Centrum voor Wiskunde en Informatica

REPORT *RAPPORT*

Dynamic Reasoning Without Variables

D.J.N. van Eijck

Information Systems (INS)

INS-R9801 January 31, 1998

Report INS-R9801
ISSN 1386-3681

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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Jan van Eijck
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT
A variable free notation for dynamic logic is proposed which takes its cue from De Bruijn's variable free notation for lambda calculus. De Bruijn indexing replaces variables by indices which indicate the distance to their binders. We propose to use reverse De Bruijn indexing, which works almost the same, only now the indices refer to the depth of the binding operator in the formula. The resulting system is analysed at length and applied to a new rational reconstruction of discourse representation theory. It is argued that the present system of dynamic logic without variables provides an explicit account of anaphoric context and yields new insight into the dynamics of anaphoric linking in reasoning. A calculus for dynamic reasoning with anaphora is presented and its soundness and completeness are established.

1991 Mathematics Subject Classification: 03B65, 68Q55,
1991 Computing Reviews Classification System: I.2.4, I.2.7
Keywords and Phrases: dynamic semantics, natural language processing, anaphora and context, dynamic reasoning with anaphora
Note: Work carried out under project P4303; paper under review.

1. Introduction

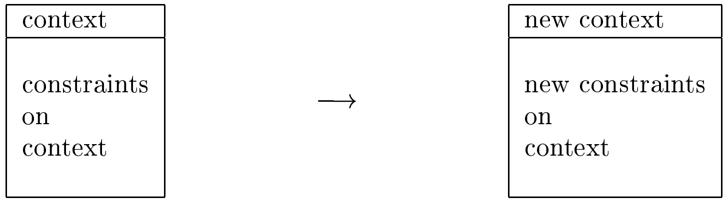
In recent developments of natural language semantics, problems of pronominal reference ana anaphoric linking have inspired logicians to a dynamic turn in natural language semantics. This started with Discourse Representation Theory (Kamp [14]) and File Change Semantics (Heim [12]), and various rational reconstructions of these proposals, with Barwise [3] and Groenendijk and Stokhof [10] as the most prominent ones.

The gist of these proposals is that the static variable binding regime from standard predicate logic gets replaced by a dynamic regime, where meanings are viewed as relations between variable states in a model.

In the original version of the 'dynamic shift', the basic ingredients are contexts and constraints on contexts. A Kamp-style representation for a piece of text (or: discourse) looks basically like this:

context
constraints on context

The informal picture of how the information conveyed by a piece of text grows is that of 'updating' of representation structures:



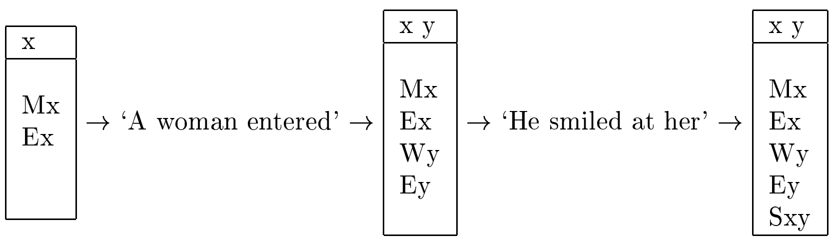
This picture can only be made to work if we make sure that the contexts are represented smartly. Contexts are essentially sets of variables: a context just is a list of dynamically bound variables. These variables represent the antecedents which are available in any extension of that context. Embedded contexts (contexts occurring inside the constraints on a given context) and extensions of contexts (representing extensions of anaphoric possibilities) should always employ fresh variables, for if they do not, existing anaphoric possibilities get blocked off by destructive value assignment.

The rational reconstructions of dynamic discourse representation given by Barwise [3] and Groenendijk and Stokhof [10] essentially represent introduction of new antecedents by means of random assignment to a variable. The meaning of $\exists x$ becomes the relation between variable states f, g with the property that f and g differ at most in their x value:

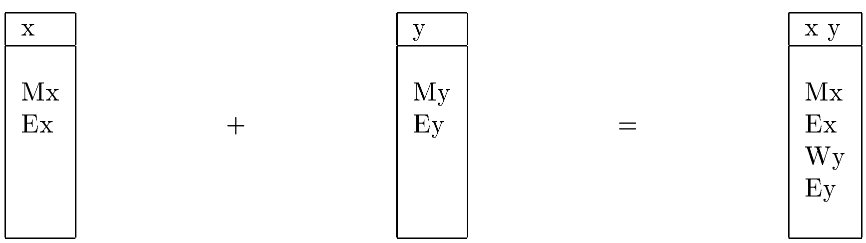
$${}_f[\![\exists x]\!]_g \text{ iff } f[x]g.$$

This does indeed solve the problem of how to use dynamic scoping of variables to account for unbounded anaphoric linkings, but it does not give a rational reconstruction of the fact that discourse representation is supposed to work incrementally.

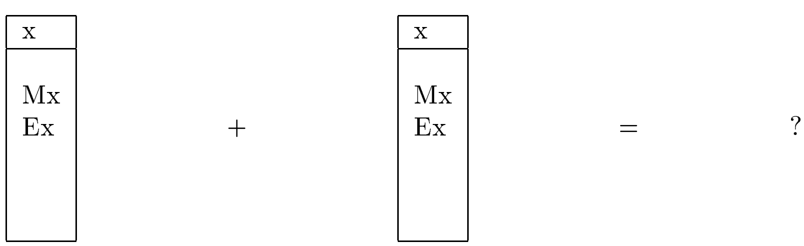
What one would like is illustrated by the following example, where we assume an initial representation for the sentence ‘A man entered’, which gets updated by subsequent processing of ‘A woman entered’, and next of ‘He smiled at her.’



In a rational reconstruction of this, one would assume that the sentences to be added to the existing representation have a representation of their own, so one would get something like:



Problem: what happens if we get a variable clash:



In Kamp's original version of discourse representation theory, and also in the extended version presented in Kamp and Reyle [15], this problem does not occur, for the algorithm presented there always parses new sentences in the context of an existing representation structure, and for any indefinite noun phrase it encounters, it simply gives the instruction: 'take a fresh variable.' In other words, Kamp never *merges* representation structures. In an algebraic reconstruction of the theory we want to be able to do so. There are various approaches to the merge problem for DRSs (see Van Eijck and Kamp [9] for an overview). These amount to various ways of avoiding destructive assignments to variables, i.e., to various ways of arriving at structures which can be interpreted monotonically in terms of an information ordering on the meanings of the representation structures.

In this paper we will argue that a variable free representation of dynamic logic leads to a very natural monotonic interpretation, and thus to a natural solution to the merge problem. We get 'fresh variables' for free if we replace the (static or dynamic) variable binding mechanism of static or dynamic predicate logic with an indexing mechanism. We will illustrate this first for standard predicate logic, and next for its dynamic relative.

2. Predicate Logic Without Variables

Predicate logics without variables have a long history. Starting point is the method of 'explaining variables away' of Quine [**?**, **?**]. Based on this, Kuhn [17] and Purdy [18] have proposed variable free representations for natural language understanding. Based on an even older approach (Peirce's existential graphs), Sanchez [20] has developed a variable free natural logic. There is also a long tradition of variable free notation in lambda calculus: combinatory logic (see [2]) and De Bruijn indices [6] come to mind here. We will take our cue from this tradition.

The De Bruijn notation for lambda calculus consists of replacing variables by indices that indicate the distance to their binding lambda operator. The lambda term $\lambda x \lambda y.(\lambda z.(y(zx))(yx)$ is written in De Bruijn notation as $\lambda\lambda.(\lambda.(2\ (1\ 3))(1\ 2)$. This approach carries over to predicate logic in a straightforward fashion. Let C be a set of individual constants and P a set of predicate constants of given arities. Assume c ranges over C and P is an n-place predicate constant from P. Then the language of De Bruijn style predicate logic without variables is given by:

$$\begin{array}{lcl} t & ::= & c \mid 1 \mid 2 \mid \cdots \\ \phi & ::= & Pt_1 \cdots t_n \mid \neg\phi \mid (\phi_1 \wedge \phi_2) \mid \exists\phi \mid \forall\phi. \end{array}$$

Here are some examples of De Bruijn style notation for predicate logic, with corresponding first order formulas in the familiar notation.

$(\exists P1 \wedge \exists Q1)$	$\exists xPx \wedge \exists xQx$
$\exists(P1 \wedge Q1)$	$\exists x(Px \wedge Qx)$
$\exists\forall R12$	$\exists x\forall yRyx$
$\forall R1\ 1$	$\forall xRxx$
$\forall R2\ 2$	$\forall xRyy$

The predicate logical formulas on the right are meant as translations modulo alphabetical variance, of course. To judge the correspondences (especially the last one, for unbound variables), an independent semantics for the language is needed.

The clauses of the truth relation $\models_{dB}$ below are stated in terms of a model $\mathrm{M} = \langle M, I\rangle$ and a sequence $\sigma \in M^{\mathbb{N}^+}$. We will use c^{M} for4 $I(c)$ and P^{M} for $I(P)$. Also, we use σ_k for the k-th element of σ. Also if $a \in M$ and $k \in \mathbb{N}^+$, $\sigma[k := a]$ denotes the sequence σ' given by $\sigma'_m = \sigma_m$ for $m \neq k$, $\sigma'_m = a$ for $m = k$.

First we define term denotations:

$$[\![t]\!]^M_\sigma := \begin{cases} t^M & \text{if } t \in C, \\ \sigma_t & \text{if } t \in \mathbb{N}^+. \end{cases}$$

In the definition of interpretations for formulas we need a function for the quantifier depth of a formula:

$$\begin{array}{lcl} d(Pt_1 \cdots t_n) & := & 0 \\ d(\neg\phi) & := & d(\phi) \\ d(\phi_1 \wedge \phi_2) & := & \max(d(\phi_1), d(\phi_2)) \\ d(\exists\phi) & := & d(\phi) + 1 \\ d(\forall\phi) & := & d(\phi) + 1. \end{array}$$

We use this in the truth definition, as follows:

$$\begin{array}{lll} \mathrm{M}, \sigma \models_{dB} Pt_1 \cdots t_n & \text{iff} & \langle [\![t_1]\!]^M_\sigma, \ldots, [\![t_n]\!]^M_\sigma \rangle \in P^{\mathrm{M}} \\ \mathrm{M}, \sigma \models_{dB} \neg\phi & \text{iff} & \mathrm{M}, \sigma \not\models_{dB} \phi \\ \mathrm{M}, \sigma \models_{dB} (\phi_1 \wedge \phi_2) & \text{iff} & \mathrm{M}, \sigma \models_{dB} \phi_1 \text{ and } \mathrm{M}, \sigma \models_{dB} \phi_2 \\ \mathrm{M}, \sigma \models_{dB} \exists\phi & \text{iff} & \text{for some } a \in M,\ \mathrm{M}, \sigma[d(\phi)+1 := a] \models_{dB} \phi \\ \mathrm{M}, \sigma \models_{dB} \forall\phi & \text{iff} & \text{for every } a \in M,\ \mathrm{M}, \sigma[d(\phi)+1 := a] \models_{dB} \phi. \end{array}$$

Assuming a set of indexed variables $V = \{x_1, x_2, \ldots\}$, a translation $^\circ$ from De Bruijn notation to standard notation is now given by:

$$\begin{array}{lcl} t^\circ & := & \begin{cases} t & \text{if } t \in C \\ x_t & \text{if } t \in \mathbb{N}^+ \end{cases} \\ (Pt_1 \cdots t_n)^\circ & := & Pt_1^\circ \cdots t_n^\circ \\ (\neg\phi)^\circ & := & \neg(\phi)^\circ \\ (\phi_1 \wedge \phi_2)^\circ & := & (\phi_1^\circ \wedge \phi_2^\circ) \\ (\exists\phi)^\circ & := & \exists x_{d(\phi)+1}\phi^\circ \\ (\forall\phi)^\circ & := & \forall x_{d(\phi)+1}\phi^\circ. \end{array}$$

If $\sigma \in M^{\mathbb{N}^+}$, we may regard σ as an assignment in M^V (given by $\sigma(x_k) = \sigma_k$). The following proposition gives the connection between De Bruijn style predicate logic and standard predicate logic.

Proposition 1 *For every $\phi \in L_0$:*

$$M, \sigma \models_{dB} \phi \text{ iff } M, \sigma \models \phi^\circ.$$

Proof. Induction on the structure of ϕ.

Here is the reasoning for the existential quantifier case. $\mathrm{M}, \sigma \models_{dB} \exists\phi$ iff there is some $a \in M$ with $\mathrm{M}, \sigma[d(\phi)+1 := a] \models_{dB} \phi$ iff there is some $a \in M$ with $\mathrm{M}, \sigma[x_{d(\phi)+1} := a] \models \phi^\circ$ iff $\mathrm{M}, \sigma \models \exists x_{d(\phi)+1}\phi^\circ$ iff $\mathrm{M}, \sigma \models (\exists\phi)^\circ$. ⊣

Surprisingly, De Bruijn style predicate logic turns out to be a notational variant of the system of variable free predicate logic proposed in Ben-Shalom [4].

As an alternative to the De Bruijn style binding regime, where the binding quantifier is found by counting from the inside out, it is also possible to count from the outside in. This is similar to the way lambdas are counted in Cartesian closed category models of the lambda calculus (see e.g. Gunter [11, Ch. 3]). Call this 'reverse De Bruijn style' (see also Aczel [1]). The language remains the same, but the truth definition becomes slightly more involved. It is given as a sequence of relations $\models_1, \models_2, \ldots$.

The term interpretation function $[\![\cdot]\!]^M_\sigma$ remains as before. The truth definition becomes ($k \in \mathbb{N}^+$):

$$\begin{array}{lll} \mathrm{M}, \sigma \models_k Pt_1 \cdots t_n & \text{iff} & \langle [\![t_1]\!]^M_\sigma, \ldots, [\![t_n]\!]^M_\sigma \rangle \in P^{\mathrm{M}} \\ \mathrm{M}, \sigma \models_k \neg\phi & \text{iff} & \mathrm{M}, \sigma \not\models_k \phi \\ \mathrm{M}, \sigma \models_k (\phi_1 \wedge \phi_2) & \text{iff} & \mathrm{M}, \sigma \models_k \phi_1 \text{ and } \mathrm{M}, \sigma \models_k \phi_2 \\ \mathrm{M}, \sigma \models_k \exists\phi & \text{iff} & \text{for some } a \in M,\ \mathrm{M}, \sigma[k := a] \models_{k+1} \phi \\ \mathrm{M}, \sigma \models_k \forall\phi & \text{iff} & \text{for every } a \in M,\ \mathrm{M}, \sigma[k := a] \models_{k+1} \phi. \end{array}$$

The reverse De Bruijn interpretation relation $\models_{rdB}$ is the relation $\models_0$.

Here is a set of translation functions $^{(k)}$ from reverse De Bruijn style predicate logic to standard first order logic. The translation $^\circ$ for terms remains as before.

$$\begin{array}{lcl}
(Pt_1 \cdots t_n)^{(k)} & := & Pt_1^\circ \cdots t_n^\circ \\
(\neg\phi)^{(k)} & := & \neg(\phi)^{(k)} \\
(\phi_1 \wedge \phi_2)^{(k)} & := & (\phi_1^{(k)} \wedge \phi_2^{(k)}) \\
(\exists\phi)^{(k)} & := & \exists x_k \phi^{(k+1)} \\
(\forall\phi)^{(k)} & := & \forall x_k \phi^{(k+1)}.
\end{array}$$

Proposition 2 *For all $\phi \in L_0$:*

$$M, \sigma \models_{rdB} \phi \text{ iff } M, \sigma \models \phi^{(0)}.$$

Proof. The proposition follows from a more general statement which we prove by induction on the complexity of ϕ. For all $\phi \in L_0$, all $k \in \mathbb{N}^+$:

$$\mathrm{M}, \sigma \models_k \phi \text{ iff } \mathrm{M}, \sigma \models \phi^{(k)}.$$

Here is the reasoning for the existential quantifier case. $\mathrm{M}, \sigma \models_k \exists\phi$ iff there is some $a \in M$ with $\mathrm{M}, \sigma[k := a] \models_{k+1} \phi$ iff there is some $a \in M$ with $\mathrm{M}, \sigma[x_k := a] \models \phi^{(k+1)}$ iff $\mathrm{M}, \sigma \models \exists x_k \phi^{(k+1)}$ iff $\mathrm{M}, \sigma \models (\exists\phi)^{(k)}$. ⊣

3. Dynamic Predicate Logic Without Variables

The language L of variable free dynamic predicate logic has $\exists$ as a formula in its own right, and replaces $\wedge$ with a connective ; for sequential composition.

$$\begin{array}{lcl}
t & ::= & c \mid 1 \mid 2 \mid \cdots \\
A & ::= & \bot \mid \exists \mid Pt_1 \cdots t_n \\
\phi & ::= & A \mid A; \phi \mid \neg(\phi).
\end{array}$$

Note that we have built into the language that ; creates a flat list structure.

We will omit unnecessary parentheses, writing $\neg(Pt_1 \cdots t_n)$ as $\neg Pt_1 \cdots t_n$, etcetera. Occasionally, we will write $\exists; \phi$ as $\exists\phi$. Also, we abbreviate $\neg\bot$ as $\top$, and $\neg(\phi_1; \cdots; \phi_n; \neg\phi_{n+1})$ as $(\phi_1; \cdots; \phi_n) \to \phi_{n+1}$.

The static interpretation is replaced by a dynamic one. Interpretation of terms as before. For the dynamic interpretation relation it is more convenient to use reverse De Bruijn style than regular De Bruijn style. The reason is the following.

The key feature of dynamic predicate logic is the ability of the existential quantifier to bind variables outside its proper scope. Consider the DPL text $\exists x; Px; \exists y; Qy; Rxy$. Here the x and y of Rxy are bound outside of the proper scope by $\exists x$ and $\exists y$ respectively, so variables can be viewed as anaphoric elements linked to a preceding existential quantifier that introduces a referent. The regular De Bruijn analogue of this text would be the following:

$$\exists; P1; \exists; Q1; R2\ 1$$

Here anaphoric coreference (or: dynamic binding) is no longer encoded by use of the same index, but the antecedent of an index has to be worked out taking the 'existential depth' of the intervening formula into account.

Regular De Bruijn style interpretation of the existential quantifier employs the following stack instruction:

$$\mathrm{M}, \sigma, \sigma' \models \exists \text{ iff } \sigma' = a\hat{\ }\sigma \text{ for some } a \in M.$$

Here $a\hat{\ }\sigma$ denotes the sequence σ' with $\sigma'_1 = a$ and $\sigma'_k = \sigma_{k-1}$ for $k > 1$.

The awkwardness in antecedent recovery can be avoided by using reverse De Bruijn indexing. As in the case of reverse indexing for variable free predicate logic, we define the truth relation in stages. We use $\sigma[k]\sigma'$ to indicate that σ and σ' differ at most at position k. We also need a function for existential depth of L formulas. The existential depth of a formula is given by:

$$\begin{array}{lcl} e(\bot) & := & 0 \\ e(\exists) & := & 1 \\ e(Pt_1 \cdots t_n) & := & 0 \\ e(A;\phi) & := & e(A) + e(\phi) \\ e(\neg(\phi)) & := & 0. \end{array}$$

This function is used in the clause for sequential composition in the following semantic stipulations:

$$\begin{array}{lll} \mathrm{M}, \sigma, \sigma' \models_k \bot & & \text{never} \\ \mathrm{M}, \sigma, \sigma' \models_k \exists & \text{iff} & \sigma[k+1]\sigma' \\ \mathrm{M}, \sigma, \sigma' \models_k Pt_1 \cdots t_n & \text{iff} & \sigma = \sigma' \text{ and } \langle [\![t_1]\!]^M_\sigma, \ldots, [\![t_n]\!]^M_\sigma \rangle \in P^{\mathrm{M}} \\ \mathrm{M}, \sigma, \sigma' \models_k (A;\phi) & \text{iff} & \text{there is a } \sigma'' \text{ with } \mathrm{M}, \sigma, \sigma'' \models_k A \text{ and } \mathrm{M}, \sigma'', \sigma' \models_{k+e(A)} \phi \\ \mathrm{M}, \sigma, \sigma' \models_k \neg(\phi) & \text{iff} & \sigma = \sigma' \text{ and there is no } \sigma'' \text{ with } \mathrm{M}, \sigma, \sigma'' \models_k \phi. \end{array}$$

As in the case of reverse De Bruijn style predicate logic, we use $\models_{rdB}$ for the relation $\models_0$.

The definition of $\models_{rdB}$ for L is in fact a straightforward adaptation of the dynamic semantics for predicate logic defined in Groenendijk and Stokhof [10], which is in turn closely related to a proposal made by Barwise in [3].

However, this semantics is *not* equivalent to the semantics given by Groenendijk and Stokhof, but has an important advantage over it. In Groenendijk and Stokhof's semantics for DPL, a repeated assignment to a single variable by means of a repeated use of the same existential quantifier-variable combination blocks off the individual introduced by the first use of the quantifier from further anaphoric reference. After $\exists x Px; \exists x Qx$, the variable x will refer to the individual introduced by $\exists x Qx$, and the individual introduced by $\exists x Px$ has become inaccessible.

In the sequence semantics proposed by Vermeulen [23] this problem is solved by making every variable refer to a stack, and interpreting an existential quantification for variable x as a push operation on the x stack. The quantification $\exists x$ now gets a counterpart xE, interpreted as a pop of the x stack.

In the $\models_{dB}$ semantics for L one uses a single infinite stack, and one doesn't allow pops (if existential quantification is to be non-destructive). In our reverse De Bruijn style semantics for L the push stack operation is replaced by a set of stack pointer operations for modifying the stack at depth k, where k depends on the existential depth of the preceding part of the formula (this is made precise in the definition of the truth relation above). Note that quantifications never can destroy previous dynamic assignments in the same formula, but they can overwrite initially given values. In

$$P1; \exists; Q1$$

the first occurrence of the index 1 refers to the first position of the input state, while the second occurrence of 1 refers to this same position after it has been reset by $\exists$. Thus, the two occurrences of the same index do not co-refer.

For a translation from L to standard DPL we use a set of functions ${}^k : L_1 \longrightarrow \text{DPL}$. The definition of these functions uses the term translation $^\circ$, and also the function e for existential depth.

$$\begin{array}{lcl} (\exists)^k & := & \exists x_{k+1} \\ (Pt_1 \cdots t_n)^k & := & Pt_1^\circ \cdots t_n^\circ \\ (A;\phi)^k & := & A^k; \phi^{k+e(A)} \\ (\neg(\phi))^k & := & \neg(\phi)^k. \end{array}$$

Proposition 3 *For all $\phi \in L_1$:*

$$M, \sigma, \sigma' \models_{rdB} \phi \textit{ iff } M, \sigma, \sigma' \models_{dpl} \phi^0.$$

Proof. Again, we will prove a stronger statement by induction on the structure of ϕ, namely:

$$\text{for all } k \in \mathbb{N} : \mathrm{M}, \sigma, \sigma' \models_k \phi \text{ iff } \mathrm{M}, \sigma, \sigma' \models_{dpl} \phi^k.$$

The existential quantifier case: $\mathrm{M}, \sigma, \sigma' \models_k \exists$ iff $\sigma[k+1]\sigma'$ iff $\mathrm{M}, \sigma, \sigma' \models_{dpl} \exists x_{k+1}$ iff $\mathrm{M}, \sigma, \sigma' \models_{dpl} (\exists)^k$.

The sequential composition case: $\mathrm{M}, \sigma, \sigma' \models_k (A; \phi)$ iff there is a σ'' with

$$\mathrm{M}, \sigma, \sigma'' \models_k A \text{ and } \mathrm{M}, \sigma'', \sigma' \models_{k+e(A)} \phi$$

iff (ind. hypothesis) there is a σ'' with

$$\mathrm{M}, \sigma, \sigma'' \models_{dpl} A^k \text{ and } \mathrm{M}, \sigma'', \sigma' \models_{dpl} \phi^{k+e(A)}$$

iff $\mathrm{M}, \sigma, \sigma' \models_{dpl} A^k; \phi^{k+e(A)}$ iff (definition of ^{k}) $\mathrm{M}, \sigma, \sigma' \models_{dpl} (A; \phi)^k$. ⊣

4. Offset Formulas

To make it into a suitable tool for the rational reconstruction of dynamic reasoning, the present variable free account of dynamic predicate logic is in need of one further enhancement. Up until now we have tacitly assumed that the stack manipulations which interpret the existential quantifier start at the first position of the input assignment. This does not completely solve the problem of destructive assignment, for in $P1; \exists; Q1$ the first occurrence of the index 1 refers to the first position of the input state, while the second index refers to this same position after it has been reset by $\exists$. Thus, the two occurrences of the same index do not co-refer. As long as we treat texts where every position is existentially quantified over (the dynamic equivalent of closed formulas), this does not matter.

But we would like to cover open texts as well: sequences of formulas in which some positions take their content from the surrounding context. After all, the process of picking up antecedents from context is the essence of anaphoric linking. All we have to do is to shift the index from the dynamic truth relation to the language itself. Let an offset formula be a pair (k, ϕ) with $k \in \mathbb{N}$ and $\phi \in L_1$. The interpretation of offset formulas is given by: $\{(\sigma, \sigma') \mid \mathrm{M}, \sigma, \sigma' \models_k \phi\}$. An example:

$$(2, \exists; R(1, 3); S(2, 3)).$$

This would correspond to the DPL formula:

$$\exists x_3; Rx_1x_3; Sx_2x_3.$$

The translation function for mapping offset L formulas to DPL formulas is of course the mapping ^{k} defined in Section 3.

The positions $1, \ldots, k$ of offset formula (k, ϕ) are not affected by the stack dynamics of the existential quantifier. The values of these positions are read from the input state; these are the anaphoric references picked up from the surrounding context. Positions higher up on the stack may be affected by an existential quantifier action inside ϕ, or their values may be destroyed later in a subsequent context. Call an offset formula *bounded* if it contains no such threatened positions (a formal definition is given below).

If $X \subseteq \mathbb{N}$ and X finite, then $\sup(X)$ is defined by:

$$\begin{aligned} \sup(\emptyset) &:= 0, \\ \sup(\{n\} \cup Y) &:= \sup(n, \sup(Y)). \end{aligned}$$

An index n occurs open in a formula (k, ϕ) if the occurrence of n is not in the dynamic scope of an occurrence of $\exists$, bound otherwise. To make this precise, we can use the function g which calculates the greatest natural number that occurs as an unbound index in (k, ϕ). The definition of $g : \mathbb{N} \times L_1 \to \mathbb{N}$ is:

$$g(k, \phi) := g'(k, 0, \phi),$$

where $g' : \mathbb{N} \times \mathbb{N} \times L_1 \to \mathbb{N}$ is given by:

$$\begin{array}{lll} g'(k,m,Pt_1 \cdots t_n) & := & \sup(\{t_i \in \mathbb{N}^+ \mid 1 \le i \le n, t_i \notin [k..k+m]\}) \\ g'(k,m,\exists) & := & 0 \\ g'(k,m,A;\phi) & := & \max(g'(k,m,A), g'(k,m+e(A),\phi)) \\ g'(k,m,\neg(\phi)) & := & g'(k,m,\phi). \end{array}$$

For example:

$$g(2, \exists; R(3,4)) = 4,$$

$$g(3, \exists; R(3,4)) = 0.$$

Note that the second argument of g' keeps track of the growth of the range of bound numbers starting from a given position.

Definition 4 *An L formula is* bounded *if every occurrence of a number in it is bound by a preceding occurrence of $\exists$. An offset L formula (k,ϕ) is* bounded *if its highest unbound number is less than or equal to k. An L formula ϕ is k-bounded if (k,ϕ) is bounded.*

It is useful to have a function for the highest unbound number in an offset L formula. Define $m : (\mathbb{N} \times L_1) \to \mathbb{N}$ by means of $m(k,\phi) := g(k,\phi) \dot{-} k$. An offset formula (k,ϕ) is bounded iff $m(k,\phi) = 0$. This generalizes the notion of boundedness for L formulas, for an L formula is bounded iff $m(0,\phi) = 0$.

What $m(k,\phi) = 0$ says is that the free indices of (k,ϕ) (indices having an occurrence in (k,ϕ) which is not dynamically bound by an occurrence of $\exists$ in (k,ϕ)) all are in $[1..k]$.

We will use $\sigma[k..m]$ for the finite sequence $(\sigma_k, \dots, \sigma_m)$ and $\sigma[k..\omega]$ for the infinite sequence $(\sigma_k, \dots)$.

Lemma 5 (Finiteness lemma) *Let an offset formula (k,ϕ) be given. Let $p := \max(k, m(k,\phi))$ and $q := \max(k + e(\phi), m(k,\phi)) + 1$. Assume $(\sigma,\sigma') \in [\![k,\phi]\!]^M$. Then the following hold:*

1. $\sigma[1..k] = \sigma'[1..k]$.
2. $\sigma[q..\omega] = \sigma'[q..\omega]$.
3. *if* $\sigma[1..p] = \tau[1..p]$, $\sigma'[p+1..q] = \tau'[p+1..q]$ *and* $\tau[q..\omega] = \tau'[q..\omega]$, *then* $(\tau,\tau') \in [\![k,\phi]\!]^M$.

Proof. 1. Immediate from the truth definition.

2. An inspection of the definition of q shows that the values $[q..\omega]$ are not affected by $\exists$ actions.

3. σ and τ agree on the values that are read by ϕ, σ' and τ' on the values that are written by ϕ, τ, τ' agree on the values that are neither read nor written. ⊣

Lemma 6 (Monotonicity lemma) *Assume (k,ϕ) is bounded and suppose $(\sigma,\sigma') \in [\![k,\phi]\!]^M$. Then the following hold:*

1. $\sigma[k+e(\phi)+1..\omega] = \sigma'[k+e(\phi)+1..\omega]$.
2. *if* $\sigma[1..k] = \tau[1..k]$, $\sigma'[k{+}1..k{+}e(\phi)] = \tau'[k{+}1..k{+}e(\phi)]$ *and* $\tau[k{+}e(\phi){+}1..\omega] = \tau'[k{+}e(\phi){+}1..\omega]$, *then* $(\tau,\tau') \in [\![k,\phi)]\!]^M$.

Proof. Immediate from the finiteness lemma and the definition of boundedness. ⊣

It follows from the finiteness lemma that one can define the semantics of bounded offset formulas in terms of finite assignments, as follows. We now take M^* as the set of assignments for model $\mathrm{M} = (M, I)$ and we use $l(\sigma)$ for the length of $\sigma \in M^*$. ϵ is the assignment of length 0. The assignment consisting only of element a is denoted a. $\sigma\hat{}\tau$ denotes the concatenation of finite assignments σ and τ.

$$
\begin{array}{rll}
(\sigma,\tau) \in [\![n,\bot]\!]^{\mathrm{M}} & & \text{never,} \\
(\sigma,\tau) \in [\![n,\exists]\!]^{\mathrm{M}} & \text{iff} & l(\sigma) = n \text{ and } \tau = a \text{ for some } a \in M, \\
(\sigma,\tau) \in [\![n, Pt_1 \cdots t_k]\!]^{\mathrm{M}} & \text{iff} & l(\sigma) = n \text{ and } \tau = \epsilon \text{ and } \langle [\![t_1]\!]^{\mathrm{M}}_{\sigma}, \ldots, [\![t_n]\!]^{\mathrm{M}}_{\sigma} \rangle \in R^{\mathrm{M}}, \\
(\sigma,\tau) \in [\![n,\neg\phi]\!]^{\mathrm{M}} & \text{iff} & l(\sigma) = n \text{ and } \tau = \epsilon \text{ and there is no } \tau' \text{ with } (\sigma,\tau') \in [\![\phi]\!]^{\mathrm{M}}, \\
(\sigma,\tau) \in [\![n,\phi;\psi]\!]^{\mathrm{M}} & \text{iff} & l(\sigma) = n, \tau_1 = \tau[1..e(\phi)], \tau_2 = \tau[e(\phi)+1..l(\tau)], \\
 & & \text{and } (\sigma,\tau_1) \in [\![n,\phi]\!]^{\mathrm{M}}, (\sigma\hat{}\tau_1, \tau_2) \in [\![n+e(\phi),\psi]\!]^{\mathrm{M}}.
\end{array}
$$

If $(\sigma,\tau) \in [\![n,\phi]\!]^{\mathrm{M}}$ then σ is the read memory and τ the write memory used in the interpretation process of n,ϕ.

Proposition 7 *For all bounded (n,ϕ), all models M, all $\sigma,\sigma' \in M^{\mathbb{N}^+}$:*

$$
\begin{array}{rcl}
M,\sigma,\sigma' \models_n \phi & \iff & (\sigma[1..n], \sigma'[n+1..n+e(\phi)]) \in [\![n,\phi]\!]^{M} \\
 & \iff & (\sigma'[1..n], \sigma'[n+1..n+e(\phi)]) \in [\![n,\phi]\!]^{M}.
\end{array}
$$

5. Variable Free Dynamic Logic and Discourse Representation

Next, we want to show that offset formulas correspond exactly to Discourse Representation Structures in the sense of Kamp [14]. DRSs are those structures defined by the following mutual recursion which satisfy a variable constraint (stated below).

$$
\begin{array}{rcl}
c & ::= & c_1 \mid c_2 \mid \cdots \\
v & ::= & x_1 \mid x_2 \mid \cdots \\
t & ::= & c \mid v \\
C & ::= & Pt_1 \cdots t_n \mid t_1 \doteq t_2 \mid \neg D \\
D & ::= & (\{v_1,\ldots,v_n\}, \{C_1,\ldots,C_m\})
\end{array}
$$

Call the structures defined here proto DRSs. To define DRSs, we need to distinguish three kinds of variable occurrences in proto DRSs: (1) fixed by the larger context, (2) fixed in the current context, and (3) fixed in a subordinate context.

To define these sets, we first define a function *var* on the atomic conditions of a DRS.

$$var(Pt_1 \cdots t_n) := \{t_i \mid 1 \le i \le n, t_i \in U\}$$

$$var(t_1 \doteq t_2) := \{t_i \mid 1 \le i \le 2, t_i \in U\}$$

Definition 8 (fix, intro, cbnd)

- $\mathit{fix}(\{v_1,\ldots,v_n\},\{C_1,\ldots,C_m\}) := \bigcup_i \mathit{fix}(C_i) - \{v_1,\ldots,v_n\}$.
- $\mathit{intro}(\{v_1,\ldots,v_n\},\{C_1,\ldots,C_m\}) := \{v_1,\ldots,v_n\}$.
- $\mathit{cbnd}(\{v_1,\ldots,v_n\},\{C_1,\ldots,C_m\}) := \bigcup_i \mathit{cbnd}(C_i)$.
- $\mathit{fix}(Pt_1 \cdots t_n) := \mathit{var}(Pt_1 \cdots t_n)$, $\mathit{intro}(Pt_1 \cdots t_n) := \emptyset$, $\mathit{cbnd}(Pt_1 \cdots t_n) := \emptyset$.
- $\mathit{fix}(t_1 \doteq t_2) := \mathit{var}(t_1 \doteq t_2)$, $\mathit{intro}(t_1 \doteq t_2) := \emptyset$, $\mathit{cbnd}(t_1 \doteq t_2) := \emptyset$.
- $\mathit{fix}(\neg D) := \mathit{fix}(D)$, $\mathit{intro}(\neg D) := \emptyset$, $\mathit{cbnd}(\neg D) := \mathit{intro}(D) \cup \mathit{cbnd}(D)$.

DRSs are the proto-DRSs D with $\mathit{intro}(D) \cap \mathit{fix}(D) = \emptyset$, (and the same condition imposed on subordinate DRSs D').

If $D = (\{v_1, \ldots, v_n\}, \{C_1, \ldots, C_m\})$ and D' are DRSs, then $D \Rightarrow D'$ abbreviates the condition

$$\neg(\{v_1, \ldots, v_n\}, \{C_1, \ldots, C_m, \neg D'\}).$$

We give a translation function $^{[k]}$ from offset formulas to DRSs (where k is the offset), as follows (using $\phi_0^{[k]}$ and $\phi_1^{[k]}$ for the first and second components of $\phi^{[k]}$):

$$\begin{array}{lcl} (Pt_1 \cdots t_n)^{[k]} & := & (\emptyset, \{Pt_1^{\circ} \cdots t_n^{\circ}\}) \\ (t_1 \doteq t_2)^{[k]} & := & (\emptyset, \{t_1^{\circ} \doteq t_2^{\circ}\}) \\ (\exists)^{[k]} & := & (\{x_{k+1}\}, \emptyset) \\ (A;\phi)^{[k]} & := & (A_0^{[k]} \cup \phi_0^{[k+e(A)]}, A_1^{[k]} \cup \phi_1^{[k+e(A)]}) \\ (\neg(\phi))^{[k]} & := & (\emptyset, \neg\phi^{[k]}). \end{array}$$

Example:

$$(R(1,2); \exists; R(1,3))^{[2]} \quad = \quad \begin{array}{|l|} \hline x_3 \\ \hline Rx_1x_2 \\ Rx_1x_3 \\ \hline \end{array}$$

Proposition 9 *If (k, ϕ) is bounded, then:*

1. *$\phi^{[k]}$ is a DRS.*
2. *σ verifies $\phi^{[k]}$ in M (in the sense of DRT) iff there is a σ' with $M, \sigma, \sigma' \models_k \phi$.*

Proof. Both claims are proved by induction on the structure of ϕ. $\dashv$

The DRS translations have the additional property that they yield *pure* DRSs: If K' is a sub-DRS K then their sets of introduced markers will be disjoint.

A special case is the case of bounded $(0, \phi)$. These correspond precisely to so-called *proper* DRSs, i.e., DRSs without 'anchored' (or 'fixed') variable occurrences.

6. Merging Offset Formulas

Suppose we want to 'merge' two offset formulas (n, ϕ) and (m, ψ) in left-to-right order, in such a way that the output of (n, ϕ) serves as input to (m, ψ). One could define a merge operation $\bullet$ as a partial operation on offset formulas, defined as follows:

$$(n, \phi) \bullet (m, \psi) := \begin{cases} (n, \phi; \psi) & \text{if } m = n + e(\phi), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In case the result of merging (n, ϕ) and (m, ψ) is undefined all is not lost, however. The undefinedness may be due to the fact that $m < n + e(\phi)$ or to the fact that $m > n + e(\phi)$. In the first case, the problem can be remedied by performing a 'write memory shift operation' on (m, ψ), as follows:

$$\frac{(m, \psi)}{(m + k, [^{m}_{+k}]\psi)}$$

Here, $[{}^{m}_{+k}]\psi$ is the index substitution which replaces every $i > m$ by $i+k$. Here is a formal definition:

$$
\begin{array}{rcl}
[{}^{m}_{+k}]t & := & \begin{cases} t & \text{if } t \in C, \\ t & \text{if } t \in \mathbb{N}, t \leq m, \\ t+k & \text{if } t \in \mathbb{N}, t > m \end{cases} \\
[{}^{m}_{+k}]\exists & := & \exists \\
[{}^{m}_{+k}]Pt_1 \cdots t_n & := & P[{}^{m}_{+k}]t_1 \cdots [{}^{m}_{+k}]t_n \\
[{}^{m}_{+k}](A;\phi) & := & [{}^{m}_{+k}]A; [{}^{m}_{+k}]\phi \\
[{}^{m}_{+k}]\neg\phi & := & \neg[{}^{m}_{+k}]\phi.
\end{array}
$$

Proposition 10 *$(\sigma, \tau) \in [\![m, \psi]\!]^M$ iff for all $\theta \in M^k$: $(\sigma\hat{\ }\theta, \tau) \in [\![m+k, [{}^{m}_{+k}]\psi]\!]^M$.*

The other case where the result of merging (n, ϕ) and (m, ψ), in that order, is undefined, is the case where $m > n + e(\phi)$. In this case we can use 'existential padding'. A useful abbreviation for this is $\exists^k$, defined recursively as follows:

$$
\begin{array}{rcl}
\exists^0 & := & \top \\
\exists^{k+1} & := & \exists; \exists^k
\end{array}
$$

Existential padding is applied as follows:

$$
\frac{(m+k, \psi)}{(m, \exists^k; \psi)}
$$

Proposition 11 *If $(\sigma, \tau) \in [\![m+k, \psi]\!]^M$ then $(\sigma[1..m], \sigma[m+1..l(\sigma)]\hat{\ }\tau) \in [\![m, \exists^k; \psi]\!]^M$.*

The rules for memory shift and existential padding are built into the calculus of Section 9.

7. The Merge Problem in Discourse Representation Theory

As Proposition 9 has shown us, the rational reconstruction of discourse representation theory or DRT [14] is not DPL but reverse De Bruijn indexing for L. In fact, the reconstruction has made us sensitive to a distinction which is left implicit in DRT: the distinction between representation structures which contain *anchors* (reference markers not introduced in the structure itself but imported from a pre-existing representation) and representation structures which do not (no reference markers are imported from outside; every marker gets introduced in the structure itself).

The variable constraint imposed in DRT avoids the destructive assignment problem from DPL, but makes it harder to look at DRT in a compositional way (see Van Eijck and Kamp [9] for discussion). A compositional perspective is readily available once we are prepared to use variable free notation.

Several possible solutions to the merge problem for DRT are discussed in Van Eijck and Kamp [9]. If one wants merge to be a total operation on DRSs, the merge of DRSs D and D', in that order, may involve substitution of the introduced variables of D'.

The present variable free perspective on dynamic logic suggests yet another way to merge representation structures. The DRS translations of offset formulas have the following general form:

$$
\begin{array}{|c|}
\hline
x_{m+1}, \ldots, x_n \\
\hline
C_1 \\
\vdots \\
C_k \\
\\
\hline
\end{array}
$$

Here it is assumed that all the markers occurring in $C_1, \dots, C_k$ are among $x_1, \dots, x_n$. The markers $x_1, \dots, x_m$ are the fixed markers of the DRS, the markers $x_{m+1} \cdots x_n$ the introduced reference markers.

Assuming that DRSs are all in this canonical form, we can merge them as follows, using substitution to avoid variable clashes:

$$\begin{array}{|c|}\hline x_{p+1}, \dots, x_{p+m} \\ \hline C_1 \\ \vdots \\ C_k \\ \hline \end{array} \quad \bullet_{[x_{p+m+1}/x_{q+1}, \dots, x_{p+m+n}/x_{q+n}]} \quad \begin{array}{|c|}\hline x_{q+1}, \dots, x_{q+n} \\ \hline C'_1 \\ \vdots \\ C'_r \\ \hline \end{array}$$

$$\Longrightarrow \quad \begin{array}{|c|}\hline x_{p+1}, \dots, x_{p+m+n} \\ \hline C_1 \\ \vdots \\ C_k \\ C'_1[x_{p+m+1}/x_{q+1}, \dots, x_{p+m+n}/x_{q+n}] \\ \vdots \\ C'_r[x_{p+m+1}/x_{q+1}, \dots, x_{p+m+n}/x_{q+n}] \\ \hline \end{array}$$

Example:

$$\begin{array}{|c|}\hline x_2, x_3 \\ \hline Rx_1x_2 \\ Sx_2x_3 \\ \hline \end{array} \quad \bullet_{[x_4/x_3, x_5/x_4]} \quad \begin{array}{|c|}\hline x_3, x_4 \\ \hline Tx_1x_4 \\ Vx_3x_4 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{|c|}\hline x_2, x_3, x_4, x_5 \\ \hline Rx_1x_2 \\ Sx_2x_3 \\ Tx_1x_5 \\ Vx_4x_5 \\ \hline \end{array}$$

This corresponds to the attempt to combine $(1, \exists; \exists; R(1,2); S(2,3))$ and $(2, \exists; \exists; T(1,4), V(3,4))$, in that order. This is possible after memory shift right of the second formula over one position, to get $(3, \exists; \exists; T(1,5), V(4,5))$. After this, the formulas $(1, \exists; \exists; R(1,2); S(2,3))$ and $(3, \exists; \exists; T(1,5), V(4,5))$ can be merged, with the following result:

$$(1, \exists; \exists; R(1,2); S(2,3); \exists; \exists; T(1,5), V(4,5).$$

The switch rules of the calculus of Section 9 permit to transform this in turn into:

$$(1, \exists; \exists; \exists; \exists; R(1,2); S(2,3); T(1,5); V(4,5),$$

which again corresponds to (a canonical representation of) the result DRS.

8. Dynamic Reasoning Without Variables

The modelling of anaphoric presupposition as a context (a piece of read-only memory) suggests a very natural consequence notion for 'reasoning under anaphoric presupposition'. The anaphoric presupposition is given by an offset n, and under this offset ϕ entails ψ iff for all models, the interpretation of ϕ under this offset is 'more informative' than that of ψ under the offset incremented with the effect of processing ϕ. This gives the following definition:

Definition 12 $n, \phi \models \psi :\Longleftrightarrow$
for all M, σ, τ*: if* $(\sigma, \tau) \in [\![n, \phi]\!]^M$ *then there is a* θ *with* $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^M$.

This consequence relation is truly dynamic in that it allows carrying anaphoric links from premiss to conclusion. For example: from 'a man walks and he talks' it follows that 'he talks':

$$0, \exists; M1; W1; T1 \models T1.$$

One of the problems with the dynamic consequence relation of DPL [10] is the fact that it is not transitive, as witnessed by Van Benthem's example:

Suppose a man owns a house. Then he owns a garden.
Suppose a man owns a garden. Then he sprinkles it.
BUT NOT: Suppose a man owns a house. Then he sprinkles it.

This is indeed a counterexample against transitivity of $\models_{dpl}$, where $\models_{dpl}$ is defined by: $\phi \models_{dpl} \psi$ iff for all M, σ, σ': if M, $\sigma, \sigma' \models_{dpl} \phi$ then there is some σ'' with M, $\sigma', \sigma'' \models \psi$.

If we use offset formulas, the situation looks a bit different. The rational reconstruction of Van Benthem's example now runs:

$$\frac{0, \exists; M1; \exists; H2; O(1,2) \models \exists; G3; O(1,3) \quad 2, \exists; G3; O(1,3) \models S(1,3)}{0, \exists; M1; \exists; H2; O(1,2) \models \exists; S(1,3)}$$

This is indeed a valid argument. Note that in the conclusion *existential padding* is used to provide an antecedent for the index 3. The conclusion should be read as:

Suppose a man owns a house. Then there is a thing which he sprinkles.

This is of course the conclusion one would expect.

9. A Calculus for Reasoning With Anaphora

In this section, we will give a set of sequent deduction rules for dynamic reasoning without variables. We will use $n, \phi \Longrightarrow \psi$ for $(n, \phi) \Longrightarrow (n + e(\phi), \psi)$, where $\Longrightarrow$ is the sequent separator. Note that $n, \phi \Longrightarrow \bot$ expresses that (n, ϕ) is inconsistent.

In the calculus we are about to present, we need some definitions for substitutions (in addition to $[{}^{m}_{+k}]$).

First a definition of $(t)_n$:

$$\begin{aligned}
(t)_n(t') \quad &:= \quad \begin{cases} n+1 & \text{if } t' = t \\ t' & \text{if } t' \in C, t' \neq t \\ t' & \text{if } t' \in \mathbb{N}, t' \neq t, t' \leq n \\ t'+1 & \text{if } t' \in \mathbb{N}, t' \neq t, t > n \end{cases} \\
(t)_n\bot \quad &:= \quad \bot \\
(t)_n\exists \quad &:= \quad \exists \\
(t)_n P t_1 \cdots t_m \quad &:= \quad P (t)_n t_1 \cdots (t)_n t_m \\
(t)_n (A; \phi) \quad &:= \quad (t)_n A; (t)_n \phi \\
(t)_n \neg\phi \quad &:= \quad \neg (t)_n \phi.
\end{aligned}$$

Note that $(n)_n$ denotes the result of incrementing all indices $> n$. In other words, we have that $(n)_n = [{}^{n}_{+1}]$. A useful abbreviation for $(n)_n$ is $(n)^+$.

Next, it is useful to have a notation for the substitution that replaces an index n by a term t and 'closes the gap'. Notation for this: $(^t/_n)^-$.

$$
\begin{array}{rcl}
(^t/_n)^-(t') & := & \begin{cases} t & \text{if } t' = n \\ t' & \text{if } t' \in C \\ t' & \text{if } t' \in \mathbb{N}, t' < n \\ t' - 1 & \text{if } t' \in \mathbb{N}, t > n \end{cases} \\
(^t/_n)^-\bot & := & \bot \\
(^t/_n)^-\exists & := & \exists \\
(^t/_n)^-Pt_1 \cdots t_m & := & P(^t/_n)^-t_1 \cdots (^t/_n)^-t_m \\
(^t/_n)^-(A;\phi) & := & (^t/_n)^-A; (^t/_n)^-\phi \\
(^t/_n)^-\neg\phi & := & \neg(^t/_n)^-\phi.
\end{array}
$$

Note that $(^n/_n)^-$ is the substitition that decrements all indices $> n$. A useful abbreviation for this is $(n)^-$.

In the calculus, we use C, with and without subscripts, as a variable over contexts (formula lists composed with ;, including the empty list). We extend the function e to contexts by stipulating that $e(C) = 0$ if C is the empty list. Substitution is extended to contexts in a similar way. In the rules below we will use T as an abbreviation of formulas ϕ with $e(\phi) = 0$ (T for *Test* formula).

Structural Rules
Test Axiom

$$\frac{}{n, T \Longrightarrow T}\ g(n, T) \leq n$$

The condition on the axiom is necessary to rule out tests that are not n bounded. For instance, it does not hold that $1, R(1,2) \Longrightarrow R(1,2)$, because the context does not provide an antecedent for the index 2.

Soundness of Test Axiom If $(\sigma, \tau) \in [\![n, T]\!]^{\mathrm{M}}$ then $\tau = \epsilon$ (because T is a test) and $(\sigma\hat{}\epsilon, \epsilon) \in [\![n, T]\!]^{\mathrm{M}}$. Thus, $n, T \models T$.

Transitivity Rule

$$\frac{n, \phi \Longrightarrow \psi \qquad n + e(\phi), \psi \Longrightarrow \chi}{n, \phi \Longrightarrow \exists^{e(\psi)}\chi}$$

Note that in the case where the 'cut' formula is a test, the rule assumes the familiar format:

$$\frac{n, \phi \Longrightarrow T \qquad n + e(\phi), T \Longrightarrow \chi}{n, \phi \Longrightarrow \chi}$$

Soundness of Transitivity Rule Suppose $n, \phi \models \psi$ and $n + e(\phi), \psi \models \chi$. Assume $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$. Then by $n, \phi \models \psi$, there is a θ with $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}$. By $n + e(\phi), \psi \models \chi$, there is a ρ with $(\sigma\hat{}\tau\hat{}\theta, \rho) \in [\![n + e(\phi) + e(\psi), \chi]\!]^{\mathrm{M}}$. Then by Proposition 11, $(\sigma\hat{}\tau, \theta\hat{}\rho) \in [\![n + e(\phi), \exists^{e(\psi)}; \chi]\!]^{\mathrm{M}}$. Thus, $n, \phi \models \exists^{e(\psi)}; \chi$.

Test Contraction Rules

$$\frac{n, C_1T; TC_2 \Longrightarrow \phi}{n, C_1TC_2 \Longrightarrow \phi} \qquad \frac{n, \phi \Longrightarrow C_1T; TC_2}{n, \phi \Longrightarrow C_1TC_2}$$

Note that contraction does in general not hold for formulas which are not tests. For instance, $\exists;\exists$ puts two elements on the stack, $\exists$ only one.

There is also a rule for left weakening. Due to the format where ; serves as the concatenation operator for formulas, the rule for ; left does double duty as an antecedent weakening rule. See below. Succedent weakening would be the step from $n, \phi \Longrightarrow \psi$ to $n, \phi \Longrightarrow \neg(\neg\psi; \neg\chi)$. This is taken care of by the negation rules.

Soundness of Test Contraction Rules Immediate from the fact that if $e(\phi) = 0$ then $(\sigma, \epsilon) \in [\![n, \phi; \phi]\!]^{\mathrm{M}}$ iff $(\sigma, \epsilon) \in [\![n, \phi]\!]^{\mathrm{M}}$.

Test Swap Rules

$$\frac{n, C_1 T_1; T_2 C_2 \Longrightarrow \phi}{n, C_1 T_2; T_1 C_2 \Longrightarrow \phi} \qquad \frac{n, \phi \Longrightarrow C_1 T_1; T_2 C_2}{n, \phi \Longrightarrow C_1 T_2; T_1 C_2}$$

Soundness of Test Swap Rules Follows from the fact that $[\![m, T_1; T_2]\!]^{\mathrm{M}} = [\![m, T_2; T_1]\!]^{\mathrm{M}}$.

$\exists$ *Swap Rules*

$$\frac{n, C_1 T; \exists\ C_2 \Longrightarrow \phi}{n, C_1 \exists; (m)^+ T C_2 \Longrightarrow \phi}\ m = n + e(C_1) \qquad \frac{n, \phi \Longrightarrow C_1 T; \exists\ C_2}{n, \phi \Longrightarrow C_1 \exists; (m)^+ T C_2}\ m = n + e(\phi) + e(C_1)$$

These rules allow us to pull $\exists$ leftward through a test T, provided we increment the appropriate indices in T.

Pulling $\exists$ through a test T in the opposite direction is allowed in those cases where $\exists$ does not bind anything in T. Now we must adjust T by decrementing the appropriate indices:

$$\frac{n, C_1 \exists; T C_2 \Longrightarrow \phi}{n, C_1 (m)^- T; \exists\ C_2 \Longrightarrow \phi}\ m = n + e(C_1), m + 1 \text{ not in } T$$

$$\frac{n, \phi \Longrightarrow C_1 \exists; T C_2}{n, \phi \Longrightarrow C_1 (m)^- T; \exists\ C_2}\ m = n + e(\phi) + e(C_1), m + 1 \text{ not in } T$$

Soundness of $\exists$ *Swap Rules* Soundness of the rules for moving $\exists$ to the left follows from the fact that $(\sigma, \epsilon\hat{}a) \in [\![m, T; \exists]\!]^{\mathrm{M}}$ iff $(\sigma, a\hat{}\epsilon) \in [\![m, \exists; (m)^+ T]\!]^{\mathrm{M}}$.

Soundness of the rules for moving $\exists$ to the right follows from the fact that if index $m + 1$ does not occur in T, then $(\sigma, a\hat{}\epsilon) \in [\![m, \exists; T]\!]^{\mathrm{M}}$ iff $(\sigma, \epsilon\hat{}a) \in [\![m, (m)^- T; \exists]\!]^{\mathrm{M}}$.

Context Rules
Memory Shift Rule

$$\frac{n, \phi \Longrightarrow \psi}{n + 1, (n)^+ \phi \Longrightarrow (n)^+ \psi}$$

Soundness of Memory Shift Rule If $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$ then for all $a \in M$, $(\sigma\hat{}a, \tau) \in [\![n + 1, (n)^+ \phi]\!]^{\mathrm{M}}$.

Context Extension

$$\frac{n, \exists; \phi \Longrightarrow \psi}{n + 1, \phi \Longrightarrow \psi}$$

The counterpart to this, context absorption, is the rule for introducing an existential quantifier in the antecedent (see the logical rules below).

What context extension and absorption express is that linking information to an outside context is equivalent, for all purposes of reasoning, to assuming that your information is existentially quantified over. (This is how one can make sense of a ongoing conversation about an unknown 'he': instead of asking questions of identification that might interrupt the flow of the gossip one simply inserts an existential quantifier and listens to what is being said.)

Soundness of Context Extension Follows from the fact that $(\sigma, \tau) \in [\![n, \exists; \phi]\!]^{\mathrm{M}}$ iff for some $a \in M$, $(\sigma\hat{\ }a, \tau) \in [\![n+1, \phi]\!]^{\mathrm{M}}$.

Logical Rules
∃ Left

$$\frac{n+1, \phi \Longrightarrow \psi}{n, \exists; \phi \Longrightarrow \psi}$$

This rule of context absorption is simpler than the rule for introducing an existential quantifier in the antecedent of a sequent in standard predicate logic, where a condition must be imposed: the variable existentially quantified over is not free in the context formulas. Such a condition is unnecessary here, as the existential quantifier binds *all* occurrences of index $n+1$.

Soundness of ∃ Left Follows from the fact that $(\sigma, \tau) \in [\![n, \exists; \phi]\!]^{\mathrm{M}}$ iff for some $a \in M$, $(\sigma\hat{\ }a, \tau) \in [\![n+1, \phi]\!]^{\mathrm{M}}$.

∃ Right

$$\frac{n, \phi \Longrightarrow ({}^{t}/_{n+e(\phi)+1})^{-}\psi}{n, \phi \Longrightarrow \exists; \psi}$$

This format is familiar from the Gentzen format of ∃-right in standard predicate logic. Here is an example application:

$$\frac{1, R(1,1) \Longrightarrow R(1,1)}{1, R(1,1) \Longrightarrow \exists; R(1,2)}$$

$R(1,1)$ equals $({}^{1}/_{2})^{-}R(1,2)$, so this is indeed a correct application of the rule.

Note that we can formulate a derived rule that looks perhaps more plausible:

$$\frac{n, \phi \Longrightarrow \psi}{n, \phi \Longrightarrow \exists; (t)_m \psi} \; m = n + e(\phi)$$

This follows from the parent version, because $({}^{t}/_{m+1})^{-}(t)_m\psi = \psi$. Easier to grasp, perhaps, but not quite general enough, witness the fact that in the above example application of the parent rule $R(1,2)$ is *not* of the form $(1)_1R(1,1)$. The snag is in the fact that substitutions are defined in a *uniform* way. (See Troelstra and Schwichtenberg [22] for extensive discussion of such issues of substitution in proof theory.)

Soundness of ∃ Right Assume a model M with input and output assignments σ, τ such that $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$. Let $m = n + e(\phi)$. Then by the soundness of the premiss there is a θ with

$$(\sigma\hat{\ }\tau, \theta) \in [\![m, ({}^{t}/_{m+1})^{-}\psi]\!]^{\mathrm{M}}.$$

Let $[\![t]\!]^{\mathrm{M}}_{\sigma\hat{\ }\tau} = a$. Then, by the definition of the substitution $({}^{t}/_{m+1})^{-}$, $(\sigma\hat{\ }\tau\hat{\ }a, \theta) \in [\![m+1, \psi]\!]^{\mathrm{M}}$. It follows that $(\sigma\hat{\ }\tau, a\hat{\ }\theta) \in [\![m, \exists; \psi]\!]^{\mathrm{M}}$. This proves $n, \phi \models \exists; \psi$.

; Left and Right

$$\frac{n + e(\phi), \psi \Longrightarrow \chi}{n, \phi; \psi \Longrightarrow \chi} \qquad \frac{n, \phi \Longrightarrow \psi \qquad n, \phi \Longrightarrow \chi}{n, \phi \Longrightarrow \psi; [{}^{m}_{+e(\psi)}]\chi} \; m = n + e(\phi)$$

The first of these does double duty as a left weakening rule. Antecedent weakening is always extension on the lefthand side. This is because extension on the righthand-side might affect the stack. Weakening with a tests is valid anywhere in the antecedent; the swap rules account for that.

An example application of the rule for ; right is:

$$\frac{1, R(1,1) \Longrightarrow \exists; R(1,2) \qquad 1, R(1,1) \Longrightarrow \exists; R(2,1)}{1, R(1,1) \Longrightarrow \exists; R(1,2); \exists; R(3,1)}$$

Soundness of ; Left Suppose $(\sigma, \tau) \in [\![n, \phi; \psi]\!]^{\mathrm{M}}$. Let $\tau_1 := \tau[1..e(\phi)]$ and $\tau_2 := \tau[e(\phi)+1..l(\tau)]$. Then $(\sigma\hat{}\tau_1, \tau_2) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}$. By the soundness of the premiss, there is a θ with $(\sigma\hat{}\tau_1\hat{}\tau_2, \theta) \in [\![n + e(\phi) + e(\psi), \chi]\!]^{\mathrm{M}}$. Thus, $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi) + e(\psi), \chi]\!]^{\mathrm{M}}$. This establishes $n, \phi; \psi \models \chi$.

Soundness of ; Right Assume $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$. Then by the soundness of the second premiss, there is a ρ with $(\sigma\hat{}\tau, \rho) \in [\![n + e(\phi), \chi]\!]^{\mathrm{M}}$. By Proposition 10, for any $\theta \in M^{e(\psi)}$,

$$(\sigma\hat{}\tau\hat{}\theta, \rho) \in [\![n + e(\phi) + e(\psi), [{}^{n+e(\phi)}_{+e(\psi)}]\chi]\!]^{\mathrm{M}}.$$

By the soundness of the first premiss, there is a $\theta \in M^{e(\psi)}$ with $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}$. It follows that $(\sigma\hat{}\tau, \theta\hat{}\rho) \in [\![n + e(\phi), \psi; [{}^{n+e(\phi)}_{+e(\psi)}]\chi]\!]^{\mathrm{M}}$. This establishes $n, \phi \models \psi; [{}^{n+e(\phi)}_{+e(\psi)}]\chi$.

$\perp$ Rule

$$\frac{n, \phi \Longrightarrow \perp; \psi}{n, \phi \Longrightarrow \perp}$$

Note that the swap rules ensure that the position of $\perp$ in the succedent does not matter.

Soundness of $\perp$ Rule The rule expresses that $\perp$ denotes the empty relation: composition with $\emptyset$ yields $\emptyset$.

$\neg$ Left and Right

$$\frac{n, \phi \Longrightarrow \psi}{n, \phi; \neg\psi \Longrightarrow \perp} \qquad \frac{n, \phi; \psi \Longrightarrow \perp}{n, \phi \Longrightarrow \neg\psi}$$

Soundness of $\neg$ Left Assume $(\sigma, \tau) \in [\![n, \phi; \neg\psi]\!]^{\mathrm{M}}$. Then $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$ and there is no θ with $(\sigma\hat{}\tau, \theta) \in [\![n+e(\phi), \psi]\!]^{\mathrm{M}}$. Contradiction with the soundness of the premiss. This establishes $n, \phi; \neg\psi \models \perp$.

Soundness of $\neg$ Right Assume $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$. Then by the soundness of the premiss, there is no θ with $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}$. This establishes $n, \phi \models \neg\psi$.

Double Negation Rules

$$\frac{n, \phi \Longrightarrow \neg\neg\psi}{n, \phi \Longrightarrow \psi} \qquad \frac{n, \phi; \neg\neg\psi \Longrightarrow \perp}{n, \phi; \psi \Longrightarrow \perp}$$

Soundness of Double Negation Rules For Double Negation Left, assume $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$. Then by the soundness of the premiss, there is no θ with

$$(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \neg\psi]\!]^{\mathrm{M}}.$$

In particular, $(\sigma\hat{}\tau, \epsilon) \notin [\![n + e(\phi), \neg\psi]\!]^{\mathrm{M}}$. Therefore, there is a θ with

$$(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}.$$

This establishes $n, \phi \models \psi$. The soundness of Double Negation Right is established similarly.

This completes the presentation of the calculus. As we have checked the soundness of every rule as we went along, we have:

Theorem 13 *The Calculus for Reasoning with Anaphora is sound.*

10. Derivable Rules for Reasoning with Anaphora

Proposition 14 (Start Rule) *The following start sequent is derivable:*

$$\frac{}{n, \phi \Longrightarrow [^{n}_{+e(\phi)}]\phi}\; g(n, \phi) \leq n$$

Proof. We show by induction on the existential depth of ϕ that there is a derivation of the start sequent for ϕ.

For the case of $e(\phi) = 0$, the test axiom provides a derivation.

Suppose we have for all n, ϕ with $e(\phi) = k$, that the relation holds (for arbitrary n). We have to show that we can derive the rule for (n, ψ) with $e(\psi) = k + 1$ (again, for arbitrary n). Let (n, ψ) be a bounded formula with $e(\psi) = k + 1$. Now any such formula can be thought of as the result of starting out with a formula $(n, \exists; \phi)$, with $e(\phi) = k$, and swapping the quantifier inwards (by means of the swap rules).

So we are done of we can show, for an arbitrary n, that $n, \exists; \phi \Longrightarrow [^{n}_{+e(\phi)+1}]\exists; \phi$, on the basis of the following induction hypothesis:

$$n + 1, \phi \Longrightarrow [^{n+1}_{+e(\phi)}]\phi.$$

This is established as follows:

$$\cfrac{\cfrac{\cfrac{}{n + 1, \phi \Longrightarrow [^{n+1}_{+e(\phi)}]\phi}\;\text{induction hypothesis}}{n, \exists; \phi \Longrightarrow [^{n+1}_{+e(\phi)}]\phi}\;\exists\text{l}}{n, \exists; \phi \Longrightarrow [^{n}_{+e(\phi)+1}]\exists; \phi}\;\exists\text{r}$$

The final step is perhaps in need of some further explanation. To understand the application of $\exists r$, observe that $[^{n+1}_{+e(\phi)}]\phi$ can be written as the result of applying substitution $(^{n+1}/_{n+e(\phi)+2})^{-}$ to the formula $[^{n}_{+e(\phi)+1}]\phi$, all in the context $n + 1 + e(\phi)$. Applying $\exists r$ to this yields $\exists; [^{n}_{+e(\phi)+1}]\phi$, which is equivalent to $[^{n}_{+e(\phi)+1}]\exists; \phi$. ⊣

Here is a concrete application of the procedure from the proof of Proposition 14.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{2, R(1, 2) \Rightarrow R(1, 2)}\;\text{test axiom}}{2, R(1, 2) \Rightarrow \exists; R(1, 3)}\;\exists\text{r}}{1, \exists; R(1, 2) \Rightarrow \exists; R(1, 3)}\;\exists\text{l}}{1, \exists; R(1, 2) \Rightarrow \exists; \exists; R(3, 4)}\;\exists\text{r}}{0, \exists; \exists; R(1, 2) \Rightarrow \exists; \exists; R(3, 4)}\;\exists\text{l}$$

Proposition 15 (Contradiction Rule) *The following rule is derivable:*

$$\frac{n,\phi;\neg\psi \Longrightarrow \neg\chi \qquad n,\phi;\neg\psi \Longrightarrow \chi}{n,\phi \Longrightarrow \psi}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n,\phi;\neg\psi \Longrightarrow \neg\chi \qquad n,\phi;\neg\psi \Longrightarrow \chi}{n,\phi;\neg\psi \Longrightarrow \neg\chi;\chi}\;;\text{r} \qquad \dfrac{\dfrac{\dfrac{}{n+e(\phi),\neg\chi \Longrightarrow \neg\chi}\text{ test axiom}}{n+e(\phi),\neg\chi;\neg\neg\chi \Longrightarrow \bot}\neg\text{l}}{n+e(\phi),\neg\chi;\chi \Longrightarrow \bot}\text{dn}}{n,\phi;\neg\psi \Longrightarrow \exists^{e(\chi)}\bot}\text{tr}}{n,\phi;\neg\psi \Longrightarrow \bot;\exists^{e(\chi)}}\text{swap}}{n,\phi;\neg\psi \Longrightarrow \bot}\bot}{n,\phi \Longrightarrow \neg\neg\psi}\neg\text{r}}{n,\phi \Longrightarrow \psi}\text{dn}$$

$\dashv$

Proposition 16 (Cases Rule) *The following rule is derivable:*

$$\frac{n,\phi;\neg\psi \Longrightarrow \chi \qquad n,\phi;\neg\neg\psi \Longrightarrow \chi}{n,\phi \Longrightarrow \chi}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{n,\phi;\neg\psi \Longrightarrow \chi}{n,\phi;\neg\psi;\neg\chi \Longrightarrow \bot}\neg\text{l}}{n,\phi;\neg\chi;\neg\psi \Longrightarrow \bot}\text{swap}}{n,\phi;\neg\chi \Longrightarrow \neg\neg\psi}\neg\text{r} \qquad \dfrac{\dfrac{\dfrac{n,\phi;\neg\neg\psi \Longrightarrow \chi}{n,\phi;\neg\neg\psi;\neg\chi \Longrightarrow \bot}\neg\text{l}}{n,\phi;\neg\chi;\neg\neg\psi \Longrightarrow \bot}\text{swap}}{n,\phi;\neg\chi \Longrightarrow \neg\neg\neg\psi}\neg\text{r}}{n,\phi \Longrightarrow \chi}\text{contrad}$$

$\dashv$

Proposition 17 (Contraposition Rule) *The following rule is derivable:*

$$\frac{n,\phi;\psi \Longrightarrow \chi}{n,\phi;\neg\exists^{e(\psi)}\chi \Longrightarrow \neg\psi}$$

Proof. In the following, let $m = n + e(\phi) + e(\psi)$. We use χ' for $(m - e(\psi))_m \cdots (m)_m\chi$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{n,\phi;\psi \Longrightarrow \chi}{n,\phi;\psi \Longrightarrow \exists^{e(\psi)}\chi'}\exists\text{r}, e(\psi)\text{ times}}{n,\phi;\psi;\neg\exists^{e(\psi)}\chi' \Longrightarrow \bot}\neg\text{l}}{n,\phi;\neg\exists^{e(\psi)}\chi;\psi \Longrightarrow \bot}\text{swap}}{n,\phi;\neg\exists^{e(\psi)}\chi \Longrightarrow \neg\psi}\neg\text{r}$$

$\dashv$

Here is a concrete example illustration:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1,\exists;R(1,2);\exists;\exists;S(2,3,4) \Longrightarrow T(3,4)}{1,\exists;R(1,2);\exists;\exists;S(2,3,4) \Longrightarrow \exists;T(3,5)}\exists\text{r}}{1,\exists;R(1,2);\exists;\exists;S(2,3,4) \Longrightarrow \exists;\exists;T(5,6)}\exists\text{r}}{1,\exists;R(1,2);\exists;\exists;S(2,3,4);\neg(\exists;\exists;T(5,6)) \Longrightarrow \bot}\neg\text{l}}{1,\exists;R(1,2);\neg(\exists;\exists;T(3,4));\exists;\exists;S(2,3,4) \Longrightarrow \bot}\text{swap}}{1,\exists;R(1,2);\neg(\exists;\exists;T(3,4)) \Longrightarrow \neg(\exists;\exists;S(2,3,4))}\neg\text{r}$$

Here is an example of this pattern of reasoning in natural language:

> Suppose a man owns a house. Suppose it has a garden. Then he sprinkles it.
> ___
> Suppose a man owns a house. Suppose he does not sprinkle anything.
> Then it (the house) does not have a garden.

Proposition 18 (Ex Falso Rule) *The following rule is derivable:*

$$\dfrac{}{n, \bot \Longrightarrow \phi}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n, \bot \Longrightarrow \bot}\ \text{start}}{n, \neg\phi; \bot \Longrightarrow \bot}\ ;\ \text{l}}{n, \bot; \neg\phi \Longrightarrow \bot}\ \text{swap}}{n, \bot \Longrightarrow \neg\neg\phi}\ \neg\text{r}}{n, \bot \Longrightarrow \phi}\ \text{dn}$$

⊣

Proposition 19 (Inconsistency Rule) *The following rule is derivable:*

$$\dfrac{n, \phi \Longrightarrow \bot}{n, \phi \Longrightarrow \psi}$$

Proof.

$$\dfrac{n, \phi \Longrightarrow \bot \qquad \dfrac{}{n + e(\phi), \bot \Longrightarrow \psi}\ \text{ex falso}}{n, \phi \Longrightarrow \psi}\ \text{tr}$$

⊣

Proposition 20 *The following rule is derivable:*

$$\dfrac{n, \phi \Longrightarrow \psi}{n, \phi; \neg\psi \Longrightarrow \chi}$$

Proof.

$$\dfrac{\dfrac{n, \phi \Longrightarrow \psi}{n, \phi; \neg\psi \Longrightarrow \bot}\ \neg\text{l}}{n, \phi; \neg\psi \Longrightarrow \chi}\ \text{incons}$$

⊣

Proposition 21 (Conditionalisation Rule) *The following rule is derivable:*

$$\dfrac{n, \phi; \psi \Longrightarrow \chi}{n, \phi \Longrightarrow \psi \to \chi}$$

Proof.

$$\dfrac{\dfrac{n, \phi; \psi \Longrightarrow \chi}{n, \phi; \psi; \neg\chi \Longrightarrow \bot}\ \neg\text{l}}{n, \phi \Longrightarrow \neg(\psi; \neg\chi)}\ \neg\text{r}$$

$\dashv$

An example of this pattern is in the following reasoning.

Suppose a man owns a house. Then he owns a garden.
Suppose there is a man. Then if he owns a house, he owns a garden.

Proposition 22 ($\neg\neg$ Weakening With $\exists$ Padding) *The following rule is derivable:*

$$\frac{n,\phi;\psi \Longrightarrow \chi}{n,\phi;\neg\neg\psi \Longrightarrow \exists^{e(\psi)}\chi}$$

Proof.

$$\cfrac{\cfrac{\cfrac{n,\phi;\psi \Longrightarrow \chi}{n,\phi;\neg\exists^{e(\phi)}\chi \Longrightarrow \neg\psi}\ \text{contrap}}{n,\phi;\neg\neg\psi \Longrightarrow \neg\neg\exists^{e(\phi)}\chi}\ \text{contrap}}{n,\phi;\neg\neg\psi \Longrightarrow \exists^{e(\phi)}\chi}\ \text{dn}$$

$\dashv$

Here is an example of this pattern:

$$\frac{1,\exists;\neg R(2,1) \Longrightarrow H(2,1)}{1,\neg\neg(\exists;\neg R(2,1)) \Longrightarrow \exists H(2,1)}$$

Note that $\neg\exists\neg$ expresses universal quantification. The natural language rendering of the example, therefore, looks like this:

Suppose there is someone who does not respect her. Then he hates her.
Suppose not everyone respects her. Then someone hates her.

Proposition 23 (Modus Ponens With $\exists$ Padding) *The following rule is derivable:*

$$\frac{n,\phi \Longrightarrow \psi \to \chi \qquad n,\phi \Longrightarrow \psi}{n,\phi \Longrightarrow \exists^{e(\psi)}\chi}$$

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{n,\phi \Longrightarrow \neg(\psi;\neg\chi)}{n,\phi;\neg\neg(\psi;\neg\chi) \Longrightarrow \bot}\ \neg\text{l}}{n,\phi;\psi;\neg\chi \Longrightarrow \bot}\ \text{dn}}{n,\phi;\psi \Longrightarrow \neg\neg\chi}\ \neg\text{r}}{n,\phi;\psi \Longrightarrow \chi}\ \text{dn}}{n,\phi;\neg\neg\psi \Longrightarrow \exists^{e(\psi)}\chi}\ \neg\neg\text{w} \qquad \cfrac{\cfrac{n,\phi \Longrightarrow \psi}{n,\phi;\neg\psi \Longrightarrow \bot}\ \neg\text{r}}{n,\phi;\neg\psi \Longrightarrow \exists^{e(\psi)}\chi}\ \text{incons}}{n,\phi \Longrightarrow \exists^{e(\psi)}\chi}\ \text{cases}$$

$\dashv$

An example of this pattern is in the following reasoning.

Suppose a man owns a house. Then if he owns a garden, he sprinkles it.
Suppose a man owns a house. Then he owns a garden.
Suppose a man owns a house. Then he sprinkles something.

11. Completeness of the Calculus

To establish the completeness of the calculus, assume that $0, \phi \not\Longrightarrow \psi$. (Because of the context extension rule, we may assume that the context is initially empty, for if it is not, and we have that $n, \phi \not\Longrightarrow \psi$ for $n \neq 0$, then by context extension also $0, \exists^n; \phi \not\Longrightarrow \psi$.)

We will construct a countermodel by a slight modification of the standard Henkin construction for the completeness of classical predicate logic. It is convenient to use k for $e(\phi)$ throughout the reasoning that follows.

Definition 24 *A set of L formulas Γ is k-bounded if every $\phi \in \Gamma$ is k-bounded. If Γ is k-bounded, we use (k, Γ) for the set of offset formulas $\{(k, \gamma) \mid \gamma \in \Gamma\}$.*

$\phi \vdash_\Gamma \psi :\Leftrightarrow$ there are $(k, \phi_1), \ldots, (k, \phi_n) \in (k, \Gamma)$ with $0, \phi; \neg\neg\phi_1; \cdots; \neg\neg\phi_n \Longrightarrow \psi$.

(k, Γ) is consistent with $(0, \phi)$ if there is an k-bounded ψ with $\phi \not\vdash_\Gamma \psi$.

(k, Γ) is negation complete with respect to $(0, \phi)$ if for every k-bounded ψ either $\phi \vdash_\Gamma \psi$ or $\phi \vdash_\Gamma \neg\psi$

(k, Γ) has witnesses for $(0, \phi)$ if for every k-bounded $\exists; \psi$ such that $\phi \vdash_\Gamma \exists; \psi$ there is a c for which $(k, \neg\neg\exists\psi \rightarrow ({}^c/_{k+1})^- \psi) \in (k, \Gamma)$.

Note that in the definition of $\phi \vdash_\Gamma \psi$ the extra premisses from Γ do not extend the 'anaphoric context': the context change potential of the premisses from Γ is blocked off by means of double negation signs. This is a key element in the canonical model construction below.

Proposition 25 *If $\phi \not\vdash_\Gamma \psi$ then at least one of $\Gamma \cup \{\psi\}$, $\Gamma \cup \{\neg\psi\}$ is consistent with $(0, \phi)$.*

Proof. Use the Cases Rule. ⊣

Let $\exists\chi_1, \ldots$ be a list of all k-bounded formulas of L that start with $\exists$. Let $C_0 := c_1^0, \ldots$ be a list of fresh individual constants. Let L_0 be $L(C_0)$ (the result of adding the constants C_0 to L).

$$\Delta_0 := \{\neg\neg\exists\chi_i \rightarrow ({}^{c_i^0}/_{k+1})^- \chi_i \mid 1 \leq i\}.$$

Let $\exists\chi_1^m, \ldots$ be a list of all k-bounded existential formulas which occur in L_m. Let $C_{m+1} := c_1^{m+1}, \ldots$ be a list of fresh individual constants. Let $L_{m+1} := L_m(C_{m+1})$.

$$\Delta_{m+1} := \{\neg\neg\exists\chi_i^{m+1} \rightarrow ({}^{c_i^{m+1}}/_{k+1})^- \chi_i^{m+1} \mid 1 \leq i\}.$$

Let $C := \bigcup_m C_m$, and let Δ be the set of $L(C)$ formulas given by:

$$\Delta := \bigcup_m \Delta_m.$$

Proposition 26 *If (k, Γ) consists of $L(C)$ formulas, and $(k, \Gamma) \supseteq (k, \Delta)$, then (k, Γ) has witnesses for $(0, \phi)$.*

Proof. Take some $(k, \exists\psi)$ with $\phi \vdash_\Gamma \exists\psi$. Then $\exists\psi \in L_m$ for some m. So there is some $c \in C$ with $\neg\neg\exists\psi \rightarrow ({}^c/_{k+1})^- \psi \in \Delta_{m+1}$. So $(k, \neg\neg\exists\psi \rightarrow ({}^c/_{k+1})^- \psi) \in (k, \Delta) \subseteq (k, \Gamma)$. ⊣

Proposition 27 *If (k, Γ) is consistent with $(0, \phi)$ then there is a $(k, \Gamma') \supseteq (k, \Gamma)$ which is consistent with $(0, \phi)$, negation complete with respect to $(0, \phi)$, and has witnesses for $(0, \phi)$.*

Proof. Assume (k, Γ) consistent with $(0, \phi)$. Let $(k, \chi_1), \ldots, (k, \chi_i), \ldots$ be an enumeration of all bounded formulas of the language $(k, L(C))$. Extend (k, Γ) as follows to a (k, Γ') with the required properties.

$$(k, \Gamma_0) := (k, \Gamma) \cup (k, \Delta)$$

$$(k, \Gamma_{m+1}) := \begin{cases} (k, \Gamma_m \cup \{\chi_m\}) & \text{if } (k, \Gamma_m \cup \{\chi_m\}) \text{ consistent with } (0, \phi), \\ (k, \Gamma_m) & \text{otherwise.} \end{cases}$$

$$(k, \Gamma') := (k, \bigcup_m \Gamma_m)$$

$(k, \Gamma') \supseteq (k, \Delta)$, so by Proposition 26 (k, Γ') has witnesses for $(0, \phi)$.

Assume (k, Γ') is inconsistent with $(0, \phi)$. Then some (k, Γ_m) has to be inconsistent with $(0, \phi)$ and contradiction with Proposition 25. So (k, Γ') is consistent with $(0, \phi)$.

Finally, (k, Γ') is negation complete by construction. ⊣

Definition 28 (Canonical Model) *Let (k, Γ) be consistent with $(0, \phi)$, be negation complete with respect to $(0, \phi)$, and have witnesses for $(0, \phi)$. Then $M_\Gamma = (D, I)$ is defined as follows. $D :=$ the set of natural numbers $\{1, \ldots, k\}$ together with the set of constants C occurring in $\Gamma \cup \{\phi\}$. For all terms of the language, let $I(t) := t$. Let $I(P) := \{\langle t_1, \ldots t_k \rangle \mid \phi \vdash_\Gamma P(t_1, \ldots, t_k)\}$ (where it is given that all the t_i are either constants or indices in the range $1, \ldots, k$).*

Lemma 29 (Satisfaction Lemma) *Let (k, Γ) be consistent with $(0, \phi)$, be negation complete with respect to $(0, \phi)$, and have witnesses for $(0, \phi)$. For all k-bounded ξ:*
$\phi \vdash_\Gamma \xi$ iff $\exists \tau$ with $(\langle 1, \ldots, k\rangle, \tau) \in [\![k, \xi]\!]^{M_\Gamma}$.

Proof. Induction on the structure of ξ.

$\phi \not\vdash_\Gamma \bot$ by the fact that $(0, \phi)$ is consistent and Γ is consistent with $(0, \phi)$.

$\phi \vdash_\Gamma Pt_1 \cdots t_n$ iff $\langle t_1, \ldots, t_n \rangle \in I(P)$ iff $(\langle 1, \ldots, k\rangle, \epsilon) \in [\![k, Pt_1 \cdots t_n]\!]^{M_\Gamma}$.

$\phi \vdash_\Gamma \neg\xi$ iff (Γ negation complete) $\phi \not\vdash_\Gamma \xi$ iff (i.h.) there is no τ with $(\langle 1, \ldots, k\rangle, \tau) \in [\![k, \xi]\!]^{M_\Gamma}$ iff (semantic clause for $\neg$) $(\langle 1, \ldots, k\rangle, \epsilon) \in [\![k, \neg\xi]\!]^{M_\Gamma}$.

$\phi \vdash_\Gamma \exists\xi$ iff (Γ has witnesses) $\phi \vdash_\Gamma ({}^c/_{k+1})^- \xi$ iff (i.h.) there is a τ with $(\langle 1, \ldots, k\rangle, \tau) \in [\![k, ({}^c/_{k+1})^- \xi]\!]^{M_\Gamma}$ iff $(\langle 1, \ldots, k\rangle, \langle c \rangle \hat{\ } \tau) \in [\![k, \exists\xi]\!]^{M_\Gamma}$.

For the case of $\xi_1; \xi_2$ we argue as follows. If ξ_1 is not a test (i.e., it is not the case that $e(\xi_1) = 0$), then ξ_1 must be of the form $\chi; \exists; \chi'$. In this case, the $\exists$ quantifier can be moved to the front using the shift rules, and dealt with as in the previous case (associativity of ; is assumed throughout). What this means is that we only have to deal with $\xi_1; \xi_2$ where ξ_1 is a test.

For this case, we have: $\phi \vdash_\Gamma \xi_1; \xi_2$ iff (ξ_1 is a test) $\phi \vdash_\Gamma \xi_1$ and $\phi \vdash_\Gamma \xi_2$ iff (i.h. twice, plus the fact that ξ_1 is a test) $(\langle 1, \ldots, k\rangle, \epsilon) \in [\![k, \xi_1]\!]^{M_\Gamma}$ and there is a τ with $(\langle 1, \ldots, k\rangle, \tau) \in [\![k, \xi_2]\!]^{M_\Gamma}$ iff there is a τ with $(\langle 1, \ldots, k\rangle, \tau) \in [\![k, \xi_1; \xi_2]\!]^{M_\Gamma}$. ⊣

Proposition 30 *Let (k, Γ) be consistent with $(0, \phi)$, be negation complete with respect to $(0, \phi)$, and have witnesses for $(0, \phi)$. Then $(\epsilon, \langle 1, \ldots, k\rangle) \in [\![0, \phi]\!]^{M_\Gamma}$.*

Proof. Let $\exists^k \phi'$ be the result of applying the rule for moving $\exists$ leftward as many times as necessary to ϕ to ensure that $e(\phi') = 0$. Then $(0, \phi)$ and the formula $(0, \exists^k \phi')$ are proof equivalent. Furthermore, we have:

$$\dfrac{\dfrac{\dfrac{}{k, \phi' \Longrightarrow \phi'}\text{ proposition 14}}{0, \exists^k \phi' \Longrightarrow \phi'}\ \exists\text{l}, k \text{ times}}{0, \phi \Longrightarrow \phi'}\text{ swap rules}$$

Therefore, $\phi \vdash_\Gamma \phi'$, and by the satisfaction lemma, $(\langle 1, \ldots, k\rangle, \epsilon) \in [\![k, \phi']\!]^{M_\Gamma}$. Also, by definition of the semantics for $\exists$, we have that $(\epsilon, \langle 1, \ldots, k\rangle) \in [\![0, \exists^k]\!]^{M_\Gamma}$. By the semantic equivalence of ϕ and $\exists^k \phi'$ we get $(\epsilon, \langle 1, \ldots, k\rangle) \in [\![0, \phi]\!]^{M_\Gamma}$. ⊣

Theorem 31 (Completeness) *If $n, \phi \models \psi$ then $n, \phi \Longrightarrow \psi$.*

Proof. Assume $n, \phi \not\Longrightarrow \psi$. By context extension, it follows from this that $0, \exists^n \phi \not\Longrightarrow \psi$. Set $\phi' := \exists^n \phi$ and set $k := e(\phi')$. Then $\{(k, \neg\psi)\}$ is consistent with $(0, \phi')$. By proposition 27, there is a $(k,\Gamma) \supseteq \{(k,\neg\psi)\}$ which is consistent with $(0,\phi')$, is negation complete with respect to $(0,\phi')$, and has witnesses for $(0,\phi')$. Construct the canonical model and apply the satisfaction lemma to get:

$$(\langle 1,\dots,k\rangle,\epsilon) \in [\![k,\neg\psi]\!]^{M_\Gamma}.$$

By the semantic clause for negation we have that for all τ:

$$(\langle 1,\dots,k\rangle,\tau) \notin [\![k,\psi]\!]^{M_\Gamma}.$$

By proposition 30:

$$(\epsilon,\langle 1,\dots,k\rangle) \in [\![0,\phi']\!]^{M_\Gamma}.$$

This proves $0,\phi' \not\models \psi$, i.e., $0,\exists^n;\phi \not\models \psi$, and therefore, $n,\phi \not\models \psi$. ⊣

12. Anaphoric Reasoning with Equality

Anaphoric linking makes extensive use of equality. See Van Eijck [8] for an in-depth analysis of the use of equality in anaphoric descriptions. An anaphoric definite description like *the garden* can be treated as a definiteness quantifier followed by a link to a contextually available index. The translation of *He sprinkles the garden* would then be something like $2,\iota : (3 \doteq 2; G3); S(1,3)$. Also, the determiner *an other* often has an implicit anaphoric element. In such cases, the treatment involves non-identity links to contextually available referents. *He met an other woman* gets a translation like $2,\exists;3 \neq 2; W3; M(1,3)$. Below we indicate how to handle equality, while leaving the axiomatisation of definiteness in the present framework for another occasion.

For the treatment of equality, add expressions $t_1 \doteq t_2$ to the language (we assume that $t_1 \neq t_2$ is an abbreviation of $\neg(t_1 \doteq t_2)$). Equalities are tests, or, in other words, $e(t_1 \doteq t_2) = 0$. To ensure that $n, t_1 = t_2$ is bounded, we assume $t_1, t_2 \in \text{Cons} \cup \{1,\dots,n\}$. The semantics of equality is as you would expect:

$$(\sigma,\tau) \in [\![n,t_1 \doteq t_2]\!]^{\text{M}} \quad \text{iff} \quad l(\sigma) = n \text{ and } \tau = \epsilon \text{ and } [\![t_1]\!]^{\text{M}}_\sigma \text{ equals } [\![t_2]\!]^{\text{M}}_\sigma.$$

The following rules must be added to the calculus to deal with equality statements:

Reflexivity Axiom

$$\frac{}{n,\phi \Longrightarrow t \doteq t}\ t \in \text{Cons} \cup \{1,\dots,n+e(\phi)\}$$

Soundness of Reflexivity Axiom The axiom expresses that equality is reflexive.

Substitution Rule For this we need the notion $({}^{t_1}/_{t_2})$ for substitution without incrementing or decrementing indices. This is defined in the obvious way.

$$\frac{n,\phi \Longrightarrow ({}^{t_1}/_{t_2})\psi}{n,\phi; t_1 \doteq t_2 \Longrightarrow \psi}\ t_1,t_2 \in \text{Cons} \cup \{1,\dots,n+e(\phi)\}$$

Example application:

$$\dfrac{\dfrac{}{0,\top \Longrightarrow a \doteq a}\ \text{refl}}{0, a \doteq b \Longrightarrow b \doteq a}\ \text{subst}$$

For the correctness of this application, note that $a \doteq a$ is of the form $({}^{a}/_{b})b \doteq a$.

$$\dfrac{\dfrac{}{0,a \doteq b \Longrightarrow a \doteq b}\ \text{test axiom}}{0, a \doteq b; b \doteq c \Longrightarrow a \doteq c}\ \text{subst}$$

For the correctness of this application, note that $a \doteq b$ is of the form $({}^{b}/_{c})a \doteq c$.

$$\dfrac{\dfrac{\overline{3, 1 \doteq 2 \Longrightarrow 1 \doteq 2}\ \text{test axiom}}{3, 1 \doteq 2; 2 \doteq 3 \Longrightarrow 1 \doteq 3}\ \text{subst}}{0, \exists; \exists; \exists; 1 \doteq 2; 2 \doteq 3 \Longrightarrow 1 \doteq 3}\ \exists\text{l}, 3\ \text{times}$$

Soundness of the Substitution Rule Assume $(\sigma, \tau) \in [\![n, \phi; t_1 \doteq t_2]\!]^{\mathrm{M}}$. Then $(\sigma, \tau) \in [\![n, \phi]\!]^{\mathrm{M}}$, and $[\![t_1]\!]^{\mathrm{M}}_{\sigma\hat{}\tau} = [\![t_2]\!]^{\mathrm{M}}_{\sigma\hat{}\tau}$. By the soundness of the premiss, there is a θ with $(\sigma\hat{}\tau, \theta) \in [\![n, ({}^{t_1}/_{t_2})\psi]\!]^{\mathrm{M}}$. Therefore, $(\sigma\hat{}\tau, \theta) \in [\![n + e(\phi), \psi]\!]^{\mathrm{M}}$. This shows $n, \phi; t_1 \doteq t_2 \models \psi$.

The completeness of the anaphoric calculus with equality is proved by modifying the Henkin construction in the usual way (taking equivalence classes of terms under provable equality as elements of the canonical model).

13. Conclusion

To wind up our story we mention some connections to related work. Via the translation to DRT in Section 5 (proposition 9) we have a proof system for (a streamlined version of) DRT. The calculus makes the discipline of using and modifying the anaphoric context and of handling dynamically bound indices fully explicit, and still is considerably simpler than existing proof systems for DRT, such as those presented in Kamp and Reyle [16] and Saurer [21].

When looking at the general picture of dynamic reconstruction proposals for DRT, what may emerge is that there is no single 'best' reconstruction, but that various reconstructions shed light on different aspects of the dynamics of text processing that all merit study in their own right. The present 'calculus of anaphora' focusses on the use of anaphoric context in reasoning, and gives a full account of the ways in which pronouns may be used to pick up antecedents from previous discourse in reasoning.

The $\exists$ of De Bruijn style classical predicate logic without variables can be viewed as a modal operator. The connection with Ben-Shalom [4] was already mentioned. There is also a straightforward connection to the cylindric algebra approach to first order logic.

Finally we mention the connections with Dekker [7], where a similar representation is proposed, but the problem of a calculus for reasoning is not addressed, with Visser and Vermeulen [24] and Visser [27, 26], and with Blackburn and Venema [5] and Hollenberg [13]. To see the connection with Hollenberg's equational axioms of dynamic negation and relational composition, note that these are all derivable in the calculus of anaphora (as of course they should be). A comparison with Visser's [25, 26] metamathematical analysis of DPL might answer the question: What are the DLWV definable relations?

We hope to have shown that anaphoric presupposition has a natural representation in terms of dynamic memory management: anaphoric presuppositions are minimum requirements on the amount of input memory needed for an update of an existing meaning representation. Dynamic memory management is represented very naturally by means of incremental indexing, in a 'variable free' language. The resulting reconstruction of dynamic reasoning is easier to handle than its predecessors, as is witnessed by the fact that it is relatively easy to axiomatize.

Acknowledgements

Thanks to Marco Hollenberg and Albert Visser for stimulating remarks and helpful discussions.

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