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Deriving Denotational Models for Bisimulation

from

Structured Operational Semantics

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We take as a starting point the notion of *labelled transition system* (LTS) in the style of the Structured Operational Semantics of Plotkin. Every LTS gives rise to a model that maps the states of the LTS (usually terms over some signature) to a representation of their bisimulation equivalence class, namely a so-called *process*. (Such a model is often called *operational*.) These processes are elements of a metric domain which was first introduced by De Bakker and Zucker. Next we show how the *transition system specification* (a set of rules for deriving transitions) by which the LTS is defined, induces a denotational model, given that it satisfies certain syntactic restrictions. Finally we prove that both models are equal by showing that they are fixed points of the same contraction, which has a unique fixed point by Banach's Theorem.

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CONTENTS

1. INTRODUCTION

The central notion in the Structured Operational Semantics (SOS) approach of Gordon Plotkin ([Pl81]) is *labelled transition system* (LTS). A LTS is a triple $<S, A, \rightarrow>$ consisting of a set S of *states*, a set A of transition labels, and a transition relation $\rightarrow \subseteq S \times A \times S$. The set of states of the systems we consider in this paper will be the set $T(\Sigma)$ of closed terms generated by a single sorted

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signature Σ. Such a signature represents a programming language; it may contain *constants* (like atomic actions or recursion variables) and function symbols (operators) of arity greater than 0 (like sequential and parallel composition). Every LTS induces the well known notion of *bisimulation* equivalence ([Pa81]): two states are *bisimilar* with respect to a LTS whenever every step starting in the first can also be made starting in the second such that the resulting states are again bisimilar, and the symmetrical property holds as well.

Every (finitely branching) LTS $\mathcal{Q} = \langle T(\Sigma), A, \rightarrow\rangle$ gives rise to a model $\mathcal{M}_{\mathcal{Q}}: T(\Sigma) \rightarrow P$. Here the semantic universe P satisfies the reflexive domain equation $P = \mathcal{P}_c(A \times P)$. It is a complete metric space, which was first introduced by De Bakker and Zucker ([BZ82]). Its elements, called *processes*, are tree-like structures. The function $\mathcal{M}_{\mathcal{Q}}$ maps each term onto the process that represents the graph obtained by unfolding this term according to the transition relation of $\mathcal{Q}$. It is characterized by the fact that it identifies all terms that are bisimilar. Because the model $\mathcal{M}_{\mathcal{Q}}$ is based on the notion of LTS, it is often called *operational*.

We exploit the metric structure of P in two ways. First, $\mathcal{M}_{\mathcal{Q}}$ is defined as the (by Banach's theorem unique) fixed point of a *contraction*. In this way, we can deal with finite and infinite behavior at the same time. Secondly, the metric on P can be conveniently used to prove the fact that $\mathcal{M}_{\mathcal{Q}}$ indeed maps terms on their bisimulation equivalence class.

An important aspect of the "structuredness" of SOS is the fact that the transition relation $\rightarrow$ of a LTS $\mathcal{Q} = \langle T(\Sigma), A, \rightarrow\rangle$ is usually induced by a set of rules (and axioms) that specify which transitions are possible. Such is set is called a *transition system specification* (TSS) for $T(\Sigma)$. In [GV88], Groote and Vaandrager introduce a restriction on the syntactic format of TSS's which implies that the bisimulation equivalence (induced by the corresponding LTS) is a congruence with respect to all the operators in the signature Σ. For completeness sake, we repeat this interesting result (without its proof) and translate it (rather obviously) into the observation that the function $\mathcal{M}_{\mathcal{Q}}$ introduced above is compositional.

The contribution of this paper consists of showing how every TSS $\mathcal{R}$ that is in the so-called *guarded structured operational semantics* (GSOS) format induces a *denotational* model $\mathcal{M}_{\mathcal{R}}: T(\Sigma) \rightarrow P$. Here denotational means that the model is compositional and that recursion is treated by means of some fixed point argument. The GSOS format (the terminology is borrowed from [BIM88]) is a restriction of the format of [GV88], but still general enough to be interesting (see Section 9 for an example). The construction of the model $\mathcal{M}_{\mathcal{R}}$ consists of the introduction of a semantic interpretation $I(\mathcal{R})$ of all the operators in the signature Σ. For each such operator $f \in \Sigma$ an interpretation $I(\mathcal{R})(f)$ is defined such that it satisfies the rules in the TSS for Σ that have f at the left hand side of the conclusion.

The main theorem of the paper (Theorem 8.3) then states that $\mathcal{M}_{\mathcal{Q}}$ and $\mathcal{M}_{\mathcal{R}}$ are equal. This implies that also $\mathcal{M}_{\mathcal{R}}$ identifies all terms that are bisimilar. Thus, from a TSS we have derived a denotational model that respects bisimulation equivalence. (A further consequence is that bisimulation equivalence is a congruence with respect to all the operators in Σ. Thus the proof of the theorem can be seen as an alternative for a restricted version of the congruence result of [GV88].) In the proof, the metric structure of the semantic universe P is again used: the equality of the models is proved by showing that both are fixed point of the same contraction. Then Banach's theorem implies the result.

This paper is mainly inspired by three earlier ones. In [Ba87], the idea of taking interpretations as fixed points of contractions induced by TSS's was first introduced. The definition given there is more complex than is really necessary and is based on the format of [Si84], which is less general than the GSOS format. In [GV88], already mentioned above, the relationship between the syntactic format of TSS's and bisimulation as congruence is systematically studied. Finally, the method of defining and comparing models with the help of contractions on complete metric spaces was first systematically applied in [KR88].

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2. Mathematical preliminaries

We assume the following notions to be known (the reader might consult [Du66] or [En77]): complete metric space, continuous function, compact subset of a metric space.

Let (M_1, d_1) and (M_2, d_2) be two complete metric spaces. A function $f: M_1 \rightarrow M_2$ is called *non-expansive* if for all $x,y \in M_1$

$$d_2(f(x), f(y)) \leqslant d_1(x,y).$$

The set of all non-expansive functions from M_1 to M_2 is denoted by $M_1 \rightarrow^1 M_2$. A function $f: M_1 \rightarrow M_2$ is called *contracting* (or a *contraction*) if there exists $\epsilon \in [0,1)$ such that for all $x,y \in M_1$

$$d_2(f(x), f(y)) \leqslant \epsilon \cdot d_1(x,y).$$

(Non-expansive functions and contractions are continuous.)

The following fact is known as Banach's Theorem: Let (M,d) be a complete metric space and $f: M \rightarrow M$ a contraction. Then f has a unique fixed point, that is, there exists a unique $x \in M$ such that $f(x) = x$.

The set $M_1 \rightarrow^1 M_2$ is a complete metric space by taking as a metric

$$d(f_1, f_2) = \sup_{x \in M_1} \{d_2(f_1(x), f_2(x))\}.$$

(All our metrics will have $[0,1]$ as their range.)

Let $\mathcal{P}_{compact}(M) = \{X\colon X \subseteq M \wedge X \text{ is compact}\}$. We can turn $\mathcal{P}_{compact}(M)$ into a complete metric space by defining a metric d_H, called the *Hausdorff distance* induced by d (the metric on M), as follows: For every $X, Y \in \mathcal{P}_{compact}(M)$

$$d_H(X,Y) = \max\{\sup_{x \in X}\{d(x,Y)\}, \sup_{y \in Y}\{d(y,X)\}\},$$

where $d(x,Z) = \inf_{z \in Z}\{d(x,z)\}$ for every $Z \subseteq M$, $x \in M$. (By convention we have $\sup \varnothing = 0$ and $\inf \varnothing = 1$.)

We call M_1 and M_2 *isometric* (notation: $M_1 \cong M_2$) if there exists a bijective mapping $f: M_1 \rightarrow M_2$ such that for all $x,y \in M_1$

$$d_2(f(x), f(y)) = d_1(x,y).$$

Finally, we introduce the set $p,q \in P$ of *processes*. (Throughout this paper we shall use the convention that the introduction of a set X with typical elements x,y is denoted by $x,y \in X$.) Let A be an arbitrary set. The process domain P is defined as the unique complete metric space that satisfies the following reflexive equation:

$$P \cong \mathcal{P}_{compact}(A \times P).$$

The equation above was, in a metric setting, first solved in [BZ82]. The unicity of the solution is proved in [AR89]. The metric d on P equals the Hausdorff metric d_H induced by the following metric on $A \times P$:

$$\bar{d}(<a_1,p_1>, <a_2,p_2>) = \begin{cases} 1 & \text{if } a_1 \neq a_2 \\ \frac{1}{2} \cdot d(p_1,p_2) & \text{if } a_1 = a_2. \end{cases}$$

The metric d_H is characterized by the following equality

$$d_H(p,q) = 2^{-\sup\{n:\, p[n] = q[n]\}}$$

where $p[n]$, the *truncation* of process p at depth n, is inductively defined by $p[0] = \varnothing$ and $p[n+1] = \{<a,p'[n]>\colon <a,p'> \in p\}$.

A process $p \in P$ can be viewed as a tree-like object. It is a set of pairs $<a,p'>$, which represents all the possible *steps* that the process p can take. The process p' in such a step is called its *resumption* and represents all steps that can possibly follow upon this one.

3. Labelled transition systems and bisimulation

DEFINITION 3.1 (LTS): A *labelled transition system* is a triple $\mathcal{Q}=(S,A,\rightarrow)$ consisting of a set of *states* S, a set of *labels* A, and a *transition relation* $\rightarrow \subseteq S \times A \times S$. We shall write $s \xrightarrow{a} s'$ for $(s,a,s') \in \rightarrow$. A LTS is called *finitely branching* if for all $s \in S$ the set $\{(a,s'): s \xrightarrow{a} s'\}$ is finite.

DEFINITION 3.2 (Bisimulation): Let $\mathcal{Q}=(S,A,\rightarrow)$ be a LTS. A relation $R \subseteq S \times S$ is called *a (strong) bisimulation* if it satisfies for all $s,t \in S$ and $a \in A$:

$$(sRt \wedge s \xrightarrow{a} s') \Rightarrow \exists t' \in S\ [t \xrightarrow{a} t' \wedge s'Rt']\ \text{and}$$

$$(sRt \wedge t \xrightarrow{a} t') \Rightarrow \exists s' \in S\ [s \xrightarrow{a} s' \wedge s'Rt'].$$

Two states are *bisimilar* in $\mathcal{Q}$, notation $s \leftrightarrow t$, if there exists a bisimulation relation R with sRt. (Note that bisimilarity is an equivalence relation on states.)

4. Models for bisimulation

DEFINITION 4.1 ($\mathfrak{M}_{\mathcal{Q}}$): Let $\mathcal{Q}=(S,A,\rightarrow)$ be a finitely branching LTS. We define a model $\mathfrak{M}_{\mathcal{Q}}: S \rightarrow P$ by

$$\mathfrak{M}_{\mathcal{Q}}[\![s]\!] = \{<a, \mathfrak{M}_{\mathcal{Q}}[\![s']\!]>: s \xrightarrow{a} s'\}.$$

(For the set A used in the definition of P in Section 2 we take the set of labels of $\mathcal{Q}$.)

We can justify this recursive definition by taking $\mathfrak{M}_{\mathcal{Q}}$ as the unique fixed point (Banach's Theorem) of a contraction $\Phi:(S \rightarrow P) \rightarrow (S \rightarrow P)$, defined by

$$\Phi(F)(s) = \{<a, F(s')>: s \xrightarrow{a} s'\}.$$

The fact that Φ is a contraction can be easily proved. The compactness of the set $\Phi(F)(s)$ is an immediate consequence of the fact that $\mathcal{Q}$ is finitely branching.

This model is of interest because it assigns to bisimilar states the same meaning. This we prove next.

THEOREM 4.2: *Let $\leftrightarrow \subseteq S \times S$ denote the bisimilarity relation induced by the labelled transition system $\mathcal{Q}=(S,A,\rightarrow)$. Then:*

$$\forall s,t \in S\ [s \leftrightarrow t \Leftrightarrow \mathfrak{M}_{\mathcal{Q}}[\![s]\!] = \mathfrak{M}_{\mathcal{Q}}[\![t]\!]].$$

PROOF

Let $s,t \in S$.

$\Leftarrow$:

Suppose $\mathfrak{M}_{\mathcal{Q}}[\![s]\!] = \mathfrak{M}_{\mathcal{Q}}[\![t]\!]$. We define a relation $\equiv \subseteq S \times S$ by

$$s' \equiv t' \Leftrightarrow \mathfrak{M}_{\mathcal{Q}}[\![s']\!] = \mathfrak{M}_{\mathcal{Q}}[\![t']\!].$$

From the definition of $\mathfrak{M}_{\mathcal{Q}}$ it is straightforward that $\equiv$ is a bisimulation relation on S: Suppose $s' \equiv t'$ and $s' \xrightarrow{a} s''$; then $<a, \mathfrak{M}_{\mathcal{Q}}[\![s'']\!]> \in \mathfrak{M}_{\mathcal{Q}}[\![s']\!] = \mathfrak{M}_{\mathcal{Q}}[\![t']\!]$; thus there exists $t'' \in S$ with $t' \xrightarrow{a} t''$ and $\mathfrak{M}_{\mathcal{Q}}[\![s'']\!] = \mathfrak{M}_{\mathcal{Q}}[\![t'']\!]$, that is, $s'' \equiv t''$. Symmetrically, the second property of a bisimulation relation holds. From the hypothesis we have $s \equiv t$. Thus we have $s \leftrightarrow t$.

$\Rightarrow$:

Let $R \subseteq S \times S$ be a bisimulation relation with sRt. We define

$$\epsilon = \sup_{s',t' \in S} \{d(\mathfrak{M}_{\mathcal{Q}}[\![s']\!], \mathfrak{M}_{\mathcal{Q}}[\![t']\!]): s'Rt'\}.$$

We prove that $\epsilon = 0$, from which $\mathfrak{M}_{\mathcal{Q}}[\![s]\!] = \mathfrak{M}_{\mathcal{Q}}[\![t]\!]$ follows, by showing that $\epsilon \leqslant \frac{1}{2} \cdot \epsilon$. We prove for all s', t' with $s'Rt'$ that $d(\mathfrak{M}_{\mathcal{Q}}[\![s']\!], \mathfrak{M}_{\mathcal{Q}}[\![t']\!]) \leqslant \frac{1}{2} \cdot \epsilon$. Consider $s',t' \in S$ with $s'Rt'$. From the definition of the Hausdorff metric on P it follows that it suffices to show

$$d(x, \mathcal{M}_{\mathcal{C}}[\![t']\!]) \leqslant \tfrac{1}{2}\cdot\epsilon \text{ and } d(y, \mathcal{M}_{\mathcal{C}}[\![s']\!]) \leqslant \tfrac{1}{2}\cdot\epsilon$$

for all $x \in \mathcal{M}_{\mathcal{C}}[\![s']\!]$ and $y \in \mathcal{M}_{\mathcal{C}}[\![t']\!]$. We shall only show the first inequality, the second being similar. Consider $<a, \mathcal{M}_{\mathcal{C}}[\![s'']\!]>$ in $\mathcal{M}_{\mathcal{C}}[\![s']\!]$ with $s' \xrightarrow{a} s''$. Because $s'Rt'$ and $s' \xrightarrow{a} s''$ there exists $t'' \in S$ with $t' \xrightarrow{a} t''$ and $s''Rt''$. Therefore

$$\begin{aligned} d(<a, \mathcal{M}_{\mathcal{C}}[\![s'']\!]>, \mathcal{M}_{\mathcal{C}}[\![t']\!]) &= d(<a, \mathcal{M}_{\mathcal{C}}[\![s'']\!]>, \{<\bar{a}, \mathcal{M}_{\mathcal{C}}[\![\bar{t}]\!]>: t' \xrightarrow{\bar{a}} \bar{t}\}) \\ &\leqslant [\text{ we have: } d(x, Y) = \inf\{d(x,y): y \in Y\}] \\ &\quad\ d(<a, \mathcal{M}_{\mathcal{C}}[\![s'']\!]>, <a, \mathcal{M}_{\mathcal{C}}[\![t'']\!]>) \\ &= \tfrac{1}{2}\cdot d(\mathcal{M}_{\mathcal{C}}[\![s'']\!], \mathcal{M}_{\mathcal{C}}[\![t'']\!]) \\ &\leqslant [\text{ because } s''Rt'']\ \tfrac{1}{2}\cdot\epsilon \end{aligned}$$

(The proof above makes conveniently use of the Hausdorff metric on P. In [CP88] and [GR89] alternative proofs are given with the use of so-called non-well-founded sets.)

5. Interpretations and compositionality

In the remaining sections of this paper we shall consider LTS's of a special format, namely, in which the set of states consists of the set of closed terms generated by a single sorted *signature*. Let us fix for the rest of this paper a signature $\Sigma = (F, r)$, which consists of a set $f \in F$ of *function names* and a *rank* function $r: F \to \mathbb{N}$ indicating for each function symbol its arity. Function symbols of arity 0 we call *constants*. Saying $f \in \Sigma$ for $f \in F$ is an abuse of language we shall sometimes indulge in. Further we introduce a set of variables $x, y \in Var$. The set of *terms* $s, t, u \in T(\Sigma, Var)$ built from Σ and Var is defined as usual; using the so-called BNF syntax, it can be given by

$$t ::= x \mid f(t_1, \ldots, t_{r(f)}).$$

Terms containing no variables are called *closed*. The set of closed terms is denoted by $T(\Sigma)$. Let $x_1, \ldots, x_k \in Var$ be distinct variables. For a term t we write $t_{(x_1, \ldots, x_k)}$ or $t_{\mathbf{x}}$ to indicate that the set of variables occurring in t is contained in the set $\{x_1, \ldots, x_k\}$. Whenever it is clear from the context what the free variables occurring in t are, these subscripts are omitted.

We have the usual syntactic substitution: We write $t_{(x_1, \ldots, x_k)}(u_1, \ldots, u_k)$, or $t_{\mathbf{x}}(\mathbf{u})$ for the term obtained by replacing every occurrence of x_i in t by u_i, for $1 \leqslant i \leqslant k$.

Definition 5.1 (Interpretations): Let V be a set. We define the set $(I \in)$ $IntPr_{\Sigma, V}$ of *interpretations* for $\Sigma = (F, r)$ as the collection of all functions

$$I: F \to \bigcup_k (V^k \to V)$$

with $I(f) \in V^{r(f)} \to V$ for every $f \in F$. (Read V for $V^0 \to V$.) Whenever Σ or V is clear from the context the corresponding subscript in $IntPr_{\Sigma, V}$ is omitted. An interpretation I induces for every term $t_{(x_1, \ldots, x_k)}$ in $T(\Sigma, Var)$ a function $t^I_{\mathbf{x}}: V^k \to V$ that is inductively given by

$$(1)\ (x_i)^I_{\mathbf{x}}(p_1, \ldots, p_k) = p_i$$

$$(2)\ f(t_1, \ldots, t_{r(f)})^I_{\mathbf{x}}(p_1, \ldots, p_k) = I(f)((t_1)^I_{\mathbf{x}}(p_1, \ldots, p_k), \ldots, (t_{r(f)})^I_{\mathbf{x}}(p_1, \ldots, p_k))$$

(We also write f^I for $I(f)$.)

Now every interpretation $I \in IntPr_V$ induces a model $\mathcal{M}_I$: $T(\Sigma) \rightarrow V$ defined by $\mathcal{M}_I[\![t]\!] = t^I$. Such a model is called *compositional* and satisfies for every term $t_{(x_1, \ldots, x_k)}$ in $T(\Sigma, Var)$ and closed terms $u_1, \ldots, u_{r(f)}$

$$\mathcal{M}_I[\![t_{\mathbf{x}}(u_1, \ldots, u_{r(f)})]\!] = t_{\mathbf{x}}^I(\mathcal{M}_I[\![u_1]\!], \ldots, \mathcal{M}_I[\![u_{r(f)}]\!])$$

DEFINITION 5.2 (Compositionality): A model $\mathcal{M}: T(\Sigma) \rightarrow V$ is called compositional if there exists an interpretation $I \in IntPr_V$ with $\mathcal{M} = \mathcal{M}_I$.

DEFINITION 5.3 (Congruence): An equivalence relation $\equiv \subseteq T(\Sigma) \times T(\Sigma)$ is called a *congruence* relation for Σ if for all $f \in \Sigma$ and closed terms $u_1, \ldots, u_{r(f)}, v_1, \ldots, v_{r(f)}$

if for all $1 \leqslant i \leqslant r(f)$ $u_i \equiv v_i$

then $f(u_1, \ldots, u_{r(f)}) \equiv f(v_1, \ldots, v_{r(f)})$

Sometimes such a relation is called *substitutive* for Σ.

The following fact is standard.

LEMMA 5.4: *Consider $\mathcal{M}: T(\Sigma) \rightarrow V$. Let $\equiv_{\mathcal{M}} \subseteq T(\Sigma) \times T(\Sigma)$ be defined by $s \equiv_{\mathcal{M}} t \Leftrightarrow \mathcal{M}[\![s]\!] = \mathcal{M}[\![t]\!]$. Then*

$\mathcal{M}$ is compositional iff $\equiv_{\mathcal{M}}$ is a congruence.

6. TRANSITION SYSTEM SPECIFICATIONS

We show in this section how LTS's that have the set of closed terms $T(\Sigma)$ over Σ as states can be specified with the help of rules (and axioms). Next we restrict these so-called *transition system specifications* to a particular syntactic format and repeat the main result from [GV88], stating that the bisimulation relation induced by (the LTS derived from) such a specification is a congruence relation for Σ. From this fact it easily follows that the model $\mathcal{M}$ induced by the LTS (as in Definition 4.1) is compositional.

DEFINITION 6.1 (TSS): A *transition system specification* (TSS) $\mathcal{R}$ for Σ is a (possibly infinite) set of *rules* R of the form

$$\frac{\{t_i \xrightarrow{a_i} t_i' : 1 \leqslant i \leqslant n\}}{t \xrightarrow{a} t'}$$

where $n \geqslant 0$, t_i, t_i', t, $t' \in T(\Sigma, Var)$, and a_i, $a \in A$, which is a given set of labels. If $n = 0$ then a rule is called an *axiom*.

A TSS is finitely branching if for every function symbol f the number of labels a and terms t with $f(\mathbf{x}) \xrightarrow{a} t$ in the conclusion of a rule is finite. In the rest of this paper only TSS's are considered that are finitely branching. (This property will be used in the construction of Definition 8.2 to ensure compactness of the resulting process.)

A rule is in *tyft* format if it is of the form

$$\frac{\{t_i \xrightarrow{a_i} y_i : 1 \leqslant i \leqslant n\}}{f(x_1, \ldots, x_{r(f)}) \xrightarrow{a} t}$$

such that

$n \geqslant 0$,

a_i, $a \in A$,

$$y_i, x_i \in Var, \quad \text{all distinct},$$

$$var(t_i) \subseteq \{x_1, \ldots, x_{r(f)}, y_1, \ldots, y_{i-1}\},$$

$$var(t) \subseteq \{x_1, \ldots, x_{r(f)}, y_1, \ldots, y_n\}$$

A TSS is in *tyft* format if all its rules are. (End of Definition 6.1.)

The *tyft* format and the restrictions it incorporates are extensively discussed in [GV88]. There a number of examples are given which show that the *tyft* format cannot be generalized in any obvious way such that Theorem 6.5 below would still hold.

(Note that we do not speak about rules in *tyxt* format nor about non-circularity, two notions that play a role in [GV88]. Our reason for not considering rules in *tyxt* form is that these can all be expressed by equivalent rules in *tyft* format (Lemma 5.9 of [GV88]). The additional requirement of non-circularity that is needed in [GV88] to prove Theorem 6.5 below is hidden in our slightly adapted definition of the *tyft* format.)

DEFINITION 6.2 (Transitions): An expression of the form $t \xrightarrow{a} t'$, with $t,t' \in T(\Sigma)$, is called a *transition*. Let $\mathfrak{R}$ be a TSS. A *proof tree* PT for a transition ψ from $\mathfrak{R}$ is defined in the usual way: It is a finite tree with root ψ such that the transition labelling a father node follows from the transitions labelling its sons by an application of (an instantiation of) a rule $R \in \mathfrak{R}$. Notation: $\mathfrak{R} \vdash_{PT} \psi$. We write $\mathfrak{R} \vdash \psi$ to express that there exists a proof tree PT with $\mathfrak{R} \vdash_{PT} \psi$. A transition may have many proof trees. We define the *degree* of a transition by

$$deg(\psi) = \min\{depth(PT)\colon \mathfrak{R} \vdash_{PT} \psi\}$$

DEFINITION 6.3 (Induced $\mathfrak{T}$): Every TSS $\mathfrak{R}$ induces a LTS $\mathfrak{T} = (T(\Sigma), A, \rightarrow)$ by taking $\rightarrow \subseteq T(\Sigma) \times A \times T(\Sigma)$ as

$$t \xrightarrow{a} t' \Leftrightarrow \mathfrak{R} \vdash t \xrightarrow{a} t'$$

DEFINITION 6.4 (Induction coefficient): Let $\mathfrak{R}$ be a TSS (which now is tacitly assumed to be finitely branching) for Σ. For $t \in T(\Sigma)$ and $a \in A$ we define

$$deg(t,a) = \max\{deg(t \xrightarrow{a} t')\colon t' \in T(\Sigma) \text{ and } \mathfrak{R} \vdash t \xrightarrow{a} t'\}$$

(We shall use this degree as an induction coefficient in the proof of Theorem 8.3.)

The main result of Groote and Vaandrager ([GV88]) is the following theorem.

THEOREM 6.5: *Let $\mathfrak{T} = \langle T(\Sigma), A, \rightarrow \rangle$ be the LTS induced by a TSS $\mathfrak{R}$ that is in tyft format, and let $\leftrightarrow$ be the bisimilarity relation induced by $\mathfrak{T}$. We have:*

$$\leftrightarrow \text{ is a congruence relation for } \Sigma$$

The proof uses induction on the depth of proof trees that are used to derive transitions.

Let $\mathfrak{R}$ and $\mathfrak{T}$ be as in the above theorem and consider the model $\mathfrak{M}_{\mathfrak{T}}\colon T(\Sigma) \rightarrow P$ that is induced by $\mathfrak{T}$ according to Definition 4.1, i.e.,

$$\mathfrak{M}_{\mathfrak{T}}[\![s]\!] = \{\langle a, \mathfrak{M}_{\mathfrak{T}}[\![s']\!]\rangle\colon s \xrightarrow{a} s'\}$$

An immediate consequence of the above theorem is that $\mathfrak{M}_{\mathfrak{T}}$ is compositional.

COROLLARY 6.6: $\mathfrak{M}_{\mathfrak{T}}$ *is compositional.*

PROOF
From Theorem 4.2 we know that $\leftrightarrow \; = \; \equiv_{\mathcal{M}_{\mathcal{T}}}$. Thus also $\equiv_{\mathcal{M}_{\mathcal{T}}}$ is a congruence for Σ, according to the above theorem. Now Lemma 5.4 implies that $\mathcal{M}_{\mathcal{T}}$ is compositional.

Since $\mathcal{M}_{\mathcal{T}}$ is compositional there exists an interpretation I for Σ such that $\mathcal{M}_{\mathcal{T}} = \mathcal{M}_I$. It can be defined as follows. Let $f \in \Sigma$ and $p_1, \ldots, p_{r(f)} \in P$. We choose $u_1, \ldots, u_{r(f)}$ with $p_i = \mathcal{M}_{\mathcal{T}}[\![u_i]\!]$ for all i. (If there do not exist such terms then $f^I(p_1, \ldots, p_{r(f)})$ can be defined arbitrarily.) We put

$$f^I(p_1, \ldots, p_{r(f)}) = \mathcal{M}_{\mathcal{T}}[\![f(u_1, \ldots, u_{r(f)})]\!]$$

Note that the definition of I does not depend on the choice of the terms u_i since $\mathcal{M}_{\mathcal{T}}$ is substitutive for Σ. Now it is straightforward to show that $\mathcal{M}_{\mathcal{T}} = \mathcal{M}_I$:

$$\begin{aligned}
\mathcal{M}_{\mathcal{T}}[\![f(u_1, \ldots, u_{r(f)})]\!] &= [\text{ definition } I \text{ }] \\
&\quad f^I(\mathcal{M}_{\mathcal{T}}[\![u_1]\!], \ldots, \mathcal{M}_{\mathcal{T}}[\![u_{r(f)}]\!]) \\
&= [\text{ induction }] \\
&\quad f^I(\mathcal{M}_I[\![u_1]\!], \ldots, \mathcal{M}_I[\![u_{r(f)}]\!]) \\
&= \mathcal{M}_I[\![f(u_1, \ldots, u_{r(f)})]\!]
\end{aligned}$$

for all $f \in \Sigma$ and $u_1, \ldots, u_{r(f)} \in T(\Sigma)$.

By definition the interpretation I introduced above is of a syntactic nature. One can give a recursive characterization of I that one might call semantic since it is formulated directly in terms of the elements of P. It depends on the fact that the LTS $\mathcal{T}$ under consideration is induced by a TSS $\mathcal{R}$ that is in *tyft* format.

THEOREM 6.7: *Let I be as above. We have for every $f \in \Sigma$ and $p_1, \ldots, p_{r(f)} \in \mathcal{M}_{\mathcal{T}}(T(\Sigma))$*

$$f^I(p_1, \ldots, p_{r(f)}) =$$
$$\{<a,\ t^I_{\mathbf{x},\mathbf{y}}(\mathbf{p},\mathbf{q})>:\ \exists R \in \mathcal{R}\, \forall i \in \{1, \ldots, n\}\ [<a_i,\ q_i> \in (t_i)^I_{\mathbf{x},\mathbf{y}}(\mathbf{p},\mathbf{q})]\}$$

where

$$R = \frac{\{t_i \xrightarrow{a_i} y_i:\ 1 \leq i \leq n\}}{f(x_1, \ldots, x_{r(f)}) \xrightarrow{a} t}$$

satisfies the conditions of Definition 6.1 and $\mathbf{p} = p_1, \ldots, p_{r(f)}$, $\mathbf{q} = q_1, \ldots, q_n$.

PROOF
Let $f \in \Sigma$ and $p_1, \ldots, p_{r(f)} \in \mathcal{M}_{\mathcal{T}}(T(\Sigma))$. Let $s_1, \ldots, s_{r(f)}$ be such that $p_i = \mathcal{M}_{\mathcal{T}}[\![s_i]\!]$ for all i. We have

$$\begin{aligned}
& f^I(p_1, \ldots, p_{r(f)}) \\
&= f^I(\mathcal{M}_{\mathcal{T}}[\![s_1]\!], \ldots, \mathcal{M}_{\mathcal{T}}[\![s_{r(f)}]\!]) \\
&= [\text{ definition } I \text{ }] \\
&\quad \mathcal{M}_{\mathcal{T}}[\![f(s_1, \ldots, s_{r(f)})]\!] \\
&= [\text{ Definition 4.1: } \mathcal{M}_{\mathcal{T}} \text{ }] \\
&\quad \{<a,\ \mathcal{M}_{\mathcal{T}}[\![t]\!]>:\ f(s_1, \ldots, s_{r(f)}) \xrightarrow{a} t\} \\
&= [\text{ the LTS } \mathcal{T} \text{ is induced by } \mathcal{R} \text{ }] \\
&\quad \{<a,\ \mathcal{M}_{\mathcal{T}}[\![t(s_1, \ldots, s_{r(f)},\ u_1, \ldots, u_n)]\!]>: \\
&\quad \exists R \in \mathcal{R}\ \forall i \in \{1, \ldots, n\}\ [\ t_i(s_1, \ldots, s_{r(f)},\ u_1, \ldots, u_n) \xrightarrow{a_i} u_i\ ]\}
\end{aligned}$$

(with R as above)

$= [$ Definition 4.1: $\mathcal{M}_{\mathfrak{F}}$ $]$

$\{<a,\ \mathcal{M}_{\mathfrak{F}}[\![t(s_1,\ldots,s_{r(f)},\ u_1,\ldots,u_n)]\!]>:\ \exists R\in\mathfrak{R}\ \forall i\in\{1,\ldots,n\}$

$[\ <a_i,\ \mathcal{M}_{\mathfrak{F}}[\![u_i]\!]>\in\mathcal{M}_{\mathfrak{F}}[\![t_i(s_1,\ldots,s_{r(f)},\ u_1,\ldots,u_n)]\!]\ ]\}$

$= [$ Corollary 6.6: $\mathcal{M}_{\mathfrak{F}} = \mathcal{M}_I$ $]$

$\{<a,\ t^I(\mathcal{M}_{\mathfrak{F}}[\![s_1]\!],\ldots,\mathcal{M}_{\mathfrak{F}}[\![s_{r(f)}]\!],\ \mathcal{M}_{\mathfrak{F}}[\![u_1]\!],\ldots,\mathcal{M}_{\mathfrak{F}}[\![u_n]\!])>:$

$\exists R\in\mathfrak{R}\ \forall i\in\{1,\ldots,n\}$

$[\ <a_i,\ \mathcal{M}_{\mathfrak{F}}[\![u_i]\!]>\in t_i^I(\mathcal{M}_{\mathfrak{F}}[\![s_1]\!],\ldots,\mathcal{M}_{\mathfrak{F}}[\![s_{r(f)}]\!],\ \mathcal{M}_{\mathfrak{F}}[\![u_1]\!],\ldots,\mathcal{M}_{\mathfrak{F}}[\![u_n]\!])\ ]\}$

$= \{<a,t^I(\mathbf{p},\mathbf{q})>:\ \exists R\in\mathfrak{R}\forall i\in\{1,\ldots,n\}\ [<a_i,q_i>\in t_i^I(\mathbf{p},\mathbf{q})]\}$

7. Denotational semantics

We introduce our definition of denotational semantics. We consider a signature Σ_{rec} in which we have a set of constants reserved for recursion; it is given by

$$\Sigma_{rec} = \Sigma \cup RecVar$$

where Σ is arbitrary and $(X\in)$ *RecVar* is a set of *recursion variables* (which are constants in the signature Σ_{rec}). The interpretation of recursion variables will be dependent on so-called *declarations*. The set of declarations is given by

$$(d\in)\ Decl = RecVar \to T(\Sigma_{rec})$$

In the sequel we shall consider only interpretations $I\in IntPr_{\Sigma_{rec},P}$ that are non-expansive: for every $f\in\Sigma$ the function $I(f)$ is non-expansive. It is straightforward to prove that $t_{\mathbf{x}}^I$ is non-expansive if I is.

Definition 7.1 (Denotational interpretation): Let $I\in IntPr_{\Sigma_{rec},P}$ be an interpretation for Σ_{rec} and let $d\in Decl$ be a declaration. We call I *denotational* (with respect to d) if there exist interpretations

$$I_0 \in IntPr_{\Sigma}$$

$$I_{rec} \in IntPr_{RecVar}$$

such that

(1) $I = I_0 \cup I_{rec}$

(2) $I_{rec} =$ fixed point Γ_{rec} where

Γ_{rec}: $IntPr_{RecVar} \to IntPr_{RecVar}$ is a contraction given by

$\Gamma_{rec}(J)(X) = (d(X))^{I_0\cup J}$, for all $J\in IntPr_{RecVar}$ and $X\in RecVar$

Note that $IntPr_{RecVar}$ (as well as $IntPr_{\Sigma}$) is a complete metric space, of which the metric is induced by the metric on P.

Definition 7.2 (Denotational semantics): (Recall that every interpretation $I\in IntPr_{\Sigma_{rec}}$ induces a compositional model $\mathcal{M}_I$: $T(\Sigma)\to P$ by $\mathcal{M}_I[\![t]\!]=t^I$ as defined in Section 5.) We call a model $\mathcal{M}$: $T(\Sigma)\to P$ denotational if there exists a denotational interpretation $I\in IntPr_{\Sigma_{rec}}$ such that $\mathcal{M}=\mathcal{M}_I$.

8. Deriving denotational interpretations from SOS

In this section we return to the notion of transition system specification, now for the signature Σ_{rec}. We shall introduce yet another TSS format, the so-called GSOS format, which is a special case of the *tyft* format introduced in Section 6. Next we shall show how a TSS in GSOS format induces a denotational interpretation for Σ_{rec}. In this section, $d \in Decl$ is a fixed declaration for the recursion variables in $\Sigma_{rec} = \Sigma \cup RecVar$.

Definition 8.1 (GSOS format): A TSS $\mathfrak{R}$ for Σ_{rec} is in *GSOS* format if

$$\mathfrak{R} = \mathfrak{R}_0 \cup \mathfrak{R}_{rec}$$

where $\mathfrak{R}_{rec}$ is a TSS for $RecVar$ given by

$$\frac{d(X) \xrightarrow{a} y}{X \xrightarrow{a} y}$$

for every $X \in RecVar$, and where $\mathfrak{R}_0$ is a TSS for Σ with all its rules in *zyft* format. A rule is in *zyft* format if it is of the form

$$\frac{\{z_i \xrightarrow{a_i} y_i :\ 1 \leqslant i \leqslant n\}}{f(x_1, \ldots, x_{r(f)}) \xrightarrow{a} t}$$

such that

$n \geqslant 0$,

$a_i,\ a \in A$, (A is the set of *labels*)

$y_i,\ x_i \in Var$, all distinct,

$z_i \in \{x_1, \ldots, x_{r(f)}\}$,

$t \in T(\Sigma)$ with $var(t) \subseteq \{x_1, \ldots, x_{r(f)}, y_1, \ldots, y_n\}$.

Every TSS in GSOS format is also in *tyft* format.

The SOS in GSOS stands for the (in no way perilous) notion of Structured Operational Semantics ([Pl81]); the G stands for Guarded, a notion about which nothing has been said in the above definition. In fact, it is not complete as it is. In Definition 8.2 below we shall formulate a semantic guardedness property that a TSS must additionally satisfy in order to be in GSOS format.

The terminology GSOS is borrowed from [BIM88]. To be precise, the above definition of GSOS corresponds to what in [GV88] is called the *positive* GSOS format, indicating that as premises no negations are allowed. We conjecture that it is possible to allow also for negations and still obtain the results presented in this section. For a discussion of the expressibility of the GSOS format and a comparison with the *tyft* format, we refer to [GV88].

We have restricted the *tyft* format of Section 6 to the above GSOS format in order to carry out the following construction. It yields for every TSS for Σ_{rec} that is in GSOS format a denotational interpretation for Σ_{rec}.

Definition 8.2 ($I(\mathfrak{R})$): Let $\mathfrak{R}$ be a TSS for Σ_{rec} in GSOS format. We define two contractions

$$\Gamma_0 : IntPr_\Sigma \rightarrow IntPr_\Sigma$$

$$\Gamma_{rec} : IntPr_{RecVar} \rightarrow IntPr_{RecVar}$$

as follows. First let $I \in IntPr_\Sigma$, $f \in \Sigma$ and $p_1, \ldots, p_{r(f)} \in P$. We put

$$\Gamma_0(I)(f)(p_1, \ldots, p_{r(f)}) =$$

$$\{<a,\ t^I_{\mathbf{x},\mathbf{y}}(\mathbf{p},\mathbf{q})>:\ \exists R \in \mathfrak{R}\ \forall i \in \{1, \ldots, n\}\ [<a_i,\ q_i> \in z^I_i(\mathbf{p})]\}$$

where

$$R = \frac{\{z_i \xrightarrow{a_i} y_i:\ 1 \leq i \leq n\}}{f(x_1, \ldots, x_{r(f)}) \xrightarrow{a} t}$$

satisfies the conditions of Definition 8.1 and $\mathbf{p}=p_1, \ldots, p_{r(f)}$, $\mathbf{q}=q_1, \ldots, q_n$. The function Γ_0 is well defined since the fact that $\mathcal{R}$ is finitely branching ensures the compactness of the set $\Gamma_0(I)(f)(p_1, \ldots, p_{r(f)})$. This we prove below. Moreover, it is straightforward to see that it is contracting and that $\Gamma_0(I)(f)$ is non-expansive. Thus we can define

$$I_0 = \text{fixed point } \Gamma_0$$

Second we define Γ_{rec} by putting

$$\Gamma_{rec}(J)(X) = (d(X))^{I_0 \cup J}, \quad \text{for all } J \in IntPr_{RecVar} \text{ and } X \in RecVar$$

In general this function need not be contracting. We simply require the TSS $\mathcal{R}_{rec}$ to be such that Γ_{rec} *is* contracting. This is the semantic formulation of the additional guardedness property for $\mathcal{R}$ that was announced above. (See Section 9 for an example.) Now we can put

$$I_{rec} = \text{fixed point } \Gamma_{rec}$$

Thus the TSS $\mathcal{R}$ induces a denotational interpretation $I(\mathcal{R})$ given by

$$I(\mathcal{R}) = I_0 \cup I_{rec}$$

We indicate the model induced by $I(\mathcal{R})$ by

$$\mathcal{M}_{\mathcal{R}}: T(\Sigma_{rec}) \rightarrow P$$

(End of Definition 8.2.)

We show that $\Gamma_0(I)(f)(p_1, \ldots, p_{r(f)}) =^{def} V$ is compact. Consider a sequence $(<a_k, r_k>)_k$ in V. We have to show that it has a converging subsequence. By the fact that $\mathcal{R}$ is finitely branching (Definition 6.1) there exists $R \in \mathcal{R}$, say with $f(\mathbf{x}) \xrightarrow{a} t$ as a conclusion, such that for infinitely many k

$$r_i = t^I(\mathbf{p}, \mathbf{q}^k) \text{ with}$$

$$\mathbf{q}^k = <q_1^k, \ldots, q_n^k> \text{ satisfying } <a_i, q_i^k> \in z_i^I(\mathbf{p}) \text{ for } 1 \leq i \leq n$$

Now $z_i^I(\mathbf{p}) \in \{p_1, \ldots, p_{r(f)}\}$, thus it is compact. Hence the sequence $(<a_i, q_i^k>)_k$ in $z_i^I(\mathbf{p})$ has a converging subsequence $(<a_i, q_i^{\mu_i(k)}>)_k$, where μ_i: $\mathbf{N} \rightarrow \mathbf{N}$ is a monotonic function. We define a monotonic function μ by

$$\mu = \mu_n \circ \cdots \circ \mu_1$$

The sequence

$$(\mathbf{q}^{\mu(k)})_k = (<q_1^{\mu(k)}, \ldots, q_n^{\mu(k)}>)_k$$

is converging. Because t^I is non-expansive also

$$(<a, t^I(\mathbf{p}, \mathbf{q}^{\mu(k)})>)_k$$

is convergent. Hence we have found a converging subsequence of $(<a_k, r_k>)_k$ in V.

The above construction cannot be generalized to the *tyft* format of Definition 6.1 by simply replacing $z_i^I(\mathbf{p})$ in the definition of Γ_0 with $t_i^I(\mathbf{p})$: if arbitrary terms are allowed at the left hand side of the premises, then Γ_0 need not be a contraction (and hence I_0 cannot be defined by taking the fixed point). This can be illustrated by the following (discouragingly) simple example: consider

$$\mathcal{R} = \left\{\frac{a \xrightarrow{a} y}{a \xrightarrow{a} y}\right\}$$

and let I_1 and I_2 be such that $I_1(a) = \varnothing$ and $I_2(a) = \{<a, \varnothing>\}$. Then

$$\begin{aligned} d(\Gamma_0(I_1), \Gamma_0(I_2)) &\geqslant d(\Gamma_0(I_1)(a), \Gamma_0(I_2)(a)) \\ &= d(\varnothing, \{<a, \varnothing>\}) \\ &= 1 \end{aligned}$$

Note that the *characterizing* equality given in Theorem 6.7 is in Definition 8.2 above used as a *defining* clause for $I(\mathfrak{R})$. In contrast to the definition of I in Section 6, the definition of $I(\mathfrak{R})$ does not use the model $\mathfrak{M}_{\mathfrak{T}}$ (Definition 4.1) and consequently does not depend on the congruence result of Theorem 6.5 (which, by the way, still applies since every TSS in GSOS format is in *tyft* format). But the model $\mathfrak{M}_{\mathfrak{R}}$ does equal $\mathfrak{M}_{\mathfrak{T}}$, as one would expect. This we prove next.

THEOREM 8.3: $\mathfrak{M}_{\mathfrak{T}} = \mathfrak{M}_{\mathfrak{R}}$

PROOF: Let Φ: $(T(\Sigma) \rightarrow P) \rightarrow (T(\Sigma) \rightarrow P)$ be the contraction used for the definition of $\mathfrak{M}_{\mathfrak{T}}$ (following Definition 4.1). Because $\Phi(\mathfrak{M}_{\mathfrak{T}}) = \mathfrak{M}_{\mathfrak{T}}$ and contractions have a unique fixed point, it is sufficient to prove $\Phi(\mathfrak{M}_{\mathfrak{R}}) = \mathfrak{M}_{\mathfrak{R}}$. We shall prove for every term $s \in T(\Sigma_{rec})$ and $a \in A$:

$$\forall q \in P\ [<a,q> \in \mathfrak{M}_{\mathfrak{R}}[\![s]\!] \Leftrightarrow \exists s' \in T(\Sigma_{rec})[s \xrightarrow{a} s' \wedge q = \mathfrak{M}_{\mathfrak{R}}[\![s']\!]]], \qquad (*)$$

which is equivalent to $\Phi(\mathfrak{M}_{\mathfrak{R}}) = \mathfrak{M}_{\mathfrak{R}}$. We use induction on the degree $deg(s,a)$ (Definition 6.4). Let $s \in T(\Sigma_{rec})$ and $a \in A$. Suppose that (*) holds for all s' and a' with $deg(s',a') < deg(s,a)$.

Case 1: $s \equiv f(u_1, \ldots, u_{r(f)})$
Consider $f(u_1, \ldots, u_{r(f)})$ with $f \in \Sigma$ and $u_1, \ldots, u_{r(f)} \in T(\Sigma_{rec})$. Below we use the following abbreviations:

$\mathbf{u}$ for $u_1, \ldots, u_{r(f)}$, $\mathbf{x}$ for $x_1, \ldots, x_{r(f)}$, $\mathbf{s}$ for $s_1, \ldots, s_n$

We show (*) for $f(\mathbf{u})$ and a. Let $q \in P$. The following statements are equivalent:

$<a,q> \in \mathfrak{M}_{\mathfrak{R}}[\![f(\mathbf{u})]\!]$

$\Leftrightarrow$ (Definition $\mathfrak{M}_{\mathfrak{R}}$)

$<a,q> \in f^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!])$

$\Leftrightarrow$ (Definition 8.2: $f^{I(\mathfrak{R})} = f^{I_0}$ and $I_0 = \Gamma_0(I_0)$)

$<a,q> \in f^{\Gamma_0(I_0)}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!])$

$\Leftrightarrow$ (Definition 8.2: $\Gamma_0(I_0)(f)$)

$\exists R = \dfrac{\{z_i \xrightarrow{a_i} y_i\colon 1 \leqslant i \leqslant n\}}{f(\mathbf{x}) \xrightarrow{a} t} \in \mathfrak{R}\ \exists q_1, \ldots, q_n \in P\ \forall i \in \{1, \ldots, n\}$

$[q = t^{I_0}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!], q_1, \ldots, q_n)$

$\wedge <a_i,q_i> \in z_i^{I_0}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!])]$

$\Leftrightarrow$ ($t^{I_0} = t^{I(\mathfrak{R})}$ since $t \in T(\Sigma)$ and $z_i^{I_0} = z_i^{I(\mathfrak{R})}$ for all i)

$\exists R = \cdots \exists q_1, \ldots, q_n \in P\ \forall i \in \{1, \ldots, n\}$

$[q = t^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!], q_1, \ldots, q_n)$

$\wedge <a_i,q_i> \in z_i^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!])]$

$\Leftrightarrow$ (Definition $\mathfrak{M}_{\mathfrak{R}}$)

$\exists R = \cdots \exists q_1, \ldots, q_n \in P\ \forall i \in \{1, \ldots, n\}$

$[q = t^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!], q_1, \ldots, q_n) \wedge \langle a_i, q_i \rangle \in \mathfrak{M}_{\mathfrak{R}}[\![z_i(\mathbf{u})]\!]]$

$\Leftrightarrow$ (induction: $deg(z_i(\mathbf{u}),a_i) < deg(f(\mathbf{u}),a)$, for all i)

$\exists R = \cdots \exists q_1, \ldots, q_n \in P \ \forall i \in \{1, \ldots, n\} \ \exists s_i \in T(\Sigma_{rec})$

$[q = t^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!], q_1, \ldots, q_n)$

$\wedge \ z_i(\mathbf{u}) \xrightarrow{a_i} s_i \ \wedge \ q_i = \mathfrak{M}_{\mathfrak{R}}[\![s_i]\!]]$

$\Leftrightarrow$ (elimination of q_i's)

$\exists R = \cdots \forall i \in \{1, \ldots, n\} \ \exists s_i \in T(\Sigma_{rec})$

$[q = t^{I(\mathfrak{R})}(\mathfrak{M}_{\mathfrak{R}}[\![u_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![u_{r(f)}]\!], \mathfrak{M}_{\mathfrak{R}}[\![s_1]\!], \ldots, \mathfrak{M}_{\mathfrak{R}}[\![s_n]\!]) \wedge z_i(\mathbf{u}) \xrightarrow{a_i} s_i]$

$\Leftrightarrow$ (Definition $\mathfrak{M}_{\mathfrak{R}}$)

$\exists R = \cdots \forall i \in \{1, \ldots, n\} \ \exists s_i \in T(\Sigma_{rec}) \ [q = \mathfrak{M}_{\mathfrak{R}}[\![t(\mathbf{u},\mathbf{s})]\!] \wedge z_i(\mathbf{u}) \xrightarrow{a_i} s_i]$

$\Leftrightarrow$ (Definition 8.2: $\rightarrow$ is induced by $\mathfrak{R}$)

$\exists s' \in T(\Sigma_{rec}) \ [q = \mathfrak{M}_{\mathfrak{R}}[\![s']\!] \wedge f(\mathbf{u}) \xrightarrow{a} s']$

Thus we have proved (*) for $f(\mathbf{u})$ and a.

Case 2: $s \equiv X$

Let $X \in RecVar$. We have

$\langle a,q \rangle \in \mathfrak{M}_{\mathfrak{R}}[\![X]\!]$

$\Leftrightarrow$ (Definition $\mathfrak{M}_{\mathfrak{R}}$)

$\langle a,q \rangle \in X^{I(\mathfrak{R})}$

$\Leftrightarrow$ (Definition 8.2: $X^{I(\mathfrak{R})} = X^{I_{rec}}$ and $I_{rec} = \Gamma_{rec}(I_{rec})$)

$\langle a,q \rangle \in X^{\Gamma_{rec}(I_{rec})}$

$\Leftrightarrow$ (Definition 8.2: $\Gamma_{rec}(I_{rec})$, using $I_0 \cup I_{rec} = I(\mathfrak{R})$)

$\langle a,q \rangle \in d(X)^{I(\mathfrak{R})}$

$\Leftrightarrow$ (Definition $\mathfrak{M}_{\mathfrak{R}}$)

$\langle a,q \rangle \in \mathfrak{M}_{\mathfrak{R}}[\![d(X)]\!]$

$\Leftrightarrow$ (induction: $deg(d(X),a) < deg(X,a)$)

$\exists s' \in T(\Sigma_{rec}) \ [q = \mathfrak{M}_{\mathfrak{R}}[\![s']\!] \wedge d(X) \xrightarrow{a} s']$

$\Leftrightarrow$ (Definition 8.2: $\rightarrow$ is induced by $\mathfrak{R}$)

$\exists s' \in T(\Sigma_{rec}) \ [q = \mathfrak{M}_{\mathfrak{R}}[\![s']\!] \wedge X \xrightarrow{a} s']$.

This proves (*) for X and a and concludes the proof of Theorem 8.3.

For the proof of Case 1 it would have been sufficient to use a simple induction on the structural complexity of terms. Obviously, this would not work for Case 2. Therefore we make use of the induction coefficient of Definition 6.4, which essentially measures for a transition the depth of the tree that is used to prove it. The proof of the above theorem is inspired on and generalizes the method of [KR88] for proving the equivalence of denotational and operational models for programming languages.

Theorem 8.3 implies that $\mathfrak{M}_{\mathfrak{T}}$ is compositional, since $\mathfrak{M}_{\mathfrak{R}}$ is, and thus that $\equiv_{\mathfrak{M}_{\mathfrak{T}}}$ is a congruence for

Σ_{rec}. Then by Theorem 4.2 also $\leftrightarrow$ is a congruence. Thus the above proof can be viewed as an alternative to that of the congruence result of Theorem 6.5 given in [GV88] for the case that the TSS is in GSOS format.

9. An example: a non-uniform language with value passing

As an example we consider a signature $\Sigma_{rec} = \langle F, r\rangle$ that is defined as follows. First we introduce three syntactic categories, viz. the set $(v \in)$ *Var* of *individual variables*, the set $(e \in)$ *Exp* of *expressions* and the set $(b \in)$ *Bexp* of *Boolean expressions*. We shall not specify a syntax for *Exp* and *Bexp*. We assume that (Boolean) expressions are of an elementary kind; in particular, they have no side effects and their evaluation always terminates. Statement variables $x, y, \ldots$ are as usual. The elements $c \in C$ will be used as part of value passing communication actions $c?v$ or $c!e$ that will be introduced in a moment. Now let the set F of function symbols be given by

$$F = Act \cup \{\,;\,, \|, +\,\} \cup RecVar$$

where $(X \in)$ *RecVar* is the set of recursion variables and the set $(act \in)$ *Act* of basic actions is given by

$$Act = Comm \cup Bexp \cup Asg$$

Here the set *Comm* of communications is defined by

$$Comm = \{c?v \colon c \in C, v \in Var\} \cup \{c!e \colon c \in C, e \in Exp\}$$

and the set *Asg*, of assignments, by

$$Asg = \{v := e : v \in Var,\ e \in Exp\}$$

The rank function r of Σ_{rec} is defined by

$$r(a) = 0, \text{ for every } a \in Act$$

$$r(X) = 0, \text{ for every } X \in RecVar$$

$$r(;) = r(\|) = r(+) = 2$$

The set $T(\Sigma_{rec})$ of closed terms over Σ_{rec} is again called a *language*. In BNF notation it can be defined as the language $(s \in)$ $\mathcal{L}$ given by

$$s ::= c!e \mid c?v \mid b \mid v := e \mid s;t \mid s\|t \mid s+t \mid X$$

The interpretation of the operators ;, $\|$ and $+$, for sequential, parallel and non-deterministic composition, respectively, is as before. There are three kinds of actions: communications, Boolean expressions and assignments. Communication actions are either send actions, indicated by $c!e$, or receive actions, indicated by $c?v$. Thus we have here a CSP like communication mechanism ([Ho85]). Boolean expressions can be used to model, e.g., if-then-else and while statements. For a more elaborate discussion of this language we refer to [BKMOZ86].

Next we define a LTS $\mathcal{T} = \langle T(\Sigma_{rec}), A, \rightarrow\rangle$. The set $(\alpha \in)$ A of labels is given by

$$A = SAct \cup \overline{SAct}$$

Here the set *SAct* of semantic actions is the domain for the semantic interpretation of the basic actions:

$$(a \in)\ SAct = (States \rightarrow States) \cup (States \rightarrow \{tt, ff\}) \cup$$
$$(C \times Var) \cup (C \times (States \rightarrow Val))$$

The set *States* of (indeed) states is

$$States = Var \to Val$$

with Val some set of values. The elements of $\overline{SAct}$ ($= \{\bar{a}\colon a \in SAct\}$) are used to indicate termination (see the rules for sequential and parallel composition below).

Note that in this example the labels are of considerable complexity, representing for every type of basic action a semantic interpretation.

The transition relation $\to$ of $\mathfrak{T}$ is induced by the following TSS $\mathfrak{R}$. The transitions for the basic actions are given by

$$act \xrightarrow{\bar{a}_{act}} \delta$$

where δ is a special element of Act denoting termination and where a_{act} is defined by

$$a_{act} = \begin{cases} \lambda\sigma\cdot\ \sigma_{v:=e} & \text{if } act = v:=e \\ \lambda\sigma\cdot\ [\![b]\!]\sigma & \text{if } act = b \in BExp \\ <c,\ v> & \text{if } act = c?v \\ <c,\ \lambda\sigma\cdot\ [\![e]\!]\sigma> & \text{if } act = c!e \end{cases}$$

where $\sigma_{v:=e}$ is like σ but for its value in v, which is $[\![e]\!]\sigma$. In this definition we have postulated the presence of two semantic evaluation functions; the values of e and b in a state σ are simply denoted by

$$[\![e]\!]\sigma,\ [\![b]\!]\sigma \in Val$$

Further rules in $\mathfrak{R}$ are

$$\frac{x_1 \xrightarrow{\bar{a}} y_1}{\begin{array}{c} x_1;x_2 \xrightarrow{a} x_2 \\ x_1\|x_2 \xrightarrow{a} x_2 \\ x_2\|x_1 \xrightarrow{a} x_2 \end{array}} \qquad \frac{x_1 \xrightarrow{a} y_1}{\begin{array}{c} x_1;x_2 \xrightarrow{a} y_1;x_2 \\ x_1\|x_2 \xrightarrow{a} y_1\|x_2 \\ x_2\|x_1 \xrightarrow{a} x_2\|y_1 \end{array}} \qquad \frac{x_1 \xrightarrow{\alpha} y_1}{\begin{array}{c} x_1+x_2 \xrightarrow{\alpha} y_1 \\ x_2+x_1 \xrightarrow{\alpha} y_1 \end{array}} \qquad \frac{x_1 \xrightarrow{<c,v>} y_1,\ x_2 \xrightarrow{<c,f>} y_2}{\begin{array}{c} x_1\|x_2 \xrightarrow{l} y_1\|y_2 \\ x_2\|x_1 \xrightarrow{l} y_2\|y_1 \end{array}}$$

where $l = \lambda\sigma\cdot\sigma'$, with σ' like σ but for its value in v, which is $f(\sigma)$. (We omit two rules that deal with communication in case either component terminates.)

Finally, we have to supply rules for the recursion variables. So let us consider a declaration $d \in Decl$. In order to let $\mathfrak{R}$ be in GSOS format, which is what we aim at, we have to make sure that the semantic guardedness condition (of Definition 8.2) will be satisfied. This is the case, as one can easily proof, when we consider a declaration that assigns every recursion variable to a so-called *guarded* statement. The set ($g \in$) $GStat \subseteq T(\Sigma_{rec})$ of guarded statements is defined, in BNF notation, by

$$g ::= c!e \mid c?v \mid b \mid v:=e \mid g;s \mid g_1\|g_2 \mid g_1+g_2$$

where $s \in T(\Sigma_{rec})$. E.g., $(v:=e;X) + c!e$ is guarded whereas $(X;v:=e) + c!e$ is not. Now the rules for recursion are as in Definition 8.1, namely

$$\frac{d(X) \xrightarrow{\alpha} y}{X \xrightarrow{\alpha} y}$$

Next we will apply the definitions and theorem of the previous section. First it can again be proved by structural induction that this $\mathfrak{R}$ is finitely branching. Thus we can use Definition 4.1 that yields a model $\mathfrak{M}_{\mathfrak{T}}\colon T(\Sigma_{rec}) \to P$ given by

$$\mathfrak{M}_{\mathfrak{T}}[\![s]\!] = \{<a, \mathfrak{M}[\![s']\!]>\colon s \xrightarrow{a} s'\}$$

Moreover, $\mathfrak{R}$ is in GSOS format. Thus it induces a denotational interpretation $I(\mathfrak{R})$ according to Definition 8.2. We have for the interpretations of the function symbols the following equalities.

$$act^{I(\mathfrak{R})} = \{<\bar{a}_{act},\ \varnothing>\} \text{ with } a_{act} \text{ as above}$$

$$p;^{I(\mathfrak{R})} q = \{<a,q>:\ <\bar{a},p'>\in p\} \cup \{<a,p';^{I(\mathfrak{R})} q>:\ <a,p'>\in p\}$$

$$\begin{aligned} p\,||^{I(\mathfrak{R})}\, q = &\ \{<a,q>:\ <\bar{a},p'>\in p\} \cup \{<a,p'||^{I(\mathfrak{R})} q>:\ <a,p'>\in p\} \cup \\ &\ \{<a,p>:\ <\bar{a},q'>\in q\} \cup \{<a,p||^{I(\mathfrak{R})} q'>:\ <a,q'>\in q\} \cup \\ &\ \{<l,p'||^{I(\mathfrak{R})}q'>:\ <<c,v>,p'>\in p \wedge <<c,f>,q'>\in q\} \cup \\ &\ \{<l,p'||^{I(\mathfrak{R})}q'>:\ <<c,f>,p'>\in p \wedge <<c,v>,q'>\in q\} \\ &\ \text{(with } l \text{ as in the rules above)} \end{aligned}$$

$$p +^{I(\mathfrak{R})} q = p \cup q$$

$$X^{I(\mathfrak{R})} = (d(X))^{I(\mathfrak{R})}$$

According to Theorem 8.3 we have $\mathcal{M}_{\mathcal{T}} = \mathcal{M}_{\mathfrak{R}}$. Thus we have for $\mathcal{M}_{\mathcal{T}}$ the following equalities.

$$\mathcal{M}_{\mathcal{T}}[\![act]\!] = \{<\bar{a}_{act},\ \varnothing>\}$$

$$\mathcal{M}_{\mathcal{T}}[\![s;t]\!] = \mathcal{M}_{\mathcal{T}}[\![s]\!] \ ;^{I(\mathfrak{R})}\ \mathcal{M}_{\mathcal{T}}[\![t]\!]$$

$$\mathcal{M}_{\mathcal{T}}[\![s||t]\!] = \mathcal{M}_{\mathcal{T}}[\![s]\!] \ ||^{I(\mathfrak{R})}\ \mathcal{M}_{\mathcal{T}}[\![t]\!]$$

$$\mathcal{M}_{\mathcal{T}}[\![s+t]\!] = \mathcal{M}_{\mathcal{T}}[\![s]\!] \ +^{I(\mathfrak{R})}\ \mathcal{M}_{\mathcal{T}}[\![t]\!]$$

$$\mathcal{M}_{\mathcal{T}}[\![X]\!] = \mathcal{M}_{\mathcal{T}}[\![d(X)]\!]$$

10. Discussion

The model $\mathcal{M}_{\mathcal{T}}$ is often called *operational* because it is formulated in terms of a transition system. For those who do not like an operational semantics to be defined as the fixed point of a function, let it be observed that an equivalent definition for $\mathcal{M}_{\mathcal{T}}$ can easily be given in terms of (finite and infinite) transition sequences only. On the other hand, $\mathcal{M}_{\mathfrak{R}}$ is by definition denotational. Summarizing we see that starting with a TSS in GSOS format we obtain an operational model $\mathcal{M}_{\mathcal{T}}$ (according to Definition 4.1) and a denotational model $\mathcal{M}_{\mathfrak{R}}$ (according to Definition 8.2) that are found to be equal and that identify all terms that are bisimilar. Thus progress is made with respect to the method described in [KR88], where operational and denotational models are defined separately and considerable effort is required to establish their equivalence.

We would like to prove a similar result for the more general *tyft* format of [GV88]. (The fact that this format is more general can, for example, be illustrated by the fact that it allows for a description of *weak* bisimulation, whereas the GSOS format does not.) This seems impossible with the use of complete metric spaces, as is illustrated by the example following Definition 8.2. At present, we are investigating the universe of non-well-founded sets ([Ac88]) as a possibly more liberal alternative to complete metric spaces.

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