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Hybrid trajectory spaces

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ABSTRACT
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Hybrid Trajectory Spaces

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Abstract

In this paper, we present a general framework for describing and studying hybrid systems. We represent the trajectories of the system as functions on a *hybrid time domain*, and the system itself by its *trajectory space*, which is the set of all possible trajectories. The trajectory space is given a natural topology, the *compact-open hybrid Skorohod topology*, and we prove the existence of limiting trajectories under uniform equicontinuity assumptions. We give a compactness result for the trajectory space of *impulse differential inclusions*, a class of nondeterministic hybrid system, and discuss how to describe *hybrid automata*, a widely-used class of hybrid system, as impulse differential inclusions. For systems with compact trajectory space, we obtain results on Zeno properties, symbolic dynamics and invariant measures. We give examples showing the application of the results obtained using the trajectory space approach.

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1 Introduction

The aim of this paper is to develop a general framework for the study of the dynamical properties of a hybrid system. Rather than focus on the defining equations of motion, our main object of study is the *trajectory space* of a system, a set of *hybrid trajectories* which give all possible behaviours of the system. The most important property of a trajectory space is that of *compactness*; it is possible to give straightforward proof of several core results on hybrid systems using compactness assumptions alone. In this article we give a characterisation of Zeno behaviour for systems with compact trajectory space, and show that symbolic descriptions of the system result in a compact shift space, and prove the existence of invariant measures.

The theory considers trajectories which are continuous functions whose domain is a *hybrid time domain*. Hybrid time domains are topologically equivalent to the *hybrid time trajectories* as used by Lygeros et. al. [16], but have the advantage of having an explicit coordinate system. The natural topology on the space of trajectories is the *(compact-open) hybrid Skorohod topology*, first introduced for hybrid systems by Broucke [4], and also studied by Broucke and Arapostathis [5] and Caspi and Benveniste [8]. We give an equivalent definition of the hybrid Skorohod topology based on the graph of the system, which is simpler to analyse when two or more discrete events occur at the same time. The approach of considering hybrid trajectories on hybrid time domains with the hybrid Skorohod topology has also been used by Goebel et. al. [13][1].

The trajectory-space gives a framework which can be applied for hybrid systems specified in a number of different ways. A popular class of hybrid system is that of *hybrid automata* [23]. A hybrid automaton is described by a tuple $(Q, E, \rho, \{X_q^{\mathrm{inv}}\}, \{f_q\}, \{X_{q,e}^{\mathrm{guard}}\}, \{r_{q,e}\})$, where (Q, E, ρ) is a discrete-event system, the differential equations $\dot{x} = f_q(x, u)$ describe the continuous dynamics in the continuous state spaces X_q^{inv}, and the *guard sets* $X_{q,e}^{\mathrm{guard}}$ and *reset maps* $r_{q,e}$ describe the interaction of the discrete and continuous systems. This description of hybrid automata is useful when modelling a system, but contains many details which are irrelevant to the system dynamics.

When analysing dynamical properties of a system, it is often preferable to consider the class of *impulse differential inclusions* introduced by Aubin et. al. [3]. An impulse differential inclusion is described by a triple (X, F, G), where X is the state space, the multivalued reset map F describes the discrete-time dynamics and the differential inclusion G the continuous-time dynamics. We show that if the differential inclusion and reset map are upper-semicontinuous with closed compact values, then the trajectory space of the resulting hybrid system is compact. A proof can be extracted from the proof of Theorem 4 of [3].

The motivation for this work was to provide a framework allowing fundamental techniques from the theory of nonlinear dynamical systems in discrete or continuous time to used for the study of hybrid-time dynamical systems, notably those of *symbolic dynamics* [14] and *ergodic theory* [20, 21]. In symbolic dynamics, one gives an abstract description the system behaviour in terms of *itineraries* of orbits, which give the order in which different regions of the state space are visited. The resulting *shift* is a system which is *simulated* by the original system, and can be computed for discrete-time systems using the techniques of *fixed-point index* [6] and *Conley index* [18] theory. In ergodic theory, one considers *invariant measures* of a system, which give a stochastic description of a system and allow the computation of the time-averaged behaviour. Compactness (or local compactness) of the state space or trajectory space is crucial for obtaining a compact shift space, application of techniques of index theory, and for proving the existence of invariant measures. In this paper we show that the results on compact shift spaces and existence of invariant measures carry over to hybrid systems with compact trajectory space. General techniques for the computation of symbolic dynamics and invariant measures are important areas of research in their own right, and beyond the scope of this paper.

The study of hybrid systems is complicated by the fact that the system evolution may not depend continuously on the initial state, even over finite time intervals. One important cause of this behaviour is due to tangential or *grazing* incidence of system trajectories with the guard sets. On one side of such a trajectory a discrete event does occur, possibly causing discontinuous evolution of the system state; on the other side no discrete event occurs, and the system keeps evolving according to the continuous dynamics. Loss of continuous dependence on initial conditions may also occur if two discrete events are enabled in the same system state, requiring an arbitrary choice as to which event occurs.

The discontinuities associated with grazing and multiple enabled events are important if one

[1] The terminology used here differs slightly from that of [9] and [13]; in [9], hybrid time domains are called *hybrid time sets*, and in [13], and the hybrid Skorohod topology is called the *graph topology*.

wishes to use topological and ergodic techniques for the study of hybrid systems. To obtain (local) compactness of the trajectory space, we require that at a point of discontinuity, all possible limiting behaviours are allowed. In the case of grazing incidence with a guard set, this means that the corresponding discrete event may or may not occur. In this way, the difficulties occurring at points of discontinuous dependence on initial conditions are tamed at the expense of requiring the study of non-deterministic systems. For this reason, and since non-determinism does not cause any technical difficulties when considering trajectory spaces, we consider non-deterministic systems from the outset.

Aside from symbolic dynamics and invariant measures, useful results on other important dynamical properties can be obtained by studying the trajectory space. In this paper, we show that a hybrid system with compact trajectory space exhibits Zeno behaviour if, and only if, it has a *Zeno state*, extending a result of Zhang et al. [24] giving conditions for Zeno behaviour in hybrid automata. Other important dynamical properties include stability, invariant and viable sets, and bifurcations. Conditions for robust stability were obtained by Goebel et. al. [13] using essentially the trajectory space approach described here. Invariant and viable sets were studied by Aubin et. al. [3] in the context of impulse differential inclusions with locally compact trajectory space. There is also a considerable body of knowledge on bifurcation theory for *piecewise-smooth* hybrid systems, most notably the original work of Nordmark on the grazing bifurcation [19] and on *period-adding routes to chaos* [7];for an overview see [15].

The paper is organised as follows:

In Section 2, we define hybrid time domains and hybrid trajectories, and define some important properties of the trajectory space. We define a number of operators on hybrid time trajectories, and discuss how to obtain the evolution on the state space from the hybrid trajectory space. We give two number of equivalent definitions of the compact-open hybrid Skorohod topology. We generalise the concept of hybrid time domain to subsets of arbitrary hybrid time axes, which can be any partially-ordered set.

In Section 3, we consider hybrid systems described by impulse differential inclusions and hybrid automata. We show that for impulse differential inclusions with upper-semicontinuous reset maps and flows, the trajectory space is compact. We also discuss how to convert hybrid automata to upper-semicontinuous impulse differential inclusions given the right semantics for the system evolution.

In Section 4, we consider properties of systems with compact trajectory space. We first give checkable conditions for a system to be Zeno, and conditions for the existence of a trajectory with no discrete events. We then show that for systems with compact trajectory space, a description in terms of symbolic dynamics always yields a compact shift space. We finally discuss the existence of invariant measures. We show that an invariant measure on the trajectory space always exists for at least one of the return operator or the time-evolution operator. Under certain cases, including that of deterministic systems, an invariant measure can be projected to give an invariant measure on the state space.

In Section 5, we present three examples illustrating some of the important issues in the dynamics of hybrid systems. We first discuss the bouncing ball system, and show that generalised hybrid time axes can be used to give a complete treatment of the dynamics. We consider a switched arrival system exhibiting Zeno behaviour, derive symbolic dynamics for the system, and give invariant measures for the return map and time-evolution map. We give an example of a heating system with periodic environmental fluctuations, which may have no invariant measures.

Finally, in Section 6 we give some conclusions and possible directions for further research.

2 The trajectory space of a hybrid system

We first give a framework for the study of hybrid systems based on the *hybrid trajectory space*, which models all possible system trajectories. We give an appropriate topology for this space, the *compact-open hybrid Skorohod topology*, and define operators modelling time-evolution.

2.1 Hybrid system trajectories

To describe the behaviour of a hybrid system, we need a notion of a time axis which includes both discrete and continuous time. This can be constructed using the times of the discrete events.

Definition 2.1. An *event time sequence* $(t_n)_{n=0}^{n_*}$ with $n_* \in \mathbb{Z}^+ \cup \{\infty\}$ is a monotone sequence with $t_0 = 0$ and $t_n \geqslant t_{n-1}$ for $1 \leqslant n \leqslant n_*$. If $n_* = \infty$ we take $t_\infty = \lim_{n\to\infty} t_n$.

The time t_n is the time of the nth discrete event. The number n_* is the *number of events*. The number of events up to (and including) time t is given by $n_t = \sup\{n \in \mathbb{Z}^+ \mid t_n \leqslant t\}$.

In [16], an event time sequence is used to define a *hybrid time trajectory* τ, which is the sequence of intervals $(I_n)_{n=0}^{n_*}$ with $I_n = [t_n, t_{n+1}]$. A hybrid trajectory is then defined to be a function from τ to X. Unfortunately, specifying points in a hybrid time trajectory is awkward, since no coordinate system is given.

Instead of using hybrid time trajectories, we use *hybrid time domains*, which are topologically equivalent but have an explicit coordinate system, and hence are easier to work with.

Definition 2.2. Let $(t_n)_{n=0}^{n_*}$ be an event time sequence with total execution time t_*. The *hybrid time domain* determined by the event time sequence is the set:

$$\mathcal{T} = \{(t,n) \mid n < n_* \text{ and } t \in [t_n, t_{n+1}], \text{ or } n = n_* \text{ and } t \in [t_{n_*}, t_*]\} \subset \mathbb{Z}^+ \times \mathbb{R}^+.$$

The topology on $\mathcal{T}$ is the subspace topology inherited from the set $\mathbb{Z}^+ \times \mathbb{R}^+$, which is called the *hybrid time axis*.

Note that the point $(0,0)$ is an element of any hybrid time domain. If $t_n = t_{n+1}$ for some n, then the nth time interval $[t_n, t_{n+1}]$ consists of a single point. Resets of the continuous state are possible since a hybrid time domain is not connected (unless there are no events.

An example of a hybrid time trajectory and hybrid time domain for the same event time sequence is shown in Figure 1.

We now give a definition of hybrid trajectory using hybrid time domains.

Definition 2.3. A *hybrid trajectory* in a topological space X is a continuous function $\xi : \mathcal{T} \to X$ for some hybrid time domain $\mathcal{T}$.

We denote the hybrid time domain of ξ by $\mathcal{T}(\xi)$ or $\mathrm{dom}(\xi)$, the number of events of ξ by $n_*(\xi)$, and the total execution time of ξ by $t_*(\xi)$. The *event times* t_n are given by $t_n(\xi) = \inf\{t \in \mathbb{R}^+ \mid (t,n) \in \mathcal{T}(\xi)\}$, and the *number of events up to time* t is given by $n_t(\xi) = \sup\{n \in \mathbb{Z}^+ \mid (t,n) \in \mathcal{T}(\xi)\}$.

A hybrid trajectory ξ is *finite* if $n_*(\xi) < \infty$ and $t_*(\xi) < \infty$, otherwise it is *infinite*. A hybrid trajectory $\xi_1 : \mathcal{T}_1 \to X$ is a *prefix* of $\xi_2 : \mathcal{T}_2 \to X$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\xi_1 = \xi_2|_{\mathcal{T}_1}$.

The *graph* of a hybrid trajectory $\xi : \mathcal{T} \to X$ is the set

$$\mathrm{Gr}(\xi) = \{(t,n,x) \in \mathbb{R}^+ \times \mathbb{Z}^+ \times X \mid (t,n) \in \mathcal{T} \text{ and } x = \xi(t,n)\}.$$

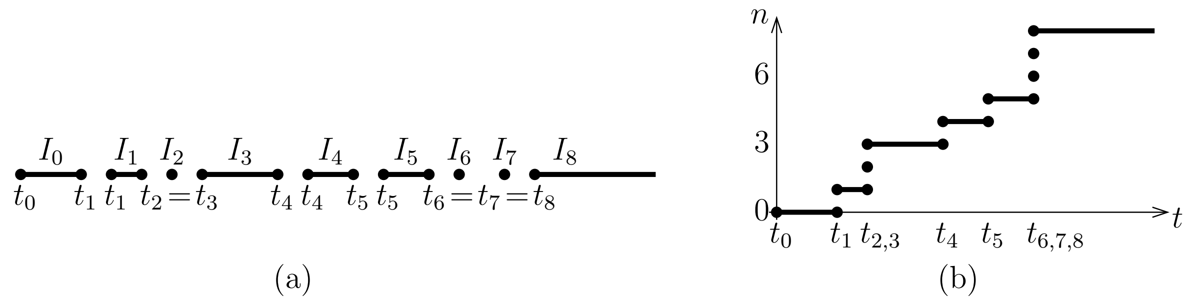

Figure 1: (a) A hybrid time trajectory, and (b) the hybrid time domain for the same event time sequence.

Notice that the definition of hybrid trajectory allows for more than one event at a given time. In particular, two trajectories are considered different even if they only vary at a single point at a time t at which two events occur. Although it may seem more natural to consider such trajectories equal, this causes difficulties when defining the topology on a set of trajectories.

The evolution of a hybrid system is defined in terms of *shift operators* on its trajectories.

Definition 2.4. Let $\mathcal{T}$ be a hybrid time domain, and $(s, m) \in \mathcal{T}$. Then we can *shift* $\mathcal{T}$ by (s, m) to obtain a new hybrid time domain

$$\hat{\sigma}_s^m \mathcal{T} = \{(t, n) \in \mathbb{R}^+ \times \mathbb{Z}^+ \mid (t + s, n + m) \in \mathcal{T}\}.$$

Similarly, if $\xi : \mathcal{T} \to X$ is a hybrid trajectory and $(s, m) \in \mathcal{T}$, then the *hybrid shift operator* $\hat{\sigma}_s^m$ is defined on ξ, and is given by

$$\hat{\sigma}_s^m \xi(t, n) = \xi(t + s, n + m) \text{ whenever } (t, n) \in \hat{\sigma}_s^m \mathcal{T}.$$

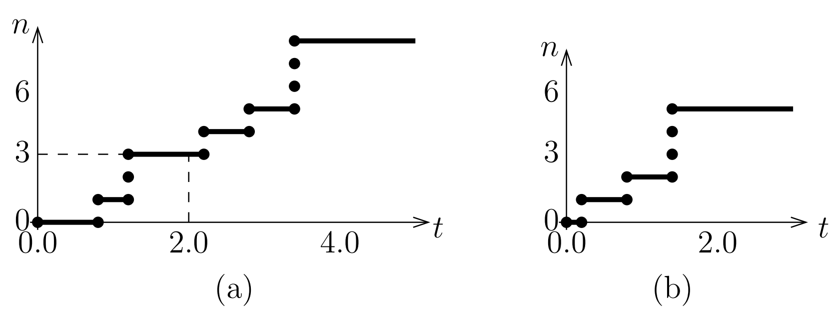

Figure 2: The shift operator on a hybrid time domain.

The shift operator on a hybrid time domain is shown in Figure 2. We will discuss shift operators further in Section 2.3.

A hybrid system can be described by the set of all its trajectories.

Definition 2.5 (Hybrid trajectory space). A *hybrid trajectory space* H in a space X is a set of hybrid trajectories in X.

A set of hybrid trajectories H is *invariant* under the hybrid shift operator if for all $\xi \in H$ and $(s, m) \in \mathcal{T}(\xi)$, the trajectory $\hat{\sigma}_m^s \xi$ is also in H. A set of hybrid trajectories H is *prefix-free* if no trajectory is a prefix of any other, and *non-blocking* if every finite trajectory is a prefix of an infinite trajectory.

The trajectory space of a hybrid system will typically be invariant with respect to the hybrid shift operator, and will usually be taken to be prefix-free.

2.2 The compact-open Skorohod topology

The natural topology for a space of trajectories H of a hybrid system is the *compact-open Skorohod topology.* The key feature of a Skorohod topology on a function space is that it allows some distortion of the domain set, and hence can be used to measure the distance between discontinuous functions. Compact-open topologies are natural for trajectory spaces of dynamical systems. The original definition of Skorohod topology and the definition of [4] were given in terms of reparameterisations of the domain of trajectories. In this paper, we instead give a definition in terms of the graphs of the trajectories, since this simpler, especially in the case that multiple events occur at the same time. For technical reasons, we assume that the trajectory space H is prefix-free. We give two definitions, one in terms of a basis of open sets, and one in terms of a metric. The hybrid Skorohod topology given here is the same as the graph topology of Goebel et. al. [13].

If ξ is a hybrid trajectory and K a compact subset of $\mathbb{R}^+ \times \mathbb{Z}^+$, define $\xi|_K$ to be the restriction of ξ to the set $\mathcal{T}(\xi) \cap K$, and $N_\epsilon(\mathrm{Gr}(\xi))$ to be the open ϵ-neighbourhood of the graph of ξ.

Definition 2.6 (Compact-open hybrid Skorohod topology). Let ξ be a hybrid trajectory, K a compact subset of $\mathbb{R}^+ \times \mathbb{Z}^+$, and $\epsilon > 0$. The (K, ϵ)-neighbourhood of ξ is the set of trajectories

$$U(\xi, K, \epsilon) = \left\{\xi' \mid \mathrm{Gr}(\xi'|_K) \subset N_\epsilon(\mathrm{Gr}(\xi))\right\}.$$

The open sets $U(\xi, K, \epsilon)$ form a basis of open sets for the Skorohod topology of H.

Here, K ensures a restriction to a compact hybrid time domain, and ϵ is a uniform error bound. The term "compact-open" refers to the property that ξ' restricted to a *compact* set lies in an *open* neighbourhood of ξ.

To obtain a metric description, we use essentially the same concept, but need to make the definition symmetric. Recall that the Hausdorff semi-distance between sets A and B is $\tilde{d}(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y)$, and that $d(A, B) = \max\{\tilde{d}(A, B), \tilde{d}(B, A)\}$ defines the *Hausdorff distance.* We need an increasing sequence of compact subsets K_N of $\mathbb{R}^+ \times \mathbb{Z}^+$ whose union equals $\mathbb{R}^+ \times \mathbb{Z}^+$; for definiteness we take $K_N = [0, N] \times \{0, \ldots, N\}$. We also need to assume that the distance d on X is bounded; this is automatic if X is compact, otherwise we can replace $d(\cdot, \cdot)$ by $\tan^{-1} d(\cdot, \cdot)$ without changing the induced topology.

Definition 2.7 (Compact-open hybrid Skorohod metric). The *hybrid Skorohod metric* is given by

$$d(\xi, \xi') = \sum_{N=0}^{\infty} \frac{\max\left\{\tilde{d}\left(\mathrm{Gr}(\xi|_{K_N}), \mathrm{Gr}(\xi')\right), \tilde{d}\left(\mathrm{Gr}(\xi'|_{K_N}), \mathrm{Gr}(\xi)\right)\right\}}{2^N}.$$

Note that we do not restrict the graph of the right-hand element in $\tilde{d}(\cdot, \cdot)$ to K_N. This is since if ξ has nth event with $(t_n(\xi), n) \in K_N$, but $(t_n(\xi'), n) \notin K_N$, we still need to compare $(t_n(\xi), n+1, \xi(t_n(\xi), n+1))$ with $(t_n(\xi'), n+1, \xi'(t_n(\xi'), n+1))$.

The distance is finite, since any point $(t, n, x) \in \mathrm{Gr}(\xi)$ with $t, n \leqslant N$ is within $2N + \mathrm{diam}(X)$ of the point $(0, 0, \xi'(0, 0)) \in \mathrm{Gr}(\xi')$, and hence $\tilde{d}\left(\mathrm{Gr}(\xi|_{K_N}), \mathrm{Gr}(\xi')\right) \leqslant 2N + \mathrm{diam}(X)$ and so the sum converges uniformly. The topology defined by the metric and that of the (K, ϵ)-neighbourhoods of trajectories can easily be shown to define the same topology on infinite trajectories.

We can also define a (K, ϵ)-neighbourhood of a hybrid time domain $\mathcal{T}$ as $\{\mathcal{T}' \mid \mathcal{T}' \cap K \subset N_\epsilon(\mathcal{T})\}$, and the distance between two hybrid time domains as

$$d(\mathcal{T}, \mathcal{T}') = \sum_{N=0}^{\infty} \frac{\max\left\{\tilde{d}\left(\mathcal{T} \cap K_N, \mathcal{T}'\right), \tilde{d}\left(\mathcal{T}' \cap K_N, \mathcal{T}\right)\right\}}{2^N}.$$

The topology generated by (K, ϵ)-neighbourhoods and the topology generated by the metric are the same. It is clear that if $\xi_i \to \xi_\infty$, then $\mathcal{T}(\xi_i) \to \mathcal{T}(\xi_\infty)$ since $d(\mathcal{T}(\xi), \mathcal{T}(\xi')) \leqslant d(\xi, \xi')$ for any two hybrid trajectories ξ, ξ'. Further, the topology on hybrid time domains is the same as that on trajectories if X is a one-point space (or all trajectories have the same constant value).

Lemma 2.8 (Convergence of hybrid time domains). *$\mathcal{T}_i \to \mathcal{T}_\infty$ if and only if $t_n(\mathcal{T}_i) \to t_n(\mathcal{T}_\infty)$ for all n (where we use the convention that $t_n(\mathcal{T}) = \infty$ if $n > n_*(\mathcal{T})$).*

Proof. Suppose $\mathcal{T}_i \to \mathcal{T}_\infty$. Suppose $n \leqslant n_*(\mathcal{T}_\infty)$ and take $N > \max\{n, t_n(\mathcal{T}_\infty)\}$. Then since $d(\mathcal{T}_i \cap K_N, \mathcal{T}_\infty) \to 0$ as $i \to \infty$, the sequences $(t_n(\mathcal{T}_i), n)$ and $(t_n(\mathcal{T}_i), n+1)$ must approach $\mathcal{T}_\infty$ as $i \to \infty$. This means that $\limsup_{i\to\infty} t_n(\mathcal{T}_i) \leqslant t_n(\mathcal{T}_\infty)$ and $\liminf_{i\to\infty} t_n(\mathcal{T}_i) \geqslant t_n(\mathcal{T}_\infty)$, hence $\lim_{i\to\infty} t_n(\mathcal{T}_i) = t_n(\mathcal{T}_\infty)$. Suppose instead $n > n_*(\mathcal{T}_\infty)$, so that $t_n(\mathcal{T}_\infty) = \infty$. Then if there exists N such that $\mathcal{T}_i \cap K_N$ contains a point (t, n) for infinitely many i, we have $d(\mathcal{T}_i \cap K_N, \mathcal{T}_\infty) > 1$ for these i, contradicting convergence. This implies $t_n(\mathcal{T}_i) \to \infty$ as $i \to \infty$.

Now suppose $t_n(\mathcal{T}_i)$ converges as $i \to \infty$ for all n. Since $t_n(\mathcal{T}_i) \leqslant t_{n+1}(\mathcal{T}_i)$ for all i, we have $\lim_{i\to\infty} t_n(\mathcal{T}_i) \leqslant \lim_{i\to\infty} t_{n+1}(\mathcal{T}_i)$ for all n. Hence the $\lim_{i\to\infty} t_n(\mathcal{T}_i)$ give the event times for some hybrid time domain $\mathcal{T}_\infty$. It remains to show that $\mathcal{T}_i \to \mathcal{T}_\infty$ as $i \to \infty$. Fix $N \in \mathbb{Z}^+$ and $\epsilon > 0$, and j such that for all $n \leqslant N$ and $i \geqslant j$, either $|t_n(\mathcal{T}_i) - t_n(\mathcal{T}_\infty)| < \epsilon$ or $t_n(\mathcal{T}_\infty) = \infty$ and $t_n(\mathcal{T}_i) > N$. Then if $(t, n) \in \mathcal{T}_i$ for some $i > j$, we have $n \leqslant n_*(\mathcal{T}_\infty)$, since otherwise we would have $t \geqslant t_n(\mathcal{T}_i) > N$. But then $t_n(\mathcal{T}_i) \leqslant t \leqslant t_{n+1}(\mathcal{T}_i)$ and $t_n(\mathcal{T}_i) - \epsilon \leqslant t_n(\mathcal{T}_\infty) \leqslant t_{n+1}(\mathcal{T}_\infty) \leqslant t_{n+1}(\mathcal{T}_i) + \epsilon$, from which we see that there exists t' with $(t', n) \in \mathcal{T}_\infty$ and $d(t, t') < \epsilon$ as required. □

Theorem 2.9 (Compactness of the set of hybrid time domains). *The set of hybrid time domains is compact.*

Proof. The space $\mathbb{R}^+ \cup \{\infty\}$ is compact (every sequence of positive real numbers has a subsequence which either has a real limit or tends to infinity), so we can find subsequences of hybrid time domains $\mathcal{T}_{i_{j,k}}$ such that $t_*(\mathcal{T}_{i_{j,k}})$ converges as $k \to \infty$ for all j, and $t_n(\mathcal{T}_{i_{j,k}})$ converges as $k \to \infty$ for all $n \leqslant j$. Taking the subsequence $\mathcal{T}_{i_{l,l}}$ we have convergence of $t_n(\mathcal{T}_{i_{l,l}})$ for all n, and hence by Lemma 2.8, $\mathcal{T}_{i_{l,l}}$ converges to some $\mathcal{T}_\infty$ as $l \to \infty$. □

The following lemma is useful in determining points of limiting trajectories.

Lemma 2.10. *Let ξ_i be a sequence of trajectories with $\lim_{i\to\infty} \xi_i = \xi_\infty$. Let K be a compact subset of $\mathbb{R}^+ \times \mathbb{Z}^+$, and $(t_i, n_i) \in \mathcal{T}(\xi_i) \cap K$ be a sequence such that $(t_i, n_i) \to (t_\infty, n_\infty)$ as $i \to \infty$. Then $(t_\infty, n_\infty) \in \mathcal{T}(\xi_\infty)$, and $\xi_\infty(t_\infty, n_\infty) = \lim_{i\to\infty} \xi_i(t_i, n_i)$.*

Proof. Since $n_i \to n_\infty$, we have $n_i = n_\infty$ for all i sufficiently large. Then $t_\infty = \lim_{i\to\infty} t_i \geqslant \lim_{i\to\infty} t_{n_\infty}(\xi_i) = t_{n_\infty}(\xi_\infty)$ and similarly $t_\infty \leqslant t_{n_\infty+1}(\xi_\infty)$, so $(t_\infty, n_\infty) \in \mathcal{T}(\xi_\infty)$. Let $x_i = \xi_i(t_i, n_i)$ and $x_\infty = \lim_{i\to\infty} x_i$. Since $\xi_i \to \xi_\infty$, we have $\tilde{d}(\mathrm{Gr}(\xi_\infty), \mathrm{Gr}(\xi_i \cap K)) \to 0$, so there exist $(t_i', n_i', x_i') \in \mathrm{Gr}(\xi_\infty)$ such that $d((t_i, n_i, x_i), (t_i', n_i', x_i')) \to 0$ as $i \to \infty$. But then $(t_i', n_i', x_i') \to (t_\infty, n_\infty, x_\infty)$ as $i \to \infty$, so $(t_\infty, n_\infty, x_\infty) \in \mathrm{Gr}(\xi_\infty)$, and hence $x_\infty = \xi_\infty(t_\infty, n_\infty)$ as required. □

The set of all hybrid trajectories is not compact in the Skorohod topology since a sequence of continuous functions on a fixed interval may converge point-wise to a function which is not continuous. Under sufficiently strong equicontinuity conditions on the trajectories, we can show H is compact.

Recall that a function f is uniformly continuous if there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that $d(f(x), f(y)) \leqslant \epsilon$ whenever $d(x, y) \leqslant \delta(\epsilon)$. Any such function δ is called a *modulus of continuity* of f. We say that f is *δ-uniformly continuous* if we need to explicitly specify the modulus of continuity. A set of functions is *δ-uniformly equicontinuous* if each function is δ-uniformly continuous.

Theorem 2.11 (Compactness of hybrid trajectory space). *Let X be a compact metric space, δ a modulus of continuity, and H be the set of all δ-uniformly continuous hybrid trajectories in X. Then H is compact.*

Proof. Let (ξ_i) be a sequence of functions in H. We need to show that some subsequence of (ξ_i) has a limit point ξ_∞. By taking a subsequence if necessary, by Lemma 2.9 we can assume that the hybrid time domains $\mathcal{T}_i = \mathcal{T}(\xi_i)$ converge to a hybrid time domain ξ_∞.

Extend the functions ξ_i from the interval $[t_n(\xi_i), t_{n+1}(\xi_i)]$ over the intervals $[t_n(\xi_\infty), t_{n+1}(\xi_\infty)]$ for each n. Since the set of δ-uniformly equicontinuous functions on a compact interval (and hence on a finite disjoint union of compact intervals) is compact, for any compact subset K of $\mathcal{T}_\infty$, we can take a subsequence such that $\lim_{i\to\infty} \xi_i(t,n)$ exists pointwise and the limiting function $\xi_\infty|_K$ is δ-uniformly continuous. The result follows by constructing ξ_∞ on an increasing family of compact subsets of $\mathcal{T}_\infty$ by taking nested subsequences. ☐

2.3 Hybrid shift operators and evolution maps

The hybrid shift operators $\hat{\sigma}_s^m$ of Section 2.1 are only defined for systems with $(s,m) \in \mathcal{T}$. It is also useful to shift a trajectory by a certain number of events or a certain time, since these operators are defined for all trajectories with sufficiently many events or sufficiently large total execution times.

Definition 2.12 (Shift operators). Define the *event-shift operators* $\hat{\rho}^m$ and the *time-shift operators* $\hat{\tau}_s$ by

$$\hat{\rho}^m \xi(t,n) = \hat{\sigma}^m_{t_m(\xi)} \xi = \xi(t + t_m(\xi), n+m) \quad \text{and} \quad \hat{\tau}_s \xi(t,n) = \hat{\sigma}^s_{n_s(\xi)} \xi = \xi(t+s, n+n_s(\xi)).$$

The shift operators form semigroups under composition. In other words,

$$\hat{\sigma}^{m_2}_{s_2} \circ \hat{\sigma}^{m_1}_{s_1} = \hat{\sigma}^{m_1+m_2}_{s_1+s_2}, \quad \hat{\rho}^{m_2} \circ \hat{\rho}^{m_1} = \hat{\rho}^{m_1+m_2} \quad \text{and} \quad \hat{\tau}_{s_2} \circ \hat{\tau}_{s_1} = \hat{\tau}_{s_1+s_2}.$$

Unfortunately, the time-shift operators are not continuous in the hybrid Skorohod topology for systems with discrete events.

Example 2.13. Consider the trajectories ξ_a with a single event at time a, and let $\xi_a(0,t) = 1$ for $t \leqslant a$ and $\xi_a(1,t) = 0$ for $t \geqslant a$. Then $d(\xi_a, \xi_b) < 2|a-b|$. However, if we take $a \leqslant s < b$, then the time shift $\hat{\tau}_s(\xi_a)$ has no events, whereas $\hat{\tau}_s(\xi_b)$ has one event at time $b-s$. Then at any time, either the number of events or the state of the system differs between ξ_a and ξ_b. Hence $d(\hat{\tau}_s\xi_a, \hat{\tau}_s\xi_b) = 2$, so $\hat{\tau}_s$ is not continuous.

This difficulty arises since under $\hat{\tau}_s$, nearby trajectories may have different numbers of events.

Though it may be possible to find a stronger topology under which $\hat{\tau}_s$ is continuous, we do not do so here. However, it is straightforward to show that the time-evolution operators are continuous on the set of trajectories with no events. The event-shift operators are better behaved, since nearby trajectories have resets at nearby times.

Theorem 2.14 (Continuity of the event-shift operator). *The event-shift operators $\hat{\rho}^m$ are continuous on their domains of definition.*

Proof. Suppose ξ_i are hybrid trajectories converging to ξ_∞ and that ξ_∞ has at least m events. Then each ξ_i has at least m events for i sufficiently large, and $t_m(\xi_i) \to t_m(\xi_\infty)$ as $i \to \infty$. Fix $\epsilon > 0$, and let $K \subset K_N$ be a compact subset of $\mathbb{R}^+ \times \mathbb{Z}^+$. Take N' sufficiently large that $N' \geqslant \max\{N+m, N+t_m(\xi)+\epsilon\}$, and let $K' = K_{N'}$. Take j sufficiently large that $|t_m(\xi_j)-t_m(\xi_\infty)| < \epsilon/2$ for all $i > j$, and $\xi_i|_{K'}$ lies within the $\epsilon/2$-neighbourhood of ξ_∞. Then for any $(t,n,x) \in \mathrm{Gr}(\hat{\sigma}^m_{t_m(\xi_i)}\xi_i|_K)$, we have $(t',n',x') \in \mathrm{Gr}(\hat{\sigma}^m_{t_m(\xi_\infty)}\xi_i)$ such that $(t+t_m(\xi_i), n+m, x))$ and $(t'+t_m(\xi_\infty), n'+m, x')$ are within $\epsilon/2$ of each other. Hence (t,n,x) and (t',n',x') are within ϵ since $|t_m(\xi_j) - t_m(\xi_\infty)| < \epsilon/2$. □

We note that the set of trajectories with at least m events is not a closed subset of the set of all hybrid trajectories, since we may have a convergent sequence of trajectories $\xi_i \to \xi_\infty$ such that $t_m(\xi_i) \to \infty$, in which case $n_*(\xi_\infty) < m$.

We can obtain purely discrete-time or purely continuous-time representations of hybrid-time trajectories using *projection operators*. The *event projection* of a hybrid trajectory $\xi : \mathcal{T} \to X$ is given by the (finite or infinite) sequence $\pi^*\xi : n_*(\xi) \to X$ with

$$\pi^*\xi\,(n) = \xi(t_n, n),$$

and represents a trajectory of some discrete-time system. There is no such natural definition of a *time projection*, since any attempt to define a time projection operator results in discontinuities. We therefore give two possible time projections, one of which is single valued and right-continuous, the other of which is multi-valued and upper-semicontinuous. The former is a function $\pi_*\xi : [0, t_*(\xi)] \to X$ with

$$\pi_*\xi\,(t) = \xi(t, n_t),$$

and the latter is a multivalued function $\pi_*\xi : [0, t_*(\xi)] \rightrightarrows X$ with

$$\pi_*\xi\,(t) = \{\xi(t,n) \mid (t,n) \in \mathcal{T}\}.$$

Given a set of hybrid trajectories, we can define multivalued evolution maps on the state space. We consider a multivalued map $\phi : X \rightrightarrows X$ to be a map on subsets of X such that for any $A \subset X$, $\phi(A) = \bigcup_{x\in A} \phi(\{x\}) \subset X$. By an abuse of notation we write $\phi(x)$ for $\phi(\{x\})$. These *evolution maps* give sets of points reachable from a given initial set after a certain number of events, or a certain time.

Definition 2.15 (Evolution maps). The *event-evolution* and *time-evolution* maps are given, respectively, by

$$\begin{aligned} \rho^m(A) &= \{x \in X \mid \exists\, \xi \in H \text{ with } \xi(0,0) \in A \text{ and } \xi(t_m, m) = x\}, \\ \tau_s(A) &= \{x \in X \mid \exists\, \xi \in H \text{ and } n \in \mathbb{Z}^+ \text{ with } \xi(0,0) \in A \text{ and } \xi(s,n) = x\}. \end{aligned}$$

The maps ρ^m are all defined if every trajectory has infinitely many events, and the maps τ_s are defined whenever there are no Zeno trajectories. Unfortunately, the image of a compact set under these maps need not be compact. Further, these maps cannot be used to integrate the system. We therefore define the following *bounded integration maps*:

$$\begin{aligned} \sigma^{\leqslant m}_{\leqslant s}(x) &= \{x \in X \mid \exists\, \xi \in H,\ n \leqslant m \text{ and } t \leqslant s \text{ with } \xi(0,0) \in A \text{ and } \xi(t,n) = x\}; \\ \rho^{m}_{\leqslant s}(A) &= \{x \in X \mid \exists\, \xi \in H \text{ with } t_m(\xi) \leqslant s,\ \xi(0,0) \in A \text{ and } \xi(t_m, m) = x\}; \\ \tau^{\leqslant m}_{s}(A) &= \{x \in X \mid \exists\, \xi \in H \text{ and } n \leqslant m \text{ with } \xi(0,0) \in A \text{ and } \xi(s,n) = x\}. \end{aligned}$$

The hybrid evolution map $\sigma^{\leqslant m}_{\leqslant s}$ gives the set of points reachable within time s and with less than m events. The map $\rho^{m}_{\leqslant s}$ gives the set of points reachable directly after the mth event, as long as this occurs before time s. The map $\tau^{\leqslant m}_{s}$ gives the set of points reachable after time s, as long as no more than m events have occurred.

Of particular interest are is the map $\nu_s^m(A) := \rho_{\leqslant s}^1(A) \cup \tau_s^{\leqslant 0}(A)$, which gives the set of points reachable either after the first event, as long as this occurs within time s, and to the set of points reachable after time s otherwise. Any point reachable from A lies in the set $\sigma_{\leqslant s}^0\left((\nu_s^1)^m(A)\right)$ for some m.

2.4 General hybrid trajectories

We now give an generalisation of the concept of hybrid trajectory on a hybrid time domain.

Let $\mathbb{T}$ be a partially-ordered set. An *interval* I in $\mathbb{T}$ is a set such that

1. I is totally ordered i.e. $\forall x, y \in I,\ x \leqslant y$ or $y \leqslant x$, and
2. if $y \in \mathbb{T}$ is such that $x \leqslant y \leqslant z$ for some $x, z \in I$, then either $y \in I$, or $I \cup \{y\}$ is not totally ordered.

If $\exists a \in I,\ \forall x \in I,\ a \leqslant x$, then a is the *minimal* element of I; similarly if $\exists b \in I, \forall x \in I,\ b \geqslant x$, then b is the *maximal* element of I. An interval is *closed* if it has a minimal and maximal element. Intuitively, an interval is a totally ordered set into which elements cannot be inserted within its bounds without destroying the total order. This notion of interval generalises the concept of an interval in $\mathbb{R}$ to arbitrary partially ordered sets.

The hybrid time domains defined in Definition 2.2 are intervals in the partially-ordered set $\mathbb{R}^+ \times \mathbb{Z}^+$ with minimal element 0. Using the concept of intervals of partially ordered sets, we can generalise the notion of hybrid time axis to include any partially-ordered set; the notion of hybrid time domain generalises accordingly.

Definition 2.16. Let the *hybrid time axis* $\mathbb{T}$ be a partially-ordered set with the order topology. A *(generalised) hybrid time domain* in $\mathbb{T}$ is an interval $\mathcal{T}$ in $\mathbb{T}$. A *(generalised) hybrid trajectory* in X is a continuous function $\xi : \mathcal{T} \to X$ for some hybrid time domain $\mathcal{T}$ in $\mathbb{T}$.

Although this definition may seem too abstract to be of much use in applications, we shall see that it enables us to provide an appropriate notion of solution for systems with certain types of Zeno behaviour, most notably the bouncing ball of Example 5.1.

Many of the concepts introduced for the hybrid time axis $\mathbb{R}^+ \times \mathbb{Z}^+$ generalise to the more abstract setting. In particular, given a projection π from $\mathbb{T}$ to a totally ordered set $\mathbb{O}$ such that π maps intervals in $\mathbb{T}$ to intervals in $\mathbb{O}$, we can define projections on hybrid trajectories by

$$\hat{\pi}\xi\,(s) = \xi(\{t \mid \pi(t) = s\}), \quad \underline{\pi}\xi\,(s) = \xi(\inf\{t \mid \pi(t) = s\}) \quad \overline{\pi}\xi\,(s) = \xi(\sup\{t \mid \pi(t) = s\}).$$

The projection $\hat{\pi}$ gives rise to multivalued trajectories on $\mathbb{O}$.

We can also shift hybrid trajectories, and catenate two hybrid trajectories. If $\sigma : \mathbb{T} \to \mathbb{T}$ is a homeomorphism, and $\xi : \mathcal{T} \to X$ is a hybrid trajectory, then $\hat{\sigma}\xi : \sigma^{-1}(\mathcal{T}) \to X$ is defined by

$$\hat{\sigma}\xi\,(t) = \xi(\sigma(t)).$$

If $\xi_1 : \mathcal{T}_1 \to X$ and $\xi_2 : \mathcal{T}_2 \to X$ are hybrid trajectories such that $\mathcal{T}_1$ has maximal element $c \in \mathbb{T}$, $\mathcal{T}_2$ has minimal element c, and $\xi_1(c) = \xi_2(c)$, then $\mathcal{T}_1 \cup \mathcal{T}_2$ is an interval in $\mathbb{T}$, and the function $\xi_1 \cdot \xi_2 : \mathcal{T}_1 \cup \mathcal{T}_2 \to X$ defined by

$$\xi_1 \cdot \xi_2(t) = \begin{cases} \xi_1(t) \text{ if } t \in \mathcal{T}_1; \\ \xi_2(t) \text{ if } t \in \mathcal{T}_2, \end{cases}$$

is continuous, so is a hybrid trajectory.

3 Hybrid system models

Hybrid systems occur in a wide variety of applications, and there are a number of modelling frameworks. In some frameworks, the "hybrid" nature is due to the interplay between discrete and continuous variables in the state space. In discrete time, the general dynamical behaviour of such systems is adequately described by that of iterated maps, though such systems may require hybrid numerical techniques. In continuous time, the state space is partitioned into subsets, and the system behaviour undergoes a discontinuous change when the state crosses from one partition element to another. Such systems often arise as simplifications of general nonlinear systems with the dynamics on each partition element being described by an affine differential equation. These systems are adequately described by the theory of differential inclusions [2, 10], with *sliding modes* [12] possible on the boundaries of partition elements. An explicit hybrid model with each subsets being represented by a different discrete state is possible, but care must be taken with the modelling to ensure that spurious Zeno behaviours are not introduced. The hybrid nature is still important in the development of numerical methods for simulation and verification, where different regions of the discrete state space must be considered separately. However, in terms of defining an appropriate solution concept and determining the dynamical properties, such systems can be considered non-hybrid.

Where the hybrid nature of the system is intrinsic in the application, such as in impacting systems, electrical circuits with switches and (ideal) diodes, fast-slow systems, systems with hysteresis effects and continuous systems with discrete (computer) controllers, the dual discrete- and continuous-time behaviour is typically more apparent. Such systems are often modelled using the framework of *hybrid automaton*, which we discuss in more detail below. In all cases, trajectories of the system can be modelled by continuous functions on hybrid time domains. As long as the set of trajectories of a system is uniformly equicontinuous, the closure of the set of trajectories is compact. Since any additional trajectories obtained by taking the closure are just limits of existing trajectories, this does not change the behaviour of the system which can be observed in practise.

3.1 Impulse differential inclusions

The results so far presented have been expressed purely in terms of hybrid trajectory spaces, with compactness of the space being a crucial property. We now discuss a class of hybrid systems for which the trajectory space is always compact. These systems are the impulse differential inclusions of [3].

Of crucial importance is the concept of *upper-semicontinuity*, which is a generalisation of continuity for multivalued maps. A multivalued map $F : X \rightrightarrows Y$ is upper-semicontinuous if $f^{-1}(B) := \{x \mid f(x) \cap B \neq \emptyset\}$ is closed whenever B is closed, and weakly upper-semicontinuous if $f^{-1}(B) := \{x \mid f(x) \cap B \neq \emptyset\}$ is closed whenever B is compact. A multivalued map $F : X \rightrightarrows Y$ with closed values (i.e. $F(x)$ is closed for all x) is weakly upper-semicontinuous if and only if its graph $\mathrm{Gr}(F) := \{(x, y) \in X \times X \mid y \in F(x)\}$ its closed.

The continuous behaviour of our hybrid systems will be described in terms of differential inclusions, a generalisation of differential equations. The subject of differential inclusions is a substantial topic in its own right. The basic results, including existence and of solutions, can be found in the book of Aubin and Cellina [2]; here we briefly summarise the relevant material.

The natural solution space for a differential inclusion is the space of *absolutely continuous functions*. Recall that a function $\xi : [a, b] \to X$ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every countable collection of disjoint subintervals $[a_k, b_k]$ with

$\sum_k (b_k - a_k) \leqslant \delta$, we have $\sum_k d(\xi(a_k), \xi(b_k)) \leqslant \epsilon$. If ξ is an absolutely continuous function, then its derivate $\dot{\xi}$ is well-defined Lebesgue almost everywhere.

A set of absolutely continuous functions is *absolutely equicontinuous* if the same modulus of continuity $\delta(\epsilon)$ applies to each. A sequence of absolutely equicontinuous functions taking values in a compact set has a convergent subsequence, and the limit is absolutely continuous with the same modulus of continuity.

A differential inclusion $\dot{x} \in F(x, t)$ always has a solution if the state space X is a closed manifold, and if the right hand side F is upper-semicontinuous in x, measurable in t, and $F(x, t)$ takes nonempty closed convex values. If further the values of $F(x, t)$ are compact, then the solution space consists of functions which are absolutely continuous. The set of solutions is compact in the compact-open topology, which means that any sequence of solutions has a subsequence which converges uniformly on compact time intervals to another solution.

If the state space X is a non-compact manifold (without boundary), then solutions may escape in finite time. In the case that X is a Banach space, this can be prevented by assuming *linear growth*; i.e. that there exists $k > 0$ such that for all t, $F(x, t) \subset B(c(1 + ||x||))$.

We now give a formal definition of a hybrid system as an impulse differential inclusion.

Definition 3.1 (Impulse differential inclusion). A hybrid system $\mathcal{H}$ is given by an *impulse differential inclusion* (X, F, G) where

- The closed manifold X is the *state space.*
- The function $F : X \rightrightarrows \mathrm{T}X$, $F(x) \subset \mathrm{T}_x X$ defines a *differential inclusion* $\dot{x} \in F(x)$.
- The function $G : X \rightrightarrows X$ defines a *reset condition* $x_{n+1} \in G(x_n)$.

A *trajectory* or *execution* of $\mathcal{H}$ is an absolutely continuous function $\xi : \mathcal{T} \to X$ such that

$$\dot{\xi}(t, n) \in F(\xi(t, n)) \text{ for almost every } t \in (t_n, t_{n+1});$$

$$\xi(t_n, n) \in G(\xi(t_n, n{-}1)).$$

The system may be non-deterministic, so there may be more than one possible trajectory for a given initial condition. A discrete event cannot occur at a state x with $G(x) = \emptyset$. An event *must* occur if there is no further continuous trajectory, such as on reaching the boundary of the set on which $F(x) \neq \emptyset$, since no continuous evolution is possible if $F(x) = \emptyset$. Otherwise a discrete event may or may not occur. In this way it is possible to model systems for which a discrete event is forced to occur at a particular state. For more generality, the differential inclusion could be replaced by a set-valued semiflow.

Control laws also have a natural interpretation in this context. A *feedback control law* for a plant described by an impulse differential inclusion $(X, F_P, G_P)$is specified by giving a system (X, F_C, G_C) describing the allowable continuous-time and discrete-time evolutions. The *closed loop dynamics* induced by coupling the plant system (X, F_P, G_P) with the controller (X, F_C, G_C) is then the system $(X, F_P \cap F_C, G_P \cap G_C)$, where we define the intersections by $(F_P \cap F_C)(x) = F_P(x) \cap F_C(x)$.

The following theorem shows that the trajectory space of an upper-semicontinuous impulse differential inclusion. It is possible to extract a proof from the proof of Theorem 4 of [3]; we give a simpler proof here.

Theorem 3.2 (Trajectory space of impulse differential inclusions). *Let $\mathcal{H} = (X, F, G)$ be an impulse differential inclusion such that X is compact, and F and G are upper-semicontinuous with compact values, and F has linear growth. Let H be the set of trajectories of $\mathcal{H}$ with initial point in some compact set X_0 of X. Then the trajectory space H is compact in the hybrid Skorohod topology.*

Proof. Standard results from the theory of differential inclusions [2] show that the solution set of $\dot{x} \in F(x)$ is uniformly equicontinuous, and further, the limit of solutions on any compact interval is also a solution. Therefore H is a set of uniformly equicontinuous functions, and by Theorem 2.11, we need only show that the solution set of the hybrid system is closed. To this end, let ξ_i be a convergent sequence of trajectories of H with limit ξ_∞. Then since $\xi_\infty(t_n(\xi_\infty), n-1) = \lim_{i\to\infty} \xi_i(t_n(\xi_i), n-1)$, $\xi_\infty(t_n(\xi_\infty), n) = \lim_{i\to\infty} \xi_i(t_n(\xi_i), n)$ and $\xi_i(t_n(\xi_i), n) \in G(\xi_i(t_n(\xi_i), n-1))$, the fact that the graph of G is closed gives $\xi_\infty(t_n(\xi_\infty), n) \in G(\xi_\infty(t_n(\xi_\infty), n-1))$. For any $t \in (t_n(\xi_\infty), t_{n+1}(\xi_\infty))$ we have $\xi_\infty(t, n) = \lim_{i\to\infty} \xi_i(t, n)$. Since the limit of ξ_i on any compact subinterval of $(t_n(\xi_\infty), t_{n+1}(\xi_\infty)) \times \{n\}$ is a solution to $\dot{x} \in F(x)$ almost everywhere, the limiting hybrid trajectory ξ_∞ solves the differential inclusion between events. Hence $\xi_\infty \in H$ as required. $\square$

3.2 Hybrid automata

A popular framework for hybrid systems is that of a *hybrid automaton* [23]. We now describe how hybrid automata can be modelled in the framework of impulse differential inclusions. This gives an indication of the generality of impulse differential differential inclusions and hybrid trajectories as a modelling framework.

The system consists of finitely many discrete states Q and discrete events E. If an event e is occurs in state q, a *discrete state transition* occurs, and the new discrete state is $q' = \rho(q, e)$, where $\rho : Q \times E \to Q$. The discrete state transitions are driven by the continuous dynamics, which are described as follows. The continuous variables lie in some space $\mathbb{R}^n$, and for any $q \in Q$ evolve according to a differential equation $\dot{x} = f_q(x, u)$, where u lies in some input space U. The continuous variables are constrained by some *invariant condition* $x \in X_q^{\text{inv}}$. Whenever some *guard condition* $x \in X_{(q,e)}^{\text{guard}}$ is satisfied in state q, the discrete transition e is enabled, and the system may jump to the discrete state $\rho(q, e)$, with the continuous variables being reset by $x' = r_{q,e}(x)$.

Such a system can be modelled by an impulse differential inclusion as follows. The state space of the system is $X = Q \times \mathbb{R}^n$, giving the set of all possible continuous and discrete states. The differential inclusion is given by $(\dot{q}, \dot{x}) \in F(q, x)$ where

$$F(q, x) = (0, f_q(x)) \text{ if } x \in X_q^{\text{inv}}, \text{ and } F(q, x) = \emptyset \text{ otherwise.}$$

The reset condition is given by a union over all events of the reset conditions for each events, so

$$G(q, x) = \bigcup\nolimits_{e \in E | x \in X_{(q,e)}^{\text{guard}}} \left(\rho(q, e), r_{(q,e)}(x)\right).$$

Note that the right-hand side of the differential inclusion is $\emptyset$ unless (q, x) satisfies the invariant $x \in X_q^{\text{inv}}$, which prohibits violation of the invariant condition in each discrete state. An alternative formulation is to include the invariant conditions directly into the state set, so that $X = \bigsqcup_{q \in Q} \{q\} \times X_q^{\text{inv}}$. In this case $F(q, x)$ does not need to be defined if $x \notin X_q^{\text{inv}}$, since then $(q, x) \notin X$.

4 Hybrid systems with compact trajectory space

We now discuss features of hybrid systems with compact trajectory space. We assume throughout that the trajectory space H of our hybrid system is invariant under the hybrid shift operators. We also assume that that every trajectory is infinite (either infinitely many events, or infinite execution time, or both), or equivalently, that the system is prefix-free and non-blocking. However, the results given extend easily to systems which are blocking. We focus on Zeno behaviour, symbolic dynamics and the existence of invariant measures.

4.1 Zeno trajectories

The existence Zeno trajectories in a hybrid system is typically due to over-simplification in the model, though it may indicate chattering behaviour in the system under consideration.

Definition 4.1 (Zeno trajectories). A hybrid trajectory ξ of a hybrid system is *Zeno* if infinitely many events occur in finite time T. (i.e. $\lim_{i\to\infty} t_i(\xi) < \infty$) A hybrid trajectory ξ is *instantaneously Zeno* if $t_i(\xi) = 0$ for all $i \in \mathbb{Z}^+$. (i.e. infinitely many events can occur without any continuous dynamics.) A state $x \in X$ is a *Zeno state* if there is an instantaneously Zeno trajectory ξ with $\xi(0,0) = x$. A set of hybrid trajectories H is *uniformly non-Zeno* if there is an integer M and a time S such that for any trajectory ξ, there are at most M discrete events in any time interval of length S.

If a hybrid system is uniformly non-Zeno and non-blocking, then all trajectories are defined for all $t \in R^+$.

We now show that for systems with compact trajectory space, the existence of a Zeno trajectory is equivalent to the existence of a trajectory with infinitely many events at a single time, which is more easily verified from the system definition since it only depends on the discrete transition map. The results here extend the setting of Proposition 6 of Zhang et. al. [24], and give a converse to by showing that the existence of a Zeno trajectory implies the existence of a Zeno state. We also show that a non-Zeno system must be uniformly non-Zeno.

Theorem 4.2 (Zeno systems). *A non-blocking hybrid system with compact trajectory space H is either uniformly non-Zeno or has an instantaneously Zeno trajectory.*

Proof. Suppose H is not uniformly non-Zeno. We first show that H has a Zeno trajectory. Let ξ_i be a sequence of trajectories such that ξ_i has at least i events in time S. Since H is compact, the ξ_i have a subsequence converging to a trajectory ξ_∞; by restricting to this subsequence we assume that the ξ_i converge. Then since $\lim_{n\to\infty} t_n(\xi_i) \leqslant S$ for all n, the limit ξ_∞ is Zeno.

Now consider the images of the Zeno trajectory ξ_∞ under the return operator. We have

$$t_n(\hat{\rho}^m \xi_\infty) = t_{n+m}(\xi_\infty) - t_m(\xi_\infty) \to t_\infty(\xi_\infty) - t_\infty(\xi_\infty) = 0$$

as $i \to \infty$. Since H is invariant, $\hat{\rho}^m \xi_\infty \in H$ for all $m \in \mathbb{Z}^+$. Since H is compact, there exists a converging subsequence $\hat{\rho}^{m_i} \xi_\infty$ with limit $\eta \in H$. Then for any n, $t_n(\eta) = \lim_{i\to\infty} t_n(\hat{\rho}^{m_i} \xi_\infty) = 0$, so η is an instantaneously Zeno trajectory. □

The significance of this result is that it shows that for systems with compact trajectory space, Zenoness can be decided by considering only the discrete-time behaviour. This is a significant advantage for systems for which the reset map is tractable e.g. piecewise-linear, but the continuous dynamics is complicated.

4.2 Bounded inter-event times

Dual to Zeno trajectories are trajectories with no events. Not surprisingly, there is an analogous result in terms of bounded inter-event times.

Definition 4.3 (Bounded inter-event time). A hybrid trajectory space H has *bounded inter-event times* if there exists $T > 0$ such that for all trajectories ξ, and all n for which $t_n(\xi)$ is defined, we have $t_n(\xi) - t_{n-1}(\xi) \leqslant T$.

Bounded inter-event time is a kind of dual to uniform non-Zenoness; instead of implying that every trajectory is defined for infinite times, it implies that every trajectory has infinitely many events.

Theorem 4.4 (Bounded inter-event time). *A non-blocking hybrid system with compact trajectory space H either has bounded inter-event times, or a trajectory with no events.*

Proof. Let ξ_i be a sequence of trajectories, and n_i a sequence of integers such that $t_{n_i+1}(\xi_i) - t_{n_i}(\xi_i) \to \infty$ as $i \to \infty$. Let $\eta_i = \hat{\sigma}_{n_i}^{t_{n_i}} \xi_i$, so $t_1(\eta_i) = t_1(\eta_i) - t_0(\eta_i) \to \infty$ as $i \to \infty$. Since H is compact, η_i has a subsequence with limit η_∞, and we have $n_*(\eta_\infty) = 0$ as required. □

Note that all trajectories of a uniformly non-Zeno system with bounded inter-event times have infinite execution times and infinitely many events.

4.3 Finite simulation and symbolic dynamics

One of the fundamental techniques for the study of system dynamics in discrete time is to represent an orbit symbolically by its *itinerary*, which gives an approximate description of an orbit in terms of the *regions* of phase space it visits. Given a set of regions, the *shift space* of a system is the set of all possible itineraries. Of particular interest are the *sofic* shift spaces, since these can be given as the possible event sequences of a finite automaton. For an introduction to the use of symbolic dynamics in the study of dynamical systems, see Kitchens [14].

An important problem of systems theory is the construction of *simulating* automata, which generate shift spaces containing the shift space of the original system. For hybrid systems, we can construct symbolic dynamics by considering the evolution maps on the trajectory space.

Definition 4.5 (Itinerary). Let $R_1, \ldots R_k$ be compact subsets of X (not necessarily disjoint or covering). A sequence (s_n) is a *itinerary* for a sequence (x_n) if $x_n \in R_{s_n}$ for all n.

Note that a sequence (x_i) may have many itineraries, or none at all. We can define itineraries for *multivalued maps* $\phi : X \rightrightarrows X$. A sequence (x_n) is an *orbit* of ϕ if $x_{n+1} \in \phi(x_n)$ for all n.

Definition 4.6 (Shift space). If $\phi : X \rightrightarrows X$ is a multivalued map, then the *shift space* of ϕ on $R_1, \ldots R_k$ is the set of all possible itineraries of orbits of ϕ.

It is straightforward to show that if $\phi : X \rightrightarrows X$ is a multivalued map with a closed graph, then the shift space of ϕ is compact. For any sequence of itineraries of ϕ has a convergent subsequence, and this limit is the itinerary of an orbit of ϕ obtained by taking a limit point of the orbits giving the sequence of itineraries. If ϕ is a multivalued map such that $\phi(A)$ is compact whenever A is compact, then the graph of ϕ is closed.

Theorem 4.7. *The shift space is compact for the maps $\sigma_{\leqslant s}^{\leqslant m}$, $\rho_{\leqslant s}^{m}$ and $\tau_{s}^{\leqslant m}$.*

Proof. If K is a compact subset of $\mathbb{R}^+ \times \mathbb{Z}^+$, H a compact set of hybrid trajectories and A is a compact subset of X, we claim that the set

$$\{x \in X \mid \exists \xi \in H, (t,n) \in K \cap \mathcal{T}(\xi) \text{ with } \xi(t,n) = x\}$$

is compact. For consider a sequence $x_i = \xi_i(t_i, n_i)$ with $(t_i, n_i) \in K$. By taking subsequence if necessary, we can assume that x_i, ξ_i, t_i and n_i converge to some x_∞, ξ_∞, t_∞ and n_∞, respectively. Then by Lemma 2.10, $x_\infty = \xi_\infty(t_\infty, n_\infty)$. By taking $K = \{(t,n) \mid t \leqslant s \text{ and } n \leqslant m\}$, we see that $\sigma^{\leqslant m}_{\leqslant s}(A)$ is compact for compact A, and by taking $K = \{(t,n) \mid t = s \text{ and } n \leqslant m\}$ we see that $\tau^{\leqslant m}_{s}(A)$ is compact for compact A.

To show that $\tau^{\leqslant m}_{s}(A)$ is compact, we suppose ξ_i is a sequence of trajectories with $t_m(\xi_i) \leqslant s$ for all m. Then ξ_i has a subsequence with limit ξ_∞, and $t_m(\xi_\infty) = \lim_{i\to\infty} t_m(\xi_i) \leqslant s$. Restricting to this subsequence, let $x_i = \xi_i(t_m(\xi_i), m)$ for all n, and $x_\infty = \lim_{i\to\infty} x_i$. Again, by Lemma 2.10, we have $x_\infty = \xi_\infty(t_m(\xi_\infty), m)$ as required. □

In general shift spaces are not Markov, since whether a region R_{s_n} follows a word $R_{s_0} \ldots R_{s_{n-1}}$ can depend on all elements of the word. An important class of shift space are the *sofic shifts*, which are described by a finite automaton. The fundamental problem of symbolic dynamics is to compute sofic shifts approximating the true shift dynamics from above and below.

4.4 Invariant measures

If $f : X \to X$ is a single-valued map, a measure μ on X is an *invariant measure* for f if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable sets A. Invariant measures are important since they allow the time-averaged behaviour of a system to be computed. In particular, if a probability measure μ gives an estimate of the initial state of the system, and is also invariant, then $\mu(A)$ gives the probability that the system state x is in the set A at any given time, and $\int f(x)\, d\mu(x)$ gives the time-averaged value of the function f.

It is a well-known result of ergodic theory (see for example [20, 21]) that any continuous map on a compact metric space has an invariant probability measure, and this result generalises to continuous flows. We would like to give a probabilistic description of a hybrid system by finding an invariant probability measure for its return map or time evolution map. However, since the return map or time evolution map of a hybrid system may be discontinuous or multivalued, we instead consider invariant measures for the return operator or time evolution operator on the trajectory space, and later project to the state space.

A map may fail to have an invariant measure if the phase space is not compact (the measure "escapes" to infinity) or if the map is discontinuous, as the following simple example shows.

Example 4.8. Let $f : [0,1] \to [0,1]$ be the function given by

$$\begin{cases} f(0) = 1 \\ f(x) = x/2 \text{ if } 0 < x \leqslant 1 \end{cases}$$

If μ were an invariant measure, then $\mu(\{0\}) = \mu\left(f^{-1}(\{0\})\right) = \mu(\emptyset) = 0$. Then since the preimage $f^{-1}\left((1/2, 1]\right) = \{0\}$, we have $\mu\left(f^{-1}(1/2, 1]\right) = \mu\left(f^{-1}(1/4, 1]\right) = \mu\{0\} = 0$. An inductive argument shows that $\mu\left((1/2^n, 1]\right) = 0$ for all $n \in \mathbb{N}$, and so $\mu\left((0,1]\right) = 0$. But then $\mu\left([0,1]\right) = 0$, and so there is no invariant probability measure.

Such situations occur naturally in hybrid systems if a discontinuity point, particularly a tangency of the continuous dynamics with the reset condition, lies on a limiting trajectory which is a stable limit cycle, but not a trajectory of the real system. Such a situation may be given by the heating system of Example 5.3.

Definition 4.9 (Invariant measures for hybrid systems). Let H be a compact invariant hybrid trajectory space. A measure μ on H is *return-invariant* if μ is an invariant measure for $\hat{\rho}^m$ for all $m \in \mathbb{Z}^+$. A measure μ on H is *time-invariant* if μ is an invariant measure for $\hat{\tau}_s$ for all $s \in \mathbb{R}^+$.

Theorem 4.10 (Existence of invariant measures). *Let H be a nonempty compact invariant set of hybrid trajectories. Then there exists a measure μ which is either return-invariant or time-invariant.*

Proof. For any $T \in \mathbb{R}^+$, the subset of H with inter-event time bounded by T is compact and invariant under the return operators $\hat{\rho}^m$. If there is a trajectory with bounded inter-event times, then there exists $T > 0$ such that the set of trajectories with inter-event time bounded by T is nonempty. Since the return operators are continuous on their domains of definition, this implies the existence of a return-invariant measure.

If there is a trajectory ξ with unbounded inter-event times, then $t_1(\hat{\rho}^{m_i}\xi) \to \infty$ as $i \to \infty$, so any limit trajectory of $\hat{\rho}^{m_i}\xi$ has no events. The set of trajectories with no events is compact and invariant under the time-evolution operators $\hat{\tau}_s$, and the time-evolution operators act continuously on this set. Hence if there is a trajectory ξ with unbounded inter-event times, then there is a time-invariant measure supported on the set of trajectories with no events. □

Any return-invariant measure must be supported on the set of trajectories which are either instantaneously Zeno, or have infinitely many events with bounded inter-event times. Any time-invariant measure must be supported on the set of trajectories which either have no events, or infinitely many events, but a bounded number in any time interval. Note that the time-evolution operators may have an invariant measure supported on the set of trajectories with infinitely many events even though they are not continuous on this set; continuity is a sufficient condition for the existence of an invariant measure, but is not necessary.

If μ is an invariant measure for an operator $\hat{\sigma}$ on the trajectory space H of a hybrid system (either $\hat{\rho}^m$ or $\hat{\tau}_s$), then we can define an induced measure $\tilde{\mu}$ on X by $\tilde{\mu}(A) = \mu(\pi^{-1}(A))$ where π is the projection to the state space given by $\pi\,\xi = \xi(0,0)$.

Theorem 4.11 (Invariant measures for deterministic systems). *Suppose H is the trajectory space for a deterministic hybrid system, and μ is an invariant measure for an operator $\hat{\sigma}$ on S (either the return operator or the time-evolution operator). Then*

$$\tilde{\mu} = \mu \circ \pi^{-1}$$

is an invariant measure for the map $\tilde{\sigma} : X \to X$ defined by $\tilde{\sigma}(\pi\xi) = \pi(\hat{\sigma}\xi)$.

Proof. We have

$$\tilde{\mu}(\tilde{\sigma}^{-1}(A)) = \mu(\pi^{-1}(\tilde{\sigma}^{-1}(A))) = \mu(\hat{\sigma}^{-1}(\pi^{-1}(A))) = \mu(\pi^{-1}(A)) = \tilde{\mu}(A)$$

as required. □

Remark 4.12. For multi-valued mappings $F : X \rightrightarrows X$, we can define

$$F^{\Leftarrow}(A) = \{x \mid F(x) \subset A\} \quad \text{and} \quad F^{-1}(A) = \{x \mid F(x) \cap A \neq \emptyset\}.$$

Then we say a measure μ is *invariant* for F if $\mu(F^{\Leftarrow}(A)) = \mu(A) = \mu(F^{-1}(A))$ for all Borel sets A. However, such a map need not have any invariant measures.

5 Examples

Example 5.1 (A bouncing ball). We first give an example illustrating how the generalised hybrid time axis can model a system which is hard to model using traditional hybrid system techniques.

A ball bouncing on a table can be modelled as follows: The state of the ball is given by the height x and the velocity $v = \dot{x}$. If the ball is not in contact with the table, the motion is governed by gravity: $(\dot{x}, \dot{v}) = (v, -g)$ if $x \geqslant 0$. When the ball bounces, a discrete event occurs, and the velocity is reset: $(x^+, v^+) = (0, -\lambda v)$ if $x = 0$, $v \leqslant 0$. If the ball is stationary, then the ball remains stationary: $(x, v) = (0, 0)$.

If the system starts at $(0, v_0)$ with $v_0 > 0$, then the next event occurs after time $t_1 = v_0/2g$, and the state at $(t_1, 1)$ is $(0, \lambda v_0/2g)$. Hence the nth event occurs at time

$$t_n = \frac{1}{2g} \sum_{i=0}^{n-1} \lambda^i$$

with limit $t_\infty = 1/2g(1-\lambda)$. The velocity just after the nth event is $v_n = \lambda^n v_0$. Therefore any trajectory starting with $x > 0$ or $x = 0$ and $v_0 > 0$ is Zeno. However, for any Zeno trajectory, $(x(t, n), v(t, n)) \to (0, 0)$ as $n \to \infty$.

To define trajectories for this system, we take as hybrid time axis the set $\mathbb{T} = \mathbb{R}^+ \times \mathbb{Z}^+_\infty$, where $\mathbb{Z}^+_\infty = \mathbb{Z}^+ \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$. We can define an infinite-time trajectory ξ on $\mathbb{Z}^+_\infty \times \mathbb{R}^+$ as follows

$$\xi(t, n) = \begin{cases} \phi(t - t_n, 0, \lambda^n v_0) \text{ if } t \in [t_n, t_{n+1}]; \\ (0, 0) \text{ if } t \in [t_\infty, \infty) \text{ and } n = \infty. \end{cases}$$

where $\phi(t, x_0, v_0) = (x_0 + v_0 t - gt^2/2,\ v_0 - gt)$. This is a continuous function on $\mathbb{Z}^+_\infty \times \mathbb{R}^+$, since $\lim_{(t,n)\to(t_\infty,\infty)} \xi(t, n) = (0, 0) = \xi(t_\infty, \infty)$. Further, after t_∞, the ball "rolls" without undergoing any more discrete events, even though $F(0, 0) = (0, 0)$ is defined, since the definition of the hybrid time domain $\mathbb{T}$ itself precludes more events!

To compactify the trajectory space, we take the stationary trajectory $\xi \equiv (0, 0)$ defined on the hybrid time domain $\mathcal{T} = \{(0, n) \mid n \in \mathbb{Z}^+\} \cup \{(t, \infty) \mid t \in \mathbb{R}^+\}$. This represents a ball which starts and stays at rest, and is a limit of trajectories starting at (x_0, v_0) as $(x_0, v_0) \to (0, 0)$. The resulting system has compact trajectory space, and models all behaviours of the ball.

Example 5.2 (A switched arrival system). We now give an example of a system exhibiting Zeno behaviour for which we can compute symbolic dynamics, and invariant measures for both the return map and the time-shift map.

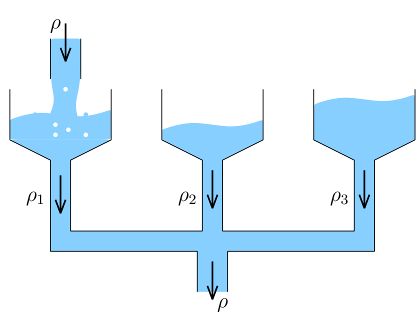

Figure 3: A switched arrival system comprising three tanks with constant outflows

Consider a system comprising three tanks T_1, T_2 and T_3 with constant outflows ρ_1, ρ_2 and ρ_3. The tanks are filled through a single inflow with capacity $\rho = \rho_1 + \rho_2 + \rho_3$. The volume of fluid in tank i is given by x_i, and the total volume is preserved, so $x_1 + x_2 + x_3$ is a constant which we can take to be 1. The continuous state can therefore be given by the (x_1, x_2) coordinates. Only only one tank can be filled at a time, yielding discrete states q_i corresponding to filling tank T_i. This system has been been studied in detail in [22, 17].

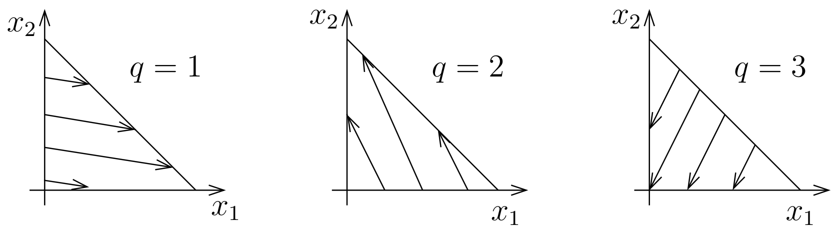

Figure 4: State-space of the switched arrival system.

Consider the switching law under which the system switches state to state q_i as soon as tank T_i empties. The system is described by a deterministic upper-semicontinuous evolution law, with Zeno trajectories occurring if two tanks empty at the same time. Instantaneously Zeno trajectories therefore exist in the system, and occur if two tanks start empty.

The order in which the tanks are filled gives a natural symbolic description of the system. It is fairly straightforward to show that the any sequence of tanks gives a possible ordering for filling. Taking $R_i = \{(q, x) \mid q = i\}$ for $i = 1, 2, 3$ gives symbolic dynamics for first return map r as

$$\Sigma = \{(s_n) \in \{1, 2, 3\}^{\mathbb{Z}^+} \mid \forall n \geqslant 0,\ s_{n+1} \neq s_n\}.$$

Indeed, assuming the system starts in a state for which one of the tanks is empty, the dynamics is determined uniquely by the order in which the tanks are filled.

There is a natural invariant measure μ for the return map of the system, which has constant distribution on each of the sets $x_i = 0$. The total measure of the set $x_1 = 0$ is given by

$$\mu(\{x_1 = 0\}) = \frac{1}{2} \frac{\rho_1(\rho_2 + \rho_3)}{\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1}$$

with analogous expressions holding for the sets $x_2 = 0$ and $x_3 = 0$. Under this measure, the set of initial conditions giving rise to Zeno trajectories has measure zero. However, the measure of the set of non-uniformly Zeno functions is also zero, which means that for almost every trajectory, each tank becomes arbitrarily close to being filled. Other measures exist, some of which are supported on uniformly non-Zeno sets.

The system also admits time-invariant measures on the state space. Since the continuous-time evolution preserves Lebesgue measure in the (x_1, x_2) space, we look for an invariant measure μ such that the measure μ_i in state q_i is a constant multiple c_i of Lebesgue measure λ. We need to ensure that measure is preserved by trajectories crossing from one discrete state to another. Since the flow across the line $x_1 = 0$ is equal to $1 - \rho_1$ in state q_1, and $-\rho_1$ otherwise, we see that in a neighbourhood of $x_1 = 0$ in state q_1, the flux is $c_1(1 - \rho_1) - c_2\rho_1 - c_3\rho_3$, which must equal zero for a time-invariant measure. We obtain similar equations for states q_2 and q_3. Solving these equations gives $c_i \propto \rho_i$, so the invariant measure is

$$\mu_i = 2\rho_i\lambda$$

in discrete state q_i. Hence the average amount of time the system spends filling tank i is proportional to the rate of outflow ρ_i.

We note that while the theory of Section 4.4 we gave ensures the existence of invariant measures for operators on the trajectory space, we are really interested in invariant measures on the state space, and these can be constructed directly. The theory ensures that at least one return-invariant measure exists since the system is deterministic.

Example 5.3 (A simple heating model). We now give an example of a system illustrating the importance of allowing non-determinism for the existence of invariant measures. This system exhibits grazing behaviour, and also the typical associated period-adding bifurcation sequence.

Consider thermal system with temperature $T(t)$ driven by external fluctuations $f(t)$ and an input $u(t)$. Suppose $f(t)$ is a periodic (sinusoidal) function given by $f(t) = T_{\text{env}} = T_{\text{av}} + A\cos(2\pi t)$, and $u(t)$ can be switched arbitrarily between two values 0 and P, corresponding to a discrete state $q \in \{\textsc{off}, \textsc{on}\}$ of a heating device. The system can then be modelled by the differential equation

$$\dot{T}(t) \quad = \quad T_{\text{av}} + A\cos(2\pi t) + P\delta(q(t) = \textsc{on}) - KT(t),$$

which can be made autonomous by introducing an auxiliary variable $t = \theta \in [0, 1]$ and setting $\dot{\theta} = 1$.

We wish to devise a feedback controller so as to maintain the temperature within an allowable range $[T_{\text{min}}, T_{\text{max}}]$. A simple control law is to turn the heater on if T reaches T_{min}, and turn the heater off if T reaches T_{max}.

The dynamics in each state is governed by a linear equation which can be solved explicitly. There is a single limit cycle in each state given by the sinusoidal orbit

$$T(t) = \frac{KA}{4\pi^2 + K^2}\cos(t) + \frac{2\pi A}{4\pi^2 + K^2}\sin(t) + \frac{T_{\text{av}} + P\delta(q = \textsc{on})}{K}.$$

Of critical importance are the values for which the control law achieves the desired control without switching, since then the invariant measures are supported on the limit cycles. In the state $q = \textsc{on}$, the temperature is maintained between T_{min} and T_{max} if

$$T_{\text{min}} \leqslant \frac{T_{\text{av}} + P}{K} - \frac{A}{\sqrt{4\pi^2 + K^2}} \quad \text{and} \quad \frac{T_{\text{av}} + P}{K} + \frac{A}{\sqrt{4\pi^2 + K^2}} \leqslant T_{\text{max}}.$$

In Figure 5, we show a simulation for the parameter values $T_{\text{av}} = 15.0$, $A = 5.0$, $P = 5.0$ and $K = 1.0$. (Note that this corresponds to the room being extremely well-insulated from the environment.) A limit cycle with $q = \textsc{on}$ can only exist if $T_{\text{max}} \geqslant 20 + 5/\sqrt{4\pi^2 + 1} \approx 20.78588$. For T_{max} smaller than this value, the temperature gradually builds over a number of days to the maximum allowed temperature, at which the heating is turned off, and the temperature falls back to the minimum temperature. Notice that as T_{max} increases, the period of the limit cycle increases by one via a *period-adding bifurcation* whenever the continuous dynamics is tangent to the switching condition. There is an invariant measure supported on this limit cycle.

For $T_{\text{max}} = (T_{\text{av}} + P)/K + A/\sqrt{4\pi^2 + K^2}$, the maximum daily temperature is an increasing sequence limiting on T_{max}, and hence the limiting behaviour as $t \to \infty$ is the periodic trajectory. However, if we were to insist on a event precisely at this value, this limiting behaviour would not be a trajectory of the system, and there would be no invariant measure.

For certain parameters, there may be times t for which $\dot{T} < 0$ with $T = T_{\text{min}}$ even for $q = \textsc{on}$. In this case, turning the heating off may result in the temperature dropping below the minimum threshold value before recovering. In Figure 6(a), we show a system with a limit cycle with $q = \textsc{on}$ with maximum value near T_{max}. Slightly reducing T_{max} results in the situation shown in Figure 6(b), in which the heater is turned off and the temperature quickly drops to T_{min}. The

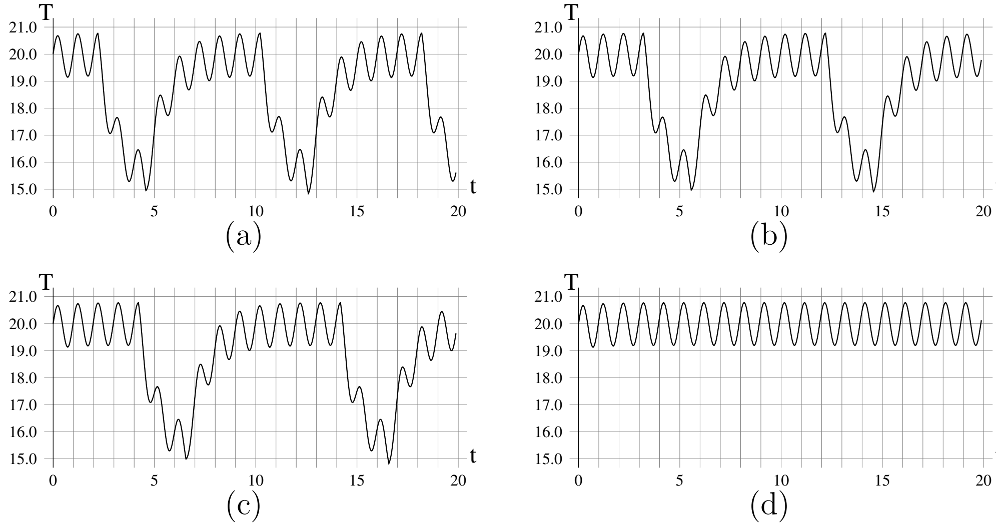

Figure 5: Simulation of the heated room system generated by CHARON[1]. $T(0) = 20.0$, $T_{\min} = 15.0$, $T_{\text{env}} = 15.0 + 5.0\cos(2\pi t)$, $P = 5.0$, $K = 1.0$. (a) $T_{\max} = 20.77$, (b) $T_{\max} = 20.78$, (c) $T_{\max} = 20.785$, and (d) $T_{\max} = 20.79$.

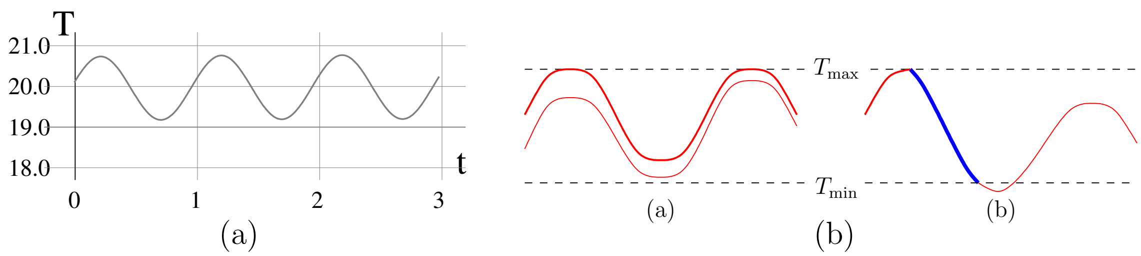

Figure 6: Turning the heater off near the maximum of the limit cycle may cause violation of system constraint $T_{\min} = 19.0$. (a) $T_{\max} = 20.80$, (b) $T_{\max} = 20.75$.

heater is then turned on, but even with $q = \text{on}$ we still have $\dot{T} < 0$ at this particular time, and the temperature falls below the minimum allowable temperature $T_{\min}$ before recovering.

This example shows that a small change in the parameters of a hybrid system can cause a rapid system failure, even if the system is operating far from the dangerous region, and hence that detecting tangencies of the continuous dynamics with the switching surfaces and grazing bifurcations is an important problem. In this example, a control law satisfying the constraints can be obtained by switching the heater on at a temperature $T_{\text{ON}} > T_{\min}$.

6 Conclusions and further research

We have presented a framework for the study of the dynamics of nonlinear hybrid systems. This framework is general enough to include most hybrid systems models, including those with sliding modes and discontinuous resets. The key concept is to consider the set of functions modelling the possible trajectories; this set consists of (absolutely) continuous functions from a hybrid time domain to the state space.

There are two natural projections from functions defined on a hybrid time domain to functions defined on the positive integers or the positive real numbers. Corresponding to these projections is a natural shift operator of the hybrid trajectory set. At least one of these shift operators has an invariant measure if the hybrid trajectory set is compact. For deterministic hybrid systems, these invariant measures on the trajectory spaces give rise to invariant measures on the state space itself. The detection of Zeno behaviour of systems with compact trajectory set is reduced to a study of the discrete events alone.

Since compactness of the trajectory space is the fundamental condition in the theory, we give natural conditions on a hybrid system for it to have a compact trajectory set. The conditions require that the reset maps be upper-semicontinuous functions, and can be imposed on general hybrid system models. Some care must be taken in the modelling to avoid Zeno behaviour; in particular, discrete events which do not cause discontinuous jumps in the trajectories are best modelled in the continuous dynamics as Filippov sliding modes.

In this paper, we have only considered pure deterministic or nondeterministic systems. However, many industrial models are given as stochastic systems, and it should be possible to extend the frameworks given here to the case of stochastic behaviour. Upper-semicontinuity of evolution maps then then arises naturally as a zero-noise limit.

The results contained in this paper provides only a theoretical framework for the study of hybrid systems, and are generally related to existence results. The most important area of further work is in the development appropriate computational tools, such as for simulation and verification of impulse differential inclusions, determining Zeno behaviour, finding symbolic dynamics, and computing invariant measures. Topological tools such as the Conley index [18] may be useful in finding symbolic dynamics; for these compactness of the trajectory space is vital, but successful application of the index requires further technical work on homotopy and homology theory for nondeterministic and hybrid systems. For continuous-time or discrete-time nonlinear systems, *particle filters* [11] are a popular technique for approximating and computing measures. Particle methods are based on iteration of measures by the *Perron-Frobenius operator*. The resulting measures can be interpreted as the long-term probability of the system being a particular set of states given the initial probability density, and hence have a more intuitive meaning than general invariant measures. Such methods are numerically straightforward to implement, and can be tuned to the problem at hand such as the estimation of (small) failure probabilities or finding state approximations.

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