Every morphism in $\mathscr{E}$ is an epimorphism.

(b2) The free T–algebra $\mathbf{TX} = (T(X), \mu_X)$ is projective with respect to $\mathscr{E}$ in $\mathtt{Alg}(\mathsf{T})$.

(b3) T preserves morphisms in $\mathscr{E}$.

(b4) There is, up to isomorphism, only a set of T–algebra morphisms in $\mathscr{E}$ with domain $\mathbf{TX}$.

We define local varieties of X–generated T–algebras as follows.

Definition 134. Let $\mathcal{D}$ be a complete category, T a monad on $\mathcal{D}$, and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (b1) and (b3). Let $X \in \mathcal{D}$. A class K of X–generated T–algebras is a *local variety of X–generated* T*–algebras* if it is closed under $\mathscr{E}$–quotients, X–generated $\mathscr{M}$–subalgebras and subdirect products.

Next, we define the notion of a local equational T–theory.

Definition 135. Let $\mathcal{D}$ be a category, T a monad on $\mathcal{D}$, $X \in \mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. A *local equational* T*–theory on* X is a T–algebra morphism $TX \xrightarrow{e_X} Q_X$ in $\mathscr{E}$ such that for any $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TX})$ there exists $g' \in \mathtt{Alg}(\mathsf{T})(\mathbf{Q_X}, \mathbf{Q_X})$ such that the following diagram commutes:

$$\begin{tikzcd} T(X) \arrow[r, "\forall g"] \arrow[d, "e_X"'] & T(X) \arrow[d, "e_X"] \\ Q_X \arrow[r, dashed, "g'"'] & Q_X \end{tikzcd}$$

Note that, in the setting of Example 103, a morphism $T(X) \xrightarrow{e_X} Q_X$ in local equational T–theory on X is characterized, up to isomorphism, by its kernel $\ker(e_X)$. In this case, the property of e_X being a local equational T–theory is exactly the property that $\ker(e_X)$ is a fully invariant congruence of $\mathbf{TX}$ [27, Definition II.14.1]. This generalizes the definition of an equational theory over X in [27, Definition II.14.9] to a categorical level.

Given a local equational T–theory $T(X) \xrightarrow{e_X} Q_X$ on X and an X–generated T–algebra $\mathbf{A}$, we say that $\mathbf{A}$ *satisfies* e_X, denoted as $\mathbf{A} \models e_X$, if every morphism $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ factors through e_X. We denote by $\mathrm{Mod}(e_X)$ the *X–generated models of* e_X, that is:

$$\mathrm{Mod}(e_X) = \{\mathbf{A} \in \mathtt{Alg}(\mathsf{T}) \mid \mathbf{A} \text{ is } X\text{–generated and } \mathbf{A} \models e_X\}.$$

A class K of X–generated T–algebras is *defined* by e_X if $K = \mathrm{Mod}(e_X)$.

With the previous definitions, we have our Birkhoff–type theorem for local varieties of T–algebras.

Theorem 136 (Birkhoff–type theorem for local varieties T–algebras)**.** *Let $\mathcal{D}$ be a complete category, T a monad on $\mathcal{D}$, $X \in \mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (b2) to (b4). Then a class K of X–generated T–algebras is a local variety of X–generated T–algebras if and only if is defined by a local equational T–theory on X. Additionally, by assuming condition (b1), local varieties of X–generated T–algebras are in one–to–one correspondence with local equational T–theories on X.* $\square$

We illustrate the previous theorem with the following example.

Example 137. Consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections and T the free group monad. Then, for local varieties V of X–generated T–algebras for the case $X = 1$ we have the following:

i) $\mathbf{TX} = \mathbf{T1}$ is isomorphic to the group $\mathbb{Z}$ of integers with addition and hence, every group in V is isomorphic to $\mathbb{Z}$ itself or isomorphic to the group $\mathbb{Z}_n$ of integers modulo n for some $n \geq 1$.

ii) Either V contains $\mathbb{Z}$, and hence every element in V is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n$, or V does not contain $\mathbb{Z}$ and hence $m := \mathtt{max}\{n \in \mathbb{N} \mid \exists G \in V, G \cong \mathbb{Z}_n\}$ exists and $G \in V$ if and only if $G \cong \mathbb{Z}_n$ for some $n|m$. In the first case $V = \mathrm{Mod}(Id_{\mathbb{Z}})$ and in the second case $V = \mathrm{Mod}(e_m)$, where $e_m : \mathbb{Z} \to \mathbb{Z}_m$ is the canonical map that sends every integer to its equivalence class modulo m. $\square$

We now provide a Birkhoff–type theorem for local varieties of finite T–algebras. We fix a complete concrete category $\mathcal{D}$ such that its forgetful functor preserves epis, monos and products, a monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{D}$, $X \in \mathcal{D}$ and a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$. We will need the following assumptions:

(b$_f$1) The factorization system $(\mathscr{E}, \mathscr{M})$ is proper.

(b$_f$2) The free T–algebra $\mathbf{TX} = (T(X), \mu_X)$ is projective with respect to $\mathscr{E}$ in $\mathtt{Alg}(\mathsf{T})$.

(b$_f$3) T preserves morphisms in $\mathscr{E}$.

We define the concept of a local pseudoequational T–theory on X as follows.

Definition 138. Let $\mathcal{D}$ be a concrete category such that its forgetful functor preserves epis and monos, T a monad on $\mathcal{D}$, $X \in \mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (b$_f$1) and (b$_f$3). A *local pseudoequational* T*–theory on* X is a nonempty collection $\mathtt{P}_X$ of T–algebra morphisms in $\mathscr{E}$ with domain $\mathbf{TX}$ and finite codomain such that:

i) For every finite set I and $f_i \in \mathtt{P}_X$, $i \in I$, there exists $f \in \mathtt{P}_X$ such that f_i factors through f, $i \in I$.

ii) For every $e \in \mathtt{P}_X$ with codomain $\mathbf{A}$ and every T–algebra morphism $e' \in \mathscr{E}$ with domain $\mathbf{A}$ we have that $e' \circ e \in \mathtt{P}_X$.

iii) For every $f \in \mathtt{P}_X$ and $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TX})$ we have that $e_{f\circ h} \in \mathtt{P}_X$ where $f \circ h = m_{f\circ h} \circ e_{f\circ h}$ is the factorization of $f \circ h$ in $\mathtt{Alg}(\mathsf{T})$.

Given an X–generated algebra $\mathbf{A} \in \mathtt{Alg}_f(\mathsf{T})$, we say that $\mathbf{A}$ *satisfies* $\mathtt{P}_X$, denoted as $\mathbf{A} \models \mathtt{P}_X$, if every $f \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A})$ factors through some morphism in $\mathtt{P}_X$. We denote by $\mathrm{Mod}_f(\mathtt{P}_X)$ the *finite X–generated models of* $\mathtt{P}_X$, that is:

$$\mathrm{Mod}_f(\mathtt{P}_X) := \{\mathbf{A} \in \mathtt{Alg}_f(\mathsf{T}) \mid \mathbf{A} \text{ is } X\text{–generated and } \mathbf{A} \models \mathtt{P}_X\}.$$

A class K of finite X–generated T–algebras is *defined* by $\mathtt{P}_X$ if $K = \mathrm{Mod}_f(\mathtt{P}_X)$.

We define the concept of a local pseudovariety of X–generated T–algebras as follows.

Definition 139. Let $\mathcal{D}$ be a complete concrete category, T a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $X \in \mathcal{D}$ a finite object. A class K of finite X–generated algebras in $\mathtt{Alg}(\mathsf{T})$ is called a *local pseudovariety of X–generated* T*–algebras* if it is closed under $\mathscr{E}$–quotients, X–generated $\mathscr{M}$–subalgebras and finite subdirect products.

From the previous definitions, we obtain our Birkhoff–type theorem for local varieties of finite T–algebras.

Theorem 140 (Birkhoff–type theorem for local varieties of finite T–algebras)**.** *Let $\mathcal{D}$ be a concrete complete category such that its forgetful functor preserves epis, monos and products, T a monad on $\mathcal{D}$, $X \in \mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (b_f1) to (b_f3). Then a class K of finite X–generated T–algebras is a local pseudovariety of X–generated T–algebras if and only if is defined by a local pseudoequational T–theory on X. Additionally, local pseudovarieties of X–generated T–algebras are in one–to–one correspondence with local pseudoequational T–theories on X.* $\square$

The following example illustrates the previous theorem.

Example 141. Consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathscr{E}$ = surjections, $\mathscr{M}$ = injections and T the free group monad. Then, for local pseudovarieties V of X–generated T–algebras for the case $X = 1$ we have the following:

i) Every morphism in a local pseudoequational T–theory $\mathtt{P}_X$ on X is of the form $e_m : \mathbb{Z} \to \mathbb{Z}_m$ for some m, where e_m sends each integer to its equivalence class modulo m. Property i) in Definition 138 is equivalent, under the assumption that $\mathtt{P}_X$ is a pseudoequational T–theory on X, to the property that for every finite family $\{e_{m_1}, \ldots, e_{m_n}\} \subseteq \mathtt{P}_X$ we have that $e_{\mathtt{l.c.m.}\{m_1,\ldots,m_n\}} \in \mathtt{P}_X$, where $\mathtt{l.c.m.}\{m_1, \ldots, m_n\}$ is the least common multiple of the family $\{m_1, \ldots, m_n\}$. Properties ii) and iii) in Definition 138 says that for every $e_m \in \mathtt{P}_X$ and $n|m$ we have $e_n \in \mathtt{P}_X$.

ii) According to i), pseudoequational T–theories are in one–to–one correspondence with proper filters of the lattice $(\mathbb{N}, \vee, \wedge)$ where $\vee$ is given by the least common multiple and $\wedge$ is the greatest common divisor. Furthermore, given such filter $F \subseteq \mathbb{N}$ its pseudovariety K_F is characterized as all the finite groups that are isomorphic to $\mathbb{Z}_n$ for some $n \in F$. □

5.4 Discussion

We presented a categorical version of a Birkhoff–type theorem for varieties of algebras over a monad. The main purpose of doing this was to get a version which is stated as a one–to–one correspondence between varieties of algebras and equational theories, which will be used in the next chapters to derive Eilenberg–type correspondences. With this in mind, we defined the concept of an equational theory based on the classical concept given in [27, Definition II.14.16] and a previous definition of variety of languages in [78, Definition 18]. We also presented similar versions for the case of pseudovarieties of T–algebras, local varieties of T–algebras and local pseudovarieties of T–algebras.

There are categorical versions of Birkhoff–type theorems in the literature such as [4, 10, 15, 17]. The main difference of the present work with those approaches is the use of equational theories and therefore the statement of the theorem as a one–to–one correspondence. The proof of the Birkhoff–type theorem for varieties of T–algebras we presented is based on the work made in [15]. We also related our concept of an equational theory with that of a monad morphism used in [17].

In the theorems we presented we used a factorization system on the base category which, under mild assumptions, is lifted to the category $\mathtt{Alg}(\mathsf{T})$ of algebras over the monad T. The main reason for this approach, instead of considering a factorization system in the category $\mathtt{Alg}(\mathsf{T})$, is to be able to use different monads for the same factorization system on the base category. We could have also considered a factorization system on $\mathtt{Alg}(\mathsf{T})$ and having essentially the same proofs for the theorems. It is worth mentioning that not every factorization system in $\mathtt{Alg}(\mathsf{T})$ comes from a factorization system in the base category, see, e.g., [17].

Birkhoff–type theorems for varieties of finite algebras, some of them also known as Reiterman's theorem, were initially proved in [14, 73, 12, 37]. Different versions such as [73, 14, 28] use topologial approaches and/or profinite techniques. The approach presented in [28] is a categorical one. The version presented in this paper is also a categorical approach and it is based on the observation made in [12, 15, 37] that pseudovarieties of algebras are characterized as directed unions of equational classes of finite algebras. We defined the concept of a pseudoequational theory based on the previous observation. As a consequence, the proof presented in this thesis does not use topological or profinite techniques, which can help to understand a Birkhoff–type theorem for varieties of finite algebras in a more basic setting and it is more accessible to some readers.

Another important aspect and parameter we used in our theorems is the use of the subcategory $\mathcal{D}_0$ of $\mathcal{D}$, which is the subcategory where we consider the objects that represent variables. In cases such as $\mathcal{D} = \mathtt{Set}$ we can also take $\mathcal{D}_0 = \mathtt{Set}$, but in cases such as $\mathcal{D} = \mathtt{Poset}$ we take $\mathcal{D}_0 =$ discrete posets (in this case, the choice $\mathcal{D}_0 = \mathtt{Poset}$ does not satisfy condition (B2) needed to prove the theorem). This choice of $\mathcal{D}_0$ will define the domain for the varieties of languages in Eilenberg–type correspondences which is one of the motivations for using it. Also, in order to deal with the case $\mathcal{D} = \mathtt{Poset}$ we needed to restrict the domain of the epimorphisms that define the equations of a variety of algebras, which is done by considering the subcategory $\mathcal{D}_0$ of $\mathcal{D}$.

It is worth mentioning that different choices of the subcategory $\mathcal{D}_0$ of $\mathcal{D}$ could work for the same $\mathcal{D}$. For instance, if we consider the monad T_τ such that $T_\tau(X)$ is the set of terms for a type of algebras τ on variables X, see [27, Definition II.10.1] or Section 1.2, then we can consider, for instance, $\mathcal{D}_0 = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets or even $\mathcal{D}_0 = \{X\}$ for any infinite set X. This property of being able to restrict the subcategory $\mathcal{D}_0$ is essentially a property that is discussed in [15, Remark 1].

Local versions of Birkhoff–type theorems are less known, and the main purpose of including them here is to derive local versions of Eilenberg–type correspondences, a work that has been done in [1, 43]. Those local versions are basically a consequence of the non–local versions by restricting to the case in which the subcategory $\mathcal{D}_0$ has only one object. For this purpose, we only dealt with X–generated algebras, where X is the only object in $\mathcal{D}_0$. In this case, the closure properties that define a local variety are modified in order to deal only with X–generated algebras.

Chapter 6

Eilenberg–type correspondences

In this chapter we show that:

Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality.

Eilenberg's theorem is an important result in algebraic language theory, stating that there is a one–to–one correspondence between certain classes of regular languages, called varieties of languages, and certain classes of monoids, called pseudovarieties of monoids [36, Theorem 34]. The concept of a regular language, which is defined in terms of deterministic automata, has an equivalent machine–independent algebraic definition, namely, a language recognized by a finite monoid. Recognizable languages on an alphabet Σ are inverse images of monoid homomorphisms with domain Σ^* and as codomain any finite monoid. This algebraic approach allows us to study various kinds of recognizable languages where the notion of homomorphism between algebras is a key ingredient.

To state Eilenberg–type theorems, which establish one–to–one correspondences between (pseudo)varieties of algebras and (pseudo)varieties of languages, one has to define and find the corresponding notion of a (pseudo)variety of languages which is, in general, a non–trivial problem. There are Eilenberg–type correspondences in the literature such as, e.g., [70] for pseudovarieties of ordered monoids and ordered semigroups, the one in [74] for pseudovarieties of finite dimensional $\mathbb{K}$–algebras, [72] for pseudovarieties of idempotent semirings and [13, Theorem 39] for varieties of monoids.

The work in this chapter has its basis in [22, 80]. We take the main idea given in [22], where algebras for a monad T on $\mathcal{D}$ are considered, to define the natural notion of a variety of T–algebras and a pseudovaritey of T–algebras. Now, based on the work of the previous chapter in which we established a one–to–one correspondence between varieties of T–algebras and equational T–theories and also a one–to–one correspondence between pseudovarieties of T–algebras and pseudoequational T–theories, we use a category $\mathcal{C}$ that is dual to $\mathcal{D}$ and Proposition 96 to define a canonical comonad B on $\mathcal{C}$ that is dual to T and lift the duality between $\mathcal{C}$ and $\mathcal{D}$ to their corresponding Eilenberg–Moore categories. With this duality, there is a canonical correspondence between (pseudo)equational T–theories and their corresponding dual, i.e., (pseudo)coequational B–theories. Our most important examples of (pseudo)coequational B–theories are those given in Eilenberg–type correspondences, namely, "varieties of languages". All in all, we get a one–to–one correspondence between (pseudo)varieties of T–algebras and (pseudo)coequational B–theories. We will show how this concept of (pseudo)coequational B–theories coincides with the different notions of "varieties of languages" in Eilenberg–type correspondences, which bring us to our slogan, Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality. As a consequence, we can summarize Eilenberg–type correspondences in the following picture:

$$\begin{array}{c}\text{(pseudo)varieties}\\ \text{of T–algebras}\end{array} \xleftrightarrow{\text{Eilenberg–type correspondences}} \left(\begin{array}{c}\text{(pseudo)equational}\\ \text{T–theories}\end{array}\right)^{op}$$

where 'op' denotes the dual operator. This easy to understand and straightforward one–to–one correspondence gives us what we called an abstract Eilenberg–type correspondence for varieties and pseudovarieties of T–algebras, Proposition 143 and 153, respectively, from which we recover and discover particular instances of Eilenberg–type correspondences for different kinds of algebraic structures, i.e., T–algebras.

It is worth mentioning that Eilenberg–type correspondences have not been fully understood for the last forty years, which can be witnessed by the numerous published results on the subject that deal with specific kinds of algebras such as [13, 36, 70, 72, 74, 90] and categorical generalizations such as [5, 22, 89, 78] in which the direct relation between "varieties of languages" and equational theories, by using duality, is not studied or explored to find and justify the defining properties of a "variety of languages".

The contributions of the present chapter can be summarized as follows:

- To unveil Eilenberg–type correspondences and show that:

  Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality.

- To show and understand where "varieties of languages" come from, that is:

  "varieties of languages" = duals of (pseudo)equational theories.

  This fact was conjectured by the author in [78].

- To provide a general and abstract Eilenberg–type correspondence theorem that encompasses existing Eilenberg–type correspondences from the literature. Not only for (local) pseudovarieties of algebras but also for (local) varieties of algebras.

- To derive Eilenberg–type correspondences without the use of syntactic algebras.
- To show that the notion of derivatives used to define the different kinds of "varieties of languages" in Eilenberg–type correspondences is exactly the coalgebraic structure of an object, which is easily derived via duality, in most of the cases, from the notion of an algebra homomorphism.

As a running example we will work with the case of monoids, additional examples and applications will be presented in the next chapter.

6.1 Eilenberg–type correspondence for varieties of T–algebras

In this section, we state an abstract Eilenberg–type correspondence for varieties of T–algebras. We start by dualizing the categorical definition of an equational T–theory, given in Definition 102, to get that of a coequational B–theory, where B is a comonad on a category $\mathcal{C}$. We will note that particular instances of coequational B–theories have been already studied in the literature under the name of "varieties of languages" to establish Eilenberg–type correspondences, e.g., [13, Theorem 39]. Thus, if we assume that $\mathcal{D}$ and $\mathcal{C}$ are dual categories and that the comonad B is the dual of the monad T, as in Proposition 96, then, by duality, we get a one–to–one correspondence between equational T–theories and coequational B–theories. All in all, we get a one–to–one correspondence between varieties of T–algebras and coequational B–theories, which is the abstract Eilenberg–type correspondence for varieties of T–algebras, Proposition 143. The main picture can be summarized as follows:

$$\begin{tikzcd}
\text{Varieties of T–algebras} \arrow[rr, Leftrightarrow, "\text{Birkhoff's thm. Theorem 105}"'] \arrow[drr, Leftrightarrow, "\text{Eilenberg–type correspondence Proposition 143}"'] & & \text{Equational T–theories} \arrow[d, Leftrightarrow, "\text{Duality}"] \\
& & \text{Coequational B–theories}
\end{tikzcd}$$

where each arrow symbolizes a one–to–one correspondence and B is the comonad that is the dual of the monad T (Proposition 96).

We dualize the definition of an equational T–theory as follows.

Definition 142. Let $\mathcal{C}$ be a category, $\mathsf{B} = (B, \epsilon, \delta)$ a comonad on $\mathcal{C}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{C}$ and $\mathcal{C}_0$ a full subcategory of $\mathcal{C}$. A *coequational* B*–theory on* $\mathcal{C}_0$ is a family of B–coalgebra morphisms $\mathtt{M} = \{S_Y \xhookrightarrow{m_Y} B(Y)\}_{Y \in \mathcal{C}_0}$ in $\mathscr{M}$ such that for any $X, Y \in \mathcal{C}_0$ and any $g \in \mathtt{Coalg}(\mathsf{B})(\mathbf{BX}, \mathbf{BY})$ there exists $g' \in \mathtt{Coalg}(\mathsf{B})(\mathbf{S_X}, \mathbf{S_Y})$ such that the following diagram commutes:

$$\begin{tikzcd} B(X) \arrow[r, "\forall g"] & B(Y) \\ S_X \arrow[u, hook, "m_X"] \arrow[r, dashed, "g'"'] & S_Y \arrow[u, hook, "m_Y"'] \end{tikzcd}$$

Intuitively, every coalgebra $\mathbf{S_Y}$ is a B–subcoalgebra of the cofree coalgebra $\mathbf{BY} = (B(Y), \delta_Y)$, and the family $\{S_Y\}_{Y \in \mathcal{C}_0}$ is closed under any coalgebra morphism, i.e., for every $g \in \texttt{Coalg}(\mathsf{B})(\mathbf{BX}, \mathbf{BY})$, $x \in S_X$ implies $g(x) \in S_Y$. As an example of coequational B–theories we have the "varieties of languages" defined in [13, Definition 35] which we describe in a simpler way in Example 144. Now, with the previous definition, Theorem 105 and duality, we have the following.

Proposition 143 (Abstract Eilenberg–type correspondence for varieties of T–algebras)**.** *Let $\mathcal{D}$ be a complete category,* T *a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$. Assume (B1) to (B5). Let $\mathcal{C}$ be a category that is dual to $\mathcal{D}$, $\mathcal{C}_0$ the corresponding dual category of $\mathcal{D}_0$ and let* B *be the comonad on $\mathcal{C}$ that is dual to* T *which is defined as in Proposition 96. Then there is a one–to–one correspondence between varieties of* T*–algebras and coequational* B*–theories on $\mathcal{C}_0$.* □

We now proceed with the continuation of Example 112 and describe the properties that characterize the coequational B–theories that correspond to varieties of monoids. In this case, the Eilenberg–type correspondence we obtain is the one in [13, Theorem 39].

Example 144 (Example 112 continued)**.** Consider the setting of Example 112, i.e., $\mathcal{D} = \mathcal{D}_0 = \texttt{Set}$, T the free monoid monad on `Set`, $\mathscr{E}$ = surjections and $\mathscr{M}$ = injections. Then we get a one–to–one correspondence between varieties of monoids and coequational B–theories on `CABA`. The latter can be characterized (see Example 169 for a detailed proof of this fact in a more general setting) as operators $\mathscr{L}$ on `Set` such that for every $X \in \texttt{Set}$:

i) $\mathscr{L}(X) \in \texttt{CABA}$ and it is a subalgebra of the complete atomic Boolean algebra $\texttt{Set}(X^*, 2)$ of subsets of X^*, i.e., every element in $\mathscr{L}(X)$ is a language on X.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of monoids $h : Y^* \to X^*$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$. □

We finish this section by showing that the previous characterization of a coequational B–theory on `CABA` is equivalent with the –more complicated– notion of a "variety of languages" in [13, Definition 35]. This one–to–one correspondence is exactly the Eilenberg–type theorem [13, Theorem 39].

In order to prove the previous fact, we recall the concept of a "variety of languages" defined in [13].

Definition 145 ([13, Definition 35]). A *variety of languages*, is an operator $\mathcal{V}$ on $\mathtt{Set}$ such that for every $X \in \mathtt{Set}$, $\mathcal{V}(X) \subseteq \mathtt{Set}(X^*, 2)$ and it satisfies the following:

i) for every $L \in \mathcal{V}(X)$ we have that $\mathtt{coeq}(X^*/\,\mathtt{eq}\langle L\rangle) \subseteq \mathcal{V}(X)$;

ii) if $\mathtt{coeq}(X^*/C_i) \subseteq \mathcal{V}(X)$, where C_i a monoid congruence of X^*, $i \in I$, then we have that $\mathtt{coeq}(X^*/\bigcap_{i\in I} C_i) \subseteq \mathcal{V}(X)$;

iii) for every $Y \in \mathtt{Set}$, if $L \in \mathcal{V}(Y)$ and $\nu_L : Y^* \to Y^*/\,\mathtt{eq}\langle L\rangle$ denotes the quotient morphism, then for each monoid morphism $\varphi : X^* \to Y^*$ we have $\mathtt{coeq}(X^*/\ker(\nu_L \circ \varphi)) \subseteq \mathcal{V}(X)$.

We have the following coalgebraic characterization of $\mathtt{coeq}(X^*/\,\mathtt{eq}\langle L\rangle)$.

Lemma 146. *Let $X \in \mathtt{Set}$ and $L \in \mathtt{Set}(X^*, 2)$, then $\mathtt{coeq}(X^*/\,\mathtt{eq}\langle L\rangle) = \langle\langle L\rangle\rangle$, where $\langle\langle L\rangle\rangle$ is the* B*–coalgebra generated by L.*

Proof. By [13, Corollary 8] we have that the monoid $X^*/\,\mathtt{eq}\langle L\rangle$ is the syntactic monoid of L. The universal property of the syntactic monoid of L is, by duality, the property that $\mathtt{coeq}(X^*/\,\mathtt{eq}\langle L\rangle) = \langle\langle L\rangle\rangle$. This property of $\langle\langle L\rangle\rangle$ being the dual of the syntactic monoid of L was also mentioned in [42, Section 6]. □

Now, we characterize each of the $\mathcal{V}(X)$ for a variety of languages $\mathcal{V}$.

Lemma 147. *Let $\mathcal{V}$ be a variety of languages and $X \in \mathtt{Set}$, then*

$$\mathcal{V}(X) = \mathtt{coeq}\left(X^*/\bigcap_{L\in\mathcal{V}(X)} \mathtt{eq}\langle L\rangle\right).$$

Proof. ($\supseteq$): Follows from properties i) and ii) of $\mathcal{V}$ being a variety of languages.

($\subseteq$): Consider the canonical morphism of monoids $e_{L'} : X^*/\bigcap_{L\in\mathcal{V}(X)} \mathtt{eq}\langle L\rangle \to X^*/\,\mathtt{eq}\langle L'\rangle$, $L' \in \mathcal{V}(X)$. Then, by duality, i.e., applying $\mathtt{coeq}$, we get the monomorphism $m_{L'} : \langle\langle L'\rangle\rangle \to \mathtt{coeq}\left(X^*/\bigcap_{L\in\mathcal{V}(X)} \mathtt{eq}\langle L\rangle\right)$ which implies that the language L' is such that $L' \in \mathtt{coeq}\left(X^*/\bigcap_{L\in\mathcal{V}(X)} \mathtt{eq}\langle L\rangle\right)$ since $L' \in \langle\langle L'\rangle\rangle$. □

We now have, according to our notation, the following result that the syntactic monoid of a language recognizes the language.

Lemma 148. *For every $X \in \mathtt{Set}$ and every $L \in \mathtt{Set}(X^*, 2)$ we have that $L = \bigcup_{w\in L} w/\,\mathtt{eq}\langle L\rangle$, where $w/\,\mathtt{eq}\langle L\rangle$ denotes the equivalence class of w in $X^*/\,\mathtt{eq}\langle L\rangle$.*

Proof. ($\subseteq$): obvious.

($\supseteq$): Let $u \in \bigcup_{w \in L} w/\operatorname{\mathtt{eq}}\langle L\rangle$, then there exists $v \in L$ such that $(u, v) \in \operatorname{\mathtt{eq}}\langle L\rangle$. In particular, $L_u = L_v$. Now, using the fact that $v \in L$ we get the following implications:

$$v \in L \Rightarrow \epsilon \in L_v = L_u \Rightarrow \epsilon \in L_u$$

i.e., $u \in L$. □

Lemma 147 says that $\mathcal{V}(X) \in \mathtt{CABA}$ for every $X \in \mathtt{Set}$, since $\mathtt{coeq}(X^*/C) \cong \mathcal{P}(X^*/C)$ for every monoid congruence C of X^* [13, Proposition 15]. Lemma 146 together with property i) of $\mathcal{V}$ being a variety of languages imply that $\mathcal{V}(X)$ is closed under left and right derivatives. That is, every variety of languages $\mathcal{V}$ satisfies properties i) and ii) of a coequational B–theory. Now we show that $\mathcal{V}$ also satisfies property iii) of a coequational B–theory.

Lemma 149. *Let $\mathcal{V}$ be a variety of languages. Then for every $X, Y \in \mathtt{Set}$, homomorphism of monoids $h : X^* \to Y^*$ and $L \in \mathcal{V}(Y)$ we have that $L \circ h \in \mathcal{V}(X)$.*

Proof. By property iii) of $\mathcal{V}$ being a variety of languages we have the inclusion $\mathtt{coeq}(X^*/\ker(\nu_L \circ h)) \subseteq \mathcal{V}(X)$. We will show that $L \circ h \in \mathtt{coeq}(X^*/\ker(\nu_L \circ h)) \subseteq \mathcal{V}(X)$. In fact,

Claim: $L \circ h = \bigcup\{w/\ker(\nu_L \circ h) \mid w \in X^* \text{ s.t. } h(w) \in L\}$.

Let $v \in X^*$, then:

($\subseteq$): $v \in L \circ h \Rightarrow h(v) \in L \Rightarrow v \in \bigcup\{w/\ker(\nu_L \circ h) \mid w \in X^* \text{ s.t. } h(w) \in L\}$.

($\supseteq$): Assume $v \in \bigcup\{w/\ker(\nu_L \circ h) \mid w \in X^* \text{ s.t. } h(w) \in L\}$, i.e., there exists $u \in X^*$ with $h(u) \in L$ such that $(v, u) \in \ker(\nu_L \circ h)$. Now, we have

$$(v, u) \in \ker(\nu_L \circ h) \Rightarrow (h(v), h(u)) \in \ker(\nu_L) = \operatorname{\mathtt{eq}}\langle L\rangle \Rightarrow h(v) \in L$$

where the last implication follows from Lemma 148 since $h(u) \in L$. Finally, from $h(u) \in L$ we get $u \in L \circ h$. This finishes the proof of the claim.

From the claim we have that $L \circ h \in \mathtt{coeq}(X^*/\ker(\nu_L \circ h)) \subseteq \mathcal{V}(X)$. □

Until now we proved the following.

Proposition 150. *Let $\mathcal{V}$ be a variety of languages. Then $\mathcal{V}$ is a coequational* B*–theory.* □

Now we prove the converse.

Proposition 151. *Let $\mathscr{L}$ be a coequational* B*–theory. Then $\mathscr{L}$ is a variety of languages.*

Proof. We have to prove that $\mathscr{L}$ satisfies properties i), ii) and iii) that define a variety of languages. In fact, let $X \in \mathtt{Set}$, then:

i) Properties i) and ii) of $\mathscr{L}$ being a coequational B–theory say that $\mathscr{L}(X)$ is a B–subcoalgebra of $\mathtt{Set}(X^*,2)$. In particular, for every $L \in \mathscr{L}(X)$ we have $\mathtt{coeq}(X^*/\,\mathtt{eq}\langle L\rangle) = \langle\!\langle L\rangle\!\rangle \subseteq \mathscr{L}(X)$.

ii) To prove property ii) we show that for a monoid congruence C_i of X^*, $i \in I$, the B–coalgebra $\mathtt{coeq}(X^*/\bigcap_{i\in I} C_i)$ is the B–subcoalgebra of $\mathtt{Set}(X^*,2)$ generated by the family $\{\mathtt{coeq}(X^*/C_i)\}_{i\in I}$. We show this by duality, i.e., in the category of monoids. We have the following setting:

$$\begin{tikzcd} & X^* \arrow[dl, "\nu_j"'] \arrow[d, "\nu"] \arrow[dr, "e_\nu"] & \\ X^*/C_j & P \arrow[l, "\pi_j"] & X^*/\bigcap_{i\in I} C_i \arrow[l, hook', "m_\nu"] \end{tikzcd}$$

where:

- $\nu_j : X^* \to X^*/C_j$ is the canonical homomorphism, $j \in I$,
- P is the product $P = \prod_{i\in I} X^*/C_i$ with projections $\pi_j : P \to X^*/C_j$, $j \in I$,
- ν is obtained from ν_j, $j \in I$, by the universal property of P, and
- $\nu = m_\nu \circ e_\nu$ is the factorization of ν, i.e., $\ker(\nu) = \bigcap_{i\in I} C_i$.

Now we prove, by duality, that $\mathtt{coeq}(X^*/\bigcap_{i\in I} C_i)$ is the least B–subcoalgebra of $\mathtt{Set}(X^*,2)$ containing each of $\mathtt{coeq}(X^*/C_i)$. Let $e : X^* \to X^*/C$ be an epimorphism of monoids such that each ν_j factors through e, $j \in I$. That is, there exists $g_j : X^*/C \to X^*/C_j$ such that $\nu_j = g_j \circ e$, $j \in I$. Therefore, $C \subseteq C_j$, $j \in I$, and hence $C \subseteq \bigcap_{i\in I} C_i$, which means that there exists $g : X^*/C \to X^*/\bigcap_{i\in I} C_i$ such that $e_\nu = g \circ e$.

Now, $\mathscr{L}$ satisfying property ii) of a variety of languages follows from the observation above. In fact, if $\mathscr{L}(X)$ contains $\mathtt{coeq}(X^*/C_i)$, $i \in I$, then, by using the fact that $\mathscr{L}(X)$ is a B–subcoalgebra of $\mathtt{Set}(X^*,2)$, it contains the least B–subcoalgebra of $\mathtt{Set}(X^*,2)$ containing each of $\mathtt{coeq}(X^*/C_i)$, $i \in I$, which is $\mathtt{coeq}(X^*/\bigcap_{i\in I} C_i)$.

iii) Let $Y \in \mathtt{Set}$, $L \in \mathscr{L}(Y)$ and $\nu_L : Y^* \to Y^*/\,\mathtt{eq}\langle L\rangle$ be the quotient morphism. Let $\varphi : X^* \to Y^*$ be a monoid morphism. We have to show that $\mathtt{coeq}(X^*/\ker(\nu_L \circ \varphi)) \subseteq \mathscr{L}(X)$. In fact, let $L' \in \mathtt{coeq}(X^*/\ker(\nu_L \circ \varphi))$, i.e., L' is of the form $L' = \bigcup_{w\in W} w/\ker(\nu_L \circ \varphi)$ for some $W \subseteq X^*$. Define L'' as $L'' = \bigcup_{w\in W} \varphi(w)/\ker(\nu_L) = \bigcup_{w\in W} \varphi(w)/\,\mathtt{eq}\langle L\rangle$. Then we have that $L'' \in \mathtt{coeq}(Y^*/\,\mathtt{eq}\langle L\rangle)$ which by i) implies that $L'' \in \mathscr{L}(Y)$, since $\mathtt{coeq}(Y^*/\,\mathtt{eq}\langle L\rangle) \subseteq \mathscr{L}(Y)$. Since $\mathscr{L}$ is a coequational B–theory then $L'' \circ \varphi \in \mathscr{L}(X)$. To finish the proof we prove the following:

Claim: $L' = L'' \circ \varphi$.

Let $u \in X^*$, then:

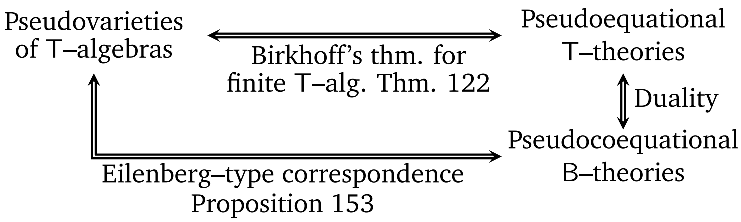
($\subseteq$): Assume that $u \in L'$. Then there exists $w \in W$ such that $(u, w) \in \ker(\nu_L \circ \varphi)$. This implies that $(\varphi(u), \varphi(w)) \in \ker(\nu_L) = \mathtt{eq}\langle L \rangle$ with $w \in W$, i.e., $\varphi(u) \in L''$ which means that $u \in L'' \circ \varphi$.

($\supseteq$): Assume that $u \in L'' \circ \varphi$, i.e., $\varphi(u) \in L''$. Then there exists $w \in W$ such that $(\varphi(u), \varphi(w)) \in \ker(\nu_L)$. This implies that $(u, w) \in \ker(\nu_L \circ \varphi)$ with $w \in W$, i.e., $u \in L'$.

□

6.2 Eilenberg–type correspondence for pseudovarieties of T–algebras

Similar to the previous section, the purpose of this section is to derive Eilenberg–type correspondences for pseudovarieties of T–algebras. This is summarized in the following picture:



As we saw in Example 130, we can derive Eilenberg–type correspondences for pseudovarieties of T–algebras from Birkhoff's theorem for finite T–algebras, Theorem 122. Eilenberg–type correspondences for pseudovariaties of T–algebras are exactly one–to–one correspondences between pseudovarieties of T–algebras and duals of pseudoequational T–theories. By dualizing the definition of a pseudoequational T–theory we get the following.

Definition 152. Let $\mathcal{C}$ be a concrete category such that its forgetful functor preserves monos, $\mathsf{B} = (B, \epsilon, \delta)$ a comonad on $\mathcal{C}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{C}$ and $\mathcal{C}_0$ a full subcategory of $\mathcal{C}$. Assume (B_f1) and that B preserves the morphisms in $\mathscr{M}$. A *pseudocoequational* B*–theory on* $\mathcal{C}_0$ is an operator $\mathtt{R}$ on $\mathcal{C}_0$ such that for every $X \in \mathcal{C}_0$, $\mathtt{R}(X)$ is a nonempty collection of B–coalgebra morphisms in $\mathscr{M}$ with codomain $\mathbf{BX}$ and finite domain and:

i) For every finite set I and $f_i \in \mathtt{R}(X)$, $i \in I$, there exists $f \in \mathtt{R}(X)$ such that every f_i factors through f, $i \in I$.

ii) For every $m \in \mathtt{R}(X)$ with domain $\mathbf{A}$ and every B–coalgebra morphism $m' \in \mathscr{M}$ with codomain $\mathbf{A}$ we have that $m \circ m' \in \mathtt{R}(X)$.

iii) For every $Y \in \mathcal{C}_0$, $f \in \mathtt{R}(X)$ and $h \in \mathtt{Coalg}(\mathsf{B})(\mathbf{BX}, \mathbf{BY})$ we have that $m_{h \circ f} \in \mathtt{R}(Y)$ where $h \circ f = m_{h \circ f} \circ e_{h \circ f}$ is the factorization of $h \circ f$ in $\mathtt{Coalg}(\mathsf{B})$ by using the factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{C}$, which is lifted to $\mathtt{Coalg}(\mathsf{B})$ by using the fact that B preserves the morphisms in $\mathscr{M}$ and the dual of Lemma 100.

With the previous definition, Theorem 122 and duality, we have the following:

Proposition 153 (Abstract Eilenberg–type correspondence for pseudovarieties of T–algebras)**.** *Let $\mathcal{D}$ be a complete concrete category such that its forgetful functor preserves epis, monos and products, T a monad on $\mathcal{D}$, $\mathcal{D}_0$ a full subcategory of $\mathcal{D}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$. Assume (B_f1) to (B_f4). Let $\mathcal{C}$ be a category that is dual to $\mathcal{D}$, let $\mathcal{C}_0$ be dual of $\mathcal{D}_0$ and B be the comonad on $\mathcal{C}$ that is dual to the monad T on $\mathcal{D}$ which is defined as in Proposition 96. Then there is a one–to–one correspondence between pseudovarieties of* T*–algebras and pseudocoequational* B*–theories on $\mathcal{C}_0$.* □

We now explain and justify how the concept of a variety of languages in Eilenberg's variety theorem [36, Theorem 34] corresponds to the dual of a pseudoequational T–theory in the setting of Example 130. For this purpose, we consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets, T the free monoid monad on $\mathtt{Set}$, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. Then, by Proposition 153 we have a one–to–one correspondence betweeen pseudovarieties of monoids and pseudoequational T–theories on $\mathcal{D}_0$.

We now show that pseudoequational T–theories on $\mathcal{D}_0$ correspond exactly to varieties of languages as defined in [36], which we call pseudovarieties of languages to avoid confusion with the concept of a variety of languages defined in Example 144.

Definition 154. A *pseudovariety of languages* is an operator $\mathscr{L}$ on finite sets such that for every finite set Σ we have:

i) $\mathscr{L}(\Sigma)$ is a subalgebra of the Boolean algebra $\mathrm{Rec}(\Sigma)$ of recognizable languages on Σ,

ii) $\mathscr{L}(\Sigma)$ is closed under left and right derivatives. That is, ${}_aL, L_a \in \mathscr{L}(\Sigma)$ for every $L \in \mathscr{L}(\Sigma)$ and $a \in \Sigma$, and

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every alphabet Γ, homomorphism of monoids $h : \Gamma^* \to \Sigma^*$ and $L \in \mathscr{L}(\Sigma)$, we have that $L \circ h \in \mathscr{L}(\Gamma)$.

We have that pseudoequational T–theories on $\mathcal{D}_0$ and pseudovarieties of languages are in one–to–one correspondence.

Lemma 155. *Consider the setting $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 =$ finite sets,* T *the free monoid monad on* Set*, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. Then there is a one–to–one correspondence between pseudoequational* T*–theories on $\mathcal{D}_0$ and pseudovarieties of languages.*

Proof. Let $\mathtt{P}$ be a pseudoequational T–theory and let $\mathscr{L}$ be a pseudovariety of languages. Then:

a) Define the operator $\mathscr{L}_{\mathtt{P}}$ on $\mathtt{Set}_f$ as $\mathscr{L}_{\mathtt{P}}(X) := \bigcup_{e\in\mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e,2))$. We claim that $\mathscr{L}_{\mathtt{P}}$ is a pseudovariety of languages. In fact, as the family $\mathtt{P}(X)$ is directed in the sense of Definition 119 i), then the union $\bigcup_{e\in\mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e,2)) \subseteq \mathtt{Set}(T(X),2)$ is a directed union of finite objects in $\mathtt{CABA}$, which is a Boolean subalgebra of $\mathtt{Set}(T(X),2)$. As each $e \in \mathtt{P}(X)$ has as codomain a finite monoid then $\mathrm{Im}(\mathtt{Set}(e,2))$ is a subset of $\mathrm{Rec}(X)$ which is closed under left and right derivatives. The previous argument shows that $\mathscr{L}_{\mathtt{P}}$ satisfies properties i) and ii) above. Now, closure under morphic preimages follows from property iii) in Definition 119. Therefore, $\mathscr{L}_{\mathtt{P}}$ is a pseudovariety of languages.

b) Define the operator $\mathtt{P}_{\mathscr{L}}$ on $\mathtt{Set}_f$ such that $\mathtt{P}_{\mathscr{L}}(X)$ is the collection of all T–algebra morphisms $e \in \mathscr{E}$ with domain $\mathbf{TX}$ and finite codomain such that $\mathrm{Im}(\mathtt{Set}(e,2)) \subseteq \mathscr{L}(X)$. We claim that $\mathtt{P}_{\mathscr{L}}$ is a pseudoequational T–theory. In fact, we have that $\mathtt{P}_{\mathscr{L}}(X)$ is nonempty since $e : \mathbf{TX} \twoheadrightarrow \mathbf{1} \in \mathtt{P}_{\mathscr{L}}(X)$, where $\mathbf{1}$ is the one–element T–algebra. By definition, we have that $\mathtt{P}_{\mathscr{L}}(X)$ satisfies property ii) in Definition 119, and, it also satisfies property iii) in Definition 119 since $\mathscr{L}$ is closed under morphic preimages. Now, consider a family $\{T(X)\overset{e_i}{\twoheadrightarrow}A_i\}_{i\in I}$ in $\mathtt{P}_{\mathscr{L}}(X)$ with I finite such that $\mathrm{Im}(\mathtt{Set}(e_i,2)) \subseteq \mathscr{L}(X)$, we need to find a morphism $e \in \mathtt{P}_{\mathscr{L}}(X)$ such that every e_i factors through e. In fact, let $\mathbf{A}$ be the product of $\prod_{i\in I}\mathbf{A}_i$ with projections $\pi_i : A \to A_i$, then, by the universal property of $\mathbf{A}$ there exists a T–algebra morphism $f : T(X) \to A$ such that $\pi_i \circ f = e_i$, for every $i \in I$. Let $f = m_f \circ e_f$ be the factorization of f in $\mathtt{Alg}(\mathsf{T})$. We claim that $e = e_f$ is a morphism in $\mathtt{P}_{\mathscr{L}}(X)$ such that every e_i factors through e. Clearly, from the construction above, each e_i factors through $e = e_f$. Now, let us prove that $\mathrm{Im}(\mathtt{Set}(e,2)) \subseteq \mathscr{L}(X)$. In fact, let $\mathbf{S}$ be the codomain of $e = e_f$ and let $g \in \mathtt{Set}(S,2)$. We have to prove that $g\circ e \in \mathscr{L}(X)$ which follows from the following straightforward identity:

$$g \circ e = \bigcup_{s\in g}\left(\bigcap_{i\in I} h_{i,s}\circ e_i\right)$$

where $h_{i,s} \in \mathtt{Set}(A_i,2)$ is the set $\{\pi_i(m_f(s))\}$ (i.e., we express the subset g of S as the union of its points and each point $s \in S$ is represented as $\bigcap_{i\in I} h_{i,s}\circ\pi_i\circ m_f$). Now, for every $s \in S$ and $i \in I$ the composition $h_{i,s}\circ e_i$ belongs to $\mathscr{L}(X)$ since $h_{i,s}\circ e_i \in \mathrm{Im}(\mathtt{Set}(e_i,2)) \subseteq \mathscr{L}(X)$. As S and I are finite then $g\circ e \in \mathscr{L}(X)$ because $\mathscr{L}(X)$ is a Boolean algebra.

c) We have that $\mathtt{P} = \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}$. In fact, for every $X \in \mathtt{Set}_f$ the inclusion $\mathtt{P}(X) \subseteq \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X)$ is obvious. Now, to prove that $\mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X) \subseteq \mathtt{P}(X)$, let $e' \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX},\mathbf{A}) \cap \mathscr{E}$ with finite codomain such that $e' \in \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X)$, i.e., we have the inclusion $\mathrm{Im}(\mathtt{Set}(e',2)) \subseteq \bigcup_{e\in\mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e,2))$. Then the previous inclusion means that for every $f \in \mathtt{Set}(A,2)$ there exists $e_f \in \mathtt{P}(X)$ and g_f such that $f\circ e' = g_f \circ e_f$. As $\{e_f \mid f \in \mathtt{Set}(A,2)\}$ is finite, then there exists $e \in \mathtt{P}(X)$ such that each e_f factors through e. We will prove that e' factors through $e \in \mathtt{P}(X)$ which will imply that $e' \in \mathtt{P}(X)$, since $\mathtt{P}$ is a pseudoequational T–theory. It is enough to show that $\ker(e) \subseteq \ker(e')$. In fact, assume that $(u,v) \in \ker(e)$ and define $f' \in \mathtt{Set}(A,2)$ as $f'(x) = 1$ iff $x = e'(u)$. Then, as $e_{f'}$ factors through e we have that $\ker(e) \subseteq \ker(e_{f'})$ which implies $(u,v) \in \ker(e_{f'})$. But $\ker(e_{f'}) \subseteq \ker(g_{f'} \circ e_{f'}) = \ker(f' \circ e')$, which implies that $(u,v) \in \ker(f' \circ e')$, i.e., $1 = f'(e'(u)) = f'(e'(v))$, but the later equality means that $e'(u) = e'(v)$ by definition of f', i.e., $(u,v) \in \ker(e')$ as desired.

d) We have that $\mathscr{L} = \mathscr{L}_{\mathtt{P}_{\mathscr{L}}}$. In fact, for every $X \in \mathtt{Set}_f$ the inclusion $\mathscr{L}_{\mathtt{P}_{\mathscr{L}}}(X) \subseteq \mathscr{L}(X)$ is obvious. Now, to prove $\mathscr{L}(X) \subseteq \mathscr{L}_{\mathtt{P}_{\mathscr{L}}}(X)$ we need to find for every $L \in \mathscr{L}(X)$ a surjective homomorphism $e : T(X) \to A$ with $\mathbf{A}$ finite such that $L \in \mathrm{Im}(\mathtt{Set}(e,2)) \subseteq \mathscr{L}(X)$. In fact, for $L \in \mathscr{L}(X)$ let $e' : T(X) \to B$ be a homomorphism with $\mathbf{B}$ finite and $g \in \mathtt{Set}(B,2)$ such that $L = g \circ e'$, this can be done by property i) above. Let $\langle\langle L \rangle\rangle$ be the subset of $\mathtt{Set}(T(X),2)$ obtained from $\{L\}$ which is closed under Boolean combinations and left and right derivatives. We show that $\langle\langle L \rangle\rangle \in \mathtt{Coalg}_f(\mathsf{B})$, that is, we show that $\langle\langle L \rangle\rangle$ is a finite object in $\mathtt{CABA}$ that is closed under left and right derivatives. In fact, $\mathrm{Im}(\mathtt{Set}(e',2)) \in \mathtt{Coalg}_f(\mathsf{B})$ is such that $\langle\langle L \rangle\rangle \subseteq \mathrm{Im}(\mathtt{Set}(e',2))$, which implies that $\langle\langle L \rangle\rangle$ is a finite Boolean algebra, i.e., an object in $\mathtt{Coalg}_f(\mathsf{B})$. By construction of $\langle\langle L \rangle\rangle$ we have that $L \in \langle\langle L \rangle\rangle \subseteq \mathscr{L}(X)$ since $\mathscr{L}$ satisfies properties i) and ii) above. Now, let $i \in \mathtt{Coalg}(\mathsf{B})(\langle\langle L \rangle\rangle, \mathtt{Set}(T(X),2))$ be the inclusion morphism, then by duality we have that the dual morphism e in $\mathtt{Alg}(\mathsf{T})$ of i is such that $L \in \mathrm{Im}(\mathtt{Set}(e,2)) \subseteq \mathscr{L}(X)$ (in fact, $\mathrm{Im}(\mathtt{Set}(e,2)) = \langle\langle L \rangle\rangle$). Note that the codomain of e is finite since it is an $\mathscr{E}$–quotient of $\mathbf{B}$ which is also finite.

□

Remark. Note that, for every language $L \in \mathtt{Set}(T(X),2)$, the object $\langle\langle L \rangle\rangle$ in d) above is the B–subcoalgebra of $\mathtt{Set}(T(X),2)$ generated by L, which implies, by duality, that its dual is the syntactic algebra $\mathbf{S}_L$ of L. Additionally, by using duality and the construction of $\langle\langle L \rangle\rangle$, we have that every language in $\mathrm{Im}(\mathtt{Set}(e,2))$ (i.e., recognized by the syntactic algebra of L) is a Boolean combination of derivatives of L, where e is the dual of the inclusion $i \in \mathtt{Coalg}(\mathsf{B})(\langle\langle L \rangle\rangle, \mathtt{Set}(T(X),2))$. □

Now, by combining Proposition 153 and the previous lemma we obtain the celebrated Eilenberg's variety theorem.

Corollary 156 (Eilenberg's variety theorem [36, Theorem 34])**.** *There is a one–to–one correspondence between pseudovarieties of monoids and pseudovarieties of languages.* □

6.3 Local Eilenberg–type correspondences

In this section, we provide abstract versions of local Eilenberg–type correspondences for local varieties of T–algebras and local pseudovarieties of T–algebras.

Local Eilenberg–type correspondences for pseudovarieties of algebras have been studied in [1, 43]. The main idea of local Eilenberg–type correspondences is to work with a fixed alphabet, which in our notation reduces to consider the case in which the category $\mathcal{D}_0$ has only one object, say X. Our local Eilenberg–type correspondences will follow from their corresponding local Birkhoff's theorem and duality. We start by dualizing the notion of a local equational T–theory.

Definition 157. Let $\mathcal{C}$ be a category, B a comonad on $\mathcal{C}$, $Y \in \mathcal{C}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{C}$. A *local coequational* B*–theory on* Y is a B–coalgebra morphism $S_Y \xhookrightarrow{m_Y} B(Y)$ in $\mathscr{M}$ such that for any $g \in \mathtt{Coalg}(\mathsf{B})(\mathbf{BY}, \mathbf{BY})$ there exists $g' \in \mathtt{Coalg}(\mathsf{B})(\mathbf{S_Y}, \mathbf{S_Y})$ such that the following diagram commutes:

$$\begin{tikzcd} B(Y) \arrow[r, "\forall g"] & B(Y) \\ S_Y \arrow[u, hook, "m_Y"] \arrow[r, dashed, "g'"'] & S_Y \arrow[u, hook, "m_Y"'] \end{tikzcd}$$

With the previous definition, Theorem 136 and duality, we have the following.

Proposition 158 (Abstract Eilenberg–type correspondence for varieties of X–generated T–algebras)**.** *Let $\mathcal{D}$ be a complete category,* T *a monad on $\mathcal{D}$, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$ and $X \in \mathcal{D}$. Assume (b1) to (b4). Let $\mathcal{C}$ be a category that is dual to $\mathcal{D}$, Y the corresponding dual object of X and let* B *be the comonad on $\mathcal{C}$ that is dual to* T *which is defined as in Proposition 96. Then there is a one–to–one correspondence between local varieties of X–generated* T*–algebras and local coequational* B*–theories on Y.* □

For the case of varieties of X–generated monoids we get the following Eilenberg–type correspondence.

Example 159 (cf. [43])**.** Let $\mathcal{D} = \mathtt{Set}$, T be the free monoid monad on $\mathtt{Set}$, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. Then, by fixing an object $X \in \mathcal{D}$, we get a one–to–one correspondence between varieties of X–generated monoids and subalgebras $\mathbf{S} \in \mathtt{CABA}$ of the complete atomic Boolean algebra $\mathtt{Set}(X^*, 2)$ such that:

i) S is closed under left and right derivatives. That is, for every $L \in S$ and $x \in X$, ${}_xL, L_x \in S$.

ii) S is closed under morphic preimages. That is, for every homomorphism of monoids $h : X^* \to X^*$ and $L \in S$, we have that $L \circ h \in S$. □

Now, we do a similar work for the case of finite X–generated T–algebras. We dualize the concept of a local pseudoequational T–theory as follows.

Definition 160. Let $\mathcal{C}$ be a concrete category such that its forgetful functor preserves monos, B a comonad on $\mathcal{C}$, $Y \in \mathcal{C}$ and $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{C}$.

Assume (b_f1) and that B preserves the morphisms in $\mathscr{M}$. A *local pseudocoequational* B*–theory on* Y is a nonempty collection $\mathtt{R}_Y$ of B–coalgebra morphisms in $\mathscr{M}$ with codomain $\mathbf{BY}$ and finite domain such that:

i) For every finite set I and $f_i \in \mathtt{R}_Y$, $i \in I$, there exists $f \in \mathtt{R}_Y$ such that f_i factors through f, $i \in I$.

ii) For every $m \in \mathtt{R}_Y$ with domain $\mathbf{A}$ and every B–coalgebra morphism $m' \in \mathscr{M}$ with codomain $\mathbf{A}$ we have that $m \circ m' \in \mathtt{R}_Y$.

iii) For every $f \in \mathtt{R}_Y$ and $h \in \mathtt{Coalg}(\mathsf{T})(\mathbf{BY}, \mathbf{BY})$ we have that $m_{h\circ f} \in \mathtt{R}_Y$ where $h \circ f = m_{h\circ f} \circ e_{h\circ f}$ is the factorization of $h \circ f$.

With the previous definition, Theorem 140 and duality, we have the following.

Proposition 161 (Abstract Eilenberg–type correspondence for pseudovarieties of X–generated T–algebras)**.** *Let $\mathcal{D}$ be a concrete complete category such that its forgetful functor preserves epis, monos and products,* T *a monad on $\mathcal{D}$,* $(\mathscr{E}, \mathscr{M})$ *a factorization system on $\mathcal{D}$ and $X \in \mathcal{D}$. Assume (*b_f1*) to (*b_f3*). Let $\mathcal{C}$ be a category that is dual to $\mathcal{D}$, Y the corresponding dual object of X and let* B *be the comonad on $\mathcal{C}$ that is dual to* T *which is defined as in Proposition 96. Then there is a one–to–one correspondence between local pseudovarieties of X–generated* T*–algebras and local pseudocoequational* B*–theories on Y.* □

For the case of pseudovarieties of X–generated monoids we get the following Eilenberg–type correspondence.

Example 162 (cf. [43])**.** Let $\mathcal{D} = \mathtt{Set}$, T be the free monoid monad on $\mathtt{Set}$, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. Then, by fixing an object $X \in \mathtt{Set}_f$, we get a one–to–one correspondence between pseudovarieties of X–generated monoids and Boolean algebras $\mathbf{S}$ that are subalgebras of the complete atomic Boolean algebra $\mathtt{Set}(X^*, 2)$ such that:

i) Every element in S is a recognizable language on X.

ii) S is closed under left and right derivatives. That is, for every $L \in S$ and $x \in X$, ${}_xL, L_x \in S$.

iii) S is closed under morphic preimages. That is, for every homomorphism of monoids $h : X^* \to X^*$ and $L \in S$, we have that $L \circ h \in S$. □

6.4 Syntactic algebras

In this section, we study the concept of syntactic algebras, a concept which is usually defined (but not necessary) when studying Eilenberg–type correspondences. Syntactic monoids played a fundamental role in the proof for Eilenberg's variety theorem [36, Theorem VII.3.4]. In fact, given a variety of languages $\mathscr{L}$, the pseudovariety of monoids that corresponds to $\mathscr{L}$ is generated by the syntactic monoids of languages in $\mathscr{L}$, and, given a pseudovariety of monoids K, the variety of languages that corresponds to K is given by the languages whose syntactic monoid is in K. This kind of technique was also used in order to prove Eilenberg–type correspondences for other kinds of algebraic structures [74, 72, 70, 90].

As we showed in the previous sections, syntactic algebras are not necessary in order to establish Eilenberg–type correspondences. Nevertheless, syntactic algebras have their own interest in language theory and categorical approaches such as [22, 89, 30, 78] have considered the same idea of using syntactic algebras in order to establish Eilenberg–type correspondences. In this section, we define the general concept of a syntactic algebra and some of its main properties. The work in this section is based on [78]. We start by defining the abstract concept of a T–language.

Definition 163. Let $\mathcal{D}$ be a category, T a monad on $\mathcal{D}$ and C an object in $\mathcal{D}$. A T*–language over* $X \in \mathcal{D}$ *with colours in* C is a morphism $L \in \mathcal{D}(T(X), C)$.

As examples of T–languages we have the following.

Example 164. Classical languages correspond to the setting $\mathcal{D} = \mathtt{Set}$, T the free monoid monad and $C = 2$. We can also make other choices of C such as $C = [0, 1]$, the closed interval from zero to one, to obtain fuzzy languages as in [26]. In general, a canonical choice of the object C is such that there is a category $\mathcal{C}$ that is dual to $\mathcal{D}$ and the contravariant functor $\mathcal{D}(\_, C) : \mathcal{D} \rtimes\!\!\longrightarrow \mathcal{C}$ is part of the duality (for the case of classical languages we have the functor $\mathtt{Set}(\_, 2) : \mathtt{Set} \rtimes\!\!\longrightarrow \mathtt{CABA}$ which is part of the duality between $\mathtt{Set}$ and $\mathtt{CABA}$) $\square$

Now, we proceed to define the concepts of syntactic algebra and syntactic morphism of a given language.

Definition 165 (Syntactic algebra and syntactic morphism)**.** Let $\mathcal{D}$ be a category, $(\mathscr{E}, \mathscr{M})$ be a factorization system on $\mathcal{D}$, T a monad on $\mathcal{D}$ and C an object in $\mathcal{D}$. Let $L \in \mathcal{D}(T(X), C)$ be a T–language over $X \in \mathcal{D}$ with colours in C. The *syntactic algebra of* L is an algebra $\mathbf{R}_L \in \mathtt{Alg}(\mathsf{T})$ together with a morphism $e_L \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{R}_L) \cap \mathscr{E}$, called its *syntactic morphism* such that:

i) L is recognized by $\mathbf{R}_L$ through e_L. That is, there exists $g \in \mathcal{D}(R_L, C)$ such that $g \circ e_L = L$.

ii) For every $\mathbf{A} \in \mathtt{Alg}(\mathsf{L})$ and $e \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A}) \cap \mathscr{E}$ such that $\mathbf{A}$ recognizes L through e, there exists a necessarily unique $g \in \mathtt{Alg}(\mathsf{T})(\mathbf{A}, \mathbf{R}_L)$ such that $e_L = g \circ e$.

Note that syntactic algebras and syntactic morphisms, when they exist, are unique up to isomorphism. The following settings for syntactic algebras have been studied in the literature.

Example 166. The following are some settings in which syntactic algebras have been studied:

i) The case $\mathcal{D} = \mathtt{Set}$, $\mathscr{E}$ = surjections, $\mathscr{M}$ = injections, T the free monoid monad and $C = 2$. In this case we get the notion of syntactic algebra for classical languages [36, Chapter VII].

ii) The case $\mathcal{D} = \mathtt{Vec}_{\mathbb{K}}$, $\mathscr{E}$ = surjective linear maps, $\mathscr{M}$ = injective linear maps, $C = \mathbb{K}$ and T the monad such that $T(V(X)) = V(X^*)$, where $X \in \mathtt{Set}$ and $V(X)$ denotes the vector space in $\mathtt{Vec}_{\mathbb{K}}$ with basis X. In this case we get syntactic algebras for power series [74].

iii) The case $\mathcal{D} = \mathtt{Poset}$, $\mathscr{E}$ = surjections, $\mathscr{M}$ = embeddings, T the free monoid monad and $C = 2$, the two–element chain. In this case we get the notion of syntactic algebra for classical languages recognized by ordered monoids [70]. □

Syntactic algebras, as in [36, 74, 70, 72, 90], have been constructed as a quotient of $\mathbf{TX}$ by a certain equivalence class (congruence) which is defined in terms of L. Categorical approaches such as [22, 89] have generalized such constructions by considering polynomials, in the case of [22], and unary representations, in the case of [89]. Here we study the abstract construction of syntactic algebras by using pushouts [78]. It is worth mentioning that syntactic algebras do not always exist, see, e.g., [22, Example 2].

We will prove that the defining properties of a syntactic algebra are exactly the ones of being a wide pushout of a special family. In order to do this, we fix a category $\mathcal{D}$, an object $X \in \mathcal{D}$, T a monad on $\mathcal{D}$ and a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$. We will use the following assumptions:

(S1) $\mathcal{D}$ has generalized pushouts and T preserves weak generalized pushouts whose arrows are all in $\mathscr{E}$.

(S2) The factorization system is proper and T preserves morphisms in $\mathscr{E}$.

(S3) There is, up to isomorphism, only a set of T–algebra morphism in $\mathscr{E}$ with domain $\mathbf{TX}$.

We will use the following fact about factorization systems (see, e.g., [4, 14.15 Proposition]).

Lemma 167. *Let $\mathcal{D}$ be a category and $(\mathscr{E}, \mathscr{M})$ be a factorization system on $\mathcal{D}$ such that every morphism in $\mathscr{M}$ is mono. Then $\mathscr{E}$ is closed under the formation of generalized pushouts.* □

We have the following lemma that gives sufficient conditions for existence of syntactic algebras.

Lemma 168. *Let $\mathcal{D}$ be a category, $(\mathscr{E}, \mathscr{M})$ a factorization system on $\mathcal{D}$, let $X, C \in \mathcal{D}$ and T a monad on $\mathcal{D}$. Assume (S1), (S2) and (S3). Let $L \in \mathcal{D}(T(X), C)$ be a T–language over X with colours in C. Then the syntactic algebra $\mathbf{R}_L$ exists.*

Proof. Consider the collection $\{T(X) \overset{e_i}{\twoheadrightarrow} A_i\}_{i\in I}$, where $e_i \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{A}_i) \cap \mathscr{E}$, up to isomorphism, that are factors of L, i.e., each of them recognizes L. The previous collection is a set by (S3) and it is nonempty since $id_{\mathbf{TX}} \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{TX})$ is one of them. Let the codomain of e_i be $\mathbf{A}_i = (A_i, a_i) \in \mathtt{Alg}(\mathsf{T})$. Let $\{A_i \overset{q_i}{\twoheadrightarrow} Q\}_{i\in I}$ be the generalized pushout of $\{T(X) \overset{e_i}{\twoheadrightarrow} A_i\}_{i\in I}$ in $\mathcal{D}$. By Lemma 167, every $q_i \in \mathscr{E}$ since every $e_i \in \mathscr{E}$. As T preserves weak generalized pushouts then $\{T(A_i) \xrightarrow{T(q_i)} T(Q)\}_{i\in I}$ is a weak generalized pushout of the family $\{TT(X) \xrightarrow{T(e_i)} T(A_i)\}_{i\in I}$. Now, since the family $\{T(A_i) \xrightarrow{q_i \circ a_i} Q\}_{i\in I}$ is such that $q_i \circ a_i \circ T(e_i) = q_j \circ a_j \circ T(e_j)$, $i, j \in I$, there exists $\alpha_Q \in \mathcal{D}(T(Q), Q)$ such that $\alpha_Q \circ T(q_i) = q_i \circ a_i$. That is we have the following situation:

$$\begin{tikzcd}
 & A_i \arrow[dr, two heads, "q_i"] & \\
T(X) \arrow[ur, two heads, "e_i"] \arrow[dr, two heads, "e_j"'] & \vdots & Q \\
 & A_j \arrow[ur, two heads, "q_j"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & T(A_i) \arrow[dr, "T(q_i)"] \arrow[drr, bend left, "q_i \circ a_i"] \arrow[rr, phantom, "\dagger"] & & \\
TT(X) \arrow[ur, "T(e_i)"] \arrow[dr, "T(e_j)"'] & \vdots & T(Q) \arrow[r, dashed, "\alpha_Q"] & Q \\
 & T(A_j) \arrow[ur, "T(q_j)"'] \arrow[urr, bend right, "q_j \circ a_j"'] & &
\end{tikzcd}$$

We prove that $\mathbf{Q} = (Q, \alpha_Q) \in \mathtt{Alg}(\mathsf{T})$. In fact, from the commutative diagram:

$$\begin{tikzcd}
T(Q) \arrow[dd, "\alpha_Q"'] & Q \arrow[l, "\eta_Q"'] \arrow[dr, phantom, "\text{nat. } \eta"] & & A_i \arrow[ll, two heads, "q_i"'] \arrow[dl, "\eta_{A_i}"] \arrow[ddll, bend left, "id_{A_i}"] \\
 & \dagger & T(A_i) \arrow[ull, "Tq_i"'] \arrow[dl, "a_i"] & \\
Q & A_i \arrow[l, two heads, "q_i"] & &
\end{tikzcd}$$

We conclude that $\alpha_Q \circ \eta_Q = id_Q$ since q_i is epi. Now, from the commutative diagram:

$$\begin{tikzcd}
T^2(Q) \arrow[r, "\mu_Q"] \arrow[dddr, bend right=40, "T(\alpha_Q)"'] \arrow[dr, phantom, "\text{nat. } \mu"] & T(Q) \arrow[r, "\alpha_Q"] \arrow[dr, phantom, "\dagger"] & Q \\
T^2(A_i) \arrow[u, "T^2(q_i)"] \arrow[r, "\mu_{A_i}"'] \arrow[dr, "T(\alpha_i)"] \arrow[d, phantom, "T\dagger"] & T(A_i) \arrow[u, "T(q_i)"'] \arrow[r, "a_i"'] & A_i \arrow[u, "q_i"'] \arrow[d, phantom, "\dagger"] \\
 & T(A_i) \arrow[ur, "a_i"'] \arrow[d, "T(q_i)"'] & {} \\
 & T(Q) \arrow[uuur, bend right=40, "\alpha_Q"'] &
\end{tikzcd}$$

We conclude that $\alpha_Q \circ \mu_Q = \alpha_Q \circ T(\alpha_Q)$ (start at Q following the external arrows and then compose with $T^2(q_i)$. Then use the fact that $T^2(q_i)$ is epi by (S2) since q_i is epi). This concludes the proof that $\mathbf{Q} = (Q, \alpha_Q) \in \mathtt{Alg}(\mathsf{T})$.

To finish the proof, take $\mathbf{R}_L = \mathbf{Q}$ and $e_L = q_i \circ e_i$, for some $i \in I$ (remember that $q_i \circ e_i = q_j \circ e_j$ since $\{A_i \xrightarrow{q_i} Q\}_{i \in I}$ is the generalized pushout of $\{T(X) \xrightarrow{e_i} A_i\}_{i \in I}$). We have $e_L \in \mathscr{E}$ by Lemma 167. The fact that L factors through e_L follows from the fact that each e_i is a factor of L and the property of $\{A_i \xrightarrow{q_i} Q\}_{i \in I}$ being the generalized pushout of $\{T(X) \xrightarrow{e_i} A_i\}_{i \in I}$. Also, we have that $e_L = q_i \circ e_i \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{R}_L)$ since we have the following commutative diagram:

$$\begin{tikzcd}
T^2(X) \arrow[r, two heads, "Te_i"] \arrow[d, "\mu_X"'] & T(A_i) \arrow[r, two heads, "T(q_i)"] \arrow[d, "a_i"'] \arrow[dr, phantom, "\dagger"] & T(Q) \arrow[d, "\alpha_Q"] \\
T(X) \arrow[r, two heads, "e_i"'] & A_i \arrow[r, two heads, "q_i"'] & Q
\end{tikzcd}$$

□

As we mentioned in the proof of Lemma 146, and also in the remark before Corollary 156, the syntactic algebra of a language $L \in \mathtt{Set}(X^*, 2)$ is the dual of the coalgebra $\langle\langle L \rangle\rangle$ generated by L, i.e., the least object in $\mathtt{CABA}$ containing L which is closed under left and right derivatives. This property was also already mentioned in [42, Section 6]. In general, under the hypothesis of Lemma 168, if we have a T–language L over X and we assume that there is a concrete category $\mathcal{C}$ that is dual to $\mathcal{D}$ such that the underlying set of the dual object of $\mathbf{TX} \in \mathtt{Alg}(\mathsf{T})$ is $\mathcal{C}(TX, C)$ (see, e.g., [29]), then the syntactic algebra $\mathbf{R}_L \in \mathtt{Alg}(\mathsf{T})$ of L is the dual of the B–coalgebra $\langle\langle L \rangle\rangle$ generated by L, where B is the comonad that is dual to the monad T (see Section 4.4.2).

Notice that the property of T preserving weak generalized pushouts whose arrows are all in $\mathscr{E}$ is, by duality, the property of B preserving weak generalized pullbacks whose arrows are all in $\mathscr{M}'$, where $(\mathscr{E}', \mathscr{M}')$ is the factorization system on $\mathcal{C}$ that is dual to $(\mathscr{E}, \mathscr{M})$ on $\mathcal{D}$. This requirement of preserving weak generalized pullbacks in order to guarantee the existence of the least subcoalgebra containing an element has been mentioned and studied in [76, 47].

Another related subject in connection with the notion of syntactic algebras is that of a subdirectly irreducible algebra [27, II.8]. In fact, for the case of the monad T_τ asssociated to a finitary signature τ it is shown in [86, 8.10 Proposition] that every subdirectly irreducible algebra is syntactic. Also, a generalization of the syntactic congruence θ_L of a language L over X is given in [86], i.e., the congruence θ_L such that $\mathbf{TX}/\theta_L$ is the syntactic algebra of L.

Additionally, for coalgebras on `Set`, the notion of conjunctly irreducible is the same as being one–generated [49, Proposition 2.4], where *conjunctly irreducible* is the dual notion of being subdirectly irreducible.

6.5 Discussion

There are some works in which categorical approaches to derive Eilenberg–type correspondences are used, notably [5, 22, 89, 78]. The work in [89] subsumes [5, 22] and the present one subsumes [78, 22]. The kind of algebras considered in [5] are algebras with a monoid structure which restricts the kind of algebras one can consider, e.g., Eilenberg's theorem [36, Theorem 34s] for pseudovarieties of semigroups cannot be derived from [5]. A different approach to get a general Eilenberg–type theorem is [22], were the algebras considered are algebras for a monad T on $\mathtt{Set}^S$, for a fixed set S. The fact that all the monads considered are on $\mathtt{Set}^S$ was not general enough to cover cases such as [70, 72] in which the varieties of languages are not necessarily Boolean algebras. The approach in [22] of considering algebras for a monad T is also considered and generalized in [78, 89] as well as in the present thesis. One of the main challenges in categorical approaches to Eilenberg–type correspondences is to define the right concept of a "variety of languages". The definition of a "variety of languages" given in [89] depends of finding what they call a "unary representation", which is a set of unary operations on a free algebra satisfying certain properties, see [89, Definition 3.7.]. From this "unary representation" one can construct syntactic algebras and define the kind of derivatives that define a "variety of languages". The definition of a "variety of languages" in the present thesis is a categorical one which avoids the explicit definition of derivatives and existence of syntactic algebras. In the present chapter, derivatives are captured coalgebraically and syntactic algebras are not used to prove the abstract Eilenberg–type correspondences theorems, but both of those concepts can be easily obtained via duality in each concrete case. Coalgebraic approaches, from which one can easily define the concept of a "variety of languages", are not used in [22, 89] and it is a new point of view and contribution in this thesis. Another important related work is [13], in which an Eilenberg–type correspondence for varieties of monoids is shown, which is an Eilenberg–type correspondence that can be derived from the present chapter but not from [5, 22, 89]. This motivates the study of Eilenberg–type correspondences for other classes of algebras different than pseudovarieties. It is worth mentioning that in [13] the duality between equations and coequations is studied for the first time in the context of an Eilenberg–type correspondence.

Another important observation and remark is regarding the use of syntactic algebras. In Eilenberg's original proof [36, Theorem 34] the use of syntactic monoids (semigroups) [36, VII.1] helped to prove his theorem. As in Eilenberg's proof, the use of syntactic algebras was also made in [22, 70, 72, 74, 89, 78] for establishing Eilenberg–type correspondences. Categorical approaches such as [22, 78, 89] generalized the concept of syntactic algebra. In [22, 89] syntactic algebras are obtained, under mild assumptions, by means of a congruence, while in [78] are obtained by using generalized pushouts, under the condition that T preserves weak generalized pushouts. As we saw in the present thesis, the use of syntactic algebras is not necessary in order to establish abstract Eilenberg–type correspondences. Nonetheless, the study of syntactic algebras has their own importance in language theory and some categorical properties and facts about them were shown in the previous section.

Applications and theorems we can derive form the abstract Eilenberg–type correspondences showed in this section are countless. Nevertheless, in the next chapter we will derive some particular Eilenberg–type correspondences to show the generality of the theorems presented in this chapter. We will discover and rediscover different kinds of Eilenberg–type correspondences.

The present chapter helped us to understand the general picture regarding Eilenberg–type correspondences. This not only gave us an answer to the question where varieties of languages come from but also gave us a more direct and general way to obtain Eilenberg–type correspondences. This understanding allowed us to be able to construct abstract theorems not only for the case of pseudovarieties of algebras but also for other kind of classes of algebras such as varieties and also local (pseudo)varieties. It is worth mentioning that other classes of algebras can be considered with this approach such as implicational classes, in which implications are considered instead of equations. The implicational case can be studied by omitting the requirement in an equational theory of having a free and projective domain. A categorical approach of implicational classes has been already studied in [15]. The study of Eilenberg–type correspondences for implicational classes is left as future work.

Another new subject that can be derived from this chapter is the study of the dual of an Eilenberg–type correspondence, a coEilenberg–type correspondence. The concept of a coEilenberg–type correspondence can be defined as a one–to–one correspondence between covarieties of coalgebras and equational theories. We also leave this as future work. The contents of this chapter were based on [82].

Chapter 7

Applications

In this chapter, we show applications of the Eilenberg–type correspondences we obtained in Chapter 6. The table below summarizes the examples given and their corresponding citation, if any, for existing results. Roughly, we show a total of 64 correspondences, less than a half of them are known in the literature and most of them were proved and published separately in at least 12 different papers. Most of the new results that we show in this chapter are for varieties and local varieties. The only variety case known in the literature is the case of varieties of monoids in [13], which motivates the study of Eilenberg–type correspondences for varieties and also for local varieties. In the case of the correspondence shown in [13], we show a more simplified, but equivalent, definition of a variety of languages. Note that in the last two rows of the table there is no variety version since the corresponding class of algebras do not form a variety.

Eilenberg–type correspondence for...	Pseudovarieties	Local pseudovarieties	Varieties	Local varieties
Semigroups	[36, Thm. 34s] ((2) after Ex. 176)	[43] (Sec. 7.3)	(1) after Ex. 169	Sec. 7.3
Ordered semigroups	[70] (Ex. 178)	[43] (Sec. 7.3)	after Ex. 170	Sec. 7.3
Monoids	[36, Thm. 34] ((1)after Ex. 176)	[43] (Sec. 7.3)	[13, Thm. 39] (Ex. 144)	Sec. 7.3
Ordered monoids	[70] (Ex. 178)	[43] (Sec. 7.3)	after Ex. 170	Sec. 7.3
Groups	(3) after Ex. 176	Sec. 7.3	(2) after Ex. 169	Sec. 7.3
Ordered groups	Ex. 178	Sec. 7.3	after Ex. 170	Sec. 7.3
Monoid actions (dynamical systems)	(4) after Ex. 176	Sec. 7.3	(3) after Ex. 169	Sec. 7.3
Semigroups with infinite exponentiation	cf. [90] ((5) after Ex. 176)	cf. [89, Thm. 6.3] (Sec. 7.3)	(4) after Ex. 169	Sec. 7.3

Eilenberg–type correspondence for...	Pseudovarieties	Local pseudovarieties	Varieties	Local varieties
$\mathbb{K}$–algebras for a finite field $\mathbb{K}$	[74] (Ex. 181)	[1] for $\mathbb{K} = \mathbb{Z}_2$ (Sec. 7.3)	Ex. 173	Sec. 7.3
Idempotent semirings	[72] (Ex. 182)	[1] (Sec. 7.3)	Ex. 174	Sec. 7.3
Algebras of type τ in a variety	[85] (Ex. 176)	Sec. 7.3	Ex. 169	Sec. 7.3
Ordered algebras of type τ in a variety	Ex. 177	Sec. 7.3	Ex. 170	Sec. 7.3
Multi–sorted algebras of type τ	[30] (Ex. 179)	Sec. 7.3	Ex. 171	Sec. 7.3
Fuzzy languages	[69] (Ex. 180)	Sec. 7.3	Ex. 172	Sec. 7.3
Commutative fuzzy languages	[6] (Ex. 180)	Sec. 7.3	Ex. 172	Sec. 7.3
Aperiodic fuzzy languages	[7] (Ex. 180)	Sec. 7.3	–	–
Commutative aperiodic fuzzy languages	Ex. 180	Sec. 7.3	–	–

7.1 Eilenberg–type correspondences for varieties of algebras

In this section, we derive Eilenberg–type correspondences for varieties of algebras by using Proposition 143. It is worth mentioning that Eilenberg–type correspondences for varieties of algebras, with the exception of the one showed in [13], have not been studied in the literature. Nevertheless, there is a similarity with Eilenberg–type correspondences for pseudovarieties of algebras, which are commonly studied, in the sense that the properties for closure under derivatives and closure under homomorphic images are the same. We start by describing explicitely Eilenberg–type correspondences for varieties of algebras of a given type τ in which each function symbol in τ has finite arity.

Example 169. Let τ be a type of algebras where each function symbol $g \in \tau$ has arity $n_g \in \mathbb{N}$ and let K be a variety of algebras of type τ. Consider the case $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, $\mathscr{E}$ = surjections, $\mathscr{M}$ = injections and let T_K be the monad such that for every $X \in \mathtt{Set}$, $T_K(X)$ is the underlying set of the free algebra in K on X generators (see [27, Definition II.10.9] and [66, VI.8]). Now, in order to derive Eilenberg–type correspondences for this case, we need to characterize the T_K–algebra morphisms in $\mathscr{E}$ with domain $\mathbf{T_K X}$, $X \in \mathtt{Set}$. For this we define the notion of a τ–congruence. A *τ–congruence on an algebra* $\mathbf{A}$ of type τ is an equivalence relation $\theta \subseteq A \times A$ on A such that for every $g \in \tau$ of arity n_g, every $1 \leq i \leq n_g$ and $a_j \in A$, $1 \leq j \leq n_g$, $j \neq i$, the property $(u, v) \in \theta$ implies $(g(a_1, \ldots a_{i-1}, u, a_{i+1}, \ldots a_{n_g}), g(a_1, \ldots a_{i-1}, v, a_{i+1}, \ldots a_{n_g})) \in \theta$. (cf. [27, Definition II.5.1]). We have that congruences on a T_K–algebra A are in one–to–one correspondence with T_K–algebra morphisms $e : A \twoheadrightarrow B$ in $\mathscr{E}$ with domain $\mathbf{A}$. In fact, every τ–congruence θ on $\mathbf{A}$ induces the canonical T_K–algebra morphism $\nu_\theta : A \twoheadrightarrow A/\theta$ such that $\nu_\theta(a) = a/\theta$ and every T_K–algebra morphisms

$e : A \twoheadrightarrow B$ in $\mathscr{E}$ with domain $\mathbf{A}$ induces the τ–congruence $\ker(e)$ of $\mathbf{A}$. Furthermore, a surjective morphism $e_X : T_K(X) \twoheadrightarrow Q_X$ between T_K–algebras is a T_K–algebra morphism if and only if for every operation symbol $g \in \tau$ and $1 \leq i \leq n_g$ the following diagram commutes:

$$\begin{tikzcd} T_K(X) \arrow[r, "e_X"] \arrow[d, "g_{T_K(X)}(t_1,\ldots,\_,\ldots,t_n)"'] \arrow[dr, phantom, "(\dagger)"] & Q_X \arrow[d, "g_{Q_X}(e_X(t_1),\ldots,\_,\ldots,e_X(t_n))"] \\ T_K(X) \arrow[r, "e_X"'] & Q_X \end{tikzcd}$$

where the parameter in the vertical arrows in in the i–th position.

Now, we have that `CABA` is dual to `Set`, so we can consider $\mathcal{C} = \mathcal{C}_0 =$ `CABA`. By using the duality between `CABA` and `Set`, each coequational B–theory can be indexed by $\mathcal{D}_0 =$ `Set` and can be presented, up to isomorphism, as a family $\{S_X \xhookrightarrow{m_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$ of injective B–coalgebra morphism, where B is the comonad on `CABA` that is dual to T_K. From this, we present coequational B–theories as operators $\mathscr{L}$ on `Set` given by $\mathscr{L}(X) := \mathrm{Im}(m_X)$. Then, we get a one–to–one correspondence between varieties of algebras in K and operators $\mathscr{L}$ on `Set` such that for every $X \in$ `Set`:

i) $\mathscr{L}(X) \in$ `CABA` and it is a subalgebra of the complete atomic Boolean algebra $\mathtt{Set}(T_K(X), 2)$ of subsets of $T_K(X)$.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to τ. That is, for every $g \in \tau$ of arity n_g, every $1 \leq i \leq n_g$, every $\tilde{t} = (t_j)_{1 \leq j \leq n_g, j \neq i}$, $t_j \in T_K(X)$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,\tilde{t})} \in \mathscr{L}(X)$ where $L^{(i)}_{(g,\tilde{t})} \in \mathtt{Set}(T_K(X), 2)$ is defined as
$$L^{(i)}_{(g,\tilde{t})}(t) = L(g(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{n_g}))$$
$t \in T_K(X)$. That is, for every function symbol $g \in \tau$ we get n_g kinds of derivatives.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in$ `Set`, homomorphism of T_K–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

In fact, for each coequational B–theory $\{S_X \xhookrightarrow{m_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$, if we define the operator $\mathscr{L}$ on `Set` as $\mathscr{L}(X) := \mathrm{Im}(m_X)$, then we have that $\mathscr{L}$ satisfies the conditions i), ii) and iii) above as follows:

a) Condition i) above follows from the fact that $\mathscr{L}(X) = \mathrm{Im}(m_X) \cong S_X \in$ `CABA` and $\mathrm{Im}(m_X) \subseteq \mathtt{Set}(T_K(X), 2)$.

b) Condition ii) above follows from lifting the duality between `Set` and `CABA` to a duality between $\mathtt{Alg}(\mathsf{T}_\mathsf{K})$ and $\mathtt{Coalg}(\mathsf{B})$. In fact, every surjective T_K–algebra morphism $e_X : T_K(X) \twoheadrightarrow Q_X$ defines the injective morphism $\mathtt{Set}(e_X, 2)$ in $\mathtt{Coalg}(\mathsf{B})$ which is defined as $\mathtt{Set}(e_X, 2)(f) = f \circ e_X$, $f \in \mathtt{Set}(Q_X, 2)$, and from this we have:

$$\mathscr{L}(X) = \mathrm{Im}(\mathtt{Set}(e_X, 2)) = \{f \circ e_X \mid f \in \mathtt{Set}(Q_X, 2)\}.$$

Closure of $\mathscr{L}(X)$ under under derivatives with respect to τ follows from the fact that e_X is a T_K–algebra morphism. In fact, from diagram (†), by duality, it follows that $\mathscr{L}(X)$ is closed under derivatives with respect to τ.

c) Condition iii) above is the commutativity of the diagram in Definition 142.

Conversely, each operator $\mathscr{L}$ on $\mathtt{Set}$ with the properties i), ii) and iii) above defines the family of injective morphisms $\{\mathscr{L}(X) \xhookrightarrow{i_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$ in $\mathtt{CABA}$ where i_X is the inclusion morphism. Now, we have that the family $\{\mathscr{L}(X) \xhookrightarrow{i_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$ is a coequational B–theory. In fact, for every $X \in \mathtt{Set}$, the morphism i_X defines, by duality, the canonical surjective function $e_{\mathscr{L}(X)} : T_K(X) \to T_K(X)/\theta_{\mathscr{L}(X)}$ where $\theta_{\mathscr{L}(X)} \subseteq T_K(X) \times T_K(X)$ is defined as:

$$\theta_{\mathscr{L}(X)} := \{(v, w) \in T_K(X) \times T_K(X) \mid \exists A \in \mathrm{At}(\mathscr{L}(X)) \text{ s.t. } A(w) = A(v) = 1\}$$

where $\mathrm{At}(\mathscr{L}(X))$ is the set of atoms of $\mathscr{L}(X)$. Clearly, $\theta_{\mathscr{L}(X)}$ is an equivalence relation on $T_K(X)$ since $\mathrm{At}(\mathscr{L}(X))$ is a partition of $T_K(X)$. We only need to show that $\theta_{\mathscr{L}(X)}$ is an τ–congruence on $\mathbf{T_K X}$. In fact, let $g \in \tau$ of arity n_g, $1 \leq i \leq n_g$, $\tilde{t} = (t_j)_{1 \leq j \leq n_g, j \neq i}$, $t_j \in T_K(X)$, and assume $(u, v) \in \theta_{\mathscr{L}(X)}$, i.e., there exists $A \in \mathrm{At}(\mathscr{L}(X))$ such that $A(u) = A(v) = 1$. Assume, towards a contradiction, that:

$$(g(t_1, \ldots t_{i-1}, u, t_{i+1}, \ldots t_{n_g}), g(t_1, \ldots t_{i-1}, v, t_{i+1}, \ldots t_{n_g})) \notin \theta_{\mathscr{L}(X)}$$

then there exists $B \in \mathrm{At}(\mathscr{L}(X))$ such that:

$$B(g(t_1, \ldots t_{i-1}, u, t_{i+1}, \ldots t_{n_g})) \neq B(g(t_1, \ldots t_{i-1}, v, t_{i+1}, \ldots t_{n_g}))$$

which means that $B^{(i)}_{(g,\tilde{t})}(u) \neq B^{(i)}_{(g,\tilde{t})}(v)$ with $B^{(i)}_{(g,\tilde{t})} \in \mathscr{L}(X)$ by closure under derivatives with respect to τ. Therefore $A \cap B^{(i)}_{(g,\tilde{t})}$ is an element in $\mathscr{L}(X)$ such that $0 < A \cap B^{(i)}_{(g,\tilde{t})} < A$ which contradicts the fact that A is an atom. This previous reasoning proves that $(g(t_1, \ldots t_{i-1}, u, t_{i+1}, \ldots t_{n_g}), g(t_1, \ldots t_{i-1}, v, t_{i+1}, \ldots t_{n_g})) \in \theta_{\mathscr{L}(X)}$, which means that $e_{\mathscr{L}(X)}$ is a surjective T_K–algebra morphism. Therefore, every i_X is a B–coalgebra morphism and hence the family $\{\mathscr{L}(X) \xhookrightarrow{i_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$ is, by closure under morphic preimages, a coequational B–theory. Note that conditions i) and ii) above are exactly the properties that $\mathscr{L}(X)$ is a B–subcoalgebra of $\mathtt{Set}(T_K(X), 2)$. Finally, the one–to–one correspondence follows from the duality between $\mathtt{Set}$ and $\mathtt{CABA}$.

It is worth mentioning that the notion of derivatives given in ii) above follows exactly from the defining properties of a τ–congruence on $\mathbf{T_K X}$ and diagram (†).

This kind of derivatives already appeared in [85] and are different than the kind of derivatives considered in [22] which are, in general, compositions of derivatives in ii) above. □

From the previous general example we can provide details for the properties i), ii) and iii) given in Example 144. In fact, for the case of monoids we have the type $\tau = \{e, \cdot\}$ where e is a nullary function symbol and $\cdot$ is a binary function symbol. We write $x \cdot y$ for $\cdot(x, y)$. By considering the variety K of monoids, we get the monad T_K such that $T_K(X) = X^*$, where X^* is the free monoid on X. Then, we have:

1) Properties i) and iii) in Example 169 trivially become properties i) and iii) in Example 144.

2) Property ii) in Example 169 does not give us any kind of derivatives for the nullary function symbol $e \in \tau$, but will give us the derivatives $L^{(1)}_{(\cdot,u)}$ and $L^{(2)}_{(\cdot,u)}$ for the binary function symbol $\cdot \in \tau$, $u \in T_K(X) = X^*$, which are defined for every $w \in X^*$ as

$$L^{(1)}_{(\cdot,u)}(w) = L(w \cdot u) = L(wu) \text{ and } L^{(2)}_{(\cdot,u)}(w) = L(u \cdot w) = L(uw).$$

The previous two derivatives are, respectively, the left and right derivatives of L with respect to u. That is, we get the familiar derivatives ${}_uL(w) = L(wu) = L^{(1)}_{(\cdot,u)}(w)$ and $L_u(w) = L(uw) = L^{(2)}_{(u,\cdot)}(w)$.

In a similar way, from Example 169, we get the following Eilenberg–type correspondences (note that the notation for derivatives is changed in order to use a more familiar notation).

(1) A one–to–one correspondence between varieties of semigroups and operators $\mathscr{L}$ on $\mathtt{Set}$ such that for every $X \in \mathtt{Set}$:

   i) $\mathscr{L}(X) \in \mathtt{CABA}$ and it is a subalgebra of the complete atomic Boolean algebra $\mathtt{Set}(X^+, 2)$ of subsets of X^+, i.e., every element in $\mathscr{L}(X)$ is a language on X not containing the empty word.

   ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^+$.

   iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of semigroups $h : Y^+ \to X^+$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(2) A one–to–one correspondence between varieties of groups and operators $\mathscr{L}$ on $\mathtt{Set}$ such that for every $X \in \mathtt{Set}$:

   i) $\mathscr{L}(X) \in \mathtt{CABA}$ and it is a subalgebra of the complete Boolean algebra $\mathtt{Set}(\mathfrak{F}_G(X), 2)$ of subsets of the free group $\mathfrak{F}_G(X)$ on X.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives and inverses. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x, L^{-1} \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$, $L_x(w) = L(xw)$ and $L^{-1}(w) = L(w^{-1})$, $w \in \mathfrak{F}_G(X)$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of groups $h : \mathfrak{F}_G(Y) \to \mathfrak{F}_G(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(3) For a fixed monoid $\mathbf{M} = (M, e, \cdot)$, a one–to–one correspondence between varieties of $\mathbf{M}$–actions, i.e., dynamical systems on $\mathbf{M}$, and operators $\mathscr{L}$ on $\texttt{Set}$ such that for every $X \in \texttt{Set}$:

i) $\mathscr{L}(X) \in \texttt{CABA}$ and it is a subalgebra of the complete atomic Boolean algebra $\texttt{Set}(M \times X, 2)$ of subsets of $M \times X$.

ii) $\mathscr{L}(X)$ is closed under translations. That is, if $L \in \mathscr{L}(X)$ and $m \in M$ then $mL \in \mathscr{L}(X)$, where $mL(n, x) = L(m \cdot n, x)$, $(n, x) \in M \times X$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of $\mathbf{M}$–actions $h : M \times Y \to M \times X$ (i.e., $h(m \cdot (n, y)) = m \cdot h(n, y)$) and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(4) Consider the type of algebras $\tau = \{\cdot, (\_)^\omega\}$ where $\cdot$ is a binary operation and $(\_)^\omega$ is a unary operation. Now, let T be the free monad on $\texttt{Set}$ for the algebras of type τ that satisfy the following equations:

$$
\begin{array}{lll}
(x \cdot y) \cdot z = x \cdot (y \cdot z) & x^\omega \cdot y = x^\omega & (y \cdot x^\omega)^\omega = y \cdot x^\omega \\
(x^n)^\omega = x^\omega,\ n \geq 1 & (x \cdot y)^\omega = x \cdot (y \cdot x)^\omega &
\end{array}
$$

Here $x \cdot y$ is the product of x and y, in that order, and x^ω represents the infinite product $x \cdot x \cdot \cdots$. Hence, for every $X \in \texttt{Set}$ the algebra $\mathbf{TX}$ has as carrier set the set $X^+ \cup X^{(\omega)}$, where $X^{(\omega)}$ represents the set of all ultimately periodic sequences in X^ω, i.e., every element in $X^{(\omega)}$ is of the form uv^ω for some $u \in X^*$ and $v \in X^+$, and $X^+ \cup X^{(\omega)}$ has the natural operations $\cdot$ of concatenation and $(\_)^\omega$ of "infinite power".
In this case, we get a one–to–one correspondence between varieties of semigroups with infinite exponentiation and operators $\mathscr{L}$ on $\texttt{Set}$ such that for every $X \in \texttt{Set}$:

i) $\mathscr{L}(X) \in \texttt{CABA}$ and it is a subalgebra of the complete atomic Boolean algebra $\texttt{Set}(X^+ \cup X^{(\omega)}, 2)$ of subsets of $X^+ \cup X^{(\omega)}$.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives and infinite exponentiation. That is, if $L \in \mathscr{L}(X)$ and $u \in X^+ \cup X^{(\omega)}$ then ${}_uL, L_u, L^\omega \in \mathscr{L}(X)$, where ${}_uL(w) = L(wu)$, $L_u(w) = L(uw)$ and $L^\omega(w) = L(w^\omega)$, $w \in X^+ \cup X^{(\omega)}$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of T–algebras $h : Y^+ \cup Y^{(\omega)} \to X^+ \cup X^{(\omega)}$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

We can do a similar work as in Example 169 to get Eilenberg–type correspondences for varieties of ordered algebras for a given type.

Example 170. Let τ be a type of algebras where each function symbol $g \in \tau$ has arity $n_g \in \mathbb{N}$ and let K be a variety of ordered algebras of type τ. Consider the case $\mathcal{D} =$ Poset, $\mathcal{D}_0 =$ discrete posets, $\mathscr{E} =$ surjections, $\mathscr{M} =$ embeddings and let T_K be the monad such that for every $\mathbf{X} = (X, \leq) \in$ Poset, $T_K(\mathbf{X}) := (T_K(X), \leq_{T_K X})$ is the underlying poset of the free ordered algebra in K on $\mathbf{X}$ generators (see, e.g., [21, Proposition 1]). We have that AlgCDL is dual to Poset, so we can consider $\mathcal{C} =$ AlgCDL, $\mathcal{C}_0 =$ CABA. Similar to Example 169, by using the duality between Poset and AlgCDL, each coequational B–theory can be indexed by Set (i.e., we consider every object $X \in$ Set as the object $(X, =) \in$ Poset, which is in $\mathcal{D}_0$) and can be presented, up to isomorphism, as a family $\{S_X \xhookrightarrow{m_X} 2^{T_K(X)}\}_{X \in \mathtt{Set}}$, then we present coequational B–theories as operators $\mathscr{L}$ on Set given by $\mathscr{L}(X) := \mathrm{Im}(m_X)$. Hence, we get a one–to–one correspondence between varieties of ordered algebras in K and operators $\mathscr{L}$ on Set such that for every $X \in$ Set:

i) $\mathscr{L}(X) \in$ AlgCDL and it is a subalgebra of the algebraic completely distributive lattice $\mathtt{Poset}(T_K(X), \mathbf{2}_c) \cong \mathtt{Set}(T_K(X), 2)$ of subsets of $T_K(X)$. Here $\mathbf{2}_c \in$ Poset is the two–element chain.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to the type τ. That is, for every $g \in \tau$ of arity n_g, every $1 \leq i \leq n_g$, every $\tilde{t} = (t_j)_{1 \leq j \leq n_g, j \neq i}$, $t_j \in T_K(X)$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,\tilde{t})} \in \mathscr{L}(X)$ where $L^{(i)}_{(g,\tilde{t})} \in \mathtt{Set}(T_K(X), 2)$ is defined as
$$L^{(i)}_{(g,\tilde{t})}(t) = L(g(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{n_g}))$$
$t \in T_K(X)$. That is, for every function symbol $g \in \tau$ we get n_g kinds of derivatives.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in$ Set, homomorphism of T_K–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$. □

From the previous example we can obtain Eilenberg–type correspondences for varieties of ordered semigroups, varieties of ordered monoids, varieties of ordered groups, and so on. For instance, for the case of varieties of ordered semigroups we can consider the type $\tau = \{\cdot\}$ where $\cdot$ is a binary function symbol and K is the variety of ordered semigroups. Then we get a one–to–one correspondence between varieties of ordered semigroups and operators $\mathscr{L}$ on Set such that for every $X \in$ Set:

i) $\mathscr{L}(X) \in$ AlgCDL and it is a subalgebra of the algebraic completely distributive lattice $\mathtt{Set}(X^+, 2)$ of subsets of X^+, i.e. every element in $\mathscr{L}(X)$ is a language on X not containing the empty word. In particular, $\mathscr{L}(X)$ is closed under unions and intersections.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^+$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of semigroups $h : Y^+ \to X^+$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

The following example shows a similar general result for many–sorted algebras.

Example 171 (cf. [30])**.** For this example we assume the reader has some knowledge on many–sorted universal algebra, see, e.g., [87] and its references. Let S be a set of *sorts* and consider the category $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}^S$ of S–sorted sets with S–sorted functions, $\mathscr{E} = S$–sorted surjections, and $\mathscr{M} = S$–sorted injections. We have that $\mathtt{CABA}^S$ is dual to $\mathtt{Set}^S$, so we can consider $\mathcal{C} = \mathcal{C}_0 = \mathtt{CABA}^S$. For any many–sorted signature τ on S, define the monad T_τ on $\mathtt{Set}^S$ such that $T_\tau(X)$ is the S–sorted set of τ–terms on X. Then we get a one–to–one correspondence between varieties of many–sorted algebras of type τ and operators $\mathscr{L}$ on $\mathtt{Set}^S$ such that for every $X \in \mathtt{Set}^S$:

i) $\mathscr{L}(X) \in \mathtt{CABA}^S$ and it is a subalgebra of the algebra $\mathtt{Set}^S(T_\tau(X), \mathbf{2}) \in \mathtt{CABA}^S$. Here $\mathbf{2} \in \mathtt{Set}^S$ is the S–sorted set such that for every sort $s \in S$ we have $\mathbf{2}_s = 2 = \{0, 1\}$.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to τ. That is, for every operation symbol $g \in \tau$ of sort (w, s') (i.e., $w \in S^*$ and $s' \in S$) with $w = s_1 \cdots s_n$, every $1 \leq i \leq n$, every $t = (t_j)_{1 \leq j \leq n, j \neq i}$ where $t_j \in T_\tau(X)_{s_j}$, $1 \leq j \leq n$, $j \neq i$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,t)} \in \mathscr{L}(X)$ where for every $s \in S$, $\left(L^{(i)}_{(g,t)}\right)_s \in \mathtt{Set}(T_K(X)_s, 2)$ is defined as

$$\left(L^{(i)}_{(g,t)}\right)_s (p) = \begin{cases} 0 & \text{if } s \neq s' \\ L_{s'}(g(t_1, \ldots, t_{i-1}, p, t_{i+1}, \ldots, t_n)) & \text{if } s = s' \end{cases}$$

$p \in T_K(X)_{s_i}$. That is, for every function symbol $g \in \tau$ we get n kinds of derivatives. Note that w and s' above depend on g (and therefore n also depends on g).

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}^S$, homomorphism of T_τ–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

Note that properties i) and iii) follow in a similar way as in Example 169. Now, closure under derivatives with respect to τ follow from the fact the morphisms $e_X : TX \to Q_X$ in an equational T–theory are homomorphisms of many–sorted algebras. In fact, since e_X is a many–sorted homomorphism then we have that the following diagram commutes:

$$\begin{tikzcd}
T_\tau(X)_{s_i} \arrow[r, "(e_X)_{s_i}"] \arrow[d, "g_{T_\tau(X)}(t_1,\ldots,_,\ldots,t_n)"'] & (Q_X)_{s_i} \arrow[d, "g_{Q_X}((e_X)_{s_1}(t_1),\ldots,_,\ldots,(e_X)_{s_n}(t_n))"] \\
T_\tau(X)_{s'} \arrow[r, "(e_X)_{s'}"'] & (Q_X)_{s'}
\end{tikzcd}$$

where the parameter in the vertical arrows is in the i–th position. From that diagram, by duality, it follows that $\mathscr{L}(X)$ is closed under derivatives with respect to τ. Conversely, if $\mathscr{L}(X)$ is closed under derivatives with respect to τ, then the dual of the inclusion $\iota : \mathscr{L}(X) \to \mathtt{Set}^S(T_\tau(X), \mathbf{2})$ is a surjective function whose kernel is a congruence. In fact, assume by contradiction that it is not a congruence, i.e., there exists an operation symbol $g \in \tau$ of sort (w, s'), with $w = s_1 \cdots s_n$, and there exists $t = (t_j)_{1 \leq j \leq n, j \neq i}$, $t_j \in T_\tau(X)_{s_j}$, and $p, p' \in T_\tau(X)_{s_i}$ such that $(p, p') \in \ker(\mathtt{CABA}^S(\iota, \mathbf{2}))$ but $(g_{T_\tau(X)}(t_1, \ldots, p, \ldots, t_n), g_{T_\tau(X)}(t_1, \ldots, p', \ldots, t_n)) \notin \ker(\mathtt{CABA}^S(\iota, \mathbf{2}))$. That is, there exists an atom $A \in \mathtt{CABA}(\iota_{s'}, 2)$ such that $A \cap \{g_{T_\tau(X)}(t_1, \ldots, p, \ldots, t_n), g_{T_\tau(X)}(t_1, \ldots, p', \ldots, t_n)\}$ has only one element. Now, we have that the object $L \in \mathtt{Set}^S(T_\tau(X), \mathbf{2})$ defined for every $s \in S$ as:

$$L_s = \begin{cases} 0 & \text{if } s \neq s' \\ A & \text{if } s = s' \end{cases}$$

is an element in $\mathscr{L}(X)$. Hence by closure under derivatives $L^{(i)}_{(g,t)} \in \mathscr{L}(X)$ and by assumption $\left(L^{(i)}_{(g,t)}\right)_{s_i}$ either contains both p and p' or none of them. But $p \in \left(L^{(i)}_{(g,t)}\right)_{s'}$ iff $L_{s'}(g_{T_\tau(X)}(t_1, \ldots, p, \ldots, t_n)) = 1$ iff $L_{s'}(g_{T_\tau(X)}(t_1, \ldots, p', \ldots, t_n)) = 0$ iff $p' \notin \left(L^{(i)}_{(g,t)}\right)_{s'}$ which is a contradiction.

This Eilenberg–type correspondence is the version shown in [30] but for the case of varieties of many–sorted algebras, instead of pseudovarieties. □

The following example shows a correspondence for fuzzy languages.

Example 172 (cf. [69]). Consider the case $\mathcal{D} = \mathcal{D}_0 = \mathtt{Set}$, $\mathscr{E}$ = surjections, $\mathscr{M}$ = injections and let T be the free monoid monad, i.e., $T(X) = X^*$, $X \in \mathtt{Set}$. Now, consider the set $S = [0, 1]$ and let $\mathcal{C} = \mathcal{C}_0$ be a category such that every object is isomorphic to an object of the form X^S and a morphism between an object $A \cong X^S$ and an object $B \cong Y^S$ is isomorphic to f^S for some $f \in \mathtt{Set}(Y, X)$. Then, each coequational B–theory can be indexed by $\mathcal{D}_0 = \mathtt{Set}$ and can be presented, up to isomorphism, as a family $\{S_X \xhookrightarrow{m_X} S^{X^*}\}_{X \in \mathtt{Set}}$ of injective B–coalgebra morphism, where B is the comonad on $\mathcal{C}$ that is dual to T. Elements in $S^{X^*} = \mathtt{Set}(X^*, S)$ are called *fuzzy languages on* X. From this, we present coequational B–theories as operators $\mathscr{L}$ on $\mathtt{Set}$ given by $\mathscr{L}(X) := \mathrm{Im}(m_X)$. Then, we get a one–to–one correspondence between varieties of monoids and operators $\mathscr{L}$ on $\mathtt{Set}$, which we call *varieties of fuzzy languages* (cf. [69]), such that for every $X \in \mathtt{Set}$:

i) $\mathscr{L}(X)$ is a subset of $\mathtt{Set}(X^*, S)$ such that $\mathbf{B}(\mathscr{L}(X)) \stackrel{def}{=} \{\mathrm{supp}(g) \mid g \in \mathscr{L}(X)\} \subseteq \mathtt{Set}(X^*, 2)$ is an object in $\mathtt{CABA}$ with the usual set–theoretic operations, and the set $\mathscr{L}(X)$ is determined by the atoms $\mathtt{CABA}(\mathbf{B}(\mathscr{L}(X)), 2)$ of $\mathbf{B}(\mathscr{L}(X))$ in the sense that

$$\mathscr{L}(X) = \left\{ g \in S^{X^*} \;\middle|\; g = \bigvee_{k \in \mathtt{CABA}(\mathbf{B}(\mathscr{L}(X)),2)} s_k k,\; s_k \in S \right\},$$

where in the expression above, $g(w) = s_k$ if and only if $w \in k$.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $g \in \mathscr{L}(X)$ and $u \in X^*$ then ${}_u g, g_u \in \mathscr{L}(X)$, where ${}_u g(v) = g(vu)$ and $g_u(v) = g(uv)$, $v \in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, every monoid homomorphism $h : Y^* \to X^*$ and $g \in \mathscr{L}(X)$, we have that $g \circ h \in \mathscr{L}(Y)$.

Note that conditions i) and ii) above are exactly the conditions that $\mathscr{L}(X) \in \mathcal{C}$ is a closed subsystem of $\mathtt{Set}(X^*, S)$ (see Lemma 62 and the comments after its proof).

Additionally, by restricting the elements in $\mathscr{L}(X)$ to commutative fuzzy languages we get an Eilenberg–type correspondence for varieties of commutative fuzzy languages. A fuzzy language $g : X^* \to [0,1]$ on X is *commutative* if for every $u, v \in X^*$ we have that $g(uv) = g(vu)$, cf. [6]. Note that a commutative version can also be obtained by considering the monad T such that $T(X)$ is the free commutative monoid on X, but in this case elements in $\mathscr{L}(X)$ are in $S^{T(X)}$ instead of S^{X^*}. □

Now, we derive correspondences for varieties of $\mathbb{K}$–algebras over a finite field $\mathbb{K}$.

Example 173 (cf. [74, Théorème III.1.1.]). Let $\mathbb{K}$ be a finite field. Consider the case $\mathcal{D} = \mathcal{D}_0 = \mathtt{Vec}_{\mathbb{K}}$, $\mathscr{E}$ = surjections, and $\mathscr{M}$ = injections. We have that $\mathtt{StVec}_{\mathbb{K}}$ is dual to $\mathtt{Vec}_{\mathbb{K}}$, so we can consider $\mathcal{C} = \mathcal{C}_0 = \mathtt{StVec}_{\mathbb{K}}$. For every set X denote by $\mathsf{V}(X)$ the $\mathbb{K}$–vector space with basis X. Consider the monad $T(\mathsf{V}(X)) = \mathsf{V}(X^*)$, where X^* is the free monoid on X. Then we get a one–to–one correspondence between varieties of $\mathbb{K}$–algebras and operators $\mathscr{L}$ on $\mathtt{Set}$ such that for every $X \in \mathtt{Set}$:

i) $\mathscr{L}(X) \in \mathtt{StVec}_{\mathbb{K}}$ and it is a subspace of the space $\mathtt{Vec}_{\mathbb{K}}(\mathsf{V}(X^*), \mathbb{K})$ where the topology on $\mathtt{Vec}_{\mathbb{K}}(\mathsf{V}(X^*), \mathbb{K})$ is the subspace topology of the product $\mathbb{K}^{\mathsf{V}(X^*)}$ and $\mathbb{K}$ has the discrete topology.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $v \in \mathsf{V}(X^*)$ then ${}_v L, L_v \in \mathscr{L}(X)$, where ${}_v L(w) = L(wv)$ and $L_v(w) = L(vw)$, $w \in \mathsf{V}(X^*)$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, $\mathbb{K}$–linear map $h : \mathsf{V}(Y^*) \to \mathsf{V}(X^*)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

Remark. Note that to consider the case in which $\mathbb{K}$ is an infinite field we need to consider $\mathcal{C} = \mathcal{C}_0$ to be the category of linearly compact spaces (see, [16]). In this case, we will only require $\mathscr{L}(X)$ to be a linearly compact subspace of $\mathtt{Vec}_{\mathbb{K}}(\mathsf{V}(X^*), \mathbb{K})$ in contition i). □

The next example, shows a correspondence for varieties of idempotent semirings.

Example 174 (cf. [72, Theorem 5 (iii)])**.** Consider the case $\mathcal{D} = \mathtt{JSL}$, $\mathcal{D}_0 =$ free join semilattices, i.e., $\mathcal{D}_0 = \{(\mathcal{P}_f(X), \cup, \emptyset) \mid X \in \mathtt{Set}\}$, where $\mathcal{P}_f(X)$ is the set of all finite subsets of X, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. We have that $\mathtt{StJSL}$ is dual to $\mathtt{JSL}$, so we can consider $\mathcal{C} = \mathtt{StJSL}$ and $\mathcal{C}_0 = \{\mathtt{JSL}((\mathcal{P}_f(X), \cup, \emptyset), 2) \mid X \in \mathtt{Set}\}$.

Define the monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathtt{JSL}$ as $T(X, \vee, 0) = (\mathcal{P}_f(X^*)/\theta, \cup_\theta, \emptyset/\theta)$ where θ is the least equivalence relation on $\mathcal{P}_f(X^*)$ such that:

i) for every $x, y \in X$ $\{x \vee y\}\theta\{x, y\}$,

ii) for every $A, B, C, D \in \mathcal{P}_f(X^*)$, $A\theta B$ and $C\theta D$ imply $AC\theta BD$, and

iii) for every $A, B, C, D \in \mathcal{P}_f(X^*)$, $A\theta B$ and $C\theta D$ imply $A \cup C\theta B \cup D$.

and $\cup_\theta$ is defined as $A/\theta \cup_\theta B/\theta = (A \cup B)/\theta$ which is well–defined by property iii). We should use a notation like $\theta_{(X,\vee,0)}$ for the relation defined above, but we will denote it by θ for simplicity. It will be clear from the context to which θ we are refering to in each case. If $h \in \mathtt{JSL}((X, \vee, 0), (Y, \vee, 0))$ then $T(h)$ is defined as

$$T(h)(\{w_1 \ldots, w_n\}/\theta) = \{h^*(w_1), \ldots, h^*(w_n)\}/\theta.$$

The unit of the monad is defined as $\eta_{(X,\vee,0)}(x) = \{x\}/\theta$ and the multiplication as:

$$\mu_{(X,\vee,0)}(\{W_1, \ldots W_n\}/\theta) = \left(\bigcup_{i=1}^{n}\left(\prod_{j=1}^{m_i} W_j^{(i)}\right)\right)\Big/\theta$$

where each $W_i \in (\mathcal{P}_f(X^*))^*$ is such that $W_i = W_1^{(i)} \cdots W_{m_i}^{(i)}$, $W_j^{(i)} \in \mathcal{P}_f(X^*)$, $1 \leq i \leq n$ and $1 \leq j \leq m_i$.

We have that $\mathtt{Alg}(\mathsf{T})$ is the category of idempotent semirings.

Lemma 175. *Consider the object $(\mathcal{P}_f(X), \cup, \emptyset) \in \mathtt{JSL}$, then $T(\mathcal{P}_f(X), \cup, \emptyset)$ is isomorphic to $(\mathcal{P}_f(X^*), \cup, \emptyset)$ in $\mathtt{JSL}$.*

Proof. By definition we have that

$$T(\mathcal{P}_f(X), \cup, \emptyset) = (\mathcal{P}_f\left(\mathcal{P}_f(X)^*\right)/\theta, \cup_\theta, \emptyset/\theta)$$

Now, every element in $\mathcal{P}_f(X)$ is of the form $\{x_1, \ldots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$, which by property i) and iii) of the definition of θ we have that:

$$\{\{x_1, \ldots, x_n\}\}\theta\{\{x_1\}, \ldots, \{x_n\}\}$$

Therefore, by using the defining properties of θ we have that every element in $\mathcal{P}_f\left(\mathcal{P}_f(X)^*\right)$ is equivalent to a unique element of the form:

$$\left\{\left\{x_1^{(1)}\right\}\cdots\left\{x_{n_1}^{(1)}\right\},\ldots,\left\{x_1^{(m)}\right\}\cdots\left\{x_{n_m}^{(m)}\right\}\right\}$$

where uniqueness follows since $(\mathcal{P}_f(X),\cup,\emptyset)$ is the free join semilattice. Hence, the join semilattice homomorphism $\varphi:(\mathcal{P}_f(X^*),\cup,\emptyset)\to T(\mathcal{P}_f(X),\cup,\emptyset)$ given by:

$$\varphi(\{x_1^{(1)}\cdots x_{n_1}^{(1)},\ldots,x_1^{(m)}\cdots x_{n_m}^{(m)}\})=\left\{\left\{x_1^{(1)}\right\}\cdots\left\{x_{n_1}^{(1)}\right\},\ldots,\left\{x_1^{(m)}\right\}\cdots\left\{x_{n_m}^{(m)}\right\}\right\}\Big/\theta$$

is an isomorphism in $\mathtt{JSL}$. □

We considered $\mathcal{D}_0=\{(\mathcal{P}_f(X),\cup,\emptyset)\mid X\in\mathtt{Set}\}$. As every semiring is an $\mathscr{E}$–quotient of $(\mathcal{P}_f(X^*),\cup,\emptyset)$, by the previous Lemma we have that condition (B3) is satisfied.

Now, an equational T–theory $\mathtt{E}=\{\mathcal{P}_f(X^*)\overset{e_X}{\twoheadrightarrow}Q_X\}_{X\in\mathtt{Set}}$ on $\mathcal{D}_0$ can be equivalently given as $\mathtt{E}=\{X^*\overset{\eta_{X^*}}{\longrightarrow}\mathcal{P}_f(X^*)\overset{e_X}{\twoheadrightarrow}Q_X\}_{X\in\mathtt{Set}}$ where η is the unit of the adjunction $\mathcal{P}_f(\_)\dashv U$ and $U:\mathtt{JSL}\to\mathtt{Set}$ is the forgetful functor. Therefore, for a given equational T–theory as above, we define the operator $\mathscr{L}$ on $\mathtt{Set}$ as $\mathscr{L}(X):=\mathrm{Im}(e_X\circ\eta_{X^*},2)\subseteq\mathtt{Set}(X^*,2)$. Note that $\mathtt{Set}(X^*,2)$ is isomorphic to $\mathtt{JSL}(\mathcal{P}_f(X^*),\mathbf{2})$ in $\mathtt{StJSL}$ under the correspondence $f\mapsto f\circ\eta_{X^*}$ and $L\mapsto L^\sharp$, $f\in\mathtt{JSL}(\mathcal{P}_f(X^*),\mathbf{2})$ and $L\in\mathtt{Set}(X^*,2)$, where η_{X^*} and $L^\sharp$ are defined as $\eta_{X^*}(w)=\{w\}$ and $L^\sharp(\{w_1,\ldots,w_n\})=\bigvee_{i=1}^n L(w_i)$.

Therefore, we get a one–to–one correspondence between varieties of idempotent semirings and operators $\mathscr{L}$ on $\mathtt{Set}$ such that for every $X\in\mathtt{Set}$:

i) $\mathscr{L}(X)\in\mathtt{StJSL}$ and it is a subspace of $\mathtt{Set}(X^*,2)$ where the topology given on $\mathtt{Set}(X^*,2)$ is the subspace topology of the product 2^{X^*} and 2 has the discrete topology. In particular, $\mathscr{L}(X)$ is closed under unions.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L\in\mathscr{L}(X)$ and $x\in X$ then ${}_xL,L_x\in\mathscr{L}(X)$, where ${}_xL(w)=L(wx)$ and $L_x(w)=L(xw)$, $w\in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y\in\mathtt{Set}$, semiring homomorphism $h:\mathcal{P}_f(Y^*)\to\mathcal{P}_f(X^*)$ and $L\in\mathscr{L}(X)$, we have that $L^\sharp\circ h\circ\eta_{Y^*}\in\mathscr{L}(Y)$. Note that the composite $L^\sharp\circ h\circ\eta_{Y^*}$ is the same as $h^{(-1)}(L)$ defined in [72]. The reason of the exponent $\sharp$ and the use of η_{Y^*} is that we are using the isomorphism:

$$\begin{aligned}\mathtt{JSL}(\mathcal{P}_f(X^*),2)&\cong\mathtt{Set}(X^*,2)\\ f&\mapsto f\circ\eta_{X^*}\\ L^\sharp&\leftarrow\!\shortmid L\end{aligned}$$

Closure of $\mathscr{L}(X)$ under left and right derivatives follows from the fact that each morphism e_X in an equational T–theory is a homomorphism of idempotent semirings. In fact, for every $v, w \in X^*$ and $f \in \mathtt{JSL}(Q_X, \mathbf{2})$ we have that

$$(f \circ e_X \circ \eta_{X^*})_v(w) = (f \circ e_X \circ \eta_{X^*})(vw) = (f_{e_X(\{v\})} \circ e_X \circ \eta_{X^*})(w)$$

where the function $f_{e_X(\{v\})} \in \mathtt{Set}(Q_X, 2)$ is defined as $f_{e_X(\{v\})}(q) = f(e_X(\{v\}) \cdot q)$, where $\cdot$ is the product operation in $\mathbf{Q_X}$, $q \in Q_X$. Note that $f_{e_X(\{v\})} \in \mathtt{JSL}(Q_X, \mathbf{2})$ since for any $p, q \in Q_X$ we have that

$$\begin{aligned} f_{e_X(\{v\})}(p \vee q) &= f(e_X(\{v\}) \cdot (p \vee q)) = f(e_X(\{v\}) \cdot p) \vee f(e_X(\{v\}) \cdot q) \\ &= f_{e_X(\{v\})}(p) \vee f_{e_X(\{v\})}(q) \end{aligned}$$

since $f \in \mathtt{JSL}(Q_X, \mathbf{2})$. Therefore, $(f \circ e_X \circ \eta_{X^*})_x = f_x \circ e_X \circ \eta_{X^*} \in \mathscr{L}(X)$, i.e., $\mathscr{L}(X)$ is closed under right derivatives. Closure under left derivatives is proved in a similar way.

Conversely, any $S \in \mathtt{StJSL}$ closed under left and right derivatives such that S is a subspace of $\mathtt{JSL}(\mathcal{P}_f(X^*), \mathbf{2}) \in \mathtt{StJSL}$ will define, by duality, the surjective function $e_S : \mathcal{P}_f(X^*) \to \mathtt{StJSL}(S, \mathbf{2})$ such that $e_S(\{w\})(L) = L(\{w\})$, $w \in X^*$ and $L \in S$, which is a morphism in $\mathtt{JSL}(\mathcal{P}_f(X^*), \mathtt{StJSL}(S, \mathbf{2}))$. We only need to show that for every $w \in X^*$ and $U, V \in \mathcal{P}_f(X^*)$ the equality $e_S(U) = e_S(V)$ implies that $e_S(\{w\}U) = e_S(\{w\}V)$ and $e_S(U\{w\}) = e_S(V\{w\})$. In fact, assume that $e_S(U) = e_S(V)$, i.e., for every $L \in S$ we have that $L(U) = L(V)$. Now, assume by contradiction that $e_S(\{w\}U) \neq e_S(\{w\}V)$, i.e., there exists $L' \in S$ such that $L'(\{w\}U) \neq L'(\{w\}V)$, i.e., $(L' \circ \eta_{X^*})_w(u) \neq (L' \circ \eta_{X^*})_w(v)$ with $(L' \circ \eta_{X^*})_w \in S$ by closure under right derivatives, which is a contradiction. The equality $e_S(U\{w\}) = e_S(V\{w\})$ is proved in a similar way by using closure under left derivatives. Therefore e_S is a T algebra morphism in $\mathscr{E}$. □

Remark. Note that Eilenberg–type correspondences for varieties of $\mathbb{K}$–algebras and idempotent semirings can also be obtained from Example 169.

7.2 Eilenberg–type correspondences for pseudovarieties of algebras

In this section, we derive Eilenberg–type correspondences for pseudovarieties of algebras by using Proposition 153. Eilenberg–type correspondences for pseudovarieties of algebras have been broadly studied in the literature, e.g., [36, 74, 72, 90, 70, 30]. There is a similarity with Eilenberg–type correspondences for the case of varieties of algebras, and the finiteness condition will only change the kind of object $\mathscr{L}(X)$ that a pseudocoequational theory has. We start by describing explicitely Eilenberg–type correspondences for pseudovarieties of algebras of a given type τ in which each function symbol in τ has finite arity.

Example 176 (cf. Example 169, cf. [85, 86]). Let τ be a type of algebras where each function symbol $g \in \tau$ has arity $n_g \in \mathbb{N}$ and let K be a variety of algebras for of type τ. Consider the case $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 = \mathtt{Set}_f$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections and let T_K be the monad such that for every $X \in \mathtt{Set}$, $T_K(X)$ is the underlying set of the free algebra in K on X generators (see [27, Definition II.10.9] and [66, VI.8]). We have that $\mathtt{CABA}$ is dual to $\mathtt{Set}$, so we can consider $\mathcal{C} = \mathtt{CABA}$ and $\mathcal{C}_0 = \mathtt{CABA}_f$. In this case, we get a one–to–one correspondence between pseudovarieties of algebras in K and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a Boolean algebra and it is a subalgebra of the complete atomic Boolean algebra $\mathtt{Set}(T_K(X), 2)$ of subsets of $T_K(X)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite algebra $\mathbf{A}$ in K, a morphism $h \in \mathtt{Alg}(\mathsf{T}_K)(\mathbf{T_K X}, \mathbf{A})$ and $L' \in \mathtt{Set}(A, 2)$ such that $L = L' \circ h$.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to the type τ. That is, for every $g \in \tau$ of arity n_g, every $1 \leq i \leq n_g$, every $\tilde{t} = (t_j)_{1 \leq j \leq n_g, j \neq i}$, $t_j \in T_K(X)$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,\tilde{t})} \in \mathscr{L}(X)$ where $L^{(i)}_{(g,\tilde{t})} \in \mathtt{Set}(T_K(X), 2)$ is defined as
$$L^{(i)}_{(g,\tilde{t})}(t) = L(g(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{n_g}))$$
$t \in T_K(X)$. That is, for every function symbol $g \in \tau$ we get n_g kinds of derivatives.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}_f$, homomorphism of T_K–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

In fact, let $\mathtt{P}$ be a pseudoequational T_K–theory on $\mathtt{Set}_f$ and let $\mathscr{L}$ be an operator on $\mathtt{Set}_f$ satisfying the three properties above. Then:

a) Define the operator $\mathscr{L}_{\mathtt{P}}$ on $\mathtt{Set}_f$ as $\mathscr{L}_{\mathtt{P}}(X) := \bigcup_{e \in \mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e, 2))$. We claim that $\mathscr{L}_{\mathtt{P}}$ satisfies the three properties above. In fact, as the family $\mathtt{P}(X)$ is directed in the sense of Definition 119 i), then the union $\bigcup_{e \in \mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e, 2)) \subseteq \mathtt{Set}(T_K(X), 2)$ is a directed union of finite objects in $\mathtt{CABA}$ which is a Boolean subalgebra of $\mathtt{Set}(T_K(X), 2)$. As each $e \in \mathtt{P}(X)$ has as codomain a finite algebra in K then $\mathrm{Im}(\mathtt{Set}(e, 2))$ is a subset of $\mathtt{Set}(T_K(X), 2)$ which is closed under derivatives with respect to the type τ (see Example 169). The previous argument shows that $\mathscr{L}_{\mathtt{P}}$ satisfies properties i) and ii) above. Now, closure under morphic preimages follows from property iii) in Definition 119. Therefore, $\mathscr{L}_{\mathtt{P}}$ satisfies the three properties above.

b) Define the operator $\mathtt{P}_{\mathscr{L}}$ on $\mathtt{Set}_f$ such that $\mathtt{P}_{\mathscr{L}}(X)$ is the collection of all T_K–algebra morphisms $e \in \mathscr{E}$ with domain $\mathbf{T_K X}$ and finite codomain such that $\mathrm{Im}(\mathtt{Set}(e, 2)) \subseteq \mathscr{L}(X)$. We claim that $\mathtt{P}_{\mathscr{L}}$ is a pseudoequational T_K–theory. In fact, we have that $\mathtt{P}_{\mathscr{L}}(X)$ is nonempty since $e : \mathbf{T_K X} \twoheadrightarrow \mathbf{1} \in \mathtt{P}_{\mathscr{L}}(X)$, where $\mathbf{1}$ is the one–element T_K–algebra. By definition, we have that $\mathtt{P}_{\mathscr{L}}(X)$ satisfies property ii) in Definition 119, and, it also satisfies property iii) in Definition 119 since $\mathscr{L}$ is closed under morphic preimages. Now, consider a family

$\{T_K(X) \xrightarrow{e_i} A_i\}_{i \in I}$ in $\mathtt{P}_{\mathscr{L}}(X)$ with I finite such that $\mathrm{Im}(\mathtt{Set}(e_i, 2)) \subseteq \mathscr{L}(X)$, we need to find a morphism $e \in \mathtt{P}_{\mathscr{L}}(X)$ such that every e_i factors through e. In fact, let $\mathbf{A}$ be the product of $\prod_{i \in I} \mathbf{A_i}$ with projections $\pi_i : A \to A_i$, then, by the universal property of $\mathbf{A}$ there exists a T_K–algebra morphism $f : T_K(X) \to A$ such that $\pi_i \circ f = e_i$, for every $i \in I$. Let $f = m_f \circ e_f$ be the factorization of f in $\mathtt{Alg}(\mathsf{T}_K)$. We claim that $e = e_f$ is a morphism in $\mathtt{P}_{\mathscr{L}}(X)$ such that every e_i factors through e. Clearly, from the construction above, each e_i factors through $e = e_f$. Now, let us prove that $\mathrm{Im}(\mathtt{Set}(e, 2)) \subseteq \mathscr{L}(X)$. In fact, let $\mathbf{S}$ be the codomain of $e = e_f$ and let $g \in \mathtt{Set}(S, 2)$. We have to prove that $g \circ e \in \mathscr{L}(X)$ which follows from the following straightforward identity:

$$g \circ e = \bigcup_{s \in g} \left(\bigcap_{i \in I} h_{i,s} \circ e_i \right)$$

where $h_{i,s} \in \mathtt{Set}(A_i, 2)$ is the set $\{\pi_i(m_f(s))\}$ (i.e., we express the subset g of S as the union of its points and each point $s \in S$ is represented as $\bigcap_{i \in I} h_{i,s} \circ \pi_i \circ m_f$). Now, for every $s \in S$ and $i \in I$ the composition $h_{i,s} \circ e_i$ belongs to $\mathscr{L}(X)$ since $h_{i,s} \circ e_i \in \mathrm{Im}(\mathtt{Set}(e_i, 2)) \subseteq \mathscr{L}(X)$. Finally, as S and I are finite then $g \circ e \in \mathscr{L}(X)$ because $\mathscr{L}(X)$ is a Boolean algebra.

c) We have that $\mathtt{P} = \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}$. In fact, for every $X \in \mathtt{Set}_f$ the inclusion $\mathtt{P}(X) \subseteq \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X)$ is obvious. Now, to prove that $\mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X) \subseteq \mathtt{P}(X)$, let $e' \in \mathtt{Alg}(\mathsf{T})(\mathbf{T_K X}, \mathbf{A}) \cap \mathscr{E}$ with finite codomain such that $e' \in \mathtt{P}_{\mathscr{L}_{\mathtt{P}}}(X)$, i.e., we have the inclusion $\mathrm{Im}(\mathtt{Set}(e', 2)) \subseteq \bigcup_{e \in \mathtt{P}(X)} \mathrm{Im}(\mathtt{Set}(e, 2))$. Then the previous inclusion means that for every $f \in \mathtt{Set}(A, 2)$ there exists $e_f \in \mathtt{P}(X)$ and g_f such that $f \circ e' = g_f \circ e_f$. As $\{e_f \mid f \in \mathtt{Set}(A, 2)\}$ is finite, then there exists $e \in \mathtt{P}(X)$ such that each e_f factors through e. We will prove that e' factors through $e \in \mathtt{P}(X)$ which will imply that $e' \in \mathtt{P}(X)$, since $\mathtt{P}$ is a pseudoequational T_K–theory. It is enough to show that $\ker(e) \subseteq \ker(e')$. In fact, assume that $(u, v) \in \ker(e)$ and define $f' \in \mathtt{Set}(A, 2)$ as $f'(x) = 1$ iff $x = e'(u)$. Then, as $e_{f'}$ factors through e we have that $\ker(e) \subseteq \ker(e_{f'})$ which implies $(u, v) \in \ker(e_{f'})$. But $\ker(e_{f'}) \subseteq \ker(g_{f'} \circ e_{f'}) = \ker(f' \circ e')$, which implies that $(u, v) \in \ker(f' \circ e')$, i.e., $1 = f'(e'(u)) = f'(e'(v))$, but the later equality means that $e'(u) = e'(v)$ by definition of f', i.e., $(u, v) \in \ker(e')$ as desired.

d) We have that $\mathscr{L} = \mathscr{L}_{\mathtt{P}_{\mathscr{L}}}$. In fact, for every $X \in \mathtt{Set}_f$ the inclusion $\mathscr{L}_{\mathtt{P}_{\mathscr{L}}}(X) \subseteq \mathscr{L}(X)$ is obvious. Now, to prove $\mathscr{L}(X) \subseteq \mathscr{L}_{\mathtt{P}_{\mathscr{L}}}(X)$ we need to find for every $L \in \mathscr{L}(X)$ a surjective homomorphism $e : T_K(X) \to A$ with $\mathbf{A} \in K$ such that $L \in \mathrm{Im}(\mathtt{Set}(e, 2)) \subseteq \mathscr{L}(X)$. In fact, for $L \in \mathscr{L}(X)$ let $e' : T_K(X) \to B$ be a homomorphism with $\mathbf{B} \in K$ and $g \in \mathtt{Set}(B, 2)$ such that $L = g \circ e'$, this can be done by property i) above. Let $\langle\!\langle L \rangle\!\rangle$ be the subset of $\mathtt{Set}(T_K(X), 2)$ obtained from $\{L\}$ which is closed under Boolean combinations and derivatives with respect to the type τ. We show that $\langle\!\langle L \rangle\!\rangle \in \mathtt{Coalg}_f(\mathsf{B})$, that is, we show that $\langle\!\langle L \rangle\!\rangle$ is a finite object in $\mathtt{CABA}$ that is closed under derivatives with respect to the type τ. In fact, $\mathrm{Im}(\mathtt{Set}(e', 2)) \in \mathtt{Coalg}_f(\mathsf{B})$ is such that $\langle\!\langle L \rangle\!\rangle \subseteq \mathrm{Im}(\mathtt{Set}(e', 2))$, which implies that $\langle\!\langle L \rangle\!\rangle$ is a finite Boolean algebra, i.e., an object in $\mathtt{Coalg}_f(\mathsf{B})$. By construction of $\langle\!\langle L \rangle\!\rangle$ we have that $L \in \langle\!\langle L \rangle\!\rangle \subseteq \mathscr{L}(X)$ since $\mathscr{L}$ satisfies properties i) and ii) above. Now, let $i \in \texttt{Coalg}(\mathsf{B})(\langle\!\langle L \rangle\!\rangle, \texttt{Set}(T_K(X), 2))$ be the inclusion morphism, then by duality we have that the dual morphism e in $\texttt{Alg}(\mathsf{T}_K)$ of i is such that $L \in \mathrm{Im}(\texttt{Set}(e, 2)) \subseteq \mathscr{L}(X)$ (in fact, $\mathrm{Im}(\texttt{Set}(e, 2)) = \langle\!\langle L \rangle\!\rangle$). Note that the codomain of e is in K since it is an $\mathscr{E}$–quotient of $\mathbf{B} \in K$.

Remark. Note that, for every "language" $L \in \texttt{Set}(T_K(X), 2)$, the object $\langle\!\langle L \rangle\!\rangle$ in d) above is the B–subcoalgebra of $\texttt{Set}(T_K(X), 2)$ generated by L which implies, by duality, that its dual is the syntactic algebra $\mathbf{S}_L$ of L. Additionally, by using duality and the construction of $\langle\!\langle L \rangle\!\rangle$, we have that every "language" in $\mathrm{Im}(\texttt{Set}(e, 2))$ (i.e., recognized by the syntactic algebra of L) is a Boolean combination of derivatives of L, where e is the dual of the inclusion $i \in \texttt{Coalg}(\mathsf{B})(\langle\!\langle L \rangle\!\rangle, \texttt{Set}(T_K(X), 2))$.

Additionally, if we consider K to be the variety of all algebras of type τ then we get the correspondence given in [85, 86]. $\square$

From the previous example, we get the following Eilenberg–type correspondences:

(1) [36, Theorem 34] A one–to–one correspondence between pseudovarieties of monoids and operators $\mathscr{L}$ on $\texttt{Set}_f$ such that for every $X \in \texttt{Set}_f$:

   i) $\mathscr{L}(X)$ is a Boolean subalgebra of $\texttt{Set}(X^*, 2)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite monoid $\mathbf{M}$, a monoid homomorphism $h : X^* \to M$ and $L' \in \texttt{Set}(M, 2)$ such that $L' \circ h = L$, i.e., L is a recognizable language on X.

   ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^*$.

   iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of monoids $h : Y^* \to X^*$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(2) [36, Theorem 34s] A one–to–one correspondence between pseudovarieties of semigroups and operators $\mathscr{L}$ on $\texttt{Set}_f$ such that for every $X \in \texttt{Set}_f$:

   i) $\mathscr{L}(X)$ is a Boolean subalgebra of $\texttt{Set}(X^+, 2)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite semigroup $\mathbf{S}$, a semigroup homomorphism $h : X^+ \to S$ and $L' \in \texttt{Set}(S, 2)$ such that $L' \circ h = L$, i.e., L is a recognizable language on X not containing the empty word.

   ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^+$.

   iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, homomorphism of semigroups $h : Y^+ \to X^+$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(3) A one–to–one correspondence between pseudovarieties of groups and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a Boolean subalgebra of $\mathtt{Set}(\mathfrak{F}_G(X), 2)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite group $\mathbf{G}$, a group homomorphism $h : \mathfrak{F}_G(X) \to G$ and $L' \in \mathtt{Set}(G, 2)$ such that $L' \circ h = L$.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives and inverses. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x, L^{-1} \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$, $L_x(w) = L(xw)$ and $L^{-1}(w) = L(w^{-1})$, $w \in \mathfrak{F}_G(X)$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of groups $h : \mathfrak{F}_G(Y) \to \mathfrak{F}_G(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(4) For a fixed monoid $\mathbf{M} = (M, e, \cdot)$, a one–to–one correspondence between pseudovarieties of $\mathbf{M}$–actions, i.e., dynamical systems on $\mathbf{M}$, and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a Boolean subalgebra of $\mathtt{Set}(M \times X, 2)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite $\mathbf{M}$–action $\mathbf{S}$, an $\mathbf{M}$–action homomorphism $h : M \times X \to S$ and $L' \in \mathtt{Set}(S, 2)$ such that $L' \circ h = L$.

ii) $\mathscr{L}(X)$ is closed under translations. That is, if $L \in \mathscr{L}(X)$ and $m \in M$ then $mL \in \mathscr{L}(X)$, where $mL(n, x) = L(m \cdot n, x)$, $(n, x) \in M \times X$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of $\mathbf{M}$–actions $h : M \times Y \to M \times X$ (i.e., $h(m \cdot (n, y)) = m \cdot h(n, y)$) and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

(5) (cf. [90]) A one–to–one correspondence between pseudovarieties of semigroups with infinite exponentiation and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a Boolean subalgebra of $\mathtt{Set}(X^+ \cup X^{(\omega)}, 2)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite semigroup with infinite exponentiation $\mathbf{S}$, a morphism $h \in \mathtt{Alg}(\mathsf{T})(\mathbf{TX}, \mathbf{S})$ and $L' \in \mathtt{Set}(S, 2)$ such that $L' \circ h = L$.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives and infinite exponentiation. That is, if $L \in \mathscr{L}(X)$ and $u \in X^+ \cup X^{(\omega)}$ then ${}_uL, L_u, L^\omega \in \mathscr{L}(X)$, where ${}_uL(w) = L(wu)$, $L_u(w) = L(uw)$ and $L^\omega(w) = L(w^\omega)$, $w \in X^+ \cup X^{(\omega)}$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of T–algebras $h : Y^+ \cup Y^{(\omega)} \to X^+ \cup X^{(\omega)}$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

In the next example we obtain an Eilenberg–type correspondence for pseudovarieties of ordered algebras of a given type τ such that each function symbol in τ has a finite arity.

Example 177 (cf. Example 170)**.** Let τ be a type of algebras where each function symbol $g \in \tau$ has arity $n_g \in \mathbb{N}$ and let K be a variety of ordered algebras of type τ. Consider the case $\mathcal{D} = \mathtt{Poset}$, $\mathcal{D}_0$ = finite discrete posets, $\mathscr{E}$ = surjections, $\mathscr{M}$ = embeddings and let T_K be the monad such that for every $\mathbf{X} = (X, \leq) \in \mathtt{Poset}$, $T_K(\mathbf{X}) := (T_K(X), \leq_{T_K X})$ is the underlying poset of the free ordered algebra in K on $\mathbf{X}$ generators (see [21, Proposition 1]). We have that $\mathtt{AlgCDL}$ is dual to $\mathtt{Poset}$, so we can consider $\mathcal{C} = \mathtt{AlgCDL}$, $\mathcal{C}_0 = \mathtt{CABA}_f$. In this case, we get a one–to–one correspondence between pseudovarieties of ordered algebras in K and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a distributive sublattice of $\mathtt{Poset}(T_K(X), \mathbf{2}_c) \cong \mathtt{Set}(T_K(X), 2)$ of subsets of $T_K(X)$ such that for every $L \in \mathscr{L}(X)$ there exists a finite ordered algebra $\mathbf{A}$ in K, a morphism $h \in \mathtt{Alg}(\mathsf{T}_K)(\mathbf{T_K X}, \mathbf{A})$ and $L' \in \mathtt{Poset}(A, \mathbf{2}_c)$ such that $L = L' \circ h$. Here $\mathbf{2}_c \in \mathtt{Poset}$ is the two–element chain.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to τ. That is, for every $g \in \tau$ of arity n_g, every $1 \leq i \leq n_g$, every $\tilde{t} = (t_j)_{1 \leq j \leq n_g, j \neq i}$, $t_j \in T_K(X)$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,\tilde{t})} \in \mathscr{L}(X)$ where $L^{(i)}_{(g,\tilde{t})} \in \mathtt{Set}(T_K(X), 2)$ is defined as
$$L^{(i)}_{(g,\tilde{t})}(t) = L(g(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{n_g}))$$
$t \in T_K(X)$. That is, for every function symbol $g \in \tau$ we get n_g kinds of derivatives.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of T_K–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$. □

Example 178 ([70, Theorem 5.8])**.** From the previous example we can obtain Eilenberg–type correspondences for pseudovarieties of ordered semigroups, pseudovarieties of ordered monoids, pseudovarieties of ordered groups, and so on. For instance, for the case of pseudovarieties of ordered monoids we can consider the type $\tau = \{e, \cdot\}$ where e is a nullary function symbol, $\cdot$ is a binary function symbol and K is the variety of ordered monoids. Then we get a one–to–one correspondence between pseudovarieties of ordered monoids and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a distributive sublattice of the distributive lattice $\mathtt{Set}(X^*, 2)$ of subsets of X^*, i.e., every element in $\mathscr{L}(X)$ is a language on X, such that every $L \in \mathscr{L}(X)$ is a regular language.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, homomorphism of monoids $h : Y^* \to X^*$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$. □

Next, we get an Eilenberg–type correspondence for pseudovarieties of many–sorted algebras.

Example 179 ([30], cf. Example 171)**.** For this example we assume the reader has some knowledge on many–sorted universal algebra, see, e.g., [87] and its references. Let S be a set of *sorts* and consider the category $\mathcal{D} = \mathtt{Set}^S$ of S–sorted sets with S–sorted functions, $\mathcal{D}_0 = \mathtt{Set}_f^S$, the finite S–sorted sets [1], $\mathscr{E} = S$–sorted surjections, and $\mathscr{M} = S$–sorted injections. We have that $\mathtt{CABA}^S$ is dual to $\mathtt{Set}^S$, so we can consider $\mathcal{C} = \mathtt{CABA}^S$ and $\mathcal{C}_0 = \mathtt{CABA}_f^S$. For any many–sorted signature τ on S, define the monad T_τ on $\mathtt{Set}^S$ such that $T_\tau(X)$ is the S–sorted set of τ–terms on X. Then we get a one–to–one correspondence between pseudovarieties of many–sorted algebras of type τ and operators $\mathscr{L}$ on $\mathtt{Set}_f^S$ such that for every $X \in \mathtt{Set}_f^S$:

i) $\mathscr{L}(X)$ is a Boolean subalgebra of the algebra $\mathtt{Set}^S(T_\tau(X), \mathbf{2}) \in \mathtt{CABA}^S$ such that for each $s \in S$ the Boolean algebra $\mathscr{L}(X)_s$ has only recognizable languages.

ii) $\mathscr{L}(X)$ is closed under derivatives with respect to τ. That is, for every operation symbol $g \in \tau$ of sort (w, s') (i.e., $w \in S^*$ and $s' \in S$) with $w = s_1 \cdots s_n$, every $1 \leq i \leq n$, every $t = (t_j)_{1\leq j\leq n, j\neq i}$ where $t_j \in T_\tau(X)_{s_j}$, $1 \leq j \leq n$, $j \neq i$, and every $L \in \mathscr{L}(X)$ we have that $L^{(i)}_{(g,t)} \in \mathscr{L}(X)$ where for every $s \in S$, $\left(L^{(i)}_{(g,t)}\right)_s \in \mathtt{Set}(T_K(X)_s, 2)$ is defined as

$$\left(L^{(i)}_{(g,t)}\right)_s(p) = \begin{cases} 0 & \text{if } s \neq s' \\ L_s(g(t_1, \ldots, t_{i-1}, p, t_{i+1}, \ldots, t_n)) & \text{if } s = s' \end{cases}$$

$p \in T_K(X)_{s_i}$. That is, for every function symbol $g \in \tau$ we get n kinds of derivatives.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}_f^S$, homomorphism of T_τ–algebras $h : T_K(Y) \to T_K(X)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$.

This Eilenberg–type correspondence is exactly the version shown in [30]. In this case, condition i) is equivalent to conditions (BPS 1) and (BPS 3) in [30], condition ii) is equivalent to condition (BPS 2) of closure under translations, and condition iii) is equivalent to condition (BPS 4). This kind of correspondence was also obtained in [89]. □

The following example shows a correspondence for fuzzy languages.

Example 180 ([69, Theorem 7])**.** Consider the case $\mathcal{D} = \mathtt{Set}$, $\mathcal{D}_0 = \mathtt{Set}_f$, $\mathscr{E} =$ surjections, $\mathscr{M} =$ injections and let T be the free monoid monad, i.e., $T(X) = X^*$,

[1] An S–sorted set $X = (X_s)_{s\in S}$ is finite if $\cup_{s\in S}\{s\} \times X_s$ is finite.

$X \in \texttt{Set}$. Now, consider the set $S = [0,1]$, let $\mathcal{C}$ be a category such that every object is isomorphic to an object of the form X^S and a morphism between an object $A \cong X^S$ and an object $B \cong Y^S$ is isomorphic to f^S for some $f \in \texttt{Set}(Y,X)$, and let $\mathcal{C}_0$ be the dual of $\mathcal{D}_0$. We use a similar argument as in Example 176 and use the correspondence between pseudoequational T–theories $\mathtt{P}$ and operators $\mathscr{L}$ on $\texttt{Set}_f$ by defining $\mathscr{L}(X) := \bigcup_{e \in \mathtt{P}(X)} \mathrm{Im}(\texttt{Set}(e,S))$. From this setting, we get a one–to–one correspondence between varieties of monoids and operators $\mathscr{L}$ on $\texttt{Set}_f$, which we call *pseudovarieties of fuzzy languages* (cf. [69]), such that for every $X \in \texttt{Set}_f$:

i) $\mathscr{L}(X)$ is a subset of $\texttt{Set}(X^*, S)$ in which for each $g \in \mathscr{L}(X)$ there exists a finite monoid $\mathbf{A}$, a monoid homomorphism $h : X^* \to A$ and $g' \in \texttt{Set}(A,S)$ such that $g = g' \circ h$, $\mathbf{B}(\mathscr{L}(X)) \overset{def}{=} \{\mathrm{supp}(g) \mid g \in \mathscr{L}(X)\} \subseteq \texttt{Set}(X^*, 2)$ is an object in $\mathtt{BA}$ with the usual set–theoretic operations, and the set $\mathscr{L}(X)$ is determined by $\mathbf{B}(\mathscr{L}(X))$ in the sense that
$$\mathscr{L}(X) = \left\{ g \in S^{X^*} \;\middle|\; g = \bigvee_{k \in K_0 \subseteq \mathbf{B}(\mathscr{L}(X))} s_k k,\ s_k \in S, K_0 \text{ finite} \right\},$$
where $\bigvee$ in the expression above is the maximum, i.e., $g(w) = \max\{s_k k(w) \mid k \in K_0\}$. Elements in $\texttt{Set}(X^*, S)$ are called *fuzzy languages on X*.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $g \in \mathscr{L}(X)$ and $u \in X^*$ then ${}_u g, g_u \in \mathscr{L}(X)$, where ${}_u g(v) = g(vu)$ and $g_u(v) = g(uv)$, $v \in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \texttt{Set}$, every monoid homomorphism $h : Y^* \to X^*$ and $g \in \mathscr{L}(X)$, we have that $g \circ h \in \mathscr{L}(Y)$.

Note that the three conditions above are equivalent to those that define a "variety of fuzzy languages" in the sense of [69]. In fact, in [69] it is shown that the operator $\mathbf{B}$ defined in i) above is a "variety of languages" in the classical sense [36], and that each "variety of languages" will induce a pseudovariety of fuzzy languages by the equality given above. Moreover, this correspondence is bijective [69, Theorem 6]. A "variety of fuzzy languages" in the sense of [69] is closed under unions, intersections, complements, multiplications by constants, quotients, inverse homomorphic images and cuts, see [69, Section 3]. We can prove those closure properties as follows:

a) Closure under unions, where by the union of $g_1, g_2 \in \mathscr{L}(X)$ is defined by taking their maximum componentwise. So, given $g_1, g_2 \in \mathscr{L}(X)$ means that there exists $e_i \in \mathtt{P}(X)$ with finite codomain A_i and $f_i \in \texttt{Set}(A_i, S)$ such that $f_i \circ e_i = g_i$, $i = 1,2$. Then, as $\mathtt{P}$ is a pseudoequational T–theory then there exists $e \in \mathtt{P}$ with finite codomain A such that e_i factors through e, say $e_i = h_i \circ e$. Define $f \in \texttt{Set}(A,S)$ as $f(a) = \max\{(f_1 \circ h_1)(a), (f_2 \circ h_2)(a)\}$, then
$$\begin{aligned}(g_1 \vee g_2)(w) &= \max\{g_1(w), g_2(w)\} = \max\{(f_1 \circ h_1 \circ e)(w), (f_2 \circ h_2 \circ e)(w)\} \\ &= f(e(w)) = (f \circ e)(w)\end{aligned}$$

that is, $g_1 \vee g_2 = f \circ e \in \mathscr{L}(X)$. Therefore, $\mathscr{L}(X)$ is closed under unions.

b) Closure under intersections, complements, multiplication by constants and cuts follow from a similar argument as in a) by defining the appropiate f as we did above. If fact, by using the notation above, for the intersection of g_1 and g_2 define $f(a) = \min\{(f_1 \circ h_1)(a), (f_2 \circ h_2)(a)\}$, for the complement of g_1 define $f(a) = 1 - (f_1 \circ h_1)(a)$, for multiplication of g_1 by a constant $c \in S$ define $f(a) = c(f_1 \circ h_1)(a)$, for the c–cut of g_1 define $f(a) = 1$ if $(f_1 \circ h_1)(a) \geq c$ and $f(a) = 0$ otherwise.

c) Closure under inverse homomorphic images is the same as condition iii) above.

d) Closure under quotients is proved in [69, Lemma 3].

Additionally, in the setting of this example, if we consider the pseudoequational T–theory that corresponds to the pseudovariety of commutative monoids satisfying some identity $x^n = x^{n+1}$, $n = 1, 2 = \ldots$, see Example 129, then we get an Eilenberg–type correspondence for commutative aperiodic fuzzy languages, cf. [7, 6]. A fuzzy language $g : X^* \to S$ on X is *commutative* if for every $u, v \in X^*$ we have that $g(uv) = g(vu)$. A fuzzy language $g : X^* \to [0, 1]$ on X is *aperiodic* if for every $u, v, w \in X^*$ we have $g(uv^n w) = g(uv^{n+1}w)$ for some $n \in \mathbb{N}^+$. In a similar way, we can get the Eilenberg–type correspondence for aperiodic fuzzy languages shown in [7]. □

The next example shows a correspondence for pseudovarieties of $\mathbb{K}$–algebras over a finite field $\mathbb{K}$.

Example 181 (cf. [74, Théorème III.1.1.] and Example 173). Let $\mathbb{K}$ be a finite field. Consider the case $\mathcal{D} = \mathtt{Vec}_{\mathbb{K}}$, $\mathcal{D}_0 =$ finite $\mathbb{K}$–vector spaces, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. We have that $\mathtt{StVec}_{\mathbb{K}}$ is dual to $\mathtt{Vec}_{\mathbb{K}}$, so we can consider $\mathcal{C} = \mathtt{StVec}_{\mathbb{K}}$ and $\mathcal{C}_0 =$ finite $\mathbb{K}$–vector spaces. For every set X denote by $\mathsf{V}(X)$ the $\mathbb{K}$–vector space with basis X. Consider the monad $T(\mathsf{V}(X)) = \mathsf{V}(X^*)$, where X^* is the free monoid on X. Then we get a one–to–one correspondence between pseudovarieties of $\mathbb{K}$–algebras and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a $\mathbb{K}$–vector space which is a subspace of $\mathtt{Vec}_{\mathbb{K}}(\mathsf{V}(X^*), \mathbb{K})$ such that every element S in $\mathscr{L}(X)$ is a *recognizable series* on X, i.e., there exists a $\mathbb{K}$–algebra morphism $h : \mathbf{TX} \to \mathbf{A}$, with $\mathbf{A}$ finite, and $S' \in \mathtt{Vec}_{\mathbb{K}}(A, \mathbb{K})$ such that $S' \circ h = S$.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $v \in \mathsf{V}(X^*)$ then ${}_vL, L_v \in \mathscr{L}(X)$, where ${}_vL(w) = L(wv)$ and $L_v(w) = L(vw)$, $w \in \mathsf{V}(X^*)$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, $\mathbb{K}$–linear map $h : \mathsf{V}(Y^*) \to \mathsf{V}(X^*)$ and $L \in \mathscr{L}(X)$, we have that $L \circ h \in \mathscr{L}(Y)$. □

Remark. Note that to consider the case in which $\mathbb{K}$ is an infinite field we need to consider $\mathcal{C}$ to be the category of linearly compact spaces and $\mathcal{C}_0$ the dual of $\mathcal{D}_0$ (see, [16]). □

The next example shows a correspondence for pseudovarieties of idempotent semirings.

Example 182 ([72, Theorem 5 (iii)] cf. Example 174)**.** Consider the case $\mathcal{D} = \mathtt{JSL}$, $\mathcal{D}_0 =$ finite free join semilattices, i.e., $\mathcal{D}_0 = \{(\mathcal{P}(X), \cup, \emptyset) \mid X \in \mathtt{Set}_f\}$, where $\mathcal{P}$ is the powerset operator, $\mathscr{E} =$ surjections and $\mathscr{M} =$ injections. We have that $\mathtt{StJSL}$ is dual to $\mathtt{JSL}$, so we can consider $\mathcal{C} = \mathtt{StJSL}$ and $\mathcal{C}_0 = \{\mathtt{JSL}((\mathcal{P}(X), \cup, \emptyset), 2) \mid X \in \mathtt{Set}_f\}$. Let T be the monad on $\mathtt{JSL}$ such that $T(S, \vee, 0)$ is the free idempotent semiring on $(S, \vee, 0) \in \mathtt{JSL}$. Then we get a one–to–one correspondence between pseudovarieties of idempotent semirings and operators $\mathscr{L}$ on $\mathtt{Set}_f$ such that for every $X \in \mathtt{Set}_f$:

i) $\mathscr{L}(X)$ is a join subsemilattice of $\mathtt{Set}(X^*, 2)$ such that every $L \in \mathscr{L}(X)$ is a regular language. In particular, $\mathscr{L}(X)$ is closed under unions.

ii) $\mathscr{L}(X)$ is closed under left and right derivatives. That is, if $L \in \mathscr{L}(X)$ and $x \in X$ then ${}_xL, L_x \in \mathscr{L}(X)$, where ${}_xL(w) = L(wx)$ and $L_x(w) = L(xw)$, $w \in X^*$.

iii) $\mathscr{L}$ is closed under morphic preimages. That is, for every $Y \in \mathtt{Set}$, semiring homomorphism $h : \mathcal{P}_f(Y^*) \to \mathcal{P}_f(X^*)$ and $L \in \mathscr{L}(X)$, we have that $L^\sharp \circ h \circ \eta_{Y^*} \in \mathscr{L}(Y)$ (see Example 174). □

Remark. Note that Eilenberg–type correspondences for pseudovarieties of $\mathbb{K}$–algebras and idempotent semirings can also be obtained from Example 176.

7.3 Local Eilenberg–type correspondences

In this section, we derive local Eilenberg–type correspondences for varieties and pseudovarieties of X–generated T–algebras. Local Eilenberg–type correspondences for pseudovarieties of algebras have been studied in [1, 43]. As we noted, this cases are particular cases for which the category $\mathcal{D}_0$ has only one object, namely X. Thus, the defining properties for their corresponding coequational theories and pseudocoequational theories are given only in terms of the object X, i.e., we consider local coequational theories and local pseudocoequational theories, respectively (see Definition 157 and Definition 160).

We can consider the same settings as in Section 7.1 to obtain local Eilenberg–type correspondences for varieties of X–generated T–algebras. That is, we fix an object $X \in \mathcal{D}$ satisfying the hypothesis of Proposition 158 to obtain the respective local versions. For example, the local version for the case of varieties of X–generated ordered groups reads as follows: There is a one–to–one correspondence between varieties of X–generated ordered groups and subalgebras $\mathbf{S} \in \mathtt{AlgCDL}$ of the completely distributive lattice $\mathtt{Set}(\mathfrak{F}(X), 2)$, where $\mathfrak{F}(X)$ is the free group on X, such that:

i) S is closed under left and right derivatives and inverses. That is, for every $L \in S$ and $x \in X$ we have that ${}_xL, L_x, L^{-1} \in S$. Here $L^{-1}(w) = L(w^{-1})$.

ii) S is closed under morphic preimages. That is, for every homomorphism of groups $h : \mathfrak{F}(X) \to \mathfrak{F}(X)$ and $L \in S$, we have that $L \circ h \in S$.

Now, in a similar way, we can consider the same settings as in Section 7.2 to obtain local Eilenberg–type correspondences for pseudovarieties of X–generated $\mathtt{T}$–algebras. For example, the local version for the case of pseudovarieties of X–generated $\mathbb{K}$–algebras reads as follows: There is a one–to–one correspondence between varieties of X–generated $\mathbb{K}$–algebras and subspaces $\mathbf{S} \in \mathtt{Vec}_{\mathbb{K}}$ of the space $\mathtt{Vec}_{\mathbb{K}}(\mathtt{V}(X^*), \mathbb{K})$ such that:

i) Every element $L \in S$ is a recognizable series on X.

ii) S is closed under left and right derivatives. That is, for every $L \in S$ and $x \in X$ we have that ${}_xL, L_x \in S$.

iii) S is closed under morphic preimages. That is, for every $\mathbb{K}$–linear map $h : \mathtt{V}(X^*) \to \mathtt{V}(X^*)$ and $L \in S$, we have that $L \circ h \in S$.

7.4 Discussion

In this chapter, we showed some applications of the results we obtained in Chapter 6 on Eilenberg–type correspondences. We mainly focused on deriving Eilenberg–type correspondences for most of the known cases studied in the literature in order to show the generality of the abstract Eilenberg–type correspondences we showed in Chapter 6, although more than a half of them are new. In our case, the main dualities on which we based our examples were the ones between `Set` and `CABA`, between `Poset` and `AlgCDL`, between `JSL` and `StJSL`, and between $\mathtt{Vec}_{\mathbb{K}}$ and $\mathtt{StVec}_{\mathbb{K}}$, where $\mathbb{K}$ is a finite field. All those dualities were described in Section 1.4.

It is worth mentioning that we can even get more correspondences from the examples we showed in the present chapter. For instance, from Example 169 we can use the class K as a parameter to get correspondences for different kinds of varieties such as, e.g., abelian groups, quasigroups, rings, modules over a fixed ring, lattices, Heyting algebras, and so on (see, e.g., [27]). A similar comment applies for the case of Example 176, which will give correspondences for the case of pseudovarieties. Furthermore, more Eilenberg–type correspondences are obtained if we consider different dualities than the ones mentioned in Section 1.4, e.g., Stone duality, Stone duality for nominal Boolean algebras [40], Priestley duality, or any other kind of duality of interest (see, e.g., [29, 32]). In this sense, the number of specific correspondences we can derive is countless.

Even though our correspondences were derived in a straightforward way, the work in this thesis shows that in order to obtain the general results shown in Chapter 6 we need to understand the general theory of equations, coequations, varieties, pseudovarieties, Birkhoff's theorem, duality and their relationship by using categorical methods. A key concept that helped us to gain more understanding was that of an equational T–theory, which is a categorical generalization of an equational theory and played an important role in this work. Our easy to understand slogan that "Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality" and other contributions such as "varieties of languages = duals of equational theories" have helped to unveil Eilenberg–type correspondences, which has been an active field during more than forty years and its full understanding had not been accomplished until the present thesis.

Final remarks

The notion of coequations, which is the dual notion of equations, has been explicitly studied in the search of a dual of Birkhoff's theorem [61, 62, 3, 2, 11, 75, 31, 46, 49, 84, 76]. In this sense, covarieties of coalgebras are characterized as classes of coalgebras described by coequations. In this thesis, besides of broading the study of coequations and showing some applications in which coequations can be used, we showed that coequations (or more precisely, coequational theories) implicitly appeared in Eilenberg–type correspondences since Eilenberg's variety theorem [36] under the name of varieties of languages.

This not only allowed us to unveil Eilenberg–type correspondences but also to obtain simpler proofs and to fully explain the defining properties of a variety of languages. All of this was done in the present thesis for the case of varieties of algebras, pesudovarieties of algebras and their corresponding local versions.

We now proceed to dicuss some of the ideas for future work that are based on the ideas presented in this thesis. They include the following:

1) To derive different Eilenberg–type correspondences than the ones shown in the last chapter. We could, for instance, consider other kinds of algebraic structures such as, e.g., abelian groups, quasigroups, rings, modules over a fixed ring, lattices, Heyting algebras, and so on (see, e.g., [27]). Note that Eilenberg–type correspondences for the just mentioned algebraic structures can be obtained from Example 169 and Example 176, but the defining properties for the corresponding operators $\mathscr{L}$ are in each case different and can give more intuition and lead to some applications if they are worked out individually. Similarly, another kind of algebras that can be considered are nominal algebras [63, 39, 41]. A different direction is to apply the results of Chapter 6 by considering different dualities than the ones we used, for instance, to consider Stone duality, Stone duality for nominal Boolean algebras [40], Priestley duality, or dualities such as the ones presented in [29, 32].

2) To establish new kinds of Eilenberg–type correspondences for different classes of algebraic structures, see, e.g., [8, 67]. A natural next step in this direction is to consider implicational classes, in which implications are considered instead of equations. The implicational case can be studied by omitting the requirement in an equational theory of having free projective domain. A categorical approach of implicational classes has been already studied in [15].

3) To study coEilenberg–type correspondences, that is, duals of Eilenberg–type correspondences. From the understanding and explanation for Eilenberg–type correspondences given in this thesis, the notion of a coEilenberg–type correspondence can naturally be defined as one–to–one correspondences between covarieties of coalgebras and equational theories. As in the case of Eilenberg–type correspondences, we can study this concept together with its finite version, local version and also a dual of the implicational case (cf. [48]).

4) To study and understand the concept of a coequational theory. Equations and defining properties of an equational theory are easily understood and have been broadly studied in the literature. The case of coequations and coequational theories is less known and some studies of this case include [61, 62, 3, 2, 11, 75, 31, 46, 49, 84, 76]. The concept of a coequational theory presented in this thesis, which is defined as the dual concept of an equational theory, is new and can be studied to get more intuition and understanding on coequations, their logic and its relation with existing work such as [3, 61, 84].

5) Connections of the present work with modal logic. Modal logic has algebraic and coalgebraic perspectives [19, Chapter 6]. The work done in this thesis, which also has an algebraic and coalgebraic flavour, can lead us to new applications either in the field of modal logic or in the field of Eilenberg–type correspondences. This can give us new perspectives, applications or developments in the area.

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Index of symbols

Index

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Amsterdam, October 2017

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Summary

An Eilenberg–type correspondence is a one–to–one correspondence between varieties of algebras and the so–called “varieties of languages”. They have been broadly studied in the literature since Eilenberg’s variety theorem in the 1970s. The concept of a variety of algebras is a well–known concept in universal algebra which, by Birkhoff’s theorem, is the same as a class of algebras satisfying some given equations. However, the concept of a variety of languages is less known and its defining properties have not been fully explained in the literature, they have been ad hoc definitions. The work in this thesis fully explains what varieties of languages are and where their defining properties come from. In fact, varieties of languages are duals of equational theories, that is, coequational theories.

The fact that varieties of languages are duals of equational theories not only allows us to fully understand Eilenberg–type correspondences, but also to get a general categorical version for Eilenberg–type correspondences from which we can derive countless instances. Some of those instances have been proved separately and published in different papers. Another advantage of this understanding is the simplification of proofs for particular Eilenberg–type corerspondences found in the literature. Most of the proofs in the literature follow the same idea as Eilenberg by proving the existence of syntactic algebras in order to obtain the desired correspondence. The work in this thesis shows that existence of syntactic algebras is not a necessary condition for the existence Eilenberg–type correspondences.

After introducing some basic concepts and fixing the notation, we start by studying two particular structures: deterministic automata and weighted automata. In both cases we show what equations and coequations for these automata are and what it means for an automaton to satisfy them. The concept of a coequation, which is the dual of an equation, is less known in the literature and we fully explain them for the cases of deterministic automata and weighted automata. Additionally, we show duality results between equations and coequations for automata.

We develop a general abstract theory of equations and coequations in Chapter 4. In the case of equations, we define equations for algebras for an endofunctor and also equations for Eilenberg–Moore algebras. We present a similar work for the case of coequations, that is, coequations for coalgebras for an endofunctor and coequations for Eilenberg–Moore coalgebras. We define categories of equations and coequations. Then, we show how dualities on the base categories can be lifted, under mild assumptions, to a duality between the categories of algebras

and coalgebras and also how the duality is lifted to categories of equations and coequations. We also obtain the previous liftings for the case of Eilenberg–Moore algebras and coalgebras.

In Chapter 5, we provide a categorical version of Birkhoff's theorem which, for the purpose of this thesis, we state as a one–to–one correspondence between varieties of algebras and equational theories. We introduce the categorical concept of an equational theory, which is a new definition whose main purpose is to explain the concept of a variety of languages used in Eilenberg–type correspondences. We also provide a Birkhoff's theorem for varieties of finite algebras, for local varieties of algebras and for local varieties of finite algebras.

In Chapter 6, we collect all general background and categorical development of equations, coequations and Birkhoff's theorem to obtain abstract theorems for Eilenberg–type correspondences. All of this is represented by the slogan

> "Eilenberg–type correspondences = Birkhoff's theorem for (finite) algebras + duality",

which explains the real nature of Eilenberg–type correspondences. We state a genera theorem for Eilenberg–type correspondences for the following four cases: varieties of algebras, pseudovarieties of algebras, local varieties of algebras and local pseudovarieties of algebras.

As an application of our general results we show, in Chapter 7, some of the particular instances we can get from our general theorems from Chapter 6. We derive a total of 64 correspondences. To the best of our knowledge, only 20 of them are known in the literature. Most of these known correspondences were proved and published separately in at least 12 different papers. The remaining 44 new correspondences that we show are for varieties and local varieties of algebras, which have been less studied in the literature.

Samenvatting

Een Eilenberg-achtige *correspondentie* is een *een-op-een-relatie* tussen variëteiten van algebra's en de zogeheten "variëteiten van talen". Deze correpondenties zijn uitgebreid bestudeerd in de literatuur sinds de variëteiten stelling van Eilenberg in de jaren 70. Het concept van een variëteit is een algemeen begrip in de universele algebra die, volgens de stelling van Birkhoff, overeenkomt met de klasse van algebra's die aan zekere vergelijkingen voldoen. Echter, het concept van een variëteit van talen is minder bekend en hun definities in de literatuur zijn ad hoc en onvoldoende gemotiveerd. Het werk in dit proefschrift beschrijft precies wat variëteiten van talen zijn en waar hun definiërende eigenschappen vandaan komen. In feite zijn variëteiten van talen de dualen van *theorieën van vergelijkingen,* ofwel variëteiten van talen zijn *theorieën van covergelijkingen*.

Het feit dat variëteiten van talen de dualen zijn van theorieën van vergelijkingen stelt ons niet alleen in staat om Eilenberg-achtige correspondenties volledig te begrijpen, maar ook om een algemene categorische versie te krijgen waaruit we talloze voorbeelden kunnen afleiden. Sommige van deze gevallen zijn afzonderlijk bewezen en in verschillende artikelen gepubliceerd. Een ander voordeel van dit begrip is de vereenvoudiging van bewijzen van bepaalde Eilenberg-achtige correspondenties die in de literatuur voorkomen. De meeste bewijzen in de literatuur volgen hetzelfde idee als Eilenberg door het bestaan *syntactische algebra's* aan te tonen om het gewenste resultaat te verkrijgen. Het werk in dit proefschrift laat zien dat het bestaan van syntactische algebra's geen noodzakelijk voorwaarde is voor het bestaan van Eilenberg-achtige correspondenties.

Na het introduceren van enkele basisbegrippen en het vastleggen van de notatie, beginnen we met het bestuderen van twee bepaalde structuren: deterministische automaten en *automaten met gewichten*. In beide gevallen laten we zien wat vergelijkingen en covergelijkingen voor deze automaten zijn en wat het voor een automaat betekent om hieraan te voldoen. Het concept van een covergelijking, ofwel de duale van een vergelijking, is minder bekend in de literatuur en we behandelen dit uitvoerig voor de gevallen van deterministische automaten en automaten met gewichten. Bovendien bespreken we resultaten over dualiteit tussen vergelijkingen en covergelijking voor automaten.

We ontwikkelen een algemene abstracte theorie van vergelijkingen en covergelijking in Hoofdstuk 4. In het geval van vergelijkingen definiëren we vergelijkingen voor algebra's over een endofunctor en ook voor vergelijkingen voor Eilenberg-

Moore-algebra's. We doen een soortgelijk werk voor het geval van covergelijking, dat wil zeggen, covergelijking voor coalgebras over een endofunctor en covergelijking voor Eilenberg-Moore coalgebras. We definiëren categorieën van vergelijkingen en covergelijkingen. Vervolgens laten we zien hoe dualiteiten op basis van categorieën, onder milde aannames, kunnen worden *uitgebreid* tot een dualiteit tussen de categorieën van algebra's en coalgebras en ook hoe de dualiteit wordt uitgebreid in categorieën van vergelijkingen en covergelijkingen. De eerdere uitbreidingen worden ook verkregen voor het geval van Eilenberg-Moore-algebra's en coalgebras.

In Hoofdstuk 5 geven we een categorische versie van de stelling van Birkhoff die we, ten behoeve van dit proefschrift, poneren als een een-op-een-correspondentie tussen variëteiten van algebra's en theorieën van vergelijkingen. We introduceren het categorische concept van een theorieën van vergelijkingen; een nieuwe definitie met als belangrijkste doel het verklaren van het concept van een variëteit van talen dat gebruikt wordt in Eilenberg-achtige correspondenties. We presenteren ook versies van de stelling van Birkhoff voor variëteiten van eindige algebra's, voor lokale variëteiten van algebra's en voor lokale variëteiten van eindige algebra's.

In Hoofdstuk 6 voegen we de algemene achtergrond en de categorische behandeling van vergelijkingen, covergelijkingen en de stelling van Birkhoff samen om abstracte stellingen voor Eilenberg-achtige correpondenties te verkrijgen. Dit alles wordt vertegenwoordigd door de slogan

> "Eilenberg-achtige correspondenties = stelling van Birkhoff voor (eindige) algebra's + dualiteit ",

wat de ware aard van Eilenberg-achtige correspondenties verklaart. We poneren een algemene stelling voor Eilenberg-achtige correspondenties voor de volgende vier gevallen: variëteiten van algebra's, pseudovariëteiten van algebra's, lokale variëteiten van algebra's en lokale pseudovariëteiten van algebra's.

Als een toepassing van onze algemene resultaten laten we in Hoofdstuk 7 enkele specifieke voorbeelden zien die we kunnen afleiden uit onze algemene stellingen uit Hoofdstuk 6. In totaal presenteren we 64 correspondenties, waarvan, voor zover bij ons bekend, slechts 20 reeds bekend zijn in de literatuur. De meeste van de bekende correspondenties zijn afzonderlijk bewezen en gepubliceerd in ten minste 12 verschillende artikelen. De resterende 44 door ons aangetoonde niewe correspondenties hebben betrekking tot variëteiten en lokale variëteiten van algebra's die minder bestudeerd zijn in de literatuur.

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