Grid approximation of a singularly perturbed boundary value problem modelling heat transfer in the case of flow over a flat plate with suction of the boundary layer[☆]

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Abstract

In the present paper we consider a boundary value problem on the semiaxis $(0, \infty)$ for a singularly perturbed parabolic equation with the two perturbation parameters ε_1 and ε_2 multiplying, respectively, the second and first derivatives with respect to the space variable. Depending on the relation between the parameters, the differential equation can be either of reaction–diffusion type or of convection–diffusion type. Correspondingly, the boundary layer can be either parabolic or regular. For this problem we consider the case when the boundary layer can be controlled by continuous suction of the fluid out of the boundary layer (model problems of this type appear in the mathematical modelling of heat transfer processes for flow past a flat plate). Errors in the approximations generated by standard numerical methods can be unsatisfactorily large for small values of the parameter ε_1. We construct a monotone finite difference scheme on piecewise uniform meshes which generates numerical solutions converging ε-uniformly with order $O(N^{-1} \ln N + N_0^{-1})$, where N_0 is the number of nodes in the time mesh and N is the number of meshpoints on a unit interval of the semiaxis in x. Although the solution of problem has a singularity only for $\varepsilon_1 \to 0$, the character of the boundary layer depends essentially on the vector-valued parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$. This prevents us from constructing an ε-uniformly convergent scheme having a transition parameter which is independent of the parameter ε_2.
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1. Introduction

Numerical analysis of laminar flows of incompressible fluid for large Reynolds and/or Péclet numbers often leads to the consideration of boundary value problems for boundary layer equations. These quasilinear equations are singularly perturbed, with two perturbation parameters ε_R and ε_P defined by $\varepsilon_R = Re^{-1}$ and $\varepsilon_P = Pe^{-1}$, where Re and Pe are the Reynolds and Péclet numbers; $Pe = Re\,Pr$, Pr is the Prandtl number. Parabolic and regular layers are typical for such problems [9,12]. Singularities of the same type occur in problems modelling heat transfer processes for flow past surfaces in the case of boundary layers controlled by suction of some of the fluid from the boundary layer (see, for example, [12, Chapter 14]).

The presence of parabolic boundary and/or interior layers in such problems results in large errors (for small values of the perturbation parameters ε_1, ε_2 multiplying the space derivatives involved in the equations) if we apply classical methods for finding numerical solutions. Thus, it is necessary to develop special numerical methods whose errors do not depend on the value of the vector-valued parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$, i.e. methods which converge ε-uniformly. Possible approaches to construct such methods and also some special schemes are given, for example, in [1–3,5,8,10,14] (see also references therein).

In the present paper we consider a boundary value problem on the semiaxis $(0, \infty)$ for a singularly perturbed parabolic equation with the two perturbation parameters ε_1 and ε_2 multiplying the derivatives with respect to the space variable. Model problems of such type appear in the mathematical modelling of heat transfer processes for flow past a flat plate with continuous suction of fluid out of the boundary layer (see, for example, Section 3). Depending on the value of the parameter ε_2 multiplying the first derivative in x, the differential equation can be either of reaction–diffusion type (for $\varepsilon_2 < \varepsilon_1^{1/2}$) or of convection–diffusion type (for $\varepsilon_2 \gg \varepsilon_1^{1/2}$). Correspondingly, the boundary layer is either parabolic or regular. Errors of classical numerical methods applied to this problem can be unsatisfactorily large for small values of the parameter ε_1. Standard methods allow us to obtain satisfactory numerical approximations to the solution only under the very restrictive condition $N^{-1} \ll \varepsilon_1(\varepsilon_1^{1/2} + \varepsilon_2^2)^{-1}$ imposed on the number of mesh points, where N is the number of nodes in the space mesh on the unit interval (see condition (4.6) in Section 4). At the same time, the technique for constructing ε-uniformly convergent schemes using a fitted operator turns out to be inapplicable to such problems due to the presence of parabolic boundary layers in the solution (see Remark 1 in Section 4). Here we construct a monotone finite difference scheme (on piecewise uniform meshes) for the problem under consideration, which generates numerical solutions converging ε-uniformly with order $O(N^{-1}\ln N + N_0^{-1})$, where N_0 is the number of nodes in the time mesh.

Note that special difference schemes for the problem studied in this paper, which generate numerical solutions converging ε-uniformly (in the maximum norm), are unknown in the literature.

2. Problem formulation. Aim of the research

2.1. On the set $\bar{G}$, where

$$\bar{G} = G \cup S, \quad G = D \times (0, T], \quad D = (0, \infty), \tag{2.1}$$

with boundary $S = S^L \cup S_0$, where S^L and S_0 are the lateral and bottom parts of the boundary S; $S^L = \Gamma \times (0, T]$, $S_0 = \bar{D} \times \{t = 0\}$, $\Gamma = \bar{D} \setminus D$.

We consider the following boundary value problem for the singularly perturbed parabolic equation

$$Lu(x, t) \equiv \left\{ \varepsilon_1 a(x, t) \frac{\partial^2}{\partial x^2} + \varepsilon_2 b(x, t) \frac{\partial}{\partial x} - c(x, t) - p(x, t) \frac{\partial}{\partial t} \right\} u(x, t)$$

$$= f(x, t), \quad (x, t) \in G,$$

$$u(x, t) = \Phi(x, t), \quad (x, t) \in S. \tag{2.2}$$

Here the parameters ε_1 and ε_2, which are the components of the vector-parameter ε (or, shortly, of the parameter ε), take arbitrary values in the half-interval $(0, 1]$ and the segment $[0, 1]$, respectively. We assume that the coefficients $a(x, t)$, $b(x, t)$, $c(x, t)$, $p(x, t)$ and the right side $f(x, t)$ are sufficiently smooth functions on $\bar{G}$ satisfying the condition [1]

$$a_0 \leqslant a(x, t) \leqslant a^0, \quad b_0 \leqslant b(x, t) \leqslant b^0, \quad 0 \leqslant c(x, t) \leqslant c^0, \quad p_0 \leqslant p(x, t) \leqslant p^0,$$

$$|f(x, t)| \leqslant M, \quad (x, t) \in \bar{G}, \quad a_0, b_0, p_0 > 0, \tag{2.3a}$$

the boundary function $\Phi(x, t) = \Phi(x, t; \varepsilon)$ for a fixed value of the parameter ε is sufficiently smooth on the sets $\bar{S}^L$ and S_0 and continuous on S, moreover

$$|\Phi(x, t)| \leqslant M, \quad (x, t) \in S. \tag{2.3b}$$

The solution of the boundary value problem is regarded as a function $u \in C^{2,1}(G) \cap C(\bar{G})$, which is bounded on $\bar{G}$ and satisfies the differential equation on G and the boundary condition on S.

For simplicity, we suppose that on the set $S^c = \bar{S}^L \cap S_0$, i.e. at the "corner" points, compatibility conditions (see, e.g., [7]) are satisfied which ensure the required smoothness of the solution of the problem for each fixed value of the parameter ε.

2.2. We now discuss more precise conditions imposed on the function $\Phi(x, t)$.

When the following conditions hold

$$\left| \frac{\partial^k}{\partial x^k} \Phi(x, t) \right| \leqslant M, \quad (x, t) \in S_0,$$

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} \Phi(x, t) \right| \leqslant M, \quad (x, t) \in \bar{S}^L, \quad k \leqslant K, \quad k_0 \leqslant K_0, \tag{2.4}$$

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[1] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters. Throughout the paper, the notation $L_{(j.k)}$ $(M_{(j.k)}, G_{h(j.k)})$ means that these operators (constants, meshes) are introduced in equation $(j.k)$.

where K, $K_0 > 0$ are sufficiently large numbers, a boundary layer appears in a neighbourhood of the set $\bar{S}^L$ as the parameter ε_1 tends to zero. This layer is parabolic if the condition $\varepsilon_2 = O(\varepsilon_1^{1/2})$ holds and regular if $\varepsilon_1 = o(\varepsilon_2^2)$.

If the derivatives of the function $\Phi(x,t)$ are ε-uniformly bounded (for example, (2.4) holds with $K = 7$, $K_0 = 2$), that is the data of the problem are sufficiently smooth, then the solution of the problem can be decomposed into a sum of regular and singular components

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \bar{G}. \tag{2.5}$$

Let the function $\Phi(x,t)$ for $t=0$ can also be written as a sum of the regular and singular components

$$\Phi(x,t) = \Phi_U(x,t) + \Phi_V(x,t), \quad (x,t) \in S_0. \tag{2.6a}$$

Moreover, the singular component $\Phi_V(x,t)$ has the same singularities as the component $V(x,t)$ for the case of boundary value problem (2.2), (2.1), (2.4) with $K \geqslant 7$, $K_0 \geqslant 2$, then for $t > 0$ the singular component $V(x,t)$ of the solution of problem (2.2), (2.1) retains the character of the singularity in $\Phi_V(x,t)$ (see, e.g., the estimates of Theorem 3 and Remark 3). This decomposition of the solution into its regular and singular components allows us, in a number of cases, to construct and to study ε-uniform numerical methods (see, e.g. [8,14] in the case of regular initial conditions).

We assume throughout that the function $\Phi(x,t)$ and its components in (2.6a) satisfy the conditions

$$\left| \frac{\partial^k}{\partial x^k} \Phi_U(x,t) \right| \leqslant M,$$

$$\left| \frac{\partial^k}{\partial x^k} \Phi_V(x,t) \right| \leqslant M \left\{ \begin{array}{ll} \varepsilon_1^{-k/2} \exp(-m_1 \varepsilon_1^{-1/2} x) & \text{for } \varepsilon_2 \leqslant M_0 \varepsilon_1^{1/2} \\ \varepsilon_2^k \varepsilon_1^{-k} \exp(-m_2 \varepsilon_2 \varepsilon_1^{-1} x) & \text{for } \varepsilon_2 > M_0 \varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t) \in S_0,$$

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} \Phi(x,t) \right| \leqslant M, \quad (x,t) \in \bar{S}^L, \quad k \leqslant K, \quad k_0 \leqslant K_0, \tag{2.6b}$$

where m_1 is an arbitrary constant, m_2 is a constant from the interval $(0, m_0)$,

$$m_0 = \min_{\bar{G}} [a^{-1}(x,t)b(x,t)],$$

and K, K_0 are sufficiently large numbers.

2.3. Our goal is to construct a finite difference scheme which is ε-uniformly convergent for the singularly perturbed boundary value problem (2.2), (2.1) with the singularly perturbed initial function satisfying condition (2.6).

Note that, for problem (2.2), (2.1) corresponding to the heat transfer problem (3.3) in the case of flow past a flat plate with suction of the boundary layer [12], we have $\varepsilon_1 = \varepsilon_T$ and $\varepsilon_2 = \varepsilon_R^{1/2} + v_0$, where $\varepsilon_T = Pe^{-1}$, $\varepsilon_R = Re^{-1}$, and $v_0 \geqslant 0$ is the intensity of the suction.

3. Motivation of the research

In this section we consider a boundary value problem for the boundary layer equations in a bounded domain, which describes heat transfer in a viscous fluid flowing past a flat plate. Let a

semi-infinite flat plate be placed on the semiaxis $P = \{(x, y): x \geq 0, \ y=0\}$. The problem is symmetric with respect to the plane $y = 0$; we examine the steady flow of an incompressible fluid on both sides of P, which is laminar and parallel to the plate. We consider the solution of this problem on the bounded set

$$\bar{G} \quad \text{where} \quad G = \{(x, y): x \in (d_1, d_2], \ y \in (0, d_0)\}, \quad d_1 > 0. \tag{3.1}$$

Let $G^0 = \{(x, y): x \in [d_1, d_2], \ y \in (0, d_0)\}$; $\bar{G}^0 = \bar{G}$. We write $S = \bar{G} \setminus G$, $S = \cup S_j$, $j = 0, 1, 2$, where

$$S_0 = \{(x, y): x \in [d_1, d_2], \ y = 0\}, \quad S_1 = \{(x, y): x = d_1, \ y \in (0, d_0]\},$$

$$S_2 = \{(x, y): x \in (d_1, d_2], \ y = d_0\}, \quad \bar{S}_0 = S_0; \quad S^0 = \bar{G} \setminus G^0 = S_0.$$

On the set $\bar{G}$, it is required to find the solution $U(x, y) = (u(x, y), v(x, y))$ of the following Prandtl problem:

$$L^1 U(x, y) \equiv \left\{ \varepsilon_R \frac{\partial^2}{\partial y^2} - u(x, y) \frac{\partial}{\partial x} - v(x, y) \frac{\partial}{\partial y} \right\} u(x, y) = 0, \quad (x, y) \in G, \tag{3.2a}$$

$$L^2 U(x, y) \equiv \frac{\partial}{\partial x} u(x, y) + \frac{\partial}{\partial y} v(x, y) = 0, \quad (x, y) \in G^0, \tag{3.2b}$$

$$u(x, y) = \varphi(x, y), \quad (x, y) \in S, \tag{3.2c}$$

$$v(x, y) = \psi(x, y), \quad (x, y) \in S^0. \tag{3.2d}$$

Here ε_R is the viscosity in the case when $U(x, y)$ and x, y are dimensional quantities, and $\varepsilon_R = Re^{-1}$ when $U(x, y)$ and x, y are dimensionless ones. The parameter ε_R takes arbitrary values in $(0, 1]$.

The solution of problem (3.2), (3.1) exists and is sufficiently smooth if the functions $\varphi(x, y)$ and $\psi(x, y)$ are sufficiently smooth and satisfy appropriate compatibility conditions, respectively, on the sets $S^* = \bar{S}_1 \cap \{S_0 \cup \bar{S}_2\}$ (i.e. at the corner points adjoining to the side $\bar{S}_1$) and $S^{0*} = \bar{S}_1 \cap S^0$ [9].

In the case of heat transfer between the plate and the fluid (under the assumptions that the buoyancy force is zero, and that the viscosity is independent of the temperature), in addition to the system of equations (3.2), we have the following heat equation with appropriate boundary conditions [12]

$$L^3 T(x, y) \equiv \left\{ \varepsilon_T \frac{\partial^2}{\partial y^2} - u(x, y) \frac{\partial}{\partial x} - v(x, y) \frac{\partial}{\partial y} \right\} T(x, y)$$

$$= -\varepsilon_R \left(\frac{\partial}{\partial y} u(x, y) \right)^2, \quad (x, y) \in G, \tag{3.3a}$$

$$T(x, y) = \varphi_T(x, y), \quad (x, y) \in S. \tag{3.3b}$$

Here ε_T is the heat conduction coefficient if the problem is considered in dimensional variables, and $\varepsilon_T = Pe^{-1}$ in the case of dimensionless variables; Pe is the Péclet number, $Pe = Pr\,Re$.

The solution of this problem in an infinite domain (including also the leading edge of the plate) for large Re and/or Pe has singularities of the boundary layer kind in a neighbourhood of the plate (for $x > 0$), and also an additional singularity in a neighbourhood of the leading edge due to the incompatibility of the problem data at the leading edge.

Since we are primarily interested in finding approximations to the solution of the problem near the surface of the plate, we consider the heat transfer problem for flow around the flat plate in a bounded subdomain which adjoins the plate and contains the boundary layer, but lies outside some neighbourhood of the leading edge.

In the absence of suction and blowing the typical singularity in the solutions of problem (3.2), (3.1) and (3.2), (3.3), (3.1) is a parabolic boundary layer. For example, in the case of a self-similar solution of the Prandtl problem for flow past an infinite plate (see [12]) the function $v(x, y)$ satisfies the estimate

$$|v(x, y)| \leqslant M\varepsilon_R^{1/2}, \quad (x, y) \in \bar{G};$$

in which case the thickness of the boundary layer is of order $\varepsilon_R^{1/2}$. Because of this estimate for the function $v(x, y)$, we can use the technique for constructing ε-uniformly convergent schemes developed in [8,14] for the case of problem (3.2), (3.1) (see, e.g., [3]).

It might seem that the same technique is also applicable for problem (3.2), (3.3), (3.1) provided that $Pr \approx 1$. However, for the problem of flow past a plate with the boundary layer controllable by suction, the function $v(x, y)$ can essentially exceed the quantity $\varepsilon_R^{1/2}$. For example, if the suction has intensity $v_0(x) = const > 0$, we obtain the following estimate for the function $u(x, y)$:

$$|u(x, y) - U_\infty| \leqslant M \exp(-mv_0\varepsilon_R^{-1}y), \quad (x, y) \in \bar{G},$$

where U_∞ is the flow velocity at infinity. Then, the thickness of the boundary layer is of order $v_0^{-1}\varepsilon_R$, which is much less (for $v_0 \gg \varepsilon_R^{1/2}$) than for the passive plate, and in this case the boundary layer is regular.

Similar behaviour of the controllable boundary layers is observed also for problem (3.3), (3.1) under the condition

$$\varepsilon_T^{1/2} \ll v_0^{-1}\varepsilon_R.$$

Therefore, it is of urgent interest to construct ε-uniformly convergent numerical methods for boundary layers which can be both parabolic and regular, depending on the parameter v_0.

4. Classical difference schemes

We first introduce a classical difference scheme for problem (2.2), (2.1) and discuss problems arising in the numerical solution for small values of the parameter ε.

On the set $\bar{G}$ we introduce the mesh

$$\bar{G}_h = \bar{\omega} \times \bar{\omega}_0, \tag{4.1}$$

where $\bar{\omega}$ and $\bar{\omega}_0$ are meshes on the sets $\bar{D}$ and $[0, T]$, respectively; $\bar{\omega}$ and $\bar{\omega}_0$ are meshes with distributions of the nodes subject only to the condition $h \leqslant MN^{-1}$, $h_t \leqslant MN_0^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \bar{\omega}$, $h_t = \max_j h_t^j$, $h_t^j = t^{j+1} - t^j$, t^j, $t^{j+1} \in \bar{\omega}_0$. Here $N + 1$ and $N_0 + 1$ are, respectively, the minimal number of nodes on an interval of unit length on the set $\bar{D}$ and the number of nodes in the mesh $\bar{\omega}_0$. It is also of interest to consider schemes on the simplest meshes

$$\bar{G}_h^u, \tag{4.2}$$

here $\bar{\omega}$ and $\bar{\omega}_0$ are *uniform* meshes with the step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$. Problem (2.2), (2.1) is approximated by the implicit difference scheme [11]

$$\Lambda z(x,t) \equiv \{\varepsilon_1 a(x,t)\delta_{\bar{x}\hat{x}} + \varepsilon_2 b(x,t)\delta_x - c(x,t) - p(x,t)\delta_{\bar{t}}\} z(x,t)$$

$$= f(x,t), \quad (x,t) \in G_h,$$

$$z(x,t) = \Phi(x,t), \quad (x,t) \in S_h. \tag{4.3}$$

Here $\delta_x z(x,t)$, $\delta_{\bar{t}} z(x,t)$ are the forward and backward first-order difference derivatives, and $\delta_{\bar{x}\hat{x}} z(x,t) = 2(h^i + h^{i-1})^{-1}\{\delta_x - \delta_{\bar{x}}\} z(x,t)$, $x = x^i$, is the second-order difference derivative.

For the difference scheme (4.3), (4.1) the maximum principle is valid [11].

Taking into account a priori estimates of the solution of problem (2.2), (2.1) (see Section 6), we obtain the following estimate for the solution of scheme (4.3), (4.1):

$$|u(x,t) - z(x,t)|$$

$$\leqslant M \left\{ \begin{array}{ll} [(\varepsilon_1^{1/2} + N^{-1})^{-1}N^{-1} + N_0^{-1}] & \text{for } \varepsilon_2 \leqslant M_0\varepsilon_1^{1/2} \\ [\varepsilon_2^3\varepsilon_1^{-2}N^{-1} + N_0^{-1}] & \text{for } \varepsilon_2 > M_0\varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t) \in \bar{G}_h. \tag{4.4}$$

On the other hand on the uniform mesh (4.2) we have the estimate

$$|u(x,t) - z(x,t)|$$

$$\leqslant M \left\{ \begin{array}{ll} [(\varepsilon_1^{1/2} + N^{-1})^{-1}N^{-1} + N_0^{-1}] & \text{for } \varepsilon_2 \leqslant M_0\varepsilon_1^{1/2} \\ [(\varepsilon_2^{-2}\varepsilon_1 + N^{-1})^{-1}N^{-1} + N_0^{-1}] & \text{for } \varepsilon_2 > M_0\varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t) \in \bar{G}_h^u; \tag{4.5}$$

which is unimprovable with respect to the expressions involving the parameters N, N_0, ε_1, ε_2. Thus, the condition

$$N^{-1} = o(\min[\varepsilon_1^{1/2}, \varepsilon_2^{-2}\varepsilon_1]), \tag{4.6}$$

is necessary and sufficient for the convergence of scheme (4.3), (4.2); schemes (4.3), (4.1) and (4.3), (4.2) do not converge ε-uniformly. These results are stated formally in the following theorem.

Theorem 1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), (2.6), *and also* $a, b, c, p, f \in C^{l_1+\alpha}(\bar{G})$, $\varphi \in C^{l_0+\alpha}(\bar{S}^L) \cap C^{l_1+\alpha}(S_0)$, *and let* $u \in C^{3+\alpha, 2+\alpha}(\bar{G})$, $K_{(2.6)} = l_1 = 7$, $K_{0(2.6)} = l_0 = 2$, $\alpha > 0$. *Then condition* (4.6) *is necessary (necessary and sufficient) for the convergence of the difference scheme* (4.3) *on mesh* (4.1) (*on mesh* (4.2)). *For the mesh solutions estimates* (4.4) *and* (4.5) *are valid; estimate* (4.5) *is unimprovable with respect to the values of* N, N_0, ε_1, ε_2.

Remark 1. To construct ε-uniformly convergent difference schemes for problem (2.2), (2.1), we could try to use a fitted operator technique (for a description see, e.g., [2,5,8,14]). But when $\varepsilon_2 = O(\varepsilon_1^{1/2})$ the solution of this problem has a singularity of parabolic layer type, and so, using the technique given in [8,13,14], we can show that there are no fitted operator schemes convergent ε-uniformly in this case.

5. Special difference scheme

In this section we use meshes condensing in a neighbourhood of the boundary layer, in order to construct schemes which are ε-uniformly convergent.

On the set $\bar{G}$ we introduce the mesh

$$\bar{G}_h = \bar{\omega}^* \times \bar{\omega}_0, \tag{5.1a}$$

where $\bar{\omega}_0 = \bar{\omega}_{0(4.2)}$, $\bar{\omega}^* = \bar{\omega}^*(\sigma)$ is a *piecewise uniform* mesh on $\bar{D}$. The step-sizes of the mesh $\bar{\omega}^*$ are constant on the sets $[0, \sigma]$ and $[\sigma, \infty)$ with $h^{(1)} = 2\sigma N^{-1}$ and $h^{(2)} = 2(1 - \sigma)N^{-1}$. The value of σ is defined by

$$\sigma = \sigma(\varepsilon, N) = \begin{cases} \min[2^{-1}, M_1 \varepsilon_1^{1/2} \ln N] & \text{for } \varepsilon_2 \leqslant M_0 \varepsilon_1^{1/2}, \\ \min[2^{-1}, M_2 \varepsilon_2^{-2} \varepsilon_1 \ln N] & \text{for } \varepsilon_2 > M_0 \varepsilon_1^{1/2}, \end{cases} \tag{5.1b}$$

where $M_1 = m_{1(2.6)}^{-1}$, $M_2 = m_{2(2.6)}^{-1}$. This completes the construction of the mesh $\bar{G}_h$.

Using the majorant function technique from [8,14], and taking into account the a priori estimates of the solution of problem (2.2), (2.1) discussed in the next section, we find the following error estimate for the solution of scheme (4.3), (5.1)

$$|u(x, t) - z(x, t)|$$
$$\leqslant M \begin{cases} [N^{-1} \min[\ln N, \varepsilon_1^{-1/2}] + N_0^{-1}] & \text{for } \varepsilon_2 \leqslant M_0 \varepsilon_1^{1/2} \\ [N^{-1} \min[\ln N, \varepsilon_2^2 \varepsilon_1^{-1}] + N_0^{-1}] & \text{for } \varepsilon_2 > M_0 \varepsilon_1^{1/2} \end{cases}, \quad (x, t) \in \bar{G}_h. \tag{5.2}$$

The following ε-uniform estimate is also valid:

$$|u(x, t) - z(x, t)| \leqslant M[N^{-1} \ln N + N_0^{-1}], \quad (x, t) \in \bar{G}_h. \tag{5.3}$$

The error estimates (5.2) and (5.3) are unimprovable with respect to the expressions involving the parameters N, N_0, ε_1, ε_2 and N, N_0, respectively. These results are stated formally in the following theorem.

Theorem 2. *Let the hypothesis of Theorem 1 be fulfilled. Then the solution of the difference scheme* (4.3), (5.1) *converges ε-uniformly. The mesh solutions satisfy the error estimates* (5.2) *and* (5.3), *which are unimprovable with respect to the values of* N, N_0, ε_1, ε_2 *and* N, N_0, *respectively.*

Remark 2. Although the solution of problem (2.2), (2.1) has a singularity only for $\varepsilon_1 \to 0$ (the solution of the problem is regular for $\varepsilon_1 \geqslant m$; see, e.g., estimates (6.8), (6.10) below), the character of the boundary layer depends essentially on the vector-parameter ε. Such behaviour of the singular component of the solution prevents us from constructing an ε-uniformly convergent scheme with a definition of $\sigma_{(5.1)}$ which is independent of the parameter ε_2.

6. A priori estimates

In this section we give a priori estimates used in the above construction; the technique from [4,6,7,14] is used to derive the estimates. Using comparison theorems, we find that

$$|u(x,t)| \leqslant M, \quad (x,t) \in \bar{G}. \tag{6.1}$$

We assume in what follows that the condition

$$\Phi(x,t) = \varphi(x,t), \quad (x,t) \in S, \tag{6.2}$$

is satisfied, where $\varphi(x,t)$ is independent of the parameter ε.

6.1. First we find estimates of the solution when

$$\varepsilon_2 \leqslant M\varepsilon_1^{1/2}; \tag{6.3}$$

in this case we use a priori estimates up to the boundary [7]. The boundary value problem (2.2), (2.1) in the new variables $\xi = \varepsilon_1^{-1/2}x$ is transformed into the problem

$$\tilde{L}\tilde{u}(\xi,t) = \tilde{f}(\xi,t), \quad (\xi,t) \in \tilde{G}, \tag{6.4a}$$

$$\tilde{u}(\xi,t) = \tilde{\varphi}(\xi,t), \quad (\xi,t) \in \tilde{S}. \tag{6.4b}$$

Here $\tilde{v}(\xi,t) = v(x(\xi),t)$, $v(x,t)$ is one of the functions $u(x,t), \ldots, \varphi(x,t)$; $\tilde{G}^0 = \{(\xi,t): \xi = \xi(x), (x,t) \in G^0\}$, G^0 is one of the sets G, S. The differential equation (6.4a) on the domain $\tilde{G}$ and the boundary condition (6.4b) on $\tilde{S}$ are regular with respect to the parameter ε_i. Using a priori estimates up to the boundary, we find that

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}} \tilde{u}(\xi,t) \right| \leqslant M, \quad (\xi,t) \in \bar{\tilde{G}}.$$

In the variables x,t this becomes

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t) \right| \leqslant M\varepsilon_1^{-k/2}, \quad (x,t) \in \bar{G}. \tag{6.5}$$

In fact we need a more accurate estimate than (6.5). We represent the solution of problem (2.2), (2.1) as a sum of the two functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \bar{G}, \tag{6.6}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution. The function $U(x,t)$ is the restriction to $\bar{G}$ of the function $U^*(x,t)$, $(x,t) \in \bar{G}^*$, where $U^*(x,t)$ is the solution of the problem

$$L^*U^*(x,t) = f^*(x,t), \quad (x,t) \in G^*, \quad U^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S^*.$$

Here $S^* = S(G^*)$; the domain G^* is the extension of G beyond the set $\bar{S}^L$, G^* contains G together with its m-neighbourhood (that is, the set of all points that are at a distance at most m from G); the coefficients of the operator L^* and the function $f^*(x,t)$ are smooth continuations of the

corresponding data of problem (2.2); $\varphi^*(x,t)$ is some smooth function, where $\varphi^*(x,t) = \varphi(x,t)$, $(x,t) \in S_0$. The function $V(x,t)$ is the solution of the problem

$$LV(x,t) = 0, \quad (x,t) \in G, \quad V(x,t) = \varphi(x,t) - U(x,t), \quad (x,t) \in S.$$

The function $U^*(x,t)$, $(x,t) \in \bar{G}^*$ can be decomposed into a sum of the functions

$$U^*(x,t) = \sum_{i=0}^{2} \varepsilon_1^{i/2} U_i^*(x,t) + v_U(x,t), \quad (x,t) \in \bar{G}^*. \tag{6.7}$$

Here the functions $U_i^*(x,t)$ are the solutions of the problems

$$L^{*0} U_0^*(x,t) \equiv \left\{ -c^*(x,t) - p^*(x,t)\frac{\partial}{\partial t} \right\} U_0^*(x,t) = f^*(x,t), \quad (x,t) \in \bar{G}^* \setminus S_0^*,$$

$$U_0^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S_0^*;$$

$$L^{*0} U_i^*(x,t) = \left\{ \varepsilon_1 a^*(x,t)\frac{\partial^2}{\partial x^2} - \varepsilon_2 b^*(x,t)\frac{\partial}{\partial x} \right\} U_{i-1}^*(x,t), \quad (x,t) \in \bar{G}^* \setminus S_0^*,$$

$$U_i^*(x,t) = 0, \quad (x,t) \in S_0^*, \quad i = 1,2.$$

Taking into account estimates for the components in (6.7), we find the following estimates for the components in representation (6.6)

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \leqslant M,$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t) \right| \leqslant M\varepsilon_1^{-k/2} \exp(-m_1\varepsilon_1^{-1}), \quad (x,t) \in \bar{G}, \quad k + k_0 = K, \quad k_0 \leqslant K_0, \tag{6.8}$$

where m_1 is any positive constant, $K = 3$, $K_0 = 2$.

6.2. We now consider the case

$$\varepsilon_2 \geqslant m\varepsilon_1^{1/2}. \tag{6.9}$$

In this case we pass to the variables $\xi = \varepsilon_1^{-1}\varepsilon_2 x$, $\tau = \varepsilon_1^{-1}\varepsilon_2 t$.

We represent the function $U^*(x,t)$, $(x,t) \in \bar{G}^*$ as a sum of functions

$$U^*(x,t) = \sum_{i=0}^{2} \varepsilon_1^i U_i^*(x,t) + v_U(x,t), \quad (x,t) \in \bar{G}^*,$$

where the functions $U_i^*(x,t)$ are the solutions of the problems

$$L^{*1} U_0^*(x,t) \equiv \left\{ \varepsilon_2 b^*(x,t)\frac{\partial}{\partial x} - c^*(x,t) - p^*(x,t)\frac{\partial}{\partial t} \right\} U_0^*(x,t) = f^*(x,t), \quad (x,t) \in \bar{G}^* \setminus S_0^*,$$

$$U_0^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S_0^*;$$

$$L^{*1} U_i^*(x,t) = -a^*(x,t)\frac{\partial^2}{\partial x^2} U_{i-1}^*(x,t), \quad (x,t) \in \bar{G}^* \setminus S_0^*,$$

$$U_i^*(x,t) = 0, \quad (x,t) \in S_0^*, \quad k = 1,2.$$

Having estimated the function $U^*(x,t)$, we obtain the estimates for the components in representation (6.6)

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \leqslant M,$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t) \right| \leqslant M\varepsilon_2^k \varepsilon_1^{-k} \exp(-m_2 \varepsilon_2 \varepsilon_1^{-1} x), \quad (x,t) \in \bar{G},$$

$$k + k_0 \leqslant K, \quad k_0 \leqslant K_0, \tag{6.10}$$

where m_2 is an arbitrary constant from the interval $(0, m_0)$,

$$m_0 = \min_{\bar{G}} [a^{-1}(x,t)b(x,t)], \quad K = 3, \quad K_0 = 2.$$

When deducing the estimates (6.8), (6.10), we supposed that the data of the boundary value problem satisfy the condition

$$a, b, c, p, f \in C^{l_1+\alpha, l_0+\alpha}(\bar{G}), \quad \varphi \in C^{l_0+\alpha}(\bar{S}^L) \cap C^{l_1+\alpha}(S_0), \quad l_0 \geqslant 2, \ l_1 \geqslant 7, \ \alpha > 0. \tag{6.11}$$

We remark that the compatibility conditions [7] on the set S^c are satisfied, which ensures the inclusion

$$u \in C^{3+\alpha, 2+\alpha}(\bar{G}) \tag{6.12}$$

for each fixed set of values of the vector parameter ε. These results are stated formally in the following theorem.

Theorem 3. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), (6.2), (6.11), *and let condition* (6.12) *be fulfilled for the solution of the problem. Then the solution of the problem and its components in representation* (6.6) *satisfy estimate* (6.1) *and also the estimates* (6.8) *and* (6.10) *in cases* (6.3) *and* (6.9), *respectively.*

Remark 3. Assume that the function $\Phi(x,t)$ has a singularity of the same type as the function $u(x,t)$ and that the function $\Phi(x,t)$ for $t = 0$ can be written as a sum of functions of form (2.6a). Furthermore, suppose that this function and its components in (2.6a) satisfy condition (2.6b), with $K = 7$, $K_0 = 2$. Then the conclusion of Theorem 3 remains valid for the solution of problem (2.2), (2.1).

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