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COMPARING STABILIZED RUNGE-KUTTA METHODS FOR SEMI-DISCRETIZED PARABOLIC AND HYPERBOLIC EQUATIONS

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Comparing stabilized Runge-Kutta methods for semi-discretized parabolic and hyperbolic equations

by

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ABSTRACT

This report is a contribution to a project in order to develop numerical software for time dependent partial differential equations. The method of lines applied to such equations yields large systems of ordinary differential equations. The eigenvalues of the Jacobian matrix of such systems are spread out over a large interval of the negative axis or the imaginary axis. Four classes of stabilized Runge-Kutta methods are given for the numerical solution of such systems. These methods are compared with respect to their efficiency and accuracy by solving a number of test problems.

KEY WORDS & PHRASES: *Numerical analysis, stabilized Runge-Kutta methods, partial differential equations, method of lines, comparison of methods.*

CONTENTS

1. INTRODUCTION

This paper is written as a contribution to a project of the Numerical Mathematics department of the MC to develop numerical software for time-dependent partial differential equations. In our approach, we have divided this project into four parts.

I. The automatic semi-discretization of the partial differential equation in order to reduce it to a system of ordinary differential equations (usually very large).

II. Selection of numerical algorithms for the integration of the initial value problem for these systems.

III. Providing the selected algorithms with automatic step size control, with a strategy for changing the order of accuracy, etc., and implementation in a machine-independent programming language.

IV. Development of an automatic package based on the software resulting from I and III, and testing it on a large number of initial-boundary value problems.

We do realize that the development of *reliable*, *robust* and *efficient* software for partial differential equations is much more difficult than the development of software for ordinary differential equations. Therefore, we have concentrated on *reliability* and *robustness*, and within the small class of methods left we have tried to economize the methods as much as possible. Taking this point of departure, we are automatically led to a well-known solution technique: reduce the problem by semi-discretization (method of lines) to a set of ordinary differential equations and apply an ordinary differential equation solver. Since the method of lines, either based on difference methods or on finite elements, is widely applicable, and since very reliable and robust ordinary differential equation solvers are available, this approach satisfies our main requirements. However, the usually very large systems of ordinary differential equations require integration methods with large stability regions which are necessarily implicit (for instance, the Curtiss-Hirschfelder formulas [3] and implicit Runge-Kutta methods [1]), and therefore in many cases very expensive; in particular, when the system originates from two- or higher dimensional problems, it may be advantageous to use explicit integration methods with an extended stability region. In a series of papers ([5], [6], [7], [12], [2]) single-step and multistep Runge-Kutta type methods with relatively large negative or imaginary stability intervals are presented. In our opinion, this type of methods could be an alternative when implicit methods break down, and should be embodied in a package based on the semi-discretization approach.

This paper in which we analyse the various stabilized Runge-Kutta type methods available, is a contribution to part II of the project described above. In particular, we are interested in the mutual efficiency and accuracy of the algorithms when applied without step control and changing order mechanism. By means of 14 test problems we shall investigate four classes of formulas: single-step and three-step Runge-Kutta methods, both for first order systems, single-step Nyström-Runge-Kutta methods and generalized Runge-Kutta methods for second order systems. In sections 2, 3, 4 and 5 these methods are defined and the stability conditions are stated. In section 6 the numerical experiments are described and results are reported. Section 7 tries to answer the question which class of methods is for what class of problems the best one. Finally, in section 8 and the appendix, additional data is given about the methods and the test-problems, respectively.

2. ONE-STEP RUNGE-KUTTA METHODS

Large classes of initial-boundary value problems for partial differential equations can be approximated by an initial value problem for a system of ordinary differential equations of the form

$$\frac{d\vec{y}}{dt} = \vec{f}(t,\vec{y}) \tag{2.1}$$

by discretizing the space variables of the partial differential equation. In order to solve the initial value problem for the usually very large system (2.1), we need an integration method with restricted storage requirements and with a relatively large stability region. In this section we define three classes of stabilized Runge-Kutta methods with either a large *negative* or a large *imaginary* stability interval (These *adaptive Runge-Kutta* formulas will be called ARK-formulas). The first two classes possess a relatively large *negative* interval of stability and are of first and second order, respectively. These methods are suitable for systems originating from parabolic partial differential equations, that is systems where the Jacobian matrix $\partial \vec{f}/\partial \vec{y}$ has more or less negative eigenvalues δ. The third class consists of second order formulas each of which has a relatively large *imaginary* interval of stability and is therefore appropriate for the integration of hyperbolic problems ($\partial \vec{f}/\partial \vec{y}$ imaginary eigenvalues δ). All formulas require only a few arrays for storage. For a derivation of the formulas we refer to [8, p. 110].

2.1 The numerical scheme for semi-discretized parabolic initial value problems

Let $\vec{y}_n$ denote the numerical solution at the point t_n. Then we may define the first order scheme

$$\vec{y}_{n+1}^{(0)} = \vec{y}_n,$$

$$\text{(2.2)} \qquad \vec{y}_{n+1}^{(j)} = \vec{y}_n + \lambda_j h_n \vec{f}(t_n + \lambda_{j-1} h_n, \vec{y}_{n+1}^{(j-1)}), \qquad j = 1,2,\ldots,m-1,$$

$$\vec{y}_{n+1} = \vec{y}_n + h_n \vec{f}(t_n + h_n, \vec{y}_{n+1}^{(m-1)}),$$

where $h_n = t_{n+1} - t_n$. This scheme becomes second order accurate for $\lambda_{m-1} = \frac{1}{2}$.

The variational equation of scheme (2.2) is of the form

$$\vec{y}_{n+1} = R(z)\vec{y}_n, \qquad z = h_n \delta,$$

where $R(z) = \sum_{j=0}^{m} \beta_j z^j$ is the so-called *stability polynomial*, the coefficients of which are related to the parameters λ_j according to

$$\text{(2.3)} \qquad \lambda_j = \frac{\beta_{m+1-j}}{\beta_{m-j}}, \qquad j = 1,2,\ldots,m-2; \qquad \lambda_{m-1} = \beta_2$$

As is well-known scheme (2.2) is called *(strongly) stable* when $|R(h_n\delta)| \le 1-\varepsilon < 1$ for all eigenvalues δ of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$. In [8, p. 90] it was shown that the polynomial

$$\tilde{R}_m^{(1)}(z) = \frac{T_m\left(w_0 + \frac{w_0+1}{\beta(m)} z\right)}{T_m(w_0)} \tag{2.4}$$

where T_m denotes the Chebyshev polynomial of degree m and

$$\beta(m) = m\sqrt{\frac{w_0+1}{w_0-1}} \tanh\left[m\ln(w_0 + \sqrt{w_0^2-1})\right], \qquad w_0 > 1, \tag{2.5}$$

assumes in the interval $[-\beta(m), -\beta(m)(w_0-1)/(w_0+1)]$ values between $\pm(1-\varepsilon)$ where ε and w_0 are related by

$$\varepsilon = \frac{T_m(w_0) - 1}{T_m(w_0)} . \tag{2.6}$$

Generally, the scheme generated by this polynomial is *first order accurate*. The polynomial $\tilde{R}_m^{(1)}$ is optional in the sense that for negative eigenvalues δ and given value of ε, the stability condition $|\tilde{R}_m^{(1)}(h_n\delta)| \le 1-\varepsilon$, or equivalently

$$0 \le h_n \le \frac{\beta(m)}{|\delta|} \tag{2.7}$$

permits a maximum integration step.

In table 8.1 the coefficients β_j of $\tilde{R}_m^{(1)}(z)$ are listed for m = 2,3,...,12 and

$$w_0 = 1 + \frac{1}{20m^2} . \tag{2.8}$$

This value of w_0 makes ε approximately a constant, for from (2.6) we deduce

$$\begin{aligned} \varepsilon &= 1 - T_m^{-1}(w_0) = 1 - \left[\cosh\left(m\ln\left(1 + \frac{1}{20m^2} + \sqrt{\frac{1}{20m^2}\left(1 + \frac{1}{20m^2}\right)}\right)\right)\right]^{-1} \\ &= 1 - \left[\cosh\left(\frac{1}{10}\sqrt{10} - \frac{3\sqrt{10}}{800m^2} + \dots\right)\right]^{-1} \\ &\cong .05 . \end{aligned} \tag{2.6'}$$

Similarly, (2.5) yields for this value of w_0 the *stability boundary*

$$\beta(m) \cong 1.93m^2, \qquad 2 \le m \le 12 \tag{2.4'}$$

Just as the polynomial $\tilde{R}_m^{(1)}(z)$ generates a stabilized Runge-Kutta formula of first order, polynomials $\tilde{R}_m^{(2)}(z)$ are constructed generating stabilized formulas of *second order*. In [8, p. 94] the coefficients β_j are given for polynomials $\tilde{R}_m^{(2)}(z)$, $m=3,4,\ldots,12$, with a damping factor $1-\varepsilon = .95$. In table 8.2 these coefficients are reproduced together with the stability boundary $\beta(m)$. In a first approximation, $\beta(m)$ can be written as

$$\beta(m) = 0.56m^2 + 0.02m^3, \qquad 3 \le m \le 12 \tag{2.9}$$

Finally, we remark that we have not used values of $m \ge 13$ in order to avoid the danger of instabilities (cf.[8,p.109]).

2.2 The numerical scheme for semi-discretized hyperbolic initial value problems

When the eigenvalues δ are imaginary one may use the following odd degree, weakly stable polynomials (cf.[8,p.99]).

$$I_m^{(2)}(z) = T_k\left(1+\frac{z^2}{2k^2}\right) + \frac{z}{k}\left(1+\frac{z^2}{4k^2}\right) U_{k-1}\left(1+\frac{z^2}{2k^2}\right), \qquad m = 2k+1, \quad k = 1,2,\ldots \tag{2.10}$$

with the stability boundary

$$\beta(m) = m - 1, \tag{2.11}$$

T_k and U_k being the Chebyshev polynomials of first and second kind. The corresponding scheme (2.2) is second order accurate. It is also possible to use even degree, *strongly stable* polynomials.

$$\tilde{I}_m^{(1)}(z) = 1 + z + \beta_2 z^2 + \ldots + \beta_m z^m, \qquad m = 2k+2, \quad k = 1,2,\ldots \tag{2.12}$$

with stability boundary

(2.13) $\beta(m) = m - 2.$

In order to determine the coefficients β_j in (2.12) we define the polynomials $C_{k+1}(z)$ and $S_k(z)$ by

$$(2.14) \quad \begin{cases} C_{k+1}(z) \overset{d}{=} 1 - \beta_2 z + \beta_4 z^2 + \ldots + (-1)^{k+1}\beta_m z^{k+1} \\ S_k\ \ (z) \overset{d}{=} 1 - \beta_3 z + \beta_5 z^2 + \ldots + (-1)^k \beta_{m-1} z^k \end{cases}$$

and prove the following theorem.

THEOREM. *The polynomials*

$$(2.14') \quad \begin{aligned} C_{k+1}(z) &\equiv (1-\varepsilon \frac{z}{\beta_{imag}^2}) T_k(1-2\frac{z}{\beta_{imag}^2}) \\ S_k(z) &\equiv \frac{1}{k}(1-\frac{z}{\beta_{imag}^2})\, U_{k-1}(1-2\frac{z}{\beta_{imag}^2}) \end{aligned}$$

generate a polynomial $\tilde{I}_m^{(1)}(z)$ *with optimal stability boundary*

$$\beta_{imag} = (m-2)\sqrt{\min_{0\le x\le 1} \frac{(1-\varepsilon x)^2}{1-x}}\,, \qquad 0 < \varepsilon < 1, \quad m = 2k+2$$

PROOF. From (2.14) and (2.14') it is clear how to obtain the coefficients β_j of $\tilde{I}_m^{(1)}(z)$. We now prove the optimality of β_{imag}.

$$\begin{aligned} |\tilde{I}_m^{(1)}(iy)|^2 &= C_{k+1}^2(y^2) + y^2 S_k^2(y^2) \\ &= (1-\varepsilon\frac{y^2}{\beta^2})^2\cos^2[k \text{ arc cos } (1-2\frac{y^2}{\beta^2})] + \\ &\quad + \frac{y^2}{k^2}(1-\frac{y^2}{\beta^2})^2 \sin^2[k \text{ arc cos } (1-2\frac{y^2}{\beta^2})][4\frac{y^2}{\beta^2}(1-\frac{y^2}{\beta^2})]^{-1} \\ &= (1-\varepsilon\frac{y^2}{\beta^2})^2\cos^2[---] + (\frac{\beta}{2k})^2(1-\frac{y^2}{\beta^2})\sin^2[---] \end{aligned}$$

$$= (1-\varepsilon\frac{y^2}{\beta^2})^2\left[\cos^2[---] + \sin^2[---](\frac{\beta}{2k})^2\frac{1-y^2/\beta^2}{(1-\varepsilon y^2/\beta^2)^2}\right]$$

$$\leq (1-\varepsilon\frac{y^2}{\beta^2})\left[\cos^2[---] + \sin^2[---](\frac{\beta}{2k})^2\frac{1}{M^2}\right]$$

$$\text{for } |y| \leq \beta \text{ with } M^2 = \min_{0\leq x\leq 1}\frac{(1-\varepsilon x)^2}{1-x}$$

so $$|\tilde{I}_m^{(1)}(iy)| \leq 1 - \varepsilon\frac{y^2}{\beta^2} \quad \text{if} \quad 0 < \varepsilon < 1, \quad |y| \leq \beta, \quad \beta \leq 2kM$$

hence the optimal value of β is $\beta_{imag} = \beta(m) = 2kM$ □

A simple calculation yields

$$\beta(m) = \begin{cases} m-2 & 0 < \varepsilon \leq \frac{1}{2} \\ (m-2)2\sqrt{\varepsilon(1-\varepsilon)} & \frac{1}{2} \leq \varepsilon < 1 \end{cases}$$

It is interesting to consider the limit case $\varepsilon\to 0$. In this case the degree of the polynomial $C_{k+1}(z)$ is decreased by one, i.e., $\tilde{I}_m^{(1)}(z)$ has degree $2k+1$ and we obtain in this way the weakly stable polynomials (2.10). Note that (2.12) generates a *first order* accurate scheme, whereas (2.10) generates a *second order* scheme. However, for ε close to zero the scheme is almost second order accurate. The coefficients $\beta_j(\varepsilon)$ of the stability polynomials $\tilde{I}_m^{(1)}(z)$ may be found in table 8.3. Taking $\varepsilon=0$ gives the coefficients of $I_m^{(2)}(z)$. Again, for reasons of internal stability the value of m is restricted, in this case to $m \leq 18$.

3. THREE-STEP RUNGE-KUTTA METHODS FOR SEMI-DISCRETIZED PARABOLIC INITIAL VALUE PROBLEMS

The class of three-step Runge-Kutta formulas (in the sequel called M3RK formulas) may be represented as follows (see [12]):

(3.1) $$\vec{y}_{n+1}^{(0)} = \vec{y}_n,$$

$$\vec{y}^{(j)}_{n+1} = (1-b_j)\vec{y}_n + b_j\vec{y}_{n-1} + c_j h_n \vec{f}(t_{n-1},\vec{y}_{n-1}) +$$

(3.1) cont'd

$$+ \lambda_j h_n \vec{f}(t_n+\mu_{j-1}h_n,\vec{y}^{(j-1)}_{n+1}), \qquad j = 1,\ldots,m,$$

$$\vec{y}_{n+1} = d\vec{y}^{(m)}_{n+1} + (1-d)\vec{y}_{n-2}, \qquad 2 \le m \le 12, \quad n = 2,3,\ldots$$

In formula (3.1) $\vec{y}_n$ denotes the numerical solution at $t=t_n$, while $h_n = t_{n+1} - t_n$. For the application of (3.1) the additional starting vectors $\vec{y}_1$ and $\vec{y}_2$ must be given. For the numerical experiments discussed in this paper, $\vec{y}_1$ and $\vec{y}_2$ were computed using stabilized one-step formulas of corresponding order (see section 2.1). The degree m of the formulas varies between 2 and 12. In order to preserve internal stability, m is chosen lower or equal to 12 (see section 2.1).

When applied to the test equation $y' = \delta y$, (3.1) yields the linear recurrence relation

(3.2) $$y_{n+1} = dS(z)y_n + dP(z)y_{n-1} + (1-d)y_{n-2}$$

when $S(z) = \sum_{i=0}^{m} s_i z^i$, $P(z) = \sum_{i=0}^{m} p_i z^i$ and $z = h_n\delta$. Thus the stability of (3.1) is governed by the parameter d, and by the coefficients s_i and p_i of the so-called stability polynomials S(z) and P(z), which are expressions in b_j, c_j and λ_j.

In case of first and second order formulas, approximations to optimal stability polynomials, i.e. polynomials yielding a maximal real boundary of absolute stability, are known (see [11]). The coefficients of these polynomials are listed in table 8.4. The values of the parameter d are

$$d = \begin{cases} 1.375, & \text{order} = 1 \\ 0.775, & \text{order} = 2. \end{cases}$$

The corresponding stability boundaries are

(3.3) $$\beta(m) = \begin{cases} 5.15m^2, & \text{order} = 1 \\ 2.29m^2, & \text{order} = 2. \end{cases}$$

The corresponding absolute stability regions contain a long narrow strip around the negative axis. The extrema of the amplifications factors of (3.2) are bounded by about 0.9. For first and second order formulas it is possible to express the integration parameters b_j, c_j and λ_j into the coefficients s_i and p_i. For our class of formulas these expressions are:

$$b_m = p_0,$$

$$c_m = \frac{(1-\tfrac{1}{2}p_0)(p_1-2p_2+2p_3+2s_3) - (\tfrac{1}{2}+\tfrac{1}{4}p_0)^2}{2 + p_1 - 2p_2 + 2p_3 + 2s_3}$$

$$\lambda_m = 1 - \tfrac{1}{2}p_0 - c_m$$

(3.4)
$$b_j = 0, \qquad j = 1,\ldots,m-2$$

$$b_{m-1} = \frac{p_1 - c_m}{\lambda_m}$$

$$c_j = \frac{p_{m+1-j}}{s_{m+1-j}}, \qquad j = 1,\ldots,m-2$$

$$c_{m-1} = \frac{p_2}{\lambda_m},$$

$$\lambda_j = \frac{s_{m+1-j}}{s_{m-j}}, \qquad j = 1,\ldots,m-2$$

$$\lambda_{m-1} = \frac{s_2}{\lambda_m}$$

The parameters μ_j are defined by

(3.5)
$$\mu_0 = 0, \quad \mu_j = -b_j + c_j + \lambda_j, \qquad j = 1,\ldots,m-1$$

To conclude this section we compare the boundaries (3.3) with the corresponding boundaries of the one-step formulas. We then see that the three-step boundaries are approximately 2.7 and 3.0 times the one-step boundaries for first and second order, respectively.

4. STABILIZED NYSTRÖM-RUNGE-KUTTA METHODS FOR SEMI-DISCRETIZED HYPERBOLIC INITIAL VALUE PROBLEMS

Although any partial differential equation containing second time derivatives can be transformed into two equations involving only first derivatives with respect to the time variable, it may be advantageous to maintain the second order form and to apply a numerical integration method to the system of second order equations resulting from the semi-discretization process. In particular, this is the case when the partial differential equation can be discretized in the form of a system without first derivatives, i.e. a system of the form

$$\frac{d^2\vec{y}}{dt^2} = \vec{f}(t,\vec{y}). \tag{4.1}$$

In this section we define a class of stabilized Nyström-Runge-Kutta formulas with reduced storage requirements for the integration of the initial value problem for equation (4.1). These formulas are second order accurate and possess a relatively large negative interval of stability. Therefore, they are suitable integration formulas when the Jacobian matrix of the right-hand side $\vec{f}$ has a negative eigenvalue spectrum. For the derivation of stabilized Nyström-Runge-Kutta formulas we refer to [6] and [7]. In this and subsequent sections we will also speak of SNRK-formulas.

Let $\vec{y}_n$ and $\vec{y}'_n$ denote numerical approximations to the solution $\vec{y}$ and its derivative $d\vec{y}/dt$ at the point t_n. Then we may define the second order scheme

$$
\begin{aligned}
\vec{y}^{(1)}_{n+1} &= \vec{y}_n + \mu_1 h_n \vec{y}'_n, \\
\vec{y}^{(j)}_{n+1} &= \vec{y}_n + \mu_j h_n \vec{y}'_n + \lambda_j h_n^2\, \vec{f}(x_n+\mu_{j-1}h_n, y^{(j-1)}_{n+1}) \qquad j = 2,3,\dots,m-1 \\
\vec{y}^{(m)}_{n+1} &= \vec{y}_n + \tfrac{1}{2} h_n \vec{y}'_n + \lambda_m h_n^2\, \vec{f}(x_n+\mu_{m-1}h_n, \vec{y}^{(m-1)}_{n+1}), \\
\vec{y}_{n+1} &= \vec{y}_n + h_n \vec{y}'_n + \tfrac{1}{2} h_n^2\, \vec{f}(x_n+\tfrac{1}{2}h_n, \vec{y}^{(m)}_{n+1}), \\
\vec{y}'_{n+1} &= \vec{y}'_n + h_n\, \vec{f}(x_n+\tfrac{1}{2}h_n, \vec{y}^{(m)}_{n+1}), \text{ where again } h_n = t_{n+1} - t_n.
\end{aligned}
\tag{4.2}
$$

Let the parameters μ_j and λ_j be defined by the relations

(4.3)
$$\mu_j = \frac{1}{2}\frac{\sigma_{m+1-j} + \pi_{m+1-j}}{\sigma_{m+1-j} - \pi_{m+1-j}}, \qquad j = 1,2,\ldots,m-1$$
$$\lambda_j = \frac{\sigma_{m-j+2} - \pi_{m-j+2}}{\sigma_{m-j+1} - \pi_{m-j+1}}, \qquad j = 2,\ldots,m$$

where σ_j and π_j are the coefficients of z^j in the stability polynomials

(4.4)
$$S(z) = 2\,\frac{T_m\left(w_0 + \frac{w_0+1}{2\beta(m)}z\right)}{T_m(w_0)}$$

and

(4.5)
$$P(z) = 1 - \varepsilon(m) + \varepsilon(m)\left[1-(m-1)\frac{z}{2\beta(m)}\right]\left[1+\frac{z}{2\beta(m)}\right]^{m-1}$$

w_0, $\beta(m)$ and $\varepsilon(m)$ being defined as in section 2.1. Then it can be proved that scheme (4.2) is (strongly) stable when h_n satisfies the condition

(4.6)
$$0 \le h_n \le \sqrt{\frac{2\beta(m)}{|\delta|}}$$

for all (negative) eigenvalues δ of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$ at the point $(t_n,\vec{y}_n)$. Moreover, the damping function is given by $P(h_n^2\delta)$.

Comparing (2.4) and (4.4) we see that we have

(4.7)
$$S(z) \equiv 2\tilde{R}_m^{(1)}(z/2)$$

and therefore

(4.8)
$$\sigma_j = 2^{1-j}\beta_j, \qquad j = 1,2,\ldots,m.$$

In our experiments we choose for β_j the values listed in table 8.1.

The coefficients π_j directly follow from the expansion

$$(4.5') \qquad P(z) = 1 - \varepsilon + \varepsilon\Big[1 - (m-1)\frac{z}{2\beta(m)}\Big]\Big[1 + \binom{m-1}{1}\frac{z}{2\beta(m)} + \binom{m-1}{2}\Big(\frac{z}{2\beta(m)}\Big)^2 + \dots + \Big(\frac{z}{2\beta(m)}\Big)^{m-1}\Big].$$

We find

$$(4.9) \qquad \pi_j = -\varepsilon\binom{m}{j}(j-1)[2\beta(m)]^{-j}, \qquad j = 1,2,\dots,m$$

In order to be consistent with the parameters σ_j, one should use for ε and $\beta(m)$ the same values as used for the calculation of the parameters β_j occurring in (4.8).

5. GENERALIZED RUNGE-KUTTA METHODS FOR SEMI-DISCRETIZED HYPERBOLIC INITIAL VALUE PROBLEMS

When the semi-discretization of a hyperbolic partial differential equation yields a system of ordinary differential equations

$$(5.1) \qquad \frac{d\vec{y}}{dt} = \begin{pmatrix} \frac{d\vec{u}}{dt} \\ \frac{d\vec{v}}{dt} \end{pmatrix} = \vec{F}(t,\vec{y}) = \begin{pmatrix} \vec{g}(t,\vec{u},\vec{v}) \\ \vec{h}(t,\vec{u}) \end{pmatrix},$$

the application of a generalized Runge-Kutta method may come into consideration. These methods (sometimes called *composite Runge-Kutta* methods and in the sequel denoted by CRK formulas) can be represented by (see [2]):

$$(5.2) \qquad \begin{aligned} \vec{y}^{(0)}_{n+1} &= \vec{y}_n \\ \vec{y}^{(j)}_{n+1} &= \vec{y}_n + h_n \sum_{\ell=0}^{j-1} \begin{pmatrix} \mu_{j\ell} I & 0 \\ 0 & \beta_{j\ell} I \end{pmatrix} \begin{pmatrix} \vec{g}(t_n+\beta_\ell h_n, \vec{u}^{(\ell)}_{n+1}, \vec{v}^{(\ell)}_{n+1}) \\ \vec{h}(t_n+\mu_\ell h_n, u^{(\ell)}_{n+1}) \end{pmatrix}, \\ \vec{y}_{n+1} &= \vec{y}^{(r)}_{n+1} \qquad\qquad j = 1,\dots,r \end{aligned}$$

Here, $h_n = t_{n+1} - t_n$ and $\vec{y}_n$ denotes the numerical approximation to the solution $\vec{y}$ at the point t_n: $\beta_\ell = \sum_{i=0}^{\ell-1} \beta_{\ell i}$ and $\mu_\ell = \sum_{i=0}^{\ell-1} \mu_{\ell i}$.

When scheme (5.2) is applied to the test equation

$$u' = cv, \qquad v' = -cu$$

we obtain a recurrence relation which is determined by the matrix

$$A_r(ih_n c) = \begin{pmatrix} \dfrac{P_r(ih_n c)}{Q_r(ih_n c)} & \dfrac{Q_r(ih_n c)}{P_r(ih_n c)} \end{pmatrix}$$

after a suitable similarity transformation. The entries of this matrix, the polynomials P_r and Q_r of degree r, have coefficients depending on $\beta_{j\ell}$ and $\mu_{j\ell}$, $j=1,\ldots,r$; $\ell=0,\ldots j-1$.

In [2] it was shown that for odd r and $k=(r-1)/2$ the polynomials

$$\text{(5.3)} \qquad \begin{aligned} P_r(z) &= \frac{1}{z}\left[(1+z+\tfrac{1}{2}z^2)\, T_k(1+\frac{z^2}{2k^2}) - 1 - \frac{1}{8}z^4\right] = p_0 + p_1 z + \ldots + p_r z^r \\ Q_r(z) &= P_r(z) - 1 - z - \tfrac{1}{2}z^2 = q_0 + q_1 z + \ldots + q_r z^r \end{aligned}$$

yield a matrix $A_r(ih_n c)$ whose eigenvalues are in modulus less than or equal to 1 if $|h_n c| \le r-1$. Thus, with these coefficients, scheme (5.2) can be called weakly stable for

$$\text{(5.4)} \qquad h_n \le \frac{r-1}{|c|}$$

In [2] strongly stable schemes are constructed too. As these formulas are considerably less efficient, they are not included in this report.

The relations between the coefficients of the polynomials P_r and Q_r and the parameters $\beta_{j\ell}$ and $\mu_{j\ell}$ are given by

$$\text{(5.5)} \qquad \begin{aligned} p_k + q_k &= \sum_{j_1}^{(1)} \mu_{r,j_1} \sum_{j_2}^{(2)} \beta_{j_1 j_2} \cdots \sum_{j_k}^{(k)} \\ p_k - q_k &= \sum_{j_1}^{(1)} \beta_{r,j_1} \sum_{j_2}^{(2)} \mu_{j_1 j_2} \cdots \sum_{j_k}^{(k)} . \end{aligned} \qquad k = 1,\ldots r$$

For r=3 a particular solution to (5.5) and (5.3) is determined by

$$\text{(5.6)} \qquad \beta_{31} = \beta_{21} = 1, \quad \mu_{32} = \mu_{30} = \mu_{20} = \mu_{10} = \tfrac{1}{2},$$

and zero-values for the other parameters. As is easily verified by writing out the resulting scheme (5.2), minimal storage (at most $1\frac{1}{2}$ array of the length of y) and only a few (1 or 2) right-hand side evaluations are required in this case.

Still more efficient schemes are obtained for larger values of r by choosing

$$
(5.7)\qquad
\begin{aligned}
\mu_{r\ell} &= \frac{1}{r-1}, &\quad& \ell = 0,\dots,r-1\\
\mu_{10} &= \frac{1}{r-1}\\
\mu_{2j+1,\ell} &= \frac{2}{r-1}, && j = 1,\dots,k-1,\quad \ell = 0,\dots,2j\\
\beta_{2j,\ell} &= \frac{2}{r-1}, && j = 1,\dots,k,\quad \ell = 0,\dots,2j-1
\end{aligned}
$$

and the other parameters equal to zero.

For these schemes, the matrices $A_r(ih_nc)$ can be written as a product of $A_3(ih_n2c/(r-1))$, so we now find as stability constraint $|h_nc| \le r-1$ at the cost of $(r+1)/2$ evaluations of g and $(r-1)/2$ evaluations of h.

Although the generalized Runge-Kutta methods are developed for the wider class of equations (5.1), the schemes can be applied to second order equations, too, after rewriting in the first order form

$$
(5.8)\qquad
\begin{aligned}
\frac{d\vec{y}}{dt} &= \vec{z},\\
\frac{d\vec{z}}{dt} &= \vec{f}(t,\vec{y}).
\end{aligned}
$$

For these equations the schemes (5.6) and (5.7) are of second order, provided that the function $\vec{f}$ does not depend on $\vec{z}$.

<u>REMARK</u>. When we apply the scheme determined by (5.7) to the system of equations (5.1), we obtain in general for different values of r a different algorithm. However, when we consider the equations (5.8), and apply the schemes (5.7) to them, we get almost the same computations for r=3 and $h_n=h$ as for r=2s+1 and $h_n=sh$ (confer table 6.2.2). These identities are invoked

because the function $\vec{g}$ does not depend on the argument $\vec{u}$ for (5.8); thus the last evaluation of $\vec{g}$ (with argument $\vec{v}_{n+1}^{(r-1)} = \vec{v}_{n+1}$) is exactly the same as the first evaluation of $\vec{g}$ (with argument $\vec{v}_{n+2}^{(0)} = \vec{v}_{n+1}$) in the next Runge-Kutta step.

<u>REMARK</u>. The number of function evaluations of $\vec{f}$,m is given by the relation $m = \frac{r+1}{2}$, so that the stability condition (5.4) becomes

(5.4') $$h_n \le \frac{2m-2}{|c|}.$$

6. NUMERICAL EXPERIMENTS

In this section we present numerical experiments with the four methods ARK, M3RK, SNRK and CRK for 7 parabolic and 7 hyperbolic problems. The problems chosen are of the form

(6.0.1) $$E(u) = F(u)$$

where E is either the operator $\partial/\partial t$ (parabolic problems) or $\partial^2/\partial t^2$ (hyperbolic problems). The operator F may be non-linear and contains differential operators with respect to the the space variable x. By semi-discretizing (6.0.1) with respect to x we obtain a system of ordinary differential equations which is either of the form

(6.0.2) $$\frac{d\vec{y}}{dt} = \vec{f}(t,\vec{y}) \quad \text{or} \quad \frac{d^2\vec{y}}{dt^2} = \vec{f}(t,\vec{y}).$$

In order to reduce the programming effort we chose the right-hand side $\vec{f}$ identical in the parabolic and hyperbolic test set. This is allowed since in both cases the eigenvalues δ of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$ are assumed to be negative. In the following subsections the parabolic and hyperbolic case are simultaneously treated. The initial condition consists of a prescription of $\vec{y}(0)$ in the first order case and the prescription of both $\vec{y}(0)$ and $(d\vec{y}/dt)(0)$ in the second order case.

In the semi-discretization process the x-interval is replaced by 101 grid points for all problems. The space derivatives are replaced by central differences defined on these grid points.

Since no exact solution of the resulting systems of ordinary differential equations was available we have computed a reference solution by means of a high order Runge-Kutta method with extreme small integration steps.

In the tables of results we compare ARK and M3RK for parabolic problems and ARK, SNRK and CRK for hyperbolic problems. In the first case we distinguish between first and second order methods. In the second case only second order methods are tested. In order to apply ARK and CRK to the equation $d^2\vec{y}/dt^2 = \vec{f}(t,\vec{y})$ one first has to convert it into the first order form

$$\frac{d\vec{y}}{dt} = \vec{z}$$

$$\frac{d\vec{z}}{dt} = \vec{f}(t,\vec{y}).$$

Note that the spectral radius of the Jacobian matrix of this first order system is the square root of that of the original second order system.

As already stated in the introduction, in this paper it is our purpose to *compare the numerical integration formulas* on which ARK, M3RK, SNRK and CRK are based and not such things as stepsize strategies, automatic order control etc. Therefore, we integrate all problems with a constant step length (in the near future we intend to test automatic versions of these methods). Having chosen the step length h, the degree m of the formula to be used (the number of right-hand side evaluations per integration step) has to satisfy the stability condition of the particular class of formulas, i.e. a condition of the form

(6.0.3) $\quad h\sigma \leq \beta(m).$

Here σ denotes the spectral radius of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$. The reason why we choose this strategy is that in automatic versions the degree of the formula is adjusted in the same way. In the various problems the value of σ is computed a priori, for instance by Gerschgorin's theorem. In nonlinear cases where σ is a function of $\vec{y}$, a safe upper bound is used for σ (in the automatic versions of the methods tested in this report, a strategy is included for the computation of σ during the integration process). Since this value may differ for the parabolic and hyperbolic case, we will indicate this value by σ_P and σ_H, respectively.

In order to minimize the number of right-hand side evaluations we choose m as low as possible but still satisfying (6.0.3). For convenience, we give these values of m as they can be derived from the stability boundaries (2.4'), (2.9), (2.11), (3.3), (4.6) and (5.4').

First order ARK for parab. eq.	$m = \text{entier } (1 + .72\sqrt{h\sigma_P})$
Second order ARK for parab. eq.	$m \approx \text{entier } (1 + 1.15\sqrt{h\sigma_P})$
Second order ARK for hyperb.eq.	$m = \text{entier } (2 + h\sqrt{\sigma_H})$
First order M3RK for parab. eq.	$m = \text{entier } (1 + .44\sqrt{h\sigma_P})$
Second order M3RK for parab. eq.	$m = \text{entier } (1 + .66\sqrt{h\sigma_P})$
Second order SNRK for hyperb. eq.	$m = \text{entier } (1 + .51 h\sqrt{\sigma_H}$
Second order CRK for hyperb. eq.	$m = \text{entier } (2 + \frac{1}{2}h\sqrt{\sigma_H})$

Note that for integer values of $h\sqrt{\sigma_H}$ the value of m can be chosen one less. For each experiment two quantities are listed: The number of correct significant digits *sd* of the least accurate component of the numerical solution at the endpoint of the integration interval, i.e.

$$sd \overset{\text{p.d.}}{=} \min_i \{-{}^{10}\log \left|1 - \frac{y_i(te)}{\tilde{y}_i(te)}\right|\}$$

and the number of right-hand side evaluations *fe* needed to reach the endpoint.

6.1 Test problems

In this subsection we specify the test problems.

PROBLEM I.

Initial-boundary value problem (simplified version of a problem given in [10]).

$$\begin{cases} E(u) = \dfrac{\partial^2 u}{\partial x^2} - 10 \sinh (10u), & 0 \le x \le 1, \quad t \ge 0 \\ u(t,0) = 0; \quad u(t,1) = 1; \quad u(0,x) = x; & \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

Semi-discretization on the points $x_j = j/100$, $j = 1,2,\ldots,99$.

$$\begin{cases} f_1(t,\vec{y}) = 10^4[-2y_1 + y_2] - 10 \sinh (10y_1) \\ f_j(t,\vec{y}) = 10^4[y_{j-1} - 2y_j + y_{j+1}] - 10 \sinh (10y_j), \quad j = 2,..98 \\ f_{99}(t,\vec{y}) = 10^4[y_{98} - 2y_{99} + 1] - 10 \sinh (10y_{99}) \end{cases}$$

Reference points: $t = 2160/\sigma_p$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_p = -1.2_{10}6 < \delta < 0 \quad \text{(parabolic case)}$$

$$-\sigma_H = -1.2_{10}6 < \delta < 0 \quad \text{(hyperbolic case)}.$$

PROBLEM II

Initial-boundary value problem ([10])

$$\begin{cases} E(u) = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), & 0 \le x \le 1, \quad t \ge 0 \\ u_x(t,0) = 0; \quad u_x(t,1) = 1.73_{10}{-9}(6.25_{10}10 - [u(t,1)]^4) & \\ u(0,x) = 600; \quad \frac{\partial u}{\partial t}(0,x) = 0 & \end{cases}$$

Semi-discretization on the points $x_j = j/100, \quad j = 0,\dots,100$

$$\begin{cases} f_0(t,\vec{y}) = 10^4[-4y_0 + 4y_1] & \\ f_j(t,\vec{y}) = 10^4[(1-\frac{1}{2j})\, y_{j-1} - 2y_j + (1+\frac{1}{2j})\, y_{j+1}], & j = 1,\dots 99 \\ f_{100}(t,\vec{y}) = 10^4[1.99y_{99} - 1.99y_{100}] + 3.46_{10}{-7}[6.25_{10}10 - y_{100}^4] & \end{cases}$$

Reference points: $t = 2160/\sigma_p$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_p = -\sigma_H = -6.75_{10}4 < \delta < 0.$$

PROBLEM III

Initial-boundary value problem (simplified version of a problem given in [10]).

$$\begin{cases} E(u) = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) - u^2, & 0 \le x \le 1, \quad t \ge 0 \\ u(t,0) = 50; \quad u_x(t,1) = 1; \quad u(0,x) = 50 + x; & \frac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

Semi-discretization on the points $x_j = j/100, \quad j = 1,2,\ldots,100$

$$\begin{cases} f_1(t,\vec{y}) = 5_{10}3[2500 - 2y_1^2 + y_2^2] - y_1^2 \\ f_j(t,\vec{y}) = 5_{10}3[y_{j-1}^2 - 2y_j^2 + y_{j+1}^2] - y_j^2, & j = 2,\ldots,99 \\ f_{100}(t,\vec{y}) = 5_{10}3[2y_{99}^2 - 2y_{100}^2 + 0.04y_{100}] - y_{100}^2 \end{cases}$$

Reference points: $t = 2160/\sigma_p$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_p = -\sigma_H = -2.5_{10}6 < \delta < 0.$$

PROBLEM IV

Initial-boundary value problem ([9, p. 80])

$$\begin{cases} E(u) = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} + G(t,x), & 0 \le x \le 1, \quad t \ge 0 \\ u_x(t,0) = 0; \quad u(t,1) = 0; \quad u(0,x) = 1 - x^2; & \frac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

with

$$G(t,x) = \exp(-t)\{[6 + (1-x^2)\pi^2 t^2 - (1-x^2)]\cos \pi x t - [(1-x^2)x + 4xt - 2t(1-x^2)/x]\,\pi \sin \pi x t\}$$

Semi-discretization on the points $x_j = j/100, \quad j = 0,\ldots,99$

$$\begin{cases} f_0(t,\vec{y}) = 10^4[-6y_0 + 6y_1] + G(t,x_0) \\ f_j(t,\vec{y}) = 10^4[(1-\frac{1}{j})y_{j-1} - 2y_j + (1+\frac{1}{j})y_{j+1}] + G(t,x_j), & j = 1,\ldots 98 \\ f_{99}(t,\vec{y}) = 10^4[(1-\frac{1}{99})y_{98} - 2y_{99}] + G(t,x_{99}) \end{cases}$$

Reference points: $t = 2160/\sigma_p$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_p = -\sigma_H = -8.5_{10}4 < \delta < 0.$$

PROBLEM V

Initial-boundary value problem ([9, p. 28])

$$\begin{cases} E(u) = \exp\{-8(x-\frac{1}{2})^2\} \dfrac{\partial^2 u}{\partial x^2} + G(t,x), & 0 \le x \le 1, \quad t \ge 0 \\ u(t,0) = u(t,1) = u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

with

$$G(t,x) = x(1-x)\sqrt{t}\, e^{-tx}[(\tfrac{3}{2} - tx) \sin \pi xt + \pi xt \cos \pi xt] -$$
$$- e^{-8(x-\frac{1}{2})^2} t^{3/2} e^{-tx}\{[t^2x(1-x)(1-\pi^2) - 2(1+t-2tx)] \sin \pi xt +$$
$$+2\pi t[(1-2x) - tx(1-x)] \cos \pi xt\}$$

Semi-discretization on the points $x_j = j/100$, $j = 1,\ldots,99$

$$\begin{cases} f_1(t,\vec{y}) = 10^4 \exp\{-8(x_1-\frac{1}{2})^2\}[-2y_1 + y_2] + G(t,x_1) \\ f_j(t,\vec{y}) = 10^4 \exp\{-8(x_j-\frac{1}{2})^2\}[y_{j-1} - 2y_j + y_{j+1}] + G(t,x_j), \quad j = 2,..98 \\ f_{99}(t,\vec{y}) = 10^4 \exp\{-8(x_{99}-\frac{1}{2})^2\}[y_{98} - 2y_{99}] + G(t,x_{99}) \end{cases}$$

Reference points: $t = 2160/\sigma_p$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_p = -\sigma_H = -4_{10}4 < \delta < 0.$$

PROBLEM VI

Initial-boundary value problem (simplified version of problem given in [4])

$$\begin{cases} E(u) = d(x,u)\, \dfrac{\partial^2 u}{\partial x^2}, & 0 \le x \le 1, \quad t \ge 0 \\ u(t,0) = 0; \quad u(t,1) = 1; \quad u(0,x) = x^2; \quad \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

with

$$d(x,u) = \left[1 + \frac{2x^2}{(x+u)^2}\right]^{-1}$$

Semi-discretization on the points $x_j = j/100, \quad j = 1,\dots,99$

$$\begin{cases} f_1(t,\vec{y}) = 10^4 * d(x_1,y_1)[-2y_1 + y_2] \\ f_j(t,\vec{y}) = 10^4 * d(x_j,y_j)[y_{j-1} - 2y_j + y_{j+1}], & j = 2,\dots,98 \\ f_{99}(t,\vec{y}) = 10^4 * d(x_{99},y_{99})[y_{98} - 2y_{99} + 1] \end{cases}$$

Reference points: $t = 2160/\sigma_P$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_P = -\sigma_H = -4_{10}4 < \delta < 0.$$

PROBLEM VII

Initial-boundary value problem ([7])

$$\begin{cases} E(u) = gh\, \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{4}\lambda^2 u + \exp(\tfrac{1}{2}\lambda t) w, & 0 \le x \le 10^5 = b, \quad t \ge 0 \\ u_x(t,0) = u_x(t,b) = u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

with $\quad g = 9.81, \quad \lambda = 25_{10}-6, \quad h = 10[2 + \cos(2_{10}-5\pi x)]$

and $\quad w(t,x) = 10^{-3}\sin(10^{-5}\pi x)$

Semi-discretization on the points $x_j = jb/100, \quad j = 0,\ldots,100$

with $\qquad h_j = 10\left[2 + \cos(2_{10}{}^{-5}\pi x_j)\right]$

$$\begin{cases} f_0(t,\vec{y}) = 10^{-6} g h_0\left[-2y_0 + 2y_1\right] + \frac{1}{4}\lambda^2 y_0 + \exp(\frac{1}{2}\lambda t) w(t,x_0) \\ f_j(t,\vec{y}) = 10^{-6} g h_j\left[y_{j-1} - 2y_j + y_{j+1}\right] + \frac{1}{4}\lambda^2 y_j + \exp(\frac{1}{2}\lambda t) w(t,x_j), \\ \qquad\qquad j = 1,\ldots 99 \\ f_{100}(t,\vec{y}) = 10^{-6} g h_{100}\left[2y_{99} - 2y_{100}\right] + \frac{1}{4}\lambda^2 y_{100} + \exp(\frac{1}{2}\lambda t) w(t,x_{100}) \end{cases}$$

Reference points: $t = 2160/\sigma_P$ (parabolic case)

$t = 160/\sqrt{\sigma_H}$ (hyperbolic case).

Reference solution: see appendix.

Spectrum of the Jacobian matrix of $\vec{f}(t,\vec{y})$

$$-\sigma_P = -\sigma_H = -12_{10}{}^{-4} < \delta < 0$$

6.2 Numerical results

We have integrated all problems with a constant stepsize ranging from $216/\sigma_P$ to $6.75/\sigma_P$ for the parabolic problems and from $16/\sqrt{\sigma_H}$ to $2/\sqrt{\sigma_H}$ for the hyperbolic ones with the proper adjustment for the degree m. All calculations have been performed on a CYBER 73-28 using 14 significant digits. The results for the parabolic problems are listed in table 6.2.1. An asterisk indicates an unstable result. Because of the rather large number of data, we also present in figure 6.2.1 up to 6.2.6 accuracy (*sd*) versus computational effort (*fe*) diagrams in order to visualize the results more clearly. The following symbols correspond to the methods used

□ : ARK first order (p=1)
○ : M3RK first order (p=1)
Δ : ARK second order (p=2)
+ : M3RK second order (p=2)

A diagram of the results of the problem P_1 is omitted because there was a lack of data due to instability.

In table 6.2.2, the results for the hyperbolic case are given. The number of data points is rather small and moreover, the methods SNRK and CRK are immediately comparable because of the same number of right-hand side evaluations fe. Therefore, diagrams of these results have been omitted.

In the next subsection we will discuss the results and put some remarks on it.

				sd = number of correct significant digits						
Method	m	$h\sigma_p$	fe	P_1	P_2	P_3	P_4	P_5	P_6	P_7
ARK, p=1	11	216	110	0.87	3.15	3.33	4.08	0.03	2.17	0.16
	8	108	160	1.17	3.54	3.67	4.39	0.31	2.50	0.35
	6	54	240	1.50	3.98	3.98	4.69	0.61	2.84	0.60
	4	27	320	1.85	4.35	4.28	4.99	0.90	3.16	0.87
	3	13.5	480	2.19	4.67	4.57	5.28	1.18	3.44	1.15
M3RK, p=1	7	216	70	*	2.41	2.78	3.53	-0.41	1.65	0.03
	5	108	100	0.39	2.93	3.10	3.82	-0.20	2.01	0.11
	4	54	160	0.73	3.48	3.41	4.12	0.06	2.28	0.24
	3	27	240	1.13	3.81	3.71	4.42	0.34	2.58	0.42
	2	13.5	320	1.54	4.11	4.00	4.71	0.63	2.88	0.65
ARK, p=2	12	108	240	1.64	3.71	4.86	5.96	1.87	2.66	1.06
	9	54	360	2.04	4.47	5.67	6.69	2.52	3.58	1.62
	6	27	480	*	5.63	6.68	7.37	3.16	5.03	2.20
	5	13.5	800	2.94	7.17	7.38	8.06	3.76	6.24	2.79
	4	6.75	1280	3.46	7.85	7.95	8.69	4.33	6.80	3.38
M3RK, p=2	7	108	140	1.41	3.74	4.58	6.34	1.36	2.74	0.44
	5	54	200	1.79	5.13	5.43	7.12	1.91	4.07	0.86
	4	27	320	2.19	5.97	6.05	7.82	2.48	5.08	1.37
	3	13.5	480	2.60	6.56	6.65	8.42	3.06	5.69	1.93
	2	6.75	640	3.05	7.15	7.23	9.01	3.63	6.30	2.50

Table 6.2.1

First and second order tests for the parabolic problems

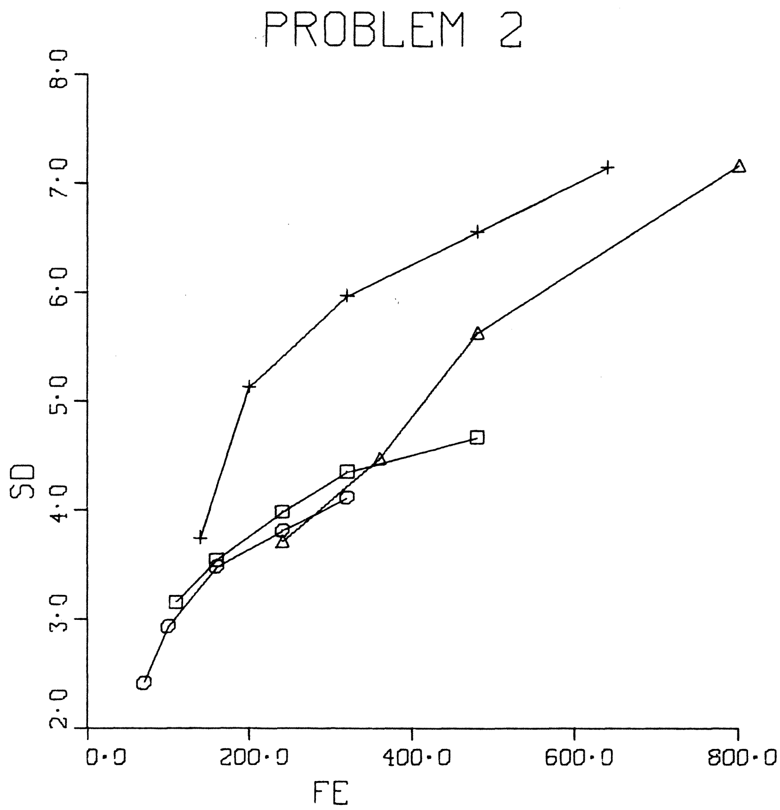


Figure 6.2.1

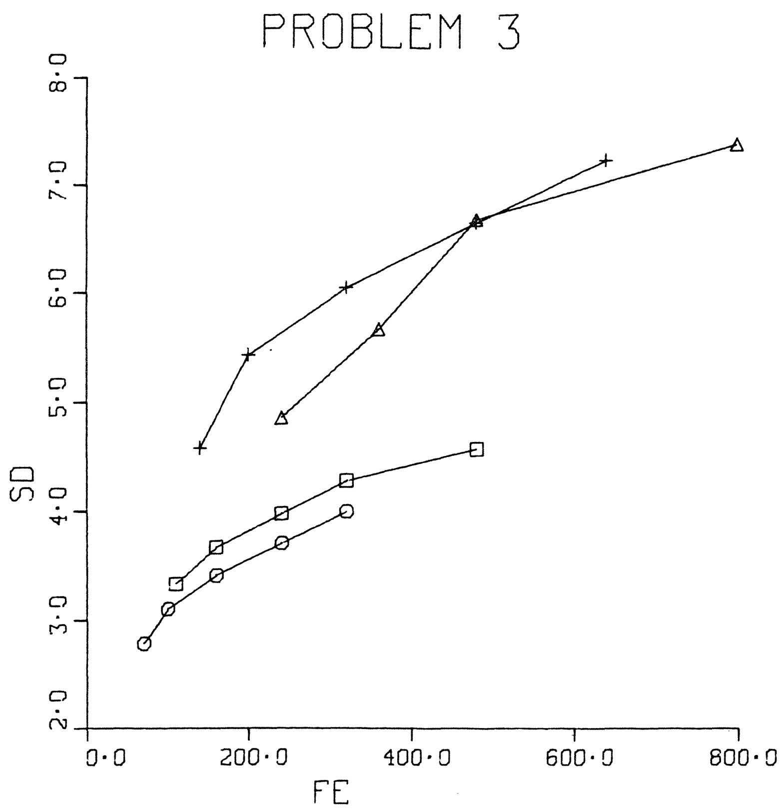


Figure 6.2.2

Figure 6.2.3

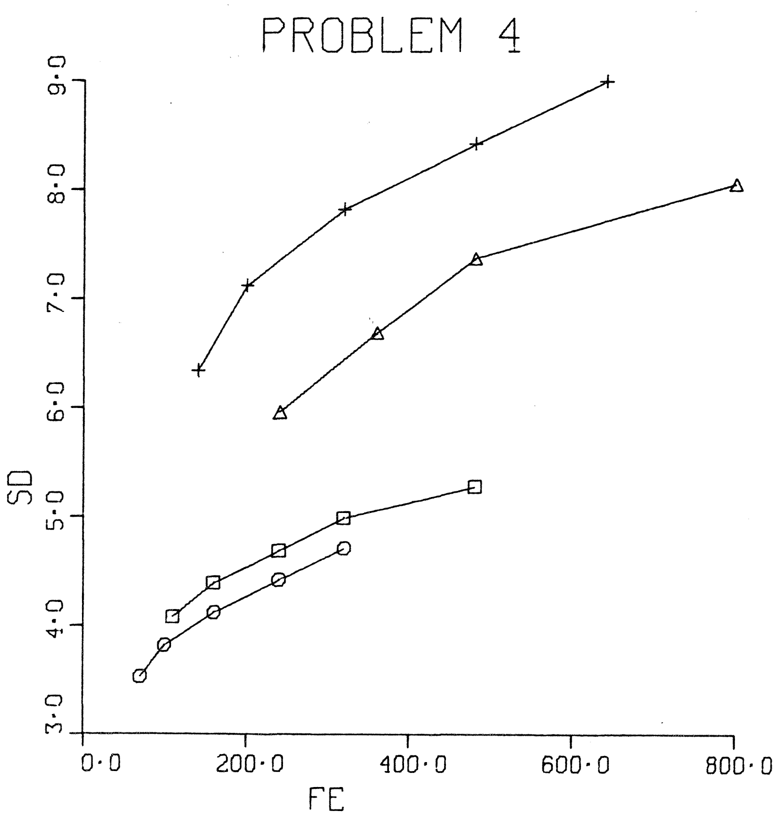
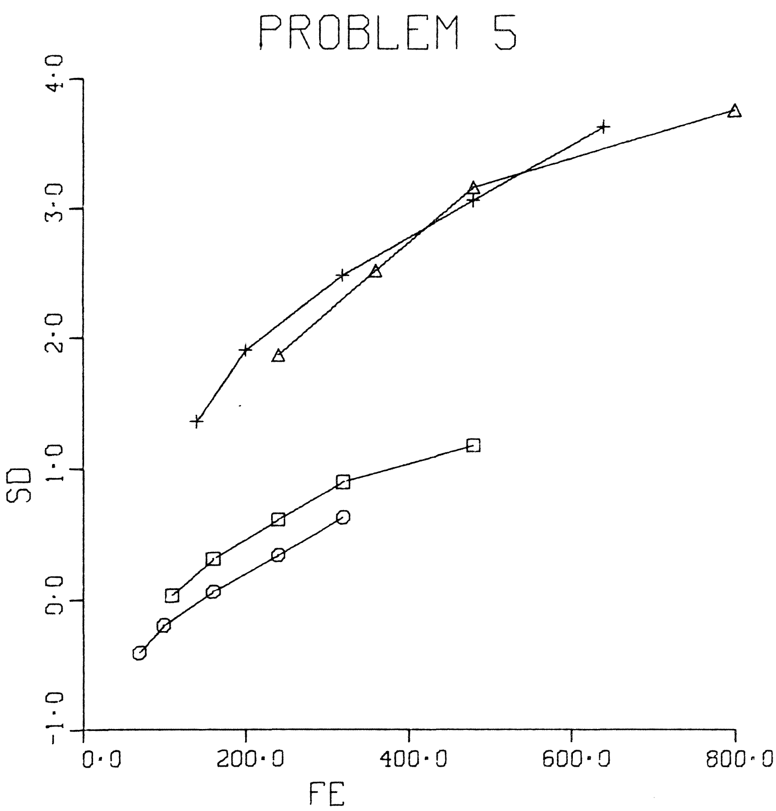


Figure 6.2.4



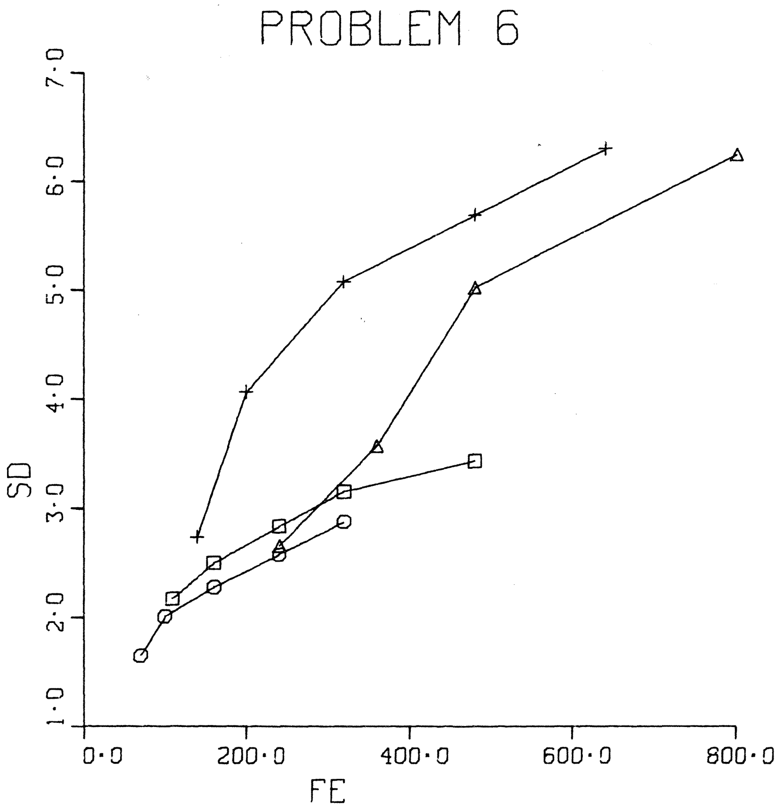


Figure 6.2.5

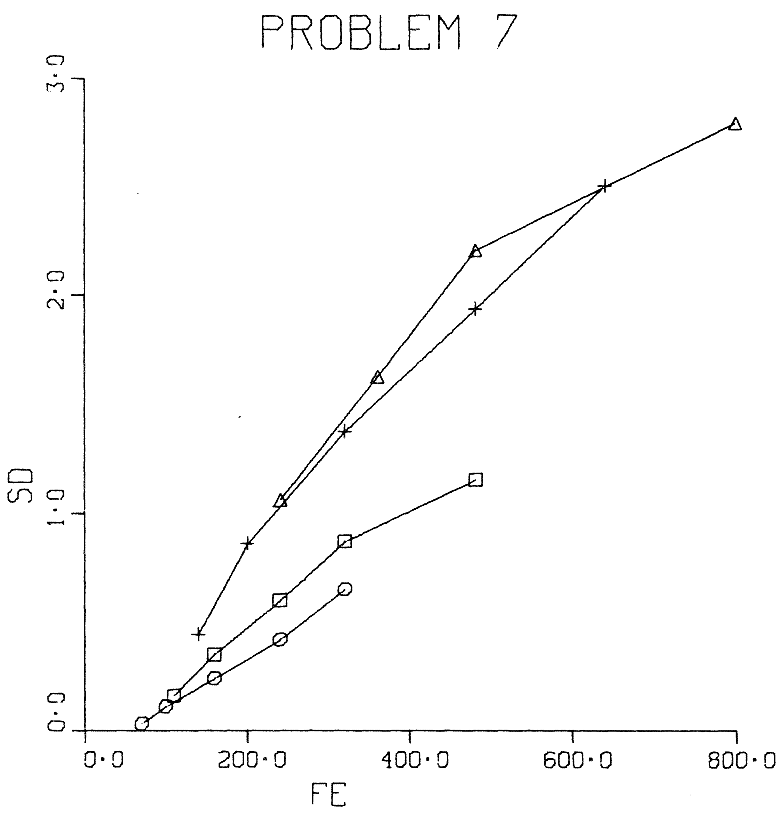


Figure 6.2.6

Method	m	$h\sqrt{\sigma_H}$	fe	sd = number of correct significant digits						
				H_1	H_2	H_3	H_4	H_5	H_6	H_7
SNRK, p=2	9	16	90	*	2.68	3.71	1.39	0.71	2.49	3.80
	5	8	100	*	3.14	4.38	3.31	1.34	3.03	4.46
	3	4	120	*	3.53	4.51	3.86	1.99	4.09	4.91
	2	2	160	*	3.66	4.72	4.36	2.70	4.29	5.18
CRK, p=2	9	16	90	-0.44	3.01	4.05	3.35	2.02	3.61	4.45
	5	8	100	-0.44	3.01	4.05	3.35	2.02	3.61	4.45
	3	4	120	-0.44	3.01	4.05	3.35	2.02	3.61	4.45
	2	2	160	-0.44	3.01	4.05	3.35	2.02	3.61	4.45
ARK, p=2	17	16	170	-1.12	3.03	3.75	1.63	0.85	3.35	4.40
	9	8	180	-1.27	2.89	3.76	2.35	1.56	3.55	4.20
	5	4	200	-1.14	2.98	3.84	3.73	2.50	3.47	4.20
	3	2	240	-0.95	2.97	3.89	3.09	1.96	3.47	4.18

Table 6.2.2

Second order tests for the hyperbolic problems

6.3 Remarks

Problem 1. In the parabolic case, first order M3RK and second order ARK are unstable for m=7 and m=6 respectively. This problem is strongly non-linear, because of the sinh-function appearing in the right-hand side. Within one integration step the Jacobian strongly varies; hence, it is always possible that the linear stability theory on which the methods are based does not hold. This may also explain the bad results obtained in the hyperbolic case.

Problems 2 through 7. For the parabolic problems the methods all performed well. All diagrams give more or less the same picture.

We remark that in a first order accurate scheme the global error is reduced by a factor 0.5 asymptotically when halving the stepsize. Hence, we may expect in our tables of results an increase of $-{}^{10}\log 0.5 = 0.3$ for the number *sd*; for second order schemes this expected increase amounts 0.6. Indeed, table 6.2.1 shows for each method this (almost) constant difference between the numbers *sd* when halving the stepsize. This also indicates that the trunction error of those formulas is more or less independent of the degree m.

For the hyperbolic problems the rather poor performance of second order ARK, when halving the stepsize, is apparent. This is *not* due to its weak stability. In order to demonstrate this we have also solved problem 3 with schemes generated by the strongly stable polynomials (2.12) with ε=0.05. The results are almost of the same accuracy (see table 6.3.1).

	ARK, p=2 weakly stable			ARK, p≈2 strongly stable ε=0.05		
$h\sqrt{\sigma_H}$	sd	fe	m	sd	fe	m
16	3.75	170	17	3.76	180	18
8	3.76	180	9	3.76	200	10
4	3.84	200	5	3.83	240	6
2	3.89	240	3	3.82	320	4

Table 6.3.1

Comparing weakly and strongly stable ARK

Again it appears that the number of significant digits does not increase with a constant. The behaviour of ARK shown in tables 6.2.2 and 6.3.1 may be explained as follows:

When a scheme with stability polynomial

$$P(z) = 1 + z + \beta_2 z^2 + \beta_3 z^3 + \ldots + \beta_m z^m,$$

is applied to a linear differential equation, the local truncation error has the form

$$\rho_n = (\beta_2-\tfrac{1}{2})h_n^2\, y''(x_n) + (\beta_3-\tfrac{1}{6})\,h_n^3\, y'''(x_n) + \dots .$$

So, by inspecting the first coefficients of the stability polynomials (2.10) and (2.12) with ε=0.05 we get an idea of the truncation error as a function of m. In table 6.3.2 the differences $\beta_2-\frac{1}{2}$ and $\beta_3-1/6$ are listed.

weakly stable			strongly stable ε=0.05		
m	$\beta_2-\frac{1}{2}$	$\beta_3-1/6$	m	$\beta_2-\frac{1}{2}$	$\beta_3-1/6$
3	0	64/768	4	3.2/256	64/768
5	0	16/768	6	0.8/256	16/768
9	0	4/768	10	0.2/256	4/768
17	0	1/768	18	0.05/256	1/768

Table 6.3.2

The differences $\beta_2-\frac{1}{2}$ and $\beta_3-1/6$

It is striking that for these second order methods an increase of m (i.e. in our case doubling the stepsize) decreases the differences $\beta_2-\frac{1}{2}$ and $\beta_3-1/6$ with a factor 4. Hence we conclude that for hyperbolic problems second order ARK has a truncation error which strongly depends on m. One might ask whether the methods do converge for fixed m. This we have tested for m=9 (weakly stable) and m=10 (strongly stable). The results, listed in table 6.3.3, show that these methods do converge, and moreover that the strongly stable methods are almost second order accurate as mentioned in section 2.2.

	ARK, p=2, m=9 weakly stable		ARK, p≈2, m=10 strongly stable ε=0.05	
$h\sqrt{\sigma_H}$	sd	fe	sd	fe
8	3.76	180	3.76	200
4	4.35	360	4.32	400
2	4.97	720	4.88	800
1	5.57	1440	5.41	1600

Table 6.3.3

Convergence of weakly and strongly stable ARK

The method SNRK is more efficient than ARK because it takes advantage of the second order form of the system.

As remarked at the end of section 5 the method CRK delivers the same accuracy whenever $h_n/m-1$ = constant. In our case this constant equals 2. This suggests that by choosing $h\sqrt{\sigma_H}=160$ and m=81 we get the same accuracy at the cost of 81 right-hand side evaluations. So it is most efficient to choose $h\sqrt{\sigma_H}$ equal to the whole integration interval with the proper value of m. Increasing the accuracy may be achieved by doubling the value of m-1. This we have tested for all problems (see table 6.3.4).

m	fe	H_1	H_2	H_3	H_4	H_5	H_6	H_7
81	81	-0.44	3.01	4.05	3.35	2.02	3.61	4.45
161	161	0.03	3.62	4.71	3.95	2.62	4.22	5.03

Table 6.3.4

Most efficient value of m for CRK

Comparing table 6.2.2 and 6.3.4 we conclude that at the cost of 160 function evaluations SNRK delivers the most accurate result.

7. CONCLUSIONS

Once more we emphasize that our conclusions are based on the performance of the methods used with a constant stepsize. In the case of *parabolic* systems the results justify the conclusion that on the basis of accuracy and computational effort second order M3RK is to be preferred to second order ARK, and also to first order M3RK and first order ARK. Because of the fact that the stability boundaries of second order M3RK are even larger that the boundaries of first order ARK, an important conclusion is that the second order M3RK schemes are always to be preferred to the ARK schemes. Thus, in spite of the fact that first order ARK does a better job than first order M3RK, the results clearly suggest a combination of the first and second order M3RK schemes. An implementation containing the M3RK schemes provided with steplength, error and order control is discussed in [13].

For *hyperbolic* problems not containing first time derivatives it is our conclusion to use SNRK and CRK rather than ARK. Comparing the results from table 6.2.2 with table 6.3.4, CRK seems to be superior for low accuracy problems, whereas the results for high accuracies are almost identical (except those for H_4). As the SNRK-formulas were specially devised for the second order hyperbolic equations tested in this report, this outcome may seem remarkable. However, the SNRK-formulas have been made strongly stable, at the cost of some efficiency, whereas the CRK schemes are weakly stable. Apparently, the strong stability property did not pay off in the short ranges of integration used in this report. When we had chosen large integration intervals, the SNRK-formulas presumably would have turned out to be superior.

For second order hyperbolic problems containing first time derivatives or first order hyperbolic problems the CRK-formulas are preferred to the ARK-formulas because just like ARK, CRK can handle general hyperbolic systems.

8. ADDITIONAL DATA

In this section, tables with the coefficients of the stability polynomials, as described in section 2 and 3 are listed.

Table 8.1 shows the coefficients β_j and the stability boundary $\beta(m)$ of the polynomials $\tilde{R}_m^{(1)}(z)$ (see (2.4)) for m=2(1)12 and ε=0.05. Since for these polynomials $\beta_0=\beta_1=1$ we have omitted these values. For the second order polynomials $\tilde{R}_m^{(2)}(z)$ we have $\beta_0=\beta_1=1$, $\beta_2=0.5$. The remaining coefficients are listed in table 8.2 together with the stability boundary $\beta(m)$ for m=3(1)12 and ε=0.05. Table 8.3 gives the coefficients (depending on ε) of the polynomials $\tilde{I}_m^{(1)}(z)$ (see (2.12)) for m=4(2)18. Taking $\varepsilon=0$ gives the coefficients of $I_m^{(2)}(z)$ (see (2.10)). Finally, in table 8.4, the coefficients of S(z) and P(z) from section 3 are listed.

M	BETA(M)	BETA[2]	BETA[3]	BETA[4]
2	7.750	+1.2820512820617"-001		
3	17.416	+1.5209292726994"-001	+5.8052440085545"-003	
4	30.948	+1.6046454425158"-001	+8.2716451387542"-003	+1.3341922092807"-004
5	48.347	+1.6434132212715"-001	+9.4897595258066"-003	+2.2395693086330"-004
6	69.613	+1.6644773456389"-001	+1.0171773332039"-002	+2.8133982060361"-004
7	94.744	+1.6771800786803"-001	+1.0589924474355"-002	+3.1882806452898"-004
8	123.743	+1.6854253378190"-001	+1.0864106957532"-002	+3.4434494934635"-004
9	156.607	+1.6910785773354"-001	+1.1053353190742"-002	+3.6238506226717"-004
10	193.339	+1.6951224645193"-001	+1.1189352606116"-002	+3.7556364956917"-004
11	233.936	+1.6981145686352"-001	+1.1290316477804"-002	+3.8546256629880"-004
12	278.400	+1.7003903575641"-001	+1.1367301227315"-002	+3.9307627607548"-004

M	BETA[5]	BETA[6]	BETA[7]	BETA[8]
5	+1.8509727522243"-006			
6	+3.5886943175509"-006	+1.7171671788509"-008		
7	+4.9303659300018"-006	+3.7818777693508"-008	+1.1398676557772"-010	
8	+5.9303079704441"-006	+5.6596912265596"-008	+2.8131199532098"-010	+5.6810883898329"-013
9	+6.6777922131312"-006	+7.2306758439164"-008	+4.5638991289791"-010	+1.5536215824106"-012
10	+7.2444547384024"-006	+8.5097856886020"-008	+6.1878099960337"-010	+2.7193189538714"-012
11	+7.6813111929097"-006	+9.5450739256292"-008	+7.6182931221042"-010	+3.9062882416951"-012
12	+8.0237680891362"-006	+1.0385461298861"-007	+8.8498679556826"-010	+5.0310926198520"-012

M	BETA[9]	BETA[10]	BETA[11]	BETA[12]
9	+2.2038421202938"-015			
10	+6.6167200170850"-015	+6.8429116974935"-018		
11	+1.2437708469848"-014	+2.2380308879804"-017	+1.7390513450627"-020	
12	+1.8892148379302"-014	+4.4989942182276"-017	+6.1549280112014"-020	+3.6840333830885"-023

Table 8.1 The coefficients β_j of $\tilde{R}_m^{(1)}(z)$ for m=2(1)12 and ε=0.05

M	BETA(M)	BETA[3]	BETA[4]	BETA[5]
3	6.143	+6.3304084512584"-002		
4	11.830	+7.9027358132064"-002	+3.7089254054902"-003	
5	19.113	+8.5605574906925"-002	+5.6773922650232"-003	+1.2758716312741"-004
6	28.004	+8.9018495840253"-002	+6.7947003356806"-003	+2.3143225848278"-004
7	38.508	+9.1025407861642"-002	+7.4822425397712"-003	+3.0567579675553"-004
8	50.625	+9.2308223521552"-002	+7.9335425847122"-003	+3.5849723292872"-004
9	64.357	+9.3178948049023"-002	+8.2451157475697"-003	+3.9680344805112"-004
10	79.703	+9.3797465070816"-002	+8.4690175645908"-003	+4.2524183220417"-004
11	96.664	+9.4252811342892"-002	+8.6352191195968"-003	+4.4683801692209"-004
12	115.240	+9.4597848198212"-002	+8.7619296097032"-003	+4.6357872334261"-004

M	BETA[6]	BETA[7]	BETA[8]	BETA[9]
6	+2.8983882217730"-006			
7	+6.0720005319056"-006	+4.6793232974677"-008		
8	+8.8001607205862"-006	+1.1108474302854"-007	+5.6487792519071"-010	
9	+1.1000300874767"-005	+1.7552366748626"-007	+1.4976733974125"-009	+5.2933108206428"-012
10	+1.2746944654676"-005	+2.3355061911107"-007	+2.5642830688663"-009	+1.5495966602985"-011
11	+1.4135169837555"-005	+2.8359078571423"-007	+3.6242801950544"-009	+2.8591261789642"-011
12	+1.5246909932379"-005	+3.2599753732482"-007	+4.6107926948091"-009	+4.2819928010952"-011

M	BETA[10]	BETA[11]	BETA[12]
10	+3.9628079917375"-014		
11	+1.2691525551095"-013	+2.4249736692668"-016	
12	+2.5110371211782"-013	+8.4319465113826"-016	+1.2357104763978"-018

Table 8.2 The coefficients β_j of $\tilde{R}_m^{(2)}(z)$ for m=3(1)12 and ε= 0.05

$\beta(m)=m-2$	$\beta_3*\beta^2$	$\beta_3*\beta^2$	$\beta_4*\beta^4$	$\beta_5*\beta^4$	$\beta_6*\beta^6$	$\beta_7*\beta^6$	$\beta_8*\beta^8$	$\beta_9*\beta^8$	$\beta_{10}*\beta^{10}$
m=4	$2+\varepsilon$	1	2ε						
m=6	$8+\varepsilon$	3	$8+8\varepsilon$	2	8ε				
m=8	$18+\varepsilon$	19/3	$48+18\varepsilon$	32/3	$32+48\varepsilon$	16/3	32ε		
m=10	$32+\varepsilon$	11	$160+32\varepsilon$	34	$256+160\varepsilon$	40	$128+256\varepsilon$	16	128ε
m=12	$50+\varepsilon$	17	$400+50\varepsilon$	416/5	$1120+400\varepsilon$	848/5	$1280+1120\varepsilon$	768/5	$512+1280\varepsilon$
m=14	$72+\varepsilon$	73/3	$840+72\varepsilon$	518/3	$3584+840\varepsilon$	1600/3	$6912+3584\varepsilon$	2432/3	$6144+6912\varepsilon$
m=16	$98+\varepsilon$	33	$1568+98\varepsilon$	320	$9408+1568\varepsilon$	9696/7	$26880+9408\varepsilon$	21760/7	$39424+26880\varepsilon$
m=18	$128+\varepsilon$	43	$2688+128\varepsilon$	546	$21504+2688\varepsilon$	3144	$84480+21504\varepsilon$	9680	$180224+84480\varepsilon$

Table 8.3

The coefficients of $\tilde{I}_m^{(1)}(z)$ for m=4(2)18

$\beta(m)=m-2$	$\beta_{11}*\beta^{10}$	$\beta_{12}*\beta^{12}$	$\beta_{13}*\beta^{12}$	$\beta_{14}*\beta^{14}$	$\beta_{15}*\beta^{14}$	$\beta_{16}*\beta^{16}$	$\beta_{17}*\beta^{16}$	$\beta_{18}*\beta^{18}$
m=12	256/5	512ε						
m=14	1792/3	$2048+6144\varepsilon$	512/3	2048ε				
m=16	26368/7	$28672+39424\varepsilon$	16384/7	$8192+28672\varepsilon$	4096/7	8192ε		
m=18	17024	$212992+180224\varepsilon$	17152	$131072+212992\varepsilon$	9216	$32768+131072\varepsilon$	2048	32768ε

Table 8.3 (continued)

The coefficients of $\tilde{I}_m^{(1)}(z)$ for m=12(2)18

```
P = 1     M =  2                                   P = 2     M =  2

P 0 = .545454545454540E+00 S 0 = .454545454545450E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
```

```
P = 1     M =  2                                   P = 2     M =  2

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32974222139503E+00 S 1 = .39753050587769E+00 P 1 = -.21559395174672E+00 S 1 = .15059165323919E+01
P 2 = .15874540963720E-01 S 2 = .19106034236683E-01 P 2 = -.30544696579492E-01 S 2 = .16978945451019E+00

P = 1     M =  3                                   P = 2     M =  3

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32957779372404E+00 S 1 = .39769493354868E+00 P 1 = -.19115223031554E+00 S 1 = .14814748109607E+01
P 2 = .18807742712872E-01 S 2 = .22675100922608E-01 P 2 = -.29473834786102E-01 S 2 = .19316031414798E+00
P 3 = .26931406295668E-03 S 3 = .32438614977749E-03 P 3 = -.10066347258135E-02 S 3 = .62334163398843E-02

P = 1     M =  4                                   P = 2     M =  4

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32959633626966E+00 S 1 = .39767639100306E+00 P 1 = -.18233307297138E+00 S 1 = .14726556536165E+01
P 2 = .19843848141588E-01 S 2 = .23921246939481E-01 P 2 = -.28236544473632E-01 S 2 = .20074218117966E+00
P 3 = .38370516974646E-03 S 3 = .46221334545758E-03 P 3 = -.12030039601232E-02 S 3 = .86336976630508E-02
P 4 = .23214008199836E-05 S 4 = .27945492539796E-05 P 4 = -.15208008334815E-04 S 4 = .11558084587197E-03

P = 1     M =  5                                   P = 2     M =  5

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32940480780399E+00 S 1 = .39786791946874E+00 P 1 = -.17833069941496E+00 S 1 = .14686532800601E+01
P 2 = .20289622974884E-01 S 2 = .24509754044867E-01 P 2 = -.28475246894827E-01 S 2 = .20498325715728E+00
P 3 = .43922728369494E-03 S 3 = .53071848010798E-03 P 3 = -.14606302030482E-02 S 3 = .99938118863291E-02
P 4 = .38881393659393E-05 S 4 = .47002589400275E-05 P 4 = -.29964111364600E-04 S 4 = .19922348036296E-03
P 5 = .12058633806519E-07 S 5 = .14585303356181E-07 P 5 = -.21343804115613E-06 S 5 = .13919201448922E-05

P = 1     M =  6                                   P = 2     M =  6

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32915730747582E+00 S 1 = .39811541979691E+00 P 1 = -.17610367098315E+00 S 1 = .14664262516283E+01
P 2 = .20529934599533E-01 S 2 = .24864589588956E-01 P 2 = -.28260676645090E-01 S 2 = .20699571533935E+00
P 3 = .47014565070180E-03 S 3 = .56998860293388E-03 P 3 = -.15322458810336E-02 S 3 = .10680275405545E-01
P 4 = .48729959290595E-05 S 4 = .59138340603510E-05 P 4 = -.36586320114212E-04 S 4 = .24933405067263E-03
P 5 = .23293337843875E-07 S 5 = .28300608926316E-07 P 5 = -.39867529427935E-06 S 5 = .26855039111666E-05
P 6 = .41768893290961E-10 S 6 = .50812598486162E-10 P 6 = -.16226665135199E-08 S 6 = .10854850713601E-07

P = 1     M =  7                                   P = 2     M =  7

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32940290556199E+00 S 1 = .39786982171073E+00 P 1 = -.17483390114911E+00 S 1 = .14651564817943E+01
P 2 = .20695785946957E-01 S 2 = .25000002282553E-01 P 2 = -.27741186226246E-01 S 2 = .20774599475456E+00
P 3 = .48959305208277E-03 S 3 = .59148858437864E-03 P 3 = -.14815956989918E-02 S 3 = .10971861052267E-01
P 4 = .55250561672575E-05 S 4 = .66757562996758E-05 P 4 = -.36301045393729E-04 S 4 = .27524368830002E-03
P 5 = .32042572866540E-07 S 5 = .38720468964770E-07 P 5 = -.44685007550778E-06 S 5 = .35403656934423E-05
P 6 = .92217187087773E-10 S 6 = .11144761163396E-09 P 6 = -.26875584507281E-08 S 6 = .22575321236761E-07
P 7 = .10431596770258E-12 S 7 = .12608153728000E-12 P 7 = -.62831231083352E-11 S 7 = .56574487882585E-10

P = 1     M =  8                                   P = 2     M =  8

P 0 = .54545454545454E+00 S 0 = .45454545454545E+00 P 0 = -.58064516129032E+00 S 0 = .15806451612903E+01
P 1 = .32904622047855E+00 S 1 = .39822650679417E+00 P 1 = -.17396071288671E+00 S 1 = .14642832935319E+01
P 2 = .20769127247467E-01 S 2 = .25183525013803E-01 P 2 = -.28684134226593E-01 S 2 = .20956213101730E+00
P 3 = .50144873108237E-03 S 3 = .60880314013113E-03 P 3 = -.17432495919146E-02 S 3 = .11509566107375E-01
P 4 = .59538312498493E-05 S 4 = .72358433014328E-05 P 4 = -.50845860092993E-04 S 4 = .31097988163828E-03
P 5 = .38413639404624E-07 S 5 = .46725317100106E-07 P 5 = -.79137773126678E-06 S 5 = .45630986133302E-05
P 6 = .13735161731261E-09 S 6 = .16719594503501E-09 P 6 = -.67374753547729E-08 S 6 = .37079641130883E-07
P 7 = .25579038177072E-12 S 7 = .31157527742621E-12 P 7 = -.29589252318632E-10 S 7 = .15682590950194E-09
P 8 = .19355325549260E-15 S 8 = .23590370482110E-15 P 8 = -.52416294063507E-13 S 8 = .26933430035588E-12
```

Table 8.4 The coefficients of P(z) and S(z) for m=2(1)8

Table 8.4 The coefficients of P(z) and S(z) for m=9(1)12 (cont'd)

P = 1	M = 9	P = 2	M = 9
P 0 = .54545454545454E+00	S 0 = .45454545454545E+00	P 0 = -.58064516129032E+00	S 0 = .15806451612903E+01
P 1 = .32902403825404E+00	S 1 = .39824868901869E+00	P 1 = -.17339878373842E+00	S 1 = .14637213643836E+01
P 2 = .20835887364795E-01	S 2 = .25265819897767E-01	P 2 = -.28044727272007E-01	S 2 = .20948465321101E+00
P 3 = .51019371265378E-03	S 3 = .61910411544859E-03	P 3 = -.16253143532453E-02	S 3 = .11536416747709E-01
P 4 = .62671263848834E-05	S 4 = .76074711148850E-05	P 4 = -.45810065724231E-04	S 4 = .31784614870548E-03
P 5 = .43273294211670E-07	S 5 = .52536344101559E-07	P 5 = -.71155788003856E-06	S 5 = .49116720047009E-05
P 6 = .17557956622083E-09	S 6 = .21317642667569E-09	P 6 = -.64049072748140E-08	S 6 = .44528196080481E-07
P 7 = .41529099936815E-12	S 7 = .50421345500406E-12	P 7 = -.33259355915939E-10	S 7 = .23505328331393E-09
P 8 = .52977760638010E-15	S 8 = .64317688539028E-15	P 8 = -.92392278190522E-13	S 8 = .66870118493809E-12
P 9 = .28162396623902E-18	S 9 = .34187163161474E-18	P 9 = -.10625147968957E-15	S 9 = .79239438466385E-15

P = 1	M = 10	P = 2	M = 10
P 0 = .54545454545454E+00	S 0 = .45454545454545E+00	P 0 = -.58064516129032E+00	S 0 = .15806451612903E+01
P 1 = .32900701770523E+00	S 1 = .39826570956750E+00	P 1 = -.17144173377009E+00	S 1 = .14617643144152E+01
P 2 = .20847709891773E-01	S 2 = .25287844091655E-01	P 2 = -.28015092938518E-01	S 2 = .21141206884584E+00
P 3 = .51474022625718E-03	S 3 = .62494425734623E-03	P 3 = -.16157222373633E-02	S 3 = .11805056288024E-01
P 4 = .64655444450446E-05	S 4 = .78540296938450E-05	P 4 = -.46227724829057E-04	S 4 = .33440743994479E-03
P 5 = .46694681553560E-07	S 5 = .56743151456408E-07	P 5 = -.75479222345544E-06	S 5 = .54414235293880E-05
P 6 = .20545752812528E-09	S 6 = .24973869169769E-09	P 6 = -.74894835017511E-08	S 6 = .53918086434111E-07
P 7 = .55985683650846E-12	S 7 = .68066567167935E-12	P 7 = -.45979053535837E-10	S 7 = .33075450191903E-09
P 8 = .92238940881933E-15	S 8 = .11216260688042E-14	P 8 = -.17060137796770E-12	S 8 = .12264057386754E-11
P 9 = .84171480799272E-18	S 9 = .10236802978533E-17	P 9 = -.35056663086597E-15	S 9 = .25181064036724E-14
P10 = .32655765072494E-21	S10 = .39720657964596E-21	P10 = -.30628886618191E-18	S10 = .21977616072810E-17

P = 1	M = 11	P = 2	M = 11
P 0 = .54545454545454E+00	S 0 = .45454545454545E+00	P 0 = -.58064516129032E+00	S 0 = .15806451612903E+01
P 1 = .32923047518706E+00	S 1 = .39804225208567E+00	P 1 = -.17300284166294E+00	S 1 = .14633254223081E+01
P 2 = .20942105514450E-01	S 2 = .25341708058167E-01	P 2 = -.27328009024214E-01	S 2 = .20916387703869E+00
P 3 = .52155114179973E-03	S 3 = .63151501325496E-03	P 3 = -.15056021545051E-02	S 3 = .11579454550239E-01
P 4 = .66698403229501E-05	S 4 = .80807554649258E-05	P 4 = -.41367453292114E-04	S 4 = .32857305529008E-03
P 5 = .49787290971999E-07	S 5 = .60354664402269E-07	P 5 = -.65884289780074E-06	S 5 = .54458388015784E-05
P 6 = .23175109073032E-09	S 6 = .28111642092611E-09	P 6 = -.65536146317341E-08	S 6 = .56371977519294E-07
P 7 = .69290153772379E-12	S 7 = .84106334638657E-12	P 7 = -.42091025159364E-10	S 7 = .37545083692440E-09
P 8 = .13309584971259E-14	S 8 = .16167280397671E-14	P 8 = -.17479977056317E-12	S 8 = .16091005950325E-11
P 9 = .15876152476718E-17	S 9 = .19299941789907E-17	P 9 = -.45368406820998E-15	S 9 = .42884398776166E-14
P10 = .10702794409632E-20	S10 = .13021732956413E-20	P10 = -.66928703208412E-18	S10 = .64665832685414E-17
P11 = .31159779644428E-24	S11 = .37944453664700E-24	P11 = -.42844311155752E-21	S11 = .42148457924105E-20

P = 1	M = 12	P = 2	M = 12
P 0 = .54545454545454E+00	S 0 = .45454545454545E+00	P 0 = -.58064516129032E+00	S 0 = .15806451612903E+01
P 1 = .32888824622955E+00	S 1 = .39838448104318E+00	P 1 = -.17229366960984E+00	S 1 = .14626162502550E+01
P 2 = .20930564082556E-01	S 2 = .25416762495946E-01	P 2 = -.28800272381574E-01	S 2 = .21134531244915E+00
P 3 = .52398048369937E-03	S 3 = .63697712076076E-03	P 3 = -.18596174430420E-02	S 3 = .12108121568554E-01
P 4 = .67865318951772E-05	S 4 = .82551983508228E-05	P 4 = -.59976978730906E-04	S 4 = .35894153745450E-03
P 5 = .51892263386503E-07	S 5 = .63149729888770E-07	P 5 = -.11090566945359E-05	S 5 = .62664422624671E-05
P 6 = .25160762750795E-09	S 6 = .30629858708888E-09	P 6 = -.12745935692238E-07	S 6 = .69193362616297E-07
P 7 = .80319349035922E-12	S 7 = .97808681916440E-12	P 7 = -.95158694657869E-10	S 7 = .50195179261835E-09
P 8 = .17105594522426E-14	S 8 = .20836563130596E-14	P 8 = -.46969406476975E-12	S 8 = .24253292036317E-11
P 9 = .24063258323529E-17	S 9 = .29320679574331E-17	P 9 = -.15218064136528E-14	S 9 = .77310365461803E-14
P10 = .21467888957759E-20	S10 = .26166497017204E-20	P10 = -.31130952108508E-17	S10 = .15613921810526E-16
P11 = .11002739569540E-23	S11 = .13415328284784E-23	P11 = -.36466961532119E-20	S11 = .18102819599155E-19
P12 = .24672233557657E-27	S12 = .30092796196223E-27	P12 = -.18644934368193E-23	S12 = .91776107476727E-23

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APPENDIX

Finally we give the reference solutions of the parabolic and hyperbolic problems 1 through 7 at the reference points.

As already mentioned in section 6 we have computed the reference solutions by means of a Runge-Kutta method with extremely small stepsize, i.e. by specifying a small error tolerance in the integration routine. In order to get an idea of the precision delivered a number of different tolerances has been specified. On the basis of our experiments we conjecture that the numbers in the next table give a lower bound of the number of correct significant digits in each component of the reference solution.

problem	parabolic	hyperbolic
1	7	4
2	10	6
3	10	8
4	9	5
5	7	7
6	9	7
7	10	7

REFERENCE SOLUTION OF PROBLEM 1 (PARABOLIC)
NUMBER OF COMPONENTS = 99

+8.2273323962316"-003	+1.6441838132550"-002	+2.4630695231499"-002
+3.2781092750105"-002	+4.0880240422760"-002	+4.8915383118601"-002
+5.6873821488260"-002	+6.4742939972941"-002	+7.2510243092344"-002
+8.0163400617283"-002	+8.7690301870764"-002	+9.5079118994208"-002
+1.0231837857454"-001	+1.0939704056835"-001	+1.1630458300211"-001
+1.2303109049638"-001	+1.2956734428482"-001	+1.3590491110221"-001
+1.4203622812659"-001	+1.4795468109884"-001	+1.5365467282309"-001
+1.5913167947764"-001	+1.6438229253159"-001	+1.6940424455093"-001
+1.7419641776078"-001	+1.7875883487422"-001	+1.8309263236185"-001
+1.8720001697953"-001	+1.9108420695701"-001	+1.9474935974377"-001
+1.9820048858815"-001	+2.0144337047652"-001	+2.0448444807843"-001
+2.0733072833565"-001	+2.0998968021343"-001	+2.1246913391916"-001
+2.1477718361061"-001	+2.1692209528726"-001	+2.1891222120681"-001
+2.2075592181497"-001	+2.2246149583903"-001	+2.2403711888520"-001
+2.2549079060755"-001	+2.2683029028643"-001	+2.2806314046893"-001
+2.2919657818239"-001	+2.3023753313080"-001	+2.3119261221991"-001
+2.3206808972363"-001	+2.3286990239785"-001	+2.3360364886171"-001
+2.3427459259625"-001	+2.3488766795184"-001	+2.3544748860467"-001
+2.3595835795569"-001	+2.3642428102030"-001	+2.3684897741177"-001
+2.3723589507393"-001	+2.3758822446844"-001	+2.3790891296817"-001
+2.3820067925009"-001	+2.3846602751960"-001	+2.3870726143205"-001
+2.3892649760945"-001	+2.3912567868013"-001	+2.3930658580333"-001
+2.3947085068385"-001	+2.3961996715206"-001	+2.3975530250997"-001
+2.3987810908313"-001	+2.3998953687662"-001	+2.4009064910295"-001
+2.4018244395864"-001	+2.4026588892366"-001	+2.4034197891321"-001
+2.4041183814029"-001	+2.4047689942994"-001	+2.4053921645892"-001
+2.4060199701334"-001	+2.4067049207623"-001	+2.4075343902825"-001
+2.4086533831801"-001	+2.4102993903852"-001	+2.4128541222698"-001
+2.4169178785726"-001	+2.4234130511755"-001	+2.4337236234380"-001
+2.4498776145880"-001	+2.4747798692138"-001	+2.5125052502397"-001
+2.5686713518204"-001	+2.6509342701238"-001	+2.7697103947266"-001
+2.9393686765571"-001	+3.1804898486636"-001	+3.5247603538658"-001
+4.0272207695651"-001	+4.8035366667801"-001	+6.1866151877797"-001

REFERENCE SOLUTION OF PROBLEM 1 (HYPERBOLIC)
NUMBER OF COMPONENTS = 99

-3.2355180000000"-004	-7.0690450000000"-004	-1.2096676000000"-003
-1.8910687000000"-003	-2.8097695000000"-003	-4.0236883000000"-003
-5.5898330000000"-003	-7.5641494000000"-003	-1.0001386000000"-002
-1.2954981300000"-002	-1.6476975300000"-002	-2.0617946600000"-002
-2.5426977000000"-002	-3.0951639200000"-002	-3.7238001900000"-002
-4.4330642500000"-002	-5.2272649700000"-002	-6.1105593200000"-002
-7.0869427900000"-002	-8.1602288100000"-002	-9.3340112100000"-002
-1.0611601750000"-001	-1.1995931970000"-001	-1.3489404400000"-001
-1.5093673210000"-001	-1.6809325990000"-001	-1.8635428250000"-001
-2.0568877680000"-001	-2.2603496910000"-001	-2.4728772770000"-001
-2.6928131050000"-001	-2.9176632690000"-001	-3.1438017960000"-001
-3.3661166340000"-001	-3.5776365050000"-001	-3.7692380110000"-001
-3.9296177560000"-001	-4.0457876190000"-001	-4.1043161820000"-001
-4.0932756520000"-001	-4.0043749220000"-001	-3.8343826310000"-001
-3.5851180560000"-001	-3.2620351350000"-001	-2.8721498880000"-001
-2.4221949920000"-001	-1.9174898320000"-001	-1.3615443140000"-001
-7.5616991500000"-002	-1.0185746200000"-002	+6.0171184800000"-002
+1.3549915850000"-001	+2.1574339360000"-001	+3.0050007370000"-001
+3.8839254210000"-001	+4.7549751390000"-001	+5.5197232160000"-001
+5.9846173940000"-001	+5.9309875240000"-001	+5.3290139780000"-001
+4.3513826250000"-001	+3.1639876450000"-001	+1.8462526670000"-001
+4.2383210200000"-002	-1.0981037360000"-001	-2.7176505650000"-001
-4.4147370010000"-001	-6.0561622620000"-001	-7.0555144030000"-001
-6.5037580520000"-001	-4.7813555390000"-001	-2.6529239180000"-001
-3.3982243300000"-002	+2.1285363110000"-001	+4.7296084130000"-001
+7.1470444290000"-001	+7.5272243150000"-001	+5.1548968380000"-001
+2.0233534760000"-001	-1.3543423810000"-001	-4.9223964870000"-001
-8.0377512420000"-001	-6.9903996810000"-001	-3.0658727750000"-001
+1.2874665320000"-001	+5.7520044670000"-001	+8.7166704520000"-001
+6.1071245940000"-001	+2.5247720640000"-001	-7.0372490500000"-002
-4.3286960130000"-001	-8.0456915650000"-001	-8.6259933290000"-001
-6.8596852370000"-001	-4.7497452760000"-001	-2.3293080280000"-001
-1.8043386190000"-001	-2.7092922230000"-001	+2.5612229170000"-001

REFERENCE SOLUTION OF PROBLEM 2 (PARABOLIC)
NUMBER OF COMPONENTS = 101

+5.9999482033444"+002	+5.9999478541956"+002	+5.9999468018422"+002
+5.9999450296005"+002	+5.9999425096192"+002	+5.9999392024673"+002
+5.9999350566598"+002	+5.9999300080563"+002	+5.9999239791222"+002
+5.9999168780462"+002	+5.9999085977137"+002	+5.9998990145274"+002
+5.9998879870738"+002	+5.9998753546285"+002	+5.9998609354962"+002
+5.9998445251785"+002	+5.9998258943660"+002	+5.9998047867468"+002
+5.9997809166293"+002	+5.9997539663708"+002	+5.9997235836104"+002
+5.9996893783019"+002	+5.9996509195417"+002	+5.9996077321923"+002
+5.9995592932994"+002	+5.9995050283025"+002	+5.9994443070424"+002
+5.9993764395675"+002	+5.9993006717472"+002	+5.9992161806971"+002
+5.9991220700286"+002	+5.9990173649340"+002	+5.9989010071236"+002
+5.9987718496319"+002	+5.9986286515147"+002	+5.9984700724612"+002
+5.9982946673495"+002	+5.9981008807746"+002	+5.9978870415864"+002
+5.9976513574714"+002	+5.9973919096235"+002	+5.9971066475461"+002
+5.9967933840338"+002	+5.9964497903848"+002	+5.9960733918992"+002
+5.9956615637180"+002	+5.9952115270622"+002	+5.9947203459354"+002
+5.9941849243488"+002	+5.9936020041358"+002	+5.9929681634190"+002
+5.9922798157943"+002	+5.9915332102962"+002	+5.9907244322077"+002
+5.9898494047739"+002	+5.9889038918791"+002	+5.9878835017412"+002
+5.9867836916736"+002	+5.9855997739607"+002	+5.9843269228848"+002
+5.9829601829396"+002	+5.9814944782544"+002	+5.9799246232445"+002
+5.9782453345007"+002	+5.9764512439121"+002	+5.9745369130137"+002
+5.9724968485368"+002	+5.9703255191283"+002	+5.9680173731963"+002
+5.9655668578257"+002	+5.9629684386965"+002	+5.9602166209272"+002
+5.9573059707514"+002	+5.9542311379288"+002	+5.9509868787769"+002
+5.9475680797022"+002	+5.9439697811004"+002	+5.9401872014843"+002
+5.9362157616944"+002	+5.9320511090367"+002	+5.9276891411894"+002
+5.9231260297168"+002	+5.9183582430239"+002	+5.9133825685867"+002
+5.9081961342918"+002	+5.9027964287222"+002	+5.8971813202308"+002
+5.8913490746448"+002	+5.8852983714565"+002	+5.8790283183600"+002
+5.8725384640054"+002	+5.8658288088533"+002	+5.8588998140252"+002
+5.8517524080593"+002	+5.8443879914946"+002	+5.8368084392274"+002
+5.8290161005912"+002	+5.8210137971439"+002	+5.8128048181394"+002
+5.8043929137195"+002	+5.7957822858102"+002	

REFERENCE SOLUTION OF PROBLEM 2 (HYPERBOLIC)
NUMBER OF COMPONENTS = 101

+6.0000000000188"+002	+6.0000000000176"+002	+6.0000000000164"+002
+6.0000000000168"+002	+6.0000000000165"+002	+6.0000000000154"+002
+6.0000000000149"+002	+6.0000000000144"+002	+6.0000000000146"+002
+6.0000000000151"+002	+6.0000000000153"+002	+6.0000000000152"+002
+6.0000000000148"+002	+6.0000000000141"+002	+6.0000000000144"+002
+6.0000000000145"+002	+6.0000000000149"+002	+6.0000000000147"+002
+6.0000000000145"+002	+6.0000000000135"+002	+6.0000000000095"+002
+5.9999999999912"+002	+5.9999999999082"+002	+5.9999999995526"+002
+5.9999999980867"+002	+5.9999999922869"+002	+5.9999999702715"+002
+5.9999998902558"+002	+5.9999996122314"+002	+5.9999986903930"+002
+5.9999957795269"+002	+5.9999870452220"+002	+5.9999622019982"+002
+5.9998954037474"+002	+5.9997261472778"+002	+5.9993234313271"+002
+5.9984273802636"+002	+5.9965719420910"+002	+5.9930168988094"+002
+5.9867576853803"+002	+5.9767164046115"+002	+5.9621923290675"+002
+5.9434966094654"+002	+5.9224152561350"+002	+5.9019281288331"+002
+5.8848391774685"+002	+5.8718715684491"+002	+5.8608297256506"+002
+5.8482657213238"+002	+5.8328395460854"+002	+5.8169308268466"+002
+5.8038932184826"+002	+5.7936103520854"+002	+5.7825217333668"+002
+5.7689702639577"+002	+5.7559291050122"+002	+5.7459150309065"+002
+5.7363693877180"+002	+5.7244958073726"+002	+5.7128482397142"+002
+5.7039707816918"+002	+5.6948301529783"+002	+5.6836824162561"+002
+5.6741337247887"+002	+5.6663302824502"+002	+5.6566112240253"+002
+5.6469714411480"+002	+5.6396701515862"+002	+5.6310630889451"+002
+5.6218598775211"+002	+5.6149866840720"+002	+5.6069771681228"+002
+5.5984477112021"+002	+5.5920949361151"+002	+5.5842669008765"+002
+5.5766572156820"+002	+5.5706826458050"+002	+5.5628658064385"+002
+5.5564923918628"+002	+5.5503208019680"+002	+5.5429728003013"+002
+5.5376581995213"+002	+5.5307970427559"+002	+5.5249099003160"+002
+5.5193499299245"+002	+5.5127823061170"+002	+5.5080202099866"+002
+5.5015440502599"+002	+5.4967686568831"+002	+5.4909944520662"+002
+5.4858275454057"+002	+5.4808629878064"+002	+5.4753635339200"+002
+5.4710145716371"+002	+5.4654016501133"+002	+5.4614334528129"+002
+5.4558884237967"+002	+5.4521479406998"+002	+5.4467576526392"+002
+5.4431823730695"+002	+5.4379599648390"+002	

REFERENCE SOLUTION OF PROBLEM 3 (PARABOLIC)
NUMBER OF COMPONENTS = 100

+4.9898179077413"+001	+4.9801024893183"+001	+4.9708419905282"+001
+4.9620245197788"+001	+4.9536380759538"+001	+4.9456705689357"+001
+4.9381098489653"+001	+4.9309437256426"+001	+4.9241599973759"+001
+4.9177464692812"+001	+4.9116909814976"+001	+4.9059814262349"+001
+4.9006057740575"+001	+4.8955520901002"+001	+4.8908085577652"+001
+4.8863634939441"+001	+4.8822053698454"+001	+4.8783228249651"+001
+4.8747046849599"+001	+4.8713399740883"+001	+4.8682179301569"+001
+4.8653280152187"+001	+4.8626599276617"+001	+4.8602036110397"+001
+4.8579492634126"+001	+4.8558873442504"+001	+4.8540085811825"+001
+4.8523039749695"+001	+4.8507648038490"+001	+4.8493826266345"+001
+4.8481492848506"+001	+4.8470569040807"+001	+4.8460978940881"+001
+4.8452649485734"+001	+4.8445510434875"+001	+4.8439494353876"+001
+4.8434536581406"+001	+4.8430575201068"+001	+4.8427550993962"+001
+4.8425407401570"+001	+4.8424090465045"+001	+4.8423548782264"+001
+4.8423733434625"+001	+4.8424597941722"+001	+4.8426098175800"+001
+4.8428192318097"+001	+4.8430840760173"+001	+4.8434006066657"+001
+4.8437652861666"+001	+4.8441747803450"+001	+4.8446259452911"+001
+4.8451158266688"+001	+4.8456416443140"+001	+4.8462007942025"+001
+4.8467908300962"+001	+4.8474094692232"+001	+4.8480545699628"+001
+4.8487241426053"+001	+4.8494163216707"+001	+4.8501293835299"+001
+4.8508617114519"+001	+4.8516118223840"+001	+4.8523783222286"+001
+4.8531599446990"+001	+4.8539554936103"+001	+4.8547638974953"+001
+4.8555841348230"+001	+4.8564153089766"+001	+4.8572565513091"+001
+4.8581071221331"+001	+4.8589662853854"+001	+4.8598334424695"+001
+4.8607079710355"+001	+4.8615893998035"+001	+4.8624772026483"+001
+4.8633710238395"+001	+4.8642704173134"+001	+4.8651751330331"+001
+4.8660847901163"+001	+4.8669992361257"+001	+4.8679181417523"+001
+4.8688414456239"+001	+4.8697688577015"+001	+4.8707004024375"+001
+4.8716358181537"+001	+4.8725752108597"+001	+4.8735183378655"+001
+4.8744653830680"+001	+4.8754161151415"+001	+4.8763707923556"+001
+4.8773291891605"+001	+4.8782916343520"+001	+4.8792579049296"+001
+4.8802283955451"+001	+4.8812028852251"+001	+4.8821818285056"+001
+4.8831650089530"+001	+4.8841529333026"+001	+4.8851453954641"+001
+4.8861429446976"+001		

REFERENCE SOLUTION OF PROBLEM 3 (HYPERBOLIC)
NUMBER OF COMPONENTS = 100

+4.9692860759935"+001	+4.9388854776442"+001	+4.9088013215281"+001
+4.8790386940331"+001	+4.8496000438545"+001	+4.8204917272732"+001
+4.7917152469737"+001	+4.7632787221142"+001	+4.7351819192428"+001
+4.7074357420117"+001	+4.6800369747656"+001	+4.6530005587219"+001
+4.6263192088866"+001	+4.6000123719090"+001	+4.5740692384822"+001
+4.5485115480642"+001	+4.5233292479916"+001	+4.4985396534546"+001
+4.4741422038569"+001	+4.4501400716440"+001	+4.4265505038167"+001
+4.4033586937815"+001	+4.3805953972615"+001	+4.3582424895890"+001
+4.3363197238666"+001	+4.3148343848627"+001	+4.2937725775605"+001
+4.2731706694479"+001	+4.2530049839511"+001	+4.2332924996413"+001
+4.2140531311288"+001	+4.1952572491139"+001	+4.1769437439078"+001
+4.1591098795731"+001	+4.1417336983854"+001	+4.1248626005471"+001
+4.1084804230417"+001	+4.0925689080143"+001	+4.0771795641002"+001
+4.0622950622756"+001	+4.0478880774284"+001	+4.0340121799463"+001
+4.0206668541745"+001	+4.0078049816003"+001	+3.9954651755185"+001
+3.9836860432372"+001	+3.9724167381830"+001	+3.9616411802095"+001
+3.9514239783490"+001	+3.9417755805218"+001	+3.9326310446802"+001
+3.9239834259160"+001	+3.9159039685291"+001	+3.9084157992484"+001
+3.9014525140244"+001	+3.8949662434514"+001	+3.8890048763220"+001
+3.8836469755538"+001	+3.8788963769291"+001	+3.8746741627816"+001
+3.8709059229947"+001	+3.8675993095523"+001	+3.8648395075327"+001
+3.8627210546792"+001	+3.8612799836408"+001	+3.8604695848803"+001
+3.8601798443523"+001	+3.8602759829849"+001	+3.8606323224266"+001
+3.8611505170532"+001	+3.8617631643808"+001	+3.8624288917633"+001
+3.8631245828718"+001	+3.8638383857583"+001	+3.8645647403959"+001
+3.8653012402468"+001	+3.8660469862636"+001	+3.8668017402634"+001
+3.8675655070506"+001	+3.8683384248585"+001	+3.8691206457499"+001
+3.8699123443260"+001	+3.8707137069846"+001	+3.8715249006002"+001
+3.8723461213601"+001	+3.8731775366356"+001	+3.8740193380387"+001
+3.8748717028733"+001	+3.8757348127285"+001	+3.8766088562125"+001
+3.8774940069093"+001	+3.8783904610526"+001	+3.8792983888823"+001
+3.8802179899996"+001	+3.8811494351132"+001	+3.8820929245680"+001
+3.8830486314452"+001	+3.8840167560334"+001	+3.8849974741334"+001
+3.8859909863869"+001		

REFERENCE SOLUTION OF PROBLEM 4 (PARABOLIC)
NUMBER OF COMPONENTS = 100

+9.7490842297000"-001	+9.7481062150932"-001	+9.7451721743826"-001
+9.7402821189650"-001	+9.7334360674906"-001	+9.7246340460699"-001
+9.7138760882730"-001	+9.7011622351302"-001	+9.6864925351316"-001
+9.6698670442272"-001	+9.6512858258267"-001	+9.6307489507994"-001
+9.6082564974740"-001	+9.5838085516389"-001	+9.5574052065414"-001
+9.5290465628882"-001	+9.4987327288445"-001	+9.4664638200348"-001
+9.4322399595418"-001	+9.3960612779068"-001	+9.3579279131292"-001
+9.3178400106662"-001	+9.2757977234330"-001	+9.2318012118022"-001
+9.1858506436037"-001	+9.1379461941240"-001	+9.0880880461065"-001
+9.0362763897512"-001	+8.9825114227140"-001	+8.9267933501066"-001
+8.8691223844959"-001	+8.8094987459042"-001	+8.7479226618084"-001
+8.6843943671400"-001	+8.6189141042841"-001	+8.5514821230797"-001
+8.4820986808190"-001	+8.4107640422470"-001	+8.3374784795610"-001
+8.2622422724104"-001	+8.1850557078958"-001	+8.1059190805690"-001
+8.0248326924324"-001	+7.9417968529384"-001	+7.8568118789888"-001
+7.7698780949350"-001	+7.6809958325761"-001	+7.5901654311598"-001
+7.4973872373811"-001	+7.4026616053817"-001	+7.3059888967496"-001
+7.2073694805186"-001	+7.1068037331674"-001	+7.0042920386195"-001
+6.8998347882418"-001	+6.7934323808445"-001	+6.6850852226804"-001
+6.5747937274440"-001	+6.4625583162709"-001	+6.3483794177370"-001
+6.2322574678583"-001	+6.1141929100891"-001	+5.9941861953221"-001
+5.8722377818876"-001	+5.7483481355522"-001	+5.6225177295182"-001
+5.4947470444229"-001	+5.3650365683376"-001	+5.2333867967669"-001
+5.0997982326474"-001	+4.9642713863473"-001	+4.8268067756648"-001
+4.6874049258280"-001	+4.5460663694931"-001	+4.4027916467437"-001
+4.2575813050900"-001	+4.1104358994672"-001	+3.9613559922348"-001
+3.8103421531754"-001	+3.6573949594936"-001	+3.5025149958145"-001
+3.3457028541830"-001	+3.1869591340618"-001	+3.0262844423312"-001
+2.8636793932868"-001	+2.6991446086385"-001	+2.5326807175092"-001
+2.3642883564336"-001	+2.1939681693561"-001	+2.0217208076298"-001
+1.8475469300150"-001	+1.6714472026772"-001	+1.4934222991861"-001
+1.3134729005132"-001	+1.1315996950309"-001	+9.4780337851017"-002
+7.6208465411896"-002	+5.7444423242042"-002	+3.8488283137099"-002
+1.9340117631845"-002		

REFERENCE SOLUTION OF PROBLEM 4 (HYPERBOLIC)
NUMBER OF COMPONENTS = 100

+8.5810313736240"-001	+8.5794650979570"-001	+8.5747665781531"-001
+8.5669369403210"-001	+8.5559780267670"-001	+8.5418924158020"-001
+8.5246834208310"-001	+8.5043550891832"-001	+8.4809122006687"-001
+8.4543602658906"-001	+8.4247055242904"-001	+8.3919549419384"-001
+8.3561162090611"-001	+8.3171977373186"-001	+8.2752086568248"-001
+8.2301588129082"-001	+8.1820587626250"-001	+8.1309197710164"-001
+8.0767538071166"-001	+8.0195735397104"-001	+7.9593923328437"-001
+7.8962242410868"-001	+7.8300840045520"-001	+7.7609870436704"-001
+7.6889494537256"-001	+7.6139879991490"-001	+7.5361201075767"-001
+7.4553638636744"-001	+7.3717380027254"-001	+7.2852619039894"-001
+7.1959555838339"-001	+7.1038396886400"-001	+7.0089354874918"-001
+6.9112648646602"-001	+6.8108503119464"-001	+6.7077149210781"-001
+6.6018823767992"-001	+6.4933769526622"-001	+6.3822235154470"-001
+6.2684475543574"-001	+6.1520752760803"-001	+6.0331338613625"-001
+5.9116520850181"-001	+5.7876616778857"-001	+5.6612000393710"-001
+5.5323150789906"-001	+5.4010727992362"-001	+5.2675673236220"-001
+5.1319311742110"-001	+4.9943412320903"-001	+4.8550148097874"-001
+4.7141932370120"-001	+4.5721181238219"-001	+4.4290136485141"-001
+4.2850877305651"-001	+4.1405505897378"-001	+3.9956310303864"-001
+3.8505708573130"-001	+3.7056034140236"-001	+3.5609445419110"-001
+3.4168077813067"-001	+3.2734180216607"-001	+3.1309998248207"-001
+2.9897614931886"-001	+2.8499041321546"-001	+2.7116355603119"-001
+2.5751593020194"-001	+2.4406633808664"-001	+2.3083333938760"-001
+2.1783569933308"-001	+2.0509091330025"-001	+1.9261558767513"-001
+1.8042648825010"-001	+1.6853938881476"-001	+1.5696900776785"-001
+1.4573006033875"-001	+1.3483629154213"-001	+1.2430048185052"-001
+1.1413526718170"-001	+1.0435218309114"-001	+9.4962054192498"-002
+8.5975319761084"-002	+7.7401258043637"-002	+6.9248721457239"-002
+6.1525765115243"-002	+5.4239537175726"-002	+4.7396731524911"-002
+4.1002991035688"-002	+3.5063428684273"-002	+2.9582291721811"-002
+2.4563061649973"-002	+2.0008546153811"-002	+1.5920647351920"-002
+1.2300652752659"-002	+9.1489317979400"-003	+6.4652052466345"-003
+4.2483206473566"-003	+2.4964206497309"-003	+1.2068209243958"-003
+3.7609017179259"-004		

REFERENCE SOLUTION OF PROBLEM 5 (PARABOLIC)
NUMBER OF COMPONENTS = 99

+2.1064820843179"-007	+8.3360687324908"-007	+1.8554528494521"-006
+3.2627889464317"-006	+5.0422440475634"-006	+7.1804732215846"-006
+9.6641578276592"-006	+1.2480005615826"-005	+1.5614750824088"-005
+1.9055154272987"-005	+2.2788003458273"-005	+2.6800112642091"-005
+3.1078322942975"-005	+3.5609502424903"-005	+4.0380546185527"-005
+4.5378376443730"-005	+5.0589942626571"-005	+5.6002221455697"-005
+6.1602217033243"-005	+6.7376960927284"-005	+7.3313512256831"-005
+7.9398957776426"-005	+8.5620411960310"-005	+9.1965017086208"-005
+9.8419943318722"-005	+1.0497238879234"-004	+1.1160957969409"-004
+1.1831877034563"-004	+1.2508724328589"-004	+1.3190230934987"-004
+1.3875130775886"-004	+1.4562160617253"-004	+1.5250060085514"-004
+1.5937571651349"-004	+1.6623440702186"-004	+1.7306415387657"-004
+1.7985247014210"-004	+1.8658689182617"-004	+1.9325499648690"-004
+1.9984436598648"-004	+2.0634265773030"-004	+2.1273747629434"-004
+2.1901659306616"-004	+2.2516759083056"-004	+2.3117840912882"-004
+2.3703655179501"-004	+2.4273017870561"-004	+2.4824668058426"-004
+2.5357444300947"-004	+2.5870077688766"-004	+2.6361422071188"-004
+2.6830211256054"-004	+2.7275301824552"-004	+2.7695442920016"-004
+2.8089483215304"-004	+2.8456194526712"-004	+2.8794414348170"-004
+2.9102936690967"-004	+2.9380591344793"-004	+2.9626189356500"-004
+2.9838558586209"-004	+3.0016521757901"-004	+3.0158909311349"-004
+3.0264552292886"-004	+3.0332286094722"-004	+3.0360948608903"-004
+3.0349381104485"-004	+3.0296427847608"-004	+3.0200936270708"-004
+3.0061756914714"-004	+2.9877743459926"-004	+2.9647752724078"-004
+2.9370644671811"-004	+2.9045282420328"-004	+2.8670532246194"-004
+2.8245263591754"-004	+2.7768349071603"-004	+2.7238664479001"-004
+2.6655088792252"-004	+2.6016504181043"-004	+2.5321796012743"-004
+2.4569852858667"-004	+2.3759566500291"-004	+2.2889831935429"-004
+2.1959547384359"-004	+2.0967614295908"-004	+1.9912937353476"-004
+1.8794424481003"-004	+1.7610986848873"-004	+1.6361538879739"-004
+1.5044998254249"-004	+1.3660285916672"-004	+1.2206326080371"-004
+1.0682046233102"-004	+9.0863771420758"-005	+7.4182528586974"-005
+5.6766107228873"-005	+3.8603913668119"-005	+1.9685387177985"-005

REFERENCE SOLUTION OF PROBLEM 5 (HYPERBOLIC)
NUMBER OF COMPONENTS = 99

-1.9292684648867"-003	-3.5125901566170"-003	-4.7637029964095"-003
-5.6962795216947"-003	-6.3238989307062"-003	-6.6600519631561"-003
-6.7181638127475"-003	-6.5116242974878"-003	-6.0538181436819"-003
-5.3581504389094"-003	-4.4380645151133"-003	-3.3070514657489"-003
-1.9786511408504"-003	-4.6644483394608"-004	+1.2159592757064"-003
+3.0549464186982"-003	+5.0369201223318"-003	+7.1483308691288"-003
+9.3757074922978"-003	+1.1705690022803"-002	+1.4125062944101"-002
+1.6620788327452"-002	+1.9180038577599"-002	+2.1790228383380"-002
+2.4439045273702"-002	+2.7114478252991"-002	+2.9804844311094"-002
+3.2498812898827"-002	+3.5185428519433"-002	+3.7854131432590"-002
+4.0494776291853"-002	+4.3097648507300"-002	+4.5653478266510"-002
+4.8153452350922"-002	+5.0589224014473"-002	+5.2952921177940"-002
+5.5237153064042"-002	+5.7435015244561"-002	+5.9540092979201"-002
+6.1546462735385"-002	+6.3448691868691"-002	+6.5241836558056"-002
+6.6921438169909"-002	+6.8483518238353"-002	+6.9924572198889"-002
+7.1241561933351"-002	+7.2431907117276"-002	+7.3493475342162"-002
+7.4424571026558"-002	+7.5223923219215"-002	+7.5890672502046"-002
+7.6424357280369"-002	+7.6824899767895"-002	+7.7092591917630"-002
+7.7228081425384"-002	+7.7232357773453"-002	+7.7106738138608"-002
+7.6852852913276"-002	+7.6472630616723"-002	+7.5968282105296"-002
+7.5342284186919"-002	+7.4597362931420"-002	+7.3736477063468"-002
+7.2762801775393"-002	+7.1679713110240"-002	+7.0490772819394"-002
+6.9199713418316"-002	+6.7810423160916"-002	+6.6326930858798"-002
+6.4753390797846"-002	+6.3094068270476"-002	+6.1353326274352"-002
+5.9535613686397"-002	+5.7645454860732"-002	+5.5687440400473"-002
+5.3666219021032"-002	+5.1586490887315"-002	+4.9453003217290"-002
+4.7270548929837"-002	+4.5043968661230"-002	+4.2778156034230"-002
+4.0478066171105"-002	+3.8148728092034"-002	+3.5795262143660"-002
+3.3422903310634"-002	+3.1037030490651"-002	+2.8643201626905"-002
+2.6247195324136"-002	+2.3855060060398"-002	+2.1473171324279"-002
+1.9108295985236"-002	+1.6767663461800"-002	+1.4459043714589"-002
+1.2190830844863"-002	+9.9721295463293"-003	+7.8128415596066"-003
+5.7237481417600"-003	+3.7165817722109"-003	+1.8040785740054"-003

REFERENCE SOLUTION OF PROBLEM 6 (PARABOLIC)
NUMBER OF COMPONENTS = 99

+3.5960290599797"-003	+7.2000178193636"-003	+1.0819904405895"-002
+1.4463583992450"-002	+1.8138887787477"-002	+2.1853562577954"-002
+2.5615250994354"-002	+2.9431472655048"-002	+3.3309606333099"-002
+3.7256873271837"-002	+4.1280321757443"-002	+4.5386813037379"-002
+4.9583008653290"-002	+5.3875359236727"-002	+5.8270094795341"-002
+6.2773216497835"-002	+6.7390489946631"-002	+7.2127439909514"-002
+7.6989346465712"-002	+8.1981242506367"-002	+8.7107912517700"-002
+9.2373892563330"-002	+9.7783471374270"-002	+1.0334069244747"-001
+1.0904935704958"-001	+1.1491302801913"-001	+1.2093503425897"-001
+1.2711847581163"-001	+1.3346622941125"-001	+1.3998095440950"-001
+1.4666509897660"-001	+1.5352090648351"-001	+1.6055042197814"-001
+1.6775549867318"-001	+1.7513780437217"-001	+1.8269882776545"-001
+1.9043988453697"-001	+1.9836212322896"-001	+2.0646653081918"-001
+2.1475393797397"-001	+2.2322502394445"-001	+2.3188032108435"-001
+2.4072021896944"-001	+2.4974496810911"-001	+2.5895468324391"-001
+2.6834934622777"-001	+2.7792880850240"-001	+2.8769279316864"-001
+2.9764089667449"-001	+3.0777259012902"-001	+3.1808722027125"-001
+3.2858401010963"-001	+3.3926205926333"-001	+3.5012034403137"-001
+3.6115771721884"-001	+3.7237290775408"-001	+3.8376452012594"-001
+3.9533103367894"-001	+4.0707080179509"-001	+4.1898205100197"-001
+4.3106288003745"-001	+4.4331125890627"-001	+4.5572502796355"-001
+4.6830189705525"-001	+4.8103944474954"-001	+4.9393511769039"-001
+5.0698623009799"-001	+5.2018996345541"-001	+5.3354336638568"-001
+5.4704335478239"-001	+5.6068671215447"-001	+5.7447009029774"-001
+5.8839001018010"-001	+6.0244286324286"-001	+6.1662491286185"-001
+6.3093229638099"-001	+6.4536102711994"-001	+6.5990699728617"-001
+6.7456598039466"-001	+6.8933363529675"-001	+7.0420550871556"-001
+7.1917704078622"-001	+7.3424356717519"-001	+7.4940032676500"-001
+7.6464246251226"-001	+7.7996503220562"-001	+7.9536300689440"-001
+8.1083128588947"-001	+8.2636469141605"-001	+8.4195798901855"-001
+8.5760587729846"-001	+8.7330301434584"-001	+8.8904400230572"-001
+9.0482341915217"-001	+9.2063579950224"-001	+9.3647566904052"-001
+9.5233752517039"-001	+9.6821586925046"-001	+9.8410519249389"-001

REFERENCE SOLUTION OF PROBLEM 6 (HYPERBOLIC)
NUMBER OF COMPONENTS = 99

+1.1747397779825"-002	+2.3495749245596"-002	+3.5245288972849"-002
+4.6997643828667"-002	+5.8752483331204"-002	+7.0511613967335"-002
+8.2275227282718"-002	+9.4044382592178"-002	+1.0581978769726"-001
+1.1760153589981"-001	+1.2939235220400"-001	+1.4118967563427"-001
+1.5299888677668"-001	+1.6481551177646"-001	+1.7664522711725"-001
+1.8848570551991"-001	+2.0033777796538"-001	+2.1220597570512"-001
+2.2408425327235"-001	+2.3598115341914"-001	+2.4789225436516"-001
+2.5981807162137"-001	+2.7176597524317"-001	+2.8372702861044"-001
+2.9570908794749"-001	+3.0771409471684"-001	+3.1973346016427"-001
+3.3177892905039"-001	+3.4384849625138"-001	+3.5593386531723"-001
+3.6804810458757"-001	+3.8018785099167"-001	+3.9233950947961"-001
+4.0451016671826"-001	+4.1669047462303"-001	+4.2884281399505"-001
+4.4093880654926"-001	+4.5294375592131"-001	+4.6478849778312"-001
+4.7641213632514"-001	+4.8780490534487"-001	+4.9898549982985"-001
+5.0997411422634"-001	+5.2079877095016"-001	+5.3148524168007"-001
+5.4202224327828"-001	+5.5236585389047"-001	+5.6249547302553"-001
+5.7244362651618"-001	+5.8225291098683"-001	+5.9192311932356"-001
+6.0142393449040"-001	+6.1074033428582"-001	+6.1987179827792"-001
+6.2880635179474"-001	+6.3754514596232"-001	+6.4613975185553"-001
+6.5465310991706"-001	+6.6310568562432"-001	+6.7150817428519"-001
+6.7989485285106"-001	+6.8827731706183"-001	+6.9663893437900"-001
+7.0498967296762"-001	+7.1334222267000"-001	+7.2167885226984"-001
+7.3000192433902"-001	+7.3832479564927"-001	+7.4663196394012"-001
+7.5492597469322"-001	+7.6321777639238"-001	+7.7149269347859"-001
+7.7975768703648"-001	+7.8801768010957"-001	+7.9626049397867"-001
+8.0449789549679"-001	+8.1272548925608"-001	+8.2093953063015"-001
+8.2914995209444"-001	+8.3734644939111"-001	+8.4553660601586"-001
+8.5371888052987"-001	+8.6189022234255"-001	+8.7005792191614"-001
+8.7821418341352"-001	+8.8636677397402"-001	+8.9451085662284"-001
+9.0264941973650"-001	+9.1078287917715"-001	+9.1890964894717"-001
+9.2703355395180"-001	+9.3515099366097"-001	+9.4326661687912"-001
+9.5137697252386"-001	+9.5948595923974"-001	+9.6759124157494"-001
+9.7569549892042"-001	+9.8379761057633"-001	+9.9189915393677"-001

REFERENCE SOLUTION OF PROBLEM 7 (PARABOLIC)
NUMBER OF COMPONENTS = 101

+6.9911310019131"+010	+7.1395986288793"+010	+7.5283822217562"+010
+8.1113259184214"+010	+8.8507059297582"+010	+9.7156068773767"+010
+1.0680621013217"+011	+1.1724803826228"+011	+1.2830833751865"+011
+1.3984334760437"+011	+1.5173329233964"+011	+1.6387795299929"+011
+1.7619308096050"+011	+1.8860748616564"+011	+2.0106067086588"+011
+2.1350090419100"+011	+2.2588365377721"+011	+2.3817030713289"+011
+2.5032712853065"+011	+2.6232440768461"+011	+2.7413576485376"+011
+2.8573758373599"+011	+2.9710854892195"+011	+3.0822926903076"+011
+3.1908197016166"+011	+3.2965024713400"+011	+3.3991886228632"+011
+3.4987358346804"+011	+3.5950105437001"+011	+3.6878869156946"+011
+3.7772460366449"+011	+3.8629752868764"+011	+3.9449678665006"+011
+4.0231224460738"+011	+4.0973429207689"+011	+4.1675382499160"+011
+4.2336223666546"+011	+4.2955141447751"+011	+4.3531374117023"+011
+4.4064209980843"+011	+4.4552988156589"+011	+4.4997099560331"+011
+4.5395988037923"+011	+4.5749151579914"+011	+4.6056143566163"+011
+4.6316573990815"+011	+4.6530110622770"+011	+4.6696480061332"+011
+4.6815468651559"+011	+4.6886923229097"+011	+4.6910751670211"+011
+4.6886923229097"+011	+4.6815468651559"+011	+4.6696480061332"+011
+4.6530110622770"+011	+4.6316573990815"+011	+4.6056143566163"+011
+4.5749151579914"+011	+4.5395988037923"+011	+4.4997099560331"+011
+4.4552988156589"+011	+4.4064209980843"+011	+4.3531374117023"+011
+4.2955141447751"+011	+4.2336223666545"+011	+4.1675382499159"+011
+4.0973429207689"+011	+4.0231224460737"+011	+3.9449678665005"+011
+3.8629752868764"+011	+3.7772460366449"+011	+3.6878869156945"+011
+3.5950105437001"+011	+3.4987358346803"+011	+3.3991886228632"+011
+3.2965024713400"+011	+3.1908197016165"+011	+3.0822926903076"+011
+2.9710854892195"+011	+2.8573758373599"+011	+2.7413576485375"+011
+2.6232440768460"+011	+2.5032712853065"+011	+2.3817030713289"+011
+2.2588365377720"+011	+2.1350090419099"+011	+2.0106067086587"+011
+1.8860748616563"+011	+1.7619308096049"+011	+1.6387795299929"+011
+1.5173329233964"+011	+1.3984334760437"+011	+1.2830833751864"+011
+1.1724803826227"+011	+1.0680621013217"+011	+9.7156068773760"+010
+8.8507059297574"+010	+8.1113259184207"+010	+7.5283822217556"+010
+7.1395986288786"+010	+6.9911310019124"+010	

REFERENCE SOLUTION OF PROBLEM 7 (HYPERBOLIC)
NUMBER OF COMPONENTS = 101

+6.8721752821010"+003	+6.8740521070120"+003	+6.8795794552161"+003
+6.8886398979534"+003	+6.9011359395546"+003	+6.9169540660936"+003
+6.9360014218621"+003	+6.9581669103648"+003	+6.9833608001040"+003
+7.0114734865961"+003	+7.0424160466778"+003	+7.0760801377969"+003
+7.1123744572678"+003	+7.1511921948252"+003	+7.1924359313624"+003
+7.2360005125566"+003	+7.2817788508888"+003	+7.3296664695622"+003
+7.3795440684724"+003	+7.4313049764087"+003	+7.4848165080232"+003
+7.5399649245333"+003	+7.5966038927458"+003	+7.6546064363095"+003
+7.7138121076856"+003	+7.7740743213188"+003	+7.8352166977629"+003
+7.8970673243247"+003	+7.9594301147776"+003	+8.0221030693088"+003
+8.0848647098958"+003	+8.1474791364385"+003	+8.2096922509602"+003
+8.2712316296608"+003	+8.3318020746371"+003	+8.3910941337694"+003
+8.4487622469712"+003	+8.5044627651798"+003	+8.5577931607685"+003
+8.6083817444458"+003	+8.6557813702118"+003	+8.6996043917968"+003
+8.7394352197073"+003	+8.7749228390848"+003	+8.8058519278046"+003
+8.8320598715882"+003	+8.8535685526734"+003	+8.8704317990868"+003
+8.8825590055822"+003	+8.8898554644466"+003	+8.8922903713432"+003
+8.8898554644466"+003	+8.8825590055820"+003	+8.8704317990867"+003
+8.8535685526734"+003	+8.8320598715882"+003	+8.8058519278046"+003
+8.7749228390848"+003	+8.7394352197071"+003	+8.6996043917968"+003
+8.6557813702118"+003	+8.6083817444455"+003	+8.5577931607686"+003
+8.5044627651797"+003	+8.4487622469711"+003	+8.3910941337694"+003
+8.3318020746371"+003	+8.2712316296606"+003	+8.2096922509602"+003
+8.1474791364384"+003	+8.0848647098956"+003	+8.0221030693088"+003
+7.9594301147776"+003	+7.8970673243248"+003	+7.8352166977628"+003
+7.7740743213187"+003	+7.7138121076856"+003	+7.6546064363093"+003
+7.5966038927458"+003	+7.5399649245330"+003	+7.4848165080235"+003
+7.4313049764086"+003	+7.3795440684724"+003	+7.3296664695620"+003
+7.2817788508887"+003	+7.2360005125566"+003	+7.1924359313624"+003
+7.1511921948252"+003	+7.1123744572677"+003	+7.0760801377968"+003
+7.0424160466776"+003	+7.0114734865960"+003	+6.9833608001039"+003
+6.9581669103647"+003	+6.9360014218620"+003	+6.9169540660937"+003
+6.9011359395544"+003	+6.8886398979531"+003	+6.8795794552160"+003
+6.8740521070119"+003	+6.8721752821007"+003	