Ordinary Differential Equations

Lecture Notes 2014–2015

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Preface

In these notes we study the basic theory of ordinary differential equations, with emphasis on initial value problems, together with some modelling aspects.

The material is covered by a number of text-books. A classical book with many examples (and amusing anecdotes) is:

- M. Braun, *Differential Equations and Their Applications : An Introduction to Applied Mathematics.* Springer Texts in Appl. Math. 11, 4th ed., Springer, 1993.

Among the books in Dutch, a good introduction is:

- J.J. Duistermaat, W. Eckhaus, *Analyse van Gewone Differentiaalvergelijkingen.* Epsilon Uitgaven, 2009.

A more advanced text, which can be downloaded from the author's web page, is:

- G. Teschl, *Ordinary Differential Equations and Dynamical Systems.* Graduate Studies in Math. 140, AMS, 2012. http://www.mat.univie.ac.at/~gerald/ftp/book-ode

Further material for these notes has been taken from Chapter I of:

- E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I – Nonstiff Problems.* Springer Series in Comp. Math. 8, 2nd ed., Springer, 1993.

This last book is mostly about numerical methods, but the first chapter gives an overview of the main developments in the theory of ordinary differential equations with many examples and historical references.

Exercises: At the end of each section, a number of exercises are given. Some of them are marked with an asterisk.⋆ These are optional and will not be used for the examination. This also applies to the numbered remarks.

Typing errors: This text may still contain a number of smaal erorrs. If you find some, please let me know (willem.hundsdorfer@cwi.nl).

Contents

Some notations: In the literature of ordinary differential equations, a number of different notations can be found. In these notes we will use $u(t)$ to denote the solution u at point t, and $u'(t)$ is the derivative.

Instead of $u(t)$ on often sees $y(x)$ or $x(t)$ in the literature, and in the latter case the derivative is sometimes denoted by $\dot{x}(t)$. This 'dot' notation, which goes back to Newton, is often used in mechanics. Newton's big rival in the field of calculus, Leibniz, used the $\frac{d}{dt}$ notation. The 'prime' notation was introduced later by Lagrange.

For given real functions φ_1, φ_2 we will use the notation

$$\varphi_1(t) \;=\; \varphi_2(t) + \mathcal{O}(t^k) \quad (t \to 0)$$

if there are δ, $C > 0$ such that $|\varphi_1(t) - \varphi_2(t)| \le C\,|t|^k$ for all $|t| < \delta$. Likewise, we will write

$$\varphi_1(t) \;=\; \varphi_2(t) + o(t^k) \quad (t \to 0)$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\varphi_1(t) - \varphi_2(t)| \le \varepsilon\,|t|^k$ for all $|t| < \delta$. So in this latter case, the difference between $\varphi_1(t)$ and $\varphi_2(t)$ tends to zero faster than $C\,|t|^k$.

1 Introduction: Models and Explicit Solutions

1.1 Ordinary Differential Equations and Initial Value Problems

In these notes we will study ordinary differential equations (ODEs), which give a relation for a function u between its function values $u(t)$ and the derivatives $u'(t) = \frac{d}{dt}u(t)$. The function may be vector valued, $u(t) \in \mathbb{R}^m$. The most common form that will be considered is

$$u'(t) = f(t, u(t)) \tag{1.1}$$

with given $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. If $m = 1$ this is called a *scalar* equation; otherwise, if $m > 1$, we have a *system* of ordinary differential equations. Often we will refer to (1.1) simply as a differential equation, but it should be noted that there are other, more general, types of differential equations, most notably partial differential equations (PDEs).

A function u, defined on an interval $\mathcal{J} \subset \mathbb{R}$ with values in $\mathbb{R}^m$, is said to be a *solution* of the differential equation on $\mathcal{J}$ if it is differentiable on this interval and satisfies relation (1.1) for all $t \in \mathcal{J}$.

For a system we will denote the components of the vector $u(t) \in \mathbb{R}^m$ by $u_j(t)$ $(1 \le j \le m)$. Written out, per component, the system of ordinary differential equations reads

$$\begin{cases} u_1'(t) \;=\; f_1(t, u_1(t), u_2(t), \ldots, u_m(t))\,, \\ u_2'(t) \;=\; f_2(t, u_1(t), u_2(t), \ldots, u_m(t))\,, \\ \qquad\vdots \\ u_m'(t) \;=\; f_m(t, u_1(t), u_2(t), \ldots, u_m(t))\,. \end{cases}$$

Often the differential equation (1.1) will be written more compactly as

$$u' = f(t, u)\,,$$

where it is then understood that u is a function of the independent variable t. Usually, t will stand for 'time', but there are also many applications where the independent variable denotes a distance in 'space', in which case it may be more natural to denote it by x.

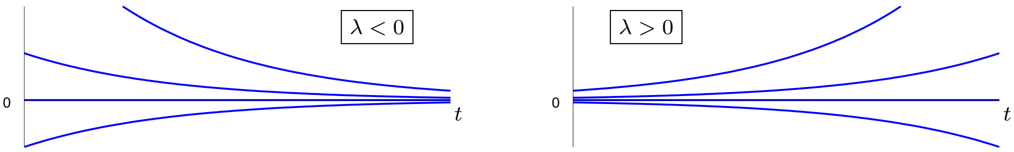
If the function f does not depend explicitly on t, the differential equation is called *autonomous*. Otherwise it is called non-autonomous. The general form of an autonomous differential equation is $u' = f(u)$. For an autonomous equation the rate of change u' is completely determined by the 'state' u, so there are no external factors (and hence the name 'autonomous').

Example 1.1 The most simple differential equation is

$$u'(t) \;=\; \lambda\, u(t)\,, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a constant. The solutions are given by $u(t) = c\, e^{\lambda t}$ with arbitrary number c.

In particular, $u = 0$ is a solution. Since this solution does not depend on t it is called a *stationary solution* or *steady state solution*. If $\lambda < 0$ then any other solution converges to this steady state solution as $t \to \infty$. This is an example of a *stable* steady state. On the other hand, if $\lambda > 0$ then all non-zero solutions will tend to infinity for increasing time, and in that case the stationary solution $u = 0$ is called unstable. (Stability will be an important issue in later sections.)



Usually we consider real valued differential equations. But sometimes – in particular with (1.2) – we will also allow complex valued equations and solutions. Here this would mean $\lambda \in \mathbb{C}$ and $u(t) \in \mathbb{C}$ with $t \in \mathbb{R}$. This scalar complex differential equation can also be written as a real equation in $\mathbb{R}^2$ by identifying $u = u_1 + iu_2 \in \mathbb{C}$ with $u = (u_1, u_2)^T \in \mathbb{R}^2$, see Exercise 1.1. ◇

Models. Ordinary differential equations arise in many applications from physics, biology, economy and numerous other fields. The differential equation is then a *mathematical model* of reality.

Example 1.2 Let $u(t)$ denote a population density of a biological species, for instance bacteria on a laboratory dish, with sufficient food available. Then the growth (or decline) of this species can be described by the differential equation $u'(t) = \alpha u(t) - \beta u(t)$ with $\alpha > 0$ the birth rate and $\beta > 0$ the natural death rate. This is the same as (1.2) with $\lambda = \alpha - \beta$,

An obvious objection to this model is that the actual population of a species will change by an integer amount. However, if the population is very large, then an increase by one individual is very small compared to the total population. In that case the continuous model may give a good correspondence to reality.

There is, however, an other objection: if $\alpha > \beta$, that is $\lambda > 0$, the population will grow beyond any bound with increasing time. This not how real populations behave, so we need a modified model. ◇

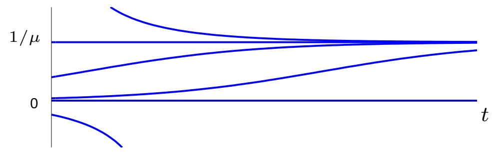
Example 1.3 To improve the population density model for $\alpha > \beta$, let us assume that the death rate is not constant, but that it increases linearly with the population, say as $\beta + \gamma u(t)$, with $\beta, \gamma > 0$. This seems reasonable if food gets scarce with an increasing population. Then the differential equation becomes $u' = (\alpha - \beta - \gamma u)u$, which we write as

$$u'(t) = \lambda\, u(t)\, (1 - \mu\, u(t)) \tag{1.3}$$

with $\lambda = \alpha - \beta$ and $\mu = \gamma/\lambda$. As we will see later, explicit expressions for the solutions of this differential equation can be found. These are $u(t) = 0$, $u(t) = 1/\mu$ and

$$u(t) \;=\; \frac{e^{\lambda t}}{c + \mu \cdot e^{\lambda t}} \tag{1.4}$$

with $c \in \mathbb{R}$ arbitrary. Solutions are plotted below, with the time axis horizontal. Of course, for a population model we should restrict ourselves to the nonnegative solutions.



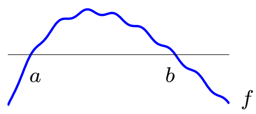
We will see in later sections more interesting examples of this type, for instance with several species or spatial migration. Those models will lead to systems of differential equations. ◇

We note already that having explicit expressions for the solutions is not so very common. For more complicated models these will generally not be known. Then a so-called qualitative analysis will be helpful.

Example 1.4 For the population model (1.3) we happen to know the solutions, but a slight modification of the model may change this. Still, it can be possible to describe the qualitative behaviour of the solutions for different starting values.

Consider a scalar problem $u' = f(u)$, $u(0) = u_0$, with f continuous and

$$\begin{aligned} &f(a) = 0\,, \quad f(b) = 0\,, \\ &f(v) > 0 \quad \text{if } a < v < b\,, \\ &f(v) < 0 \quad \text{if } v < a \text{ or } v > b\,. \end{aligned}$$



Then $u_0 = a$ or $u_0 = b$ give steady state solutions. For the other cases the sign of f tells us in which direction the solution will move as time advances. If $u_0 > b$ then the solution will be decreasing towards b, whereas if $u_0 < a$ the solution will decrease towards $-\infty$ (see Exercise 1.3). If $a < u_0 < b$ we will get an monotonically increasing solution, and even if we start just a little bit above a the solution will eventually approach b. Therefore the qualitative picture is similar as for (1.3). ◇

Initial value problems. As we saw in the examples, solutions are not completely determined by a differential equation. In these notes we will mainly consider *initial value problems* where the solution is specified at some time point t_0.

The common form of an initial value problem (for a first-order differential equation) is

$$u'(t) = f(t, u(t))\,, \tag{1.5a}$$

$$u(t_0) = u_0\,. \tag{1.5b}$$

Here $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ are given. Unless specified otherwise it will always assumed that f is continuous. Let $\mathcal{I}$ be an interval in $\mathbb{R}$ containing t_0. A function $u : \mathcal{I} \to \mathbb{R}^m$ is called a *solution* of the initial value problem on $\mathcal{I}$ if it is differentiable with $u(t_0) = u_0$, and (1.5a) is satisfied for all $t \in \mathcal{I}$.

Usually we will take $t_0 = 0$; this is merely the instant where we press in the 'stopwatch'. The interval $\mathcal{I}$ will often be taken as $[0, T]$ with an end time $T > 0$ (the derivatives at $t = 0$ and $t = T$ are then one-sided). Further, it may happen that f is only defined on $\mathcal{I} \times \mathcal{D}$ with a subset $\mathcal{D} \subset \mathbb{R}^m$. In that case we have to ensure that $u(t)$ stays in $\mathcal{D}$. (For the population models it would be quite natural to define f only for nonnegative arguments u.)

Some initial value problems can be solved explicitly. If that is not possible we may use numerical methods to approximate a solution. Sometimes, for $m = 1$ and $m = 2$, it is possible to get already a good idea how a solution will behave qualitatively by looking at the direction of $u'(t)$; see e.g. Example 1.4.

We will see in the next section that, under (weak) smoothness assumptions on f, the initial value problem (1.5) has a unqiue solution on some interval $[t_0, T]$. It may happen that a solution does not exist for arbitrary large intervals. If the norm of $u(t)$ tends to ∞ as $t \uparrow T$ for some finite T we will say that the solution *blows up in finite time*. The time interval is then taken as $[t_0, T)$. An example is provided by (1.4) with $\lambda, \mu > 0$: if $u_0 < 0$ then $c < -\mu$ and the solution will only exist up to time $T = \frac{1}{\lambda} \log(|c|/\mu)$.

Higher-order equations. The differential equation (1.1) is called a *first-order* differential equation because only the first derivative appears. Many problems from mechanics arise as *second-order* equations because of Newton's law $F = Ma$, where F is the force on a particle or solid body, M is its mass and a is the acceleration. This acceleration is the second derivative of the position, and the force may depend on velocity and position. Denoting the position by w and setting $g = \frac{1}{M} F$, we thus get a second-order equation

$$w''(t) = g(t, w(t), w'(t)) \,. \tag{1.6}$$

If $w(t) \in \mathbb{R}^n$ we can transform this second-order equation to a first-order equation in $\mathbb{R}^m$, $m = 2n$. Writing $v(t) = w'(t)$ for the velocity, we obtain

$$\begin{cases} w'(t) = v(t) \,, \\ v'(t) = g(t, w(t), v(t)) \,. \end{cases} \tag{1.7a}$$

This is a system of the form (1.1) with

$$u = \begin{pmatrix} w \\ v \end{pmatrix}, \qquad f(t, u) = \begin{pmatrix} v \\ g(t, w, v) \end{pmatrix}. \tag{1.7b}$$

So by introducing extra variables a second-order equation can be brought into first-order form. For higher-order differential equations this is similar. Therefore we can confine ourselves to studying only first-order differential equations.

Example 1.5 Suppose a stone is dropped from a tower, or an apple drops from a tree, with height h at time $t_0 = 0$. Let w be the height above the ground. The initial condition is $w(0) = h$, $w'(0) = 0$ and the motion is described by

$$w''(t) = -\gamma\,, \tag{1.8a}$$

where γ denotes the gravity constant. This equation is easily solved and the solution is $w(t) = h - \frac{1}{2}\gamma t^2$ until the time $T = \sqrt{2h/\gamma}$ when the object hits earth.

Again, this is just a simple model for the physical reality. An obvious issue that has not been taken into account is air resistance. If we assume this resistance is proportional to the velocity we get an improved model

$$w''(t) = -\gamma - \rho\, w'(t)\,, \tag{1.8b}$$

where $\rho > 0$ is the resistance coefficient. For this equation it is still quite easy to find explicit solutions; see Exercise 1.6

A further refinement is found by not taking the gravity constant. It is actually gravitational attraction between our object and the earth. If R is the radius of the earth then the distance to the center of the earth is $w + R$, and the force on the object will be proportional to $(w + R)^{-2}$. This gives

$$w''(t) = -\frac{\gamma R^2}{(w(t) + R)^2} - \rho w'(t)\,, \tag{1.8c}$$

where the scaling factor γR^2 is chosen such that (1.8b) is retrieved for $w = 0$. Of course, this modification will only be relevant if the initial position is very far above the ground. But then we should also incorporate the fact that the air resistance will be proportional to the air density which varies with the height, leading again to a more complicated description. $\diamond$

We see that even a simple problem – free fall of an object – can become rather complicated if more and more refined models are used to describe the physical reality. In this case the simplest model is easily solved exactly, but this no longer holds for very accurate models.

Still there is room for a mathematical analysis. For instance the qualitative behaviour of solutions might be investigated, or maybe it can be shown that solutions of a complicated model do not differ much from the solutions of a more simple model. (It seems obvious that for apples falling from a tree the refinement (1.8c) will not be relevant.)

1.2 Explicit Solutions

For some ordinary differential equations and initial value problems we can write down explicit expressions for the solutions. This was undertaken by the pioneers in this field, such as Newton, Leibnitz and the Bernoulli brothers, Jacob and Johann, in the second half of the 17-th century. In this section some examples are presented for classes of scalar problems that can be solved explicitly (with some overlap between the classes). As before, smoothness of the given functions is tacitly assumed.

Separable variables. Consider a scalar differential equation of the form

$$u' = \alpha(t)g(u) \tag{1.9}$$

with $g(u) \neq 0$. In this equation the independent variable t and the dependent variable u are said to be separated. Let $\beta(u) = 1/g(u)$. Then

$$\beta(u)u' = \alpha(t)\,. \tag{1.10}$$

If A, B are primitive functions for α, β, respectively, then the left-hand side equals $\frac{d}{dt}B(u) = \beta(u)u'$, by the chain rule. Hence we get $\frac{d}{dt}B(u) = \frac{d}{dt}A(t)$ and therefore the solution satisfies

$$B(u(t)) = c + A(t) \tag{1.11}$$

with arbitrary integration constant $c \in \mathbb{R}$.

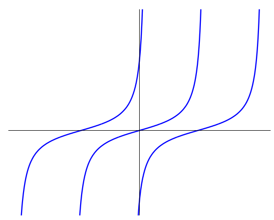
If we assume that $g(u_0) > 0$ then also $\beta(u_0) > 0$, so $B(v)$ will be monotonically increasing for arguments v near u_0. Therefore, in principle, we can locally invert this function to obtain $u(t) = B^{-1}(c+A(t))$ for t near t_0. If $g(u_0) < 0$ the situation is similar. The case $g(u_0) = 0$ leads to a stationary solution $u(t) = u_0$ for all t.

Example 1.6 Consider the initial value problem

$$u' = 1 + u^2\,, \quad u(0) = u_0\,.$$

By writing it as $(1 + u^2)^{-1}u' = 1$ we find by integration from $t_0 = 0$ to t that

$$\begin{aligned} t &= \textstyle\int_0^t ds = \int_0^t \frac{1}{1+u(s)^2}u'(s)\,ds \\ &= \textstyle\int_{u_0}^{u(t)} \frac{1}{1+v^2}\,dv = \arctan(u(t)) - c \end{aligned}$$



with $c = \arctan(u_0)$. This gives $u(t) = \tan(t+c)$ in the interval $(-\frac{1}{2}\pi - c, \frac{1}{2}\pi - c)$. At the end points of this interval the solution blows up. $\diamond$

Linear equations. A scalar differential equation of the form

$$u' = a(t)u + b(t) \tag{1.12}$$

is called *linear*. If $b = 0$ the equation is called *homogeneous.*

To solve (1.12), let us first assume that $b = 0$. Of course, $u = 0$ is then a solution. Non-zero solutions with $u(t_0) = u_0$ are found by the above procedure with separation of variables, for which the equation is written as

$$\frac{1}{u}u' = a(t)\,.$$

Let us suppose that $u > 0$ (otherwise consider $v = -u$). Then the left-hand side equals $\frac{d}{dt}\log(u)$. Therefore, integrating from t_0 to t gives $\log(u(t)) - \log(u_0) = \sigma(t)$ with $\sigma(t) = \int_{t_0}^{t} a(s)\,ds$. The solution for this homogeneous case is thus found to be

$$u(t) \;=\; u_0\, e^{\sigma(t)} \,. \tag{1.13}$$

To solve the general inhomogeneous linear differential equation (1.12) we use the substitution $u(t) = c(t)e^{\sigma(t)}$; this trick is called 'variation of constants'. Insertion into (1.12) gives

$$c'(t)e^{\sigma(t)} + c(t)e^{\sigma(t)}\sigma'(t) \;=\; a(t)c(t)e^{\sigma(t)} + b(t)\,.$$

Since $\sigma'(t) = a(t)$, we get $c'(t) = e^{-\sigma(t)}b(t)$ and $c(t) = c(t_0) + \int_{t_0}^{t} e^{-\sigma(s)}b(s)\,ds$. Thus we obtain the expression

$$u(t) \;=\; e^{\sigma(t)}u_0 \;+\; \int_{t_0}^{t} e^{\left(\sigma(t)-\sigma(s)\right)}b(s)\,ds\,. \tag{1.14}$$

This is known as the *variation of constants formula.*

Example 1.7 For constant a the variation of constants formula becomes

$$u(t) \;=\; e^{a\,(t-t_0)}\,u_0 \;+\; \int_{t_0}^{t} e^{a\,(t-s)}b(s)\,ds\,. \tag{1.15}$$

If b is also constant we get

$$u(t) \;=\; \begin{cases} e^{a\,(t-t_0)}\,u_0 \;+\; \frac{1}{a}\,(e^{a\,(t-t_0)} - 1)\,b & \text{if } a \neq 0\,, \\ u_0 \;+\; (t-t_0)\,b & \text{if } a = 0\,. \end{cases}$$

◇

Change of variables. There are several classes of differential equations that can be brought in linear or separable form by substitutions or changing variables.

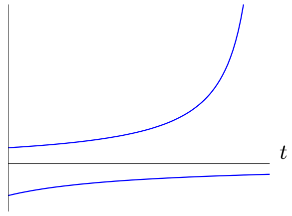
As an example, we consider the *Bernoulli equations*, which are differential equations of the type

$$u' = p(t)u + q(t)u^r\,, \qquad r \neq 1\,. \tag{1.16}$$

Solutions $u \geq 0$ are found by introducing $v(t) = u(t)^{1-r}$. Differentiation of v and a little manipulation gives $v' = (1-r)p(t)v + (1-r)q(t)$. But this is now a linear differential equation and we already know how to solve that.

Example 1.8 For the initial value problem $u' = |u|^{\kappa+1}$, $u(0) = u_0$, with constant $\kappa > 0$, the cases $u \geq 0$ and $u < 0$ can be treated separately. In both cases we get a differential equation of the type (1.16), where we consider $-u$ instead of u for the negative solutions. The solutions are given by

$$u(t) = \begin{cases} u_0(1-\kappa t u_0^\kappa)^{-1/\kappa} & \text{if } u_0 \geq 0\,, \\ u_0(1+\kappa t|u_0|^\kappa)^{-1/\kappa} & \text{if } u_0 < 0\,. \end{cases}$$



For $u_0 > 0$ this follows by taking $v = u^{-\kappa}$, giving $v' = -\kappa u^{-(\kappa+1)}u' = -\kappa$. Hence $v(t) = v_0 - \kappa t$, which leads to the above formula. The negative solutions are found in a similar way, but now with $v = (-u)^{-\kappa}$.

We see that for any $u_0 > 0$ the solution will blow up in finite time, whereas for $u_0 \leq 0$ the solutions exist for all $t > 0$. It should be noted that the solutions for this equation can also easily be found with separation of variables. $\diamond$

As an other example where a substitution leads to a familiar form is given by equations of the type[1]

$$u' = g\Big(\frac{u}{t}\Big) \qquad \text{for } t \neq 0\,, \tag{1.17}$$

Setting $v(t) = u(t)/t$, leads to the seperable equation $v' = \frac{1}{t}(g(v) - v)$, which we know how to solve (in principle).

Exact equations and integrating factors. Let E be a twice continuously differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$. A differential equation of the form

$$\frac{d}{dt}E(t,u) = 0 \tag{1.18}$$

is called *exact*. For any solution of such a differential equation we have $E(t,u) = c$ with some constant c, which is an equation between u and t from which we may now solve u as function of t.

We can write (1.18) in a form that is closer to the standard form (1.1) by defining $\alpha(t,v) = \frac{\partial}{\partial t}E(t,v)$ and $\beta(t,v) = \frac{\partial}{\partial v}E(t,v)$ for $t, v \in \mathbb{R}$. By applying the chain rule, we see that (1.18) can be written as

$$\alpha(t,u) + \beta(t,u)u' = 0\,. \tag{1.19a}$$

For the functions α, β we know that

$$\frac{\partial}{\partial v}\alpha(t,v) = \frac{\partial}{\partial t}\beta(t,v) \qquad (\text{for all } t, v \in \mathbb{R})\,, \tag{1.19b}$$

because $\frac{\partial}{\partial v}\alpha(t,v) = \frac{\partial^2}{\partial v \partial t}E(t,v)$ and $\frac{\partial}{\partial t}\beta(t,v) = \frac{\partial^2}{\partial t \partial v}E(t,v)$. So, an exact differential equation can be written in the form (1.19a) with α, β satisfying (1.19b).

[1] Equations (1.17) are called *homogeneous* differential equations. This is not to be confused with the term homogeneous for the linear equation (1.12) with $b = 0$. The term homogeneous for (1.17) arises from the fact that a function $f(t,v)$ is called homogeneous (with degree 0) if $f(t,v) = f(ct,cv)$ for all $c \in \mathbb{R}$, and setting $g(v/t) = f(1, v/t)$ leads to (1.17).

This works also the other way around. Consider the differential equation (1.19a) with given continuously differentiable functions α, β. We can define

$$E(t,v) \;=\; E_0 + \textstyle\int_{t_0}^{t} \alpha(s,v)\, ds + \int_{u_0}^{v} \beta(t_0,w)\, dw\,.$$

Then

$$\tfrac{\partial}{\partial t} E(t,v) = \alpha(t,v)\,, \qquad \tfrac{\partial}{\partial v} E(t,v) = \textstyle\int_{t_0}^{t} \frac{\partial}{\partial v}\alpha(s,v)\, ds + \beta(t_0,v)\,.$$

Assuming (1.19b) this gives

$$\tfrac{\partial}{\partial v} E(t,v) = \textstyle\int_{t_0}^{t} \frac{\partial}{\partial s}\beta(s,v)\, ds + \beta(t_0,v) = \beta(t,v)\,.$$

Hence we obtain again the form (1.18). In conclusion: any differential equation (1.19a) for which (1.19b) is satisfied corresponds to an exact differential equation.

It is easy to see that an autonomous differential equation in the standard form $u' = f(u)$ cannot be exact (except in the trivial case of constant f). Likewise, an equation with separated variables is not exact in the form (1.9). However, if we write it as (1.10) then it becomes exact. This is an example where we can bring a differential equation in exact form by multiplying the equation by a suitable function.

Consider $a(t,u) + b(t,u)u' = 0$. A function $\mu(t,v) \neq 0$ is called an *integrating factor* for this differential equation if $\mu(t,u)a(t,u) + \mu(t,u)b(t,u)u' = 0$ is exact, that is $\alpha(t,u) = \mu(t,u)a(t,u)$ and $\beta(t,u) = \mu(t,u)b(t,u)$ satisfy relation (1.19b). Finding an integrating factor is in general very difficult, and we will not pursue this topic, but sometimes it is possible with a suitable ansatz (see for instance Exercise 1.9).

1.3 Exercises

Exercise 1.1. Consider the complex scalar equation $u' = \lambda u$ with $\lambda = \alpha + i\beta$.
(a) Discuss growth or decline of the modulus of solutions for the cases $\mathrm{Re}\,\lambda < 0$, $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Re}\,\lambda = 0$. Discuss the role of $\mathrm{Im}\lambda$ for the trajectories $\{u(t) : t \geq 0\}$ in the complex plane.
(b) Write the complex scalar equation $u' = \lambda u$ with $\lambda = \alpha + i\beta$ as a real system $u' = Au$ in $\mathbb{R}^2$. Give the matrix $A \in \mathbb{R}^{2\times 2}$ and its eigenvalues.

Exercise 1.2. Often we will consider the initial value problem (1.5) with a time interval $[0,T]$ (or with $[0,T)$ if the solution blows up at T).
(a) Transform the problem $v' = g(s,v)$, $v(s_0) = v_0$ with arbitrary $s_0 \in \mathbb{R}$ to (1.5) with $t_0 = 0$.
(b) Sometimes we want to know what happened in the past, given the present state. Consider $v' = g(s,v)$, $v(s_0) = v_0$ with $s \in (-\infty, s_0]$. Transform this to (1.5) with time interval $[0,\infty)$.

Exercise 1.3. Consider a scalar differential equation $u' = f(u)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and assume that for any given initial condition $u(t_0) = u_0$ the solution $u(t)$ exists for all $t \geq t_0$.
(a) Suppose $v_* \in \mathbb{R}$ and $\varepsilon, \gamma > 0$ are such that $|f(v)| \geq \gamma$ for all $v \in [v_* - \varepsilon, v_* + \varepsilon]$. Show that if $|u(t) - v_*| \leq \varepsilon$ for $t_0 \leq t \leq t_1$, then $|u(t_1) - u(t_0)| \geq \gamma \cdot (t_1 - t_0)$.
(b) Let $u_* \in \mathbb{R}$. Show that:

$$\lim_{t\to\infty} u(t) = u_* \qquad \Longrightarrow \qquad f(u_*) = 0\,.$$

Exercise 1.4. Find explicit solutions of $u' = \rho(u-a)(u-b)$, with constants $\rho \neq 0$ and $a < b$, by transforming it to a Bernoulli type equation. Use this to verify that (1.3) has the solutions (1.4).

Exercise 1.5. Discuss qualitatively the solutions of $u' = \rho(u-a)(u-b)(u-c)$ with $a < b < c$ for the cases $\rho > 0$ and $\rho < 0$. (Assume that for any $(t_0, u_0) \in \mathbb{R}^2$ there is a unique solution that passes through this point.)

Exercise 1.6. Solve the differential equation (1.8b) with $w(0) = h$, $w'(0) = 0$. Observe that the velocity will remain bounded, in contrast to (1.8a).

Exercise 1.7. Find explicit solutions of $u' = 2tu$, and for $u' = 2tu + t$.

Exercise 1.8. Solutions may cease to exist because $u'(t)$ tends to infinity without blow-up of u.
(a) Derive the solutions of $u' = \lambda/u$. On what interval $\mathcal{I}$ do they exist? Make a sketch of the solutions in the (t,u)-plane for the cases $\lambda > 0$ and $\lambda < 0$.
(b) Do the same for the differential equation $u' = \lambda t/(u-1)$.

Exercise 1.9.* Consider the differential equation $3t - 2u + tu' = 0$. Solve this equation by finding a suitable integrating factor of the form $\mu(t)$, depending only on t. Hints: use (1.19b) to get a differential equation for μ, solve this equation (by separation of variables or by educated guessing), and then find $E(t,u)$ for the resulting exact equation by requiring $\frac{\partial}{\partial t}E(t,v) = \alpha(t,v)$ and $\frac{\partial}{\partial v}E(t,v) = \beta(t,v)$.

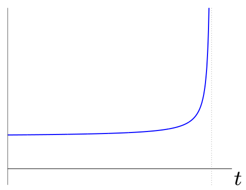
2 Existence and Uniqueness

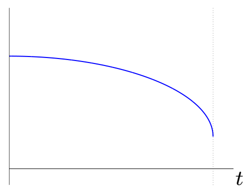
In this section we will discuss existence and uniqueness of solutions of an initial value problem

$$u'(t) = f(t, u(t))\,, \qquad u(t_0) = u_0 \tag{2.1}$$

with given $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^m$ and $f : \mathcal{E} \subset \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. Recall that, if $\mathcal{J}$ is an interval in $\mathbb{R}$ containing t_0, then u is said to be a solution of the initial value problem on $\mathcal{J}$ if u is differentiable on this interval, $(t, u(t)) \in \mathcal{E}$ and $u'(t) = f(t, u(t))$ for all $t \in \mathcal{J}$, and $u(t_0) = u_0$.

As we saw in the previous section, solutions of initial value problems may cease to exist after some time because $u(t)$ or $u'(t)$ become infinite; see Example 1.8 and Exercise 1.8.





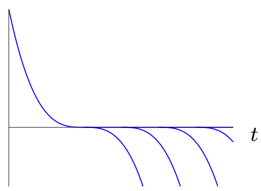
There is another troublesome situation: solutions may cease to be unique.

Example 2.1 Consider, for $t \geq 0$, the following scalar problem:

$$u'(t) \;=\; -3\,\sqrt[3]{u(t)^2}\,, \qquad u(0) = 1\,.$$

A solution is given by $u(t) = (1-t)^3$ and this is the unique solution up to $t = 1$. However, after $t = 1$ the solution is no longer unique: for any $c \geq 1$ the continuously differentiable function

$$u(t) = \begin{cases} (1-t)^3 & \text{for } 0 \leq t \leq 1\,, \\ 0 & \text{for } 1 \leq t \leq c\,, \\ (c-t)^3 & \text{for } t \geq c \end{cases}$$



is also a solution of the initial value problem. Likewise we can take $u(t) = 0$ for all $t \geq 1$. $\diamond$

As we will see, the behaviour in the last example is caused by the fact that the function $f(v) = -3\sqrt[3]{v^2}$ is not differentiable at 0. We will also see in this section that existence and uniqueness of solutions can be guaranteed under some weak smoothness assumptions of the function f.

2.1 Preliminaries

The results in this section will be presented for systems in $\mathbb{R}^m$ ($m \geq 1$). This requires some basic concepts from calculus and linear algebra, which are briefly reviewed here. For the study of scalar equations ($m = 1$) these paragraphs can be omitted, reading for $\|v\|$ the modulus of v in the sequel. The essential points of the results and their derivations are already covered by this scalar case.

Matrices. Linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^m$ will be identified with matrices $A = (a_{ij}) \in \mathbb{R}^{m\times m}$. For a given norm $\|\cdot\|$ on $\mathbb{R}^m$, the *induced matrix norm* for an $m \times m$ matrix A is defined by $\|A\| = \max_{v\neq 0} \|Av\| \,/\, \|v\|$. It is the smallest number α such that $\|Av\| \leq \alpha\|v\|$ for all $v \in \mathbb{R}^m$. (This matrix norm will be discussed in more detail in later sections.) The most common norms for vectors $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$ are the Euclidian norm $\|v\|_2 = \sqrt{v^T v}$ and the maximum norm $\|v\|_\infty = \max_{1\leq j\leq m} |v_j|$.

Integration. Integrals of vector valued functions are defined component-wise: if $u(t) \in \mathbb{R}^m$ has components $u_j(t)$ $(1 \leq j \leq m)$ for $t \in [t_0, T]$, then the integral $\int_{t_0}^t u(s)\,ds$ is the vector in $\mathbb{R}^m$ with components $\int_{t_0}^t u_j(s)\,ds$ $(1 \leq j \leq m)$. We have

$$\Big\| \int_{t_0}^t u(s)\,ds \Big\| \;\leq\; \int_{t_0}^t \|u(s)\|\,ds\,. \tag{2.2}$$

This follows by writing the integrals as a limit of Riemann sums, together with application of the triangle inequality for norms.

Differentiation. Consider a function $g : \mathbb{R}^n \to \mathbb{R}^m$ and suppose $\mathcal{D} \subset \mathbb{R}^n$ consists of an open set together with some of its boundary points. (The sets $\mathcal{D}$ in this text are always assumed to be of such type.) Then g is called k times continuously differentiable on $\mathcal{D}$ if all partial derivatives up to order k exist on the interior of $\mathcal{D}$ and can be continuously extended to $\mathcal{D}$. This is then denoted as $g \in C^k(\mathcal{D})$. For $k = 0$ we have continuity of g. If $g \in C^1(\mathcal{D})$, then $g'(v)$ stands for the $m \times n$ matrix with entries $\frac{\partial}{\partial v_j} g_i(v)$,

$$g(v) = \begin{pmatrix} g_1(v) \\ \vdots \\ g_m(v) \end{pmatrix}, \qquad g'(v) = \begin{pmatrix} \frac{\partial g_1(v)}{\partial v_1} & \cdots & \frac{\partial g_1(v)}{\partial v_n} \\ \vdots & & \vdots \\ \frac{\partial g_m(v)}{\partial v_1} & \cdots & \frac{\partial g_m(v)}{\partial v_n} \end{pmatrix}$$

for $v = (v_1, \ldots, v_n)^T \in \mathcal{D} \subset \mathbb{R}^n$. If $n = 1$ then $g'(v)$ is a row-vector, whereas for $n = m$ it is a square matrix.

Mean-value estimate. Let $g : \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable on $\mathcal{D} \subset \mathbb{R}^m$. Suppose $\mathcal{D}$ is convex, that is, for any $v, \tilde{v} \in \mathcal{D}$ and $\theta \in [0, 1]$ we have $w(\theta) = \tilde{v} + \theta(v - \tilde{v}) \in \mathcal{D}$. Denote $\varphi(\theta) = g(w(\theta))$. Then $\varphi'(\theta) = g'(w(\theta))(v - \tilde{v})$, by the chain-rule. Moreover $\varphi(1) - \varphi(0) = g(v) - g(\tilde{v})$ and $\varphi(1) - \varphi(0) = \int_0^1 \varphi'(\theta)\,d\theta$. Hence

$$g(v) - g(\tilde{v}) \;=\; \int_0^1 g'(\tilde{v} + \theta(v - \tilde{v})) \cdot (v - \tilde{v})\,d\theta\,.$$

In particular it is seen that

$$\|g(v) - g(\tilde{v})\| \;\leq\; \sup_{w\in\mathcal{D}} \|g'(w)\| \cdot \|v - \tilde{v}\|\,. \tag{2.3}$$

Gronwall's lemma. The following lemma will be very useful in this section. It is one of the variants of Gronwall's lemma.

Lemma 2.2 Let $\alpha, \mu : [t_0, T] \to \mathbb{R}$ with μ continuous, α continuously differentiable, and $\beta \geq 0$. Suppose that

$$\mu(t) \;\leq\; \alpha(t) + \beta \int_{t_0}^{t} \mu(s)\, ds \qquad (\text{for } \; t_0 \leq t \leq T)\,.$$

Then

$$\mu(t) \;\leq\; e^{\beta(t-t_0)}\alpha(t_0) + \int_{t_0}^{t} e^{\beta(t-s)}\alpha'(s)\, ds \qquad (\text{for } \; t_0 \leq t \leq T)\,.$$

Proof. Set $\varphi(t) = \int_{t_0}^{t} \mu(s)\, ds$. Then

$$\mu(t) \;=\; \varphi'(t) \;\leq\; \alpha(t) + \beta\,\varphi(t) \qquad (t_0 \leq t \leq T)\,.$$

Multiplying the inequality with the integrating factor $e^{-\beta t}$ shows that

$$\tfrac{d}{dt}\left(e^{-\beta t}\varphi(t)\right) = e^{-\beta t}(\varphi'(t) - \beta\varphi(t)) \;\leq\; e^{-\beta t}\alpha(t)\,.$$

Since $\varphi(t_0) = 0$ we see by integration from t_0 to t that $\varphi(t) \leq \int_{t_0}^{t} e^{\beta(t-s)}\alpha(s)\, ds$. Using again $\mu(t) \leq \alpha(t) + \beta\varphi(t)$, we obtain

$$\mu(t) \;\leq\; \alpha(t) + \beta \textstyle\int_{t_0}^{t} e^{\beta(t-s)}\alpha(s)\, ds\,.$$

Applying partial integration completes the proof. □

2.2 Picard Iteration and Global Existence and Uniqueness

Let $f : [t_0, T] \times \mathcal{D} \to \mathbb{R}^m$ be continuous and $u_0 \in \mathcal{D} \subset \mathbb{R}^m$. Along with the initial value problem (2.1) we also consider the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds\,. \tag{2.4}$$

If u is a solution of (2.1) we see by integration from t_0 to t that (2.4) will be satisfied. On the other hand, if u is a continuous function for which (2.4) holds, then the integral $\int_{t_0}^{t} f(s, u(s))\, ds$ is differentiable with respect to t, with continuous derivative $f(t, u(t))$. Therefore u is a solution of (2.1).

The solution will be approximated by a sequence of functions $v_0, v_1, v_2, \ldots$, where $v_0(t) = u_0$ and

$$v_k(t) \;=\; u_0 + \int_{t_0}^{t} f(s, v_{k-1}(s))\, ds \qquad (\text{for } t \in [t_0, T] \;\;,\;\; k = 1, 2, \ldots)\,. \tag{2.5}$$

This is called *Picard iteration*. As we will see shortly, it provides a tool to establish existence and uniqueness of the solution.

For a given set $\mathcal{D} \subset \mathbb{R}^m$, we consider the *Lipschitz condition*

$$\|f(t,v) - f(t,\tilde v)\| \le L\,\|v - \tilde v\| \qquad (\text{for all } t \in [t_0,T] \text{ and } v, \tilde v \in \mathcal{D}), \tag{2.6}$$

where $L > 0$ is called the Lipschitz constant. Note that this Lipschitz condition guarantees continuity for f with respect to v on $\mathcal{D}$. It will also be assumed that f depends continuously on its first argument t.

For the following result −which is known as *Picard's theorem* or the *Picard-Lindelöf theorem*− it will first be assumed that the Lipschitz condition is satisfied on the whole $\mathbb{R}^m$. Local versions are considered thereafter.

Theorem 2.3 Suppose f is continuous on $[t_0,T]\times\mathbb{R}^m$ and the Lipschitz condition (2.6) holds with $\mathcal{D} = \mathbb{R}^m$. Then the initial value problem (2.1) has a unique solution on $[t_0,T]$.

Proof. The proof is divided into three parts. First it will be shown that the sequence $\{v_j\}$ is uniformly convergent. Then it is shown that the limit function is a solution of the initial value problem. Finally, uniqueness is demonstrated.

1. Let $\mu_j(t) = \|v_{j+1}(t) - v_j(t)\|$ for $t \in [t_0,T]$. By considering (2.5) with $j = k$ and $j = k+1$, we see that

$$\mu_{j+1}(t) \le L \int_{t_0}^{t} \mu_j(s)\,ds \qquad (j = 0,1,2,\ldots).$$

For $j = 0$ this gives $\mu_1(t) \le L\int_{t_0}^{t}\mu_0(s)\,ds \le \gamma L(t-t_0)$ where $\gamma = \max_{[t_0,T]}\mu_0(t)$. Next, with $j = 1,2,\ldots$, it follows by induction that

$$\mu_j(t) \le \frac{1}{j!}\gamma\,\big(L\,(t-t_0)\big)^j \qquad (j = 1,2,\ldots).$$

Further we have

$$\begin{aligned}
\|v_k(t) - v_{k+n}(t)\| &\le \mu_k(t) + \cdots + \mu_{k+n-1}(t) \\
&\le \gamma\Big(\frac{1}{k!}\big(L\,(T-t_0)\big)^k + \frac{1}{(k+1)!}\big(L\,(T-t_0)\big)^{k+1} + \cdots\Big) \\
&= \gamma\Big(e^{L(T-t_0)} - \textstyle\sum_{j=0}^{k-1}\frac{1}{j!}\big(L\,(T-t_0)\big)^j\Big) \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

According to the Cauchy criterion for uniform convergence we know that the sequence $\{v_k\}$ converges uniformly on $[t_0,T]$ to a continuous limit function v_*.

2. For the limit function v_* we have

$$\begin{aligned}
&\|v_*(t) - u_0 - \textstyle\int_{t_0}^{t} f(s,v_*(s))\,ds\| \\
&\quad = \|v_*(t) - v_{k+1}(t) - \textstyle\int_{t_0}^{t} f(s,v_*(s))\,ds + \int_{t_0}^{t} f(s,v_k(s))\,ds\| \\
&\quad \le \|v_*(t) - v_{k+1}(t)\| + L(t-t_0)\max_{t_0\le s\le t}\|v_*(s) - v_k(s)\| \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

We thus see that v_* is a solution of the integral equation (2.4), and hence it is also a solution of the initial value problem (2.1).

3. To show uniqueness, suppose that u and $\tilde{u}$ are two solutions of (2.1). Then for $\mu(t) = \|u(t) - \tilde{u}(t)\|$ we obtain as above

$$\mu(t) \;\le\; L \int_{t_0}^{t} \mu(s)\, ds\,.$$

From Lemma 2.2 we see that $\mu(t) = 0$ for all $t \in [t_0, T]$, that is $u = \tilde{u}$. □

Remark 2.4 From the above proof it can also be shown that

$$\|u(t) - v_k(t)\| \;\le\; \Big(\textstyle\sum_{j=k}^{\infty} \frac{1}{j!} \big(L(t-t_0)\big)^j \Big) \cdot \gamma_0(t) \qquad (k = 0, 1, \ldots)\,,$$

with $\gamma_0(t) = (t - t_0) \max_{t_0 \le s \le t} \|f(s, u_0)\|$. Therefore, $\|u(t) - v_k(t)\|$ quickly becomes small for increasing k if t is close to t_0. Nevertheless, Picard iteration is not so often used to find numerical approximations. Other methods, such as Runge-Kutta methods, are easier to program and require less computer memory. (Note that to evaluate the integral in (2.5) accurately with numerical quadrature, many values of $v_{k-1}(s)$ for different $s \in [t_0, T]$ need to be available.) ◇

2.3 Local Existence and Uniqueness

The global Lipschitz condition (2.6) with $\mathcal{D} = \mathbb{R}^m$ excludes many interesting nonlinear problems. Therefore we consider a local version, assuming (2.6) to hold with a ball $\mathcal{D} = \mathcal{D}_0$,

$$\mathcal{D}_0 \;=\; \{ v \in \mathbb{R}^m : \|v - v_0\| \le R_0 \}\,, \tag{2.7}$$

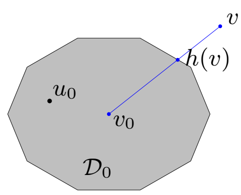
containing u_0 in its interior, $\|u_0 - v_0\| < R_0$. If f is continuous on $[t_0, T] \times \mathcal{D}_0$, then we know that f is bounded on this compact set: there is a $\gamma > 0$ such that

$$\|f(t, v)\| \;\le\; \gamma \qquad (\text{for all } t \in [t_0, T],\ v \in \mathcal{D}_0)\,. \tag{2.8}$$

We consider a function $\bar{f}$ that coincides with f on $[t_0, T] \times \mathcal{D}_0$ and is such that it will satisfy a global Lipschitz condition. This function is defined as

$$\bar{f}(t, v) = f(t, h(v)) \quad \text{with} \quad h(v) \;=\; \begin{cases} v & \text{if } v \in \mathcal{D}_0\,, \\ v_0 + \frac{R_0}{\|v - v_0\|}(v - v_0) & \text{if } v \notin \mathcal{D}_0\,. \end{cases} \tag{2.9}$$

It is clear that $\bar{f}(t, v) = f(t, v)$ for $v \in \mathcal{D}_0$, and $\|\bar{f}(t, v)\| \le \gamma$ for all $v \in \mathbb{R}^m$ and $t \in [t_0, T]$. Moreover, we have $\|h(v) - h(\tilde{v})\| \le 2\|v - \tilde{v}\|$ for any pair $v, \tilde{v} \in \mathbb{R}^m$; see Exercise 2.8. It follows that $\bar{f}$ does satisfy the global Lipschitz condition on $\mathbb{R}^m$ with constant $2L$:



$$\|\bar{f}(t, v) - \bar{f}(t, \tilde{v})\| = \|f(t, h(v)) - f(t, h(\tilde{v}))\| \;\le\; L\|h(v) - h(\tilde{v})\| \;\le\; 2L\|v - \tilde{v}\|$$

for all $t \in [t_0, T]$ and $v, \tilde{v} \in \mathbb{R}^m$.

We therefore know that the solution of the initial value problem

$$\bar{u}'(t) = \bar{f}(t, \bar{u}(t)), \qquad \bar{u}(t_0) = u_0, \tag{2.10}$$

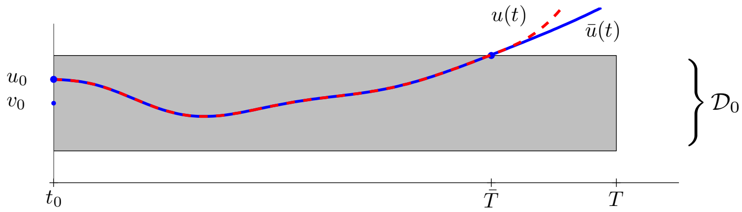
has a unique solution $\bar{u}$ on $[t_0, T]$, and we also have

$$\|\bar{u}(t) - v_0\| \le \|u_0 - v_0\| + \int_{t_0}^{t} \|\bar{f}(s, \bar{u}(s))\| \, ds \le \|u_0 - v_0\| + \gamma(t - t_0). \tag{2.11}$$

Let $\bar{T}$ be the largest number in $[t_0, T]$ such that $\bar{u}(t) \in \mathcal{D}_0$ for $t \in [t_0, \bar{T}]$. From (2.11) it follows that

$$\bar{T} - t_0 \ge \min \left\{ T - t_0, \, \frac{1}{\gamma} \left(R_0 - \|u_0 - v_0\| \right) \right\} > 0.$$

Since $\bar{f} = f$ on $[t_0, T] \times \mathcal{D}_0$, we see that $\bar{u}$ is also a solution of our original problem (2.1) on the interval $[t_0, \bar{T}]$. Conversely, any solution u of (2.1) is a solution of (2.10) as long as it stays in $\mathcal{D}_0$, so it must coincide with $\bar{u}$. Consequently, (2.1) has a unique solution on $[t_0, \bar{T}]$.



In summary, we have obtained the following result on local existence and uniqueness of solutions of initial value problems:

Theorem 2.5 Let $\mathcal{D}_0 = \{v \in \mathbb{R}^m : \|v - v_0\| \le R_0\}$. Assume $\|u_0 - v_0\| < R_0$, f is continuous on $[t_0, T] \times \mathcal{D}_0$, and the Lipschitz condition (2.6) is satisfied with $\mathcal{D} = \mathcal{D}_0$. Then the initial value problem (2.1) has a unique solution on an interval $[t_0, \bar{T}]$, $\bar{T} > t_0$, where either $\bar{T} = T$ or $\|u(\bar{T}) - v_0\| = R_0$. □

We know, by the mean-value estimate (2.3), that if f is continuously differentiable on $[t_0, T] \times \mathcal{D}_0$, then it satisfies the Lipschitz condition (2.6) with some $L > 0$. (Actually, we only need continuity of $f(t, v)$ with respect to t.) Local existence and uniqueness of solutions of the initial value problem is then guaranteed by the above theorem. Instead of $t \ge t_0$, we can also consider $t \le t_0$. This gives the following result:

Corollary 2.6 Assume that f is continuously differentiable on an open set $\mathcal{E}$ around $(t_0, u_0) \in \mathbb{R} \times \mathbb{R}^m$. Then the the initial value problem (2.1) has a unique solution on some interval $[t_0 - \delta, t_0 + \delta]$, $\delta > 0$.

Proof. For $T, R > 0$ small enough the set $\mathcal{C} = \{(t, v) : |t - t_0| \le T, \|v - u_0\| \le R\}$ is contained in $\mathcal{E}$. From the mean-value estimate (2.3) it follows that f will fulfil a

Lipschitz condition on $\mathcal{C}$ with constant $L = \max_{(t,v)\in\mathcal{C}} \|(\frac{\partial f_i(t,v)}{\partial v_j})\|$. Consequently, the initial value problem has a unique solution u_a on an interval $[t_0, t_0 + \delta_a]$ with $\delta_a > 0$.

To deal with $t < t_0$, we can introduce $\bar{t} = 2t_0 - t$ and $\bar{f}(\bar{t}, v) = -f(t, v)$. This function $\bar{f}$ will also fulfil the Lipschitz condition on $\mathcal{C}$ with constant L. Therefore the initial value problem $\bar{u}'(\bar{t}) = \bar{f}(\bar{t}, \bar{u}(\bar{t}))$, $\bar{u}(t_0) = u_0$ has a unique solution $\bar{u}$ on an interval $[t_0, t_0 + \delta_b]$, $\delta_b > 0$. But then $u_b(t) = \bar{u}(2t_0 - t)$ is seen to be the unique solution of the original initial value problem (2.1) on $[t_0 - \delta_b, t_0]$.

By combining these two pieces, setting $u(t) = u_a(t)$ on $[t_0, t_0 + \delta_a]$ and $u(t) = u_b(t)$ on $[t_0 - \delta_b, t_0]$, it is now seen that this u is the unique solution on the interval $[t_0 - \delta_b, t_0 + \delta_a]$. Taking $\delta = \min\{\delta_a, \delta_b\}$ completes the proof. $\square$

The maximal interval on which the solution of the initial value problem (2.1) will exist, as well as the interval where the solution will be unique, may depend on the starting point t_0 and initial value u_0. The following two examples illustrate this.

Example 2.7 Consider the differential equation

$$u'(t) = t^2\, u(t)^2\,.$$

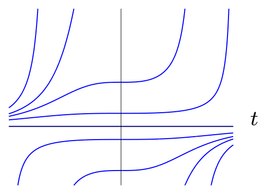
If we specify $u(t_0) = u_0$ with given $t_0, u_0 \in \mathbb{R}$, we get an initial value problem.

The function $f(t, v) = t^2 v^2$ is continuously differentiable on $\mathbb{R} \times \mathbb{R}$. Furthermore, this function will satisfy a Lipschitz condition on any bounded set $[t_0 - T, t_0 + T] \times [u_0 - R, u_0 + R]$ with $T, R > 0$, but not on the strip $[t_0 - T, t_0 + T] \times \mathbb{R}$. Consequently, by Corollary 2.6, for any $t_0, u_0 \in \mathbb{R}$, the initial value problem will have locally a unique solution on some interval $[t_0 - \delta, t_0 + \delta]$, but we do not know yet whether a solution will exist on the whole real line.

The local uniqueness property implies that the graphs of two solutions of the differential equation cannot intersect. Further we see immediately that $u = 0$ is a solution, and for any other solution we have $u'(t) > 0$ if $tu(t) \neq 0$. This already gives insight in the qualitative behaviour of solutions.

In fact, explicit expressions for the solutions can be derived quite easily because the equation has separable variables. For $u > 0$, $u = 0$ and $u < 0$, respectively, the following solutions of the differential equation are found:

$$\begin{aligned} u(t) &= 3/(\alpha - t^3) && \text{if} \quad t^3 < \alpha\,, \\ u(t) &= 0 && \text{if} \quad t \in \mathbb{R}\,, \\ u(t) &= -3/(t^3 - \beta) && \text{if} \quad t^3 > \beta \end{aligned}$$



with arbitrary constants $\alpha, \beta \in \mathbb{R}$.

Consequently, for any $u_0 > 0$, the initial value problem has a unique solution on the interval $(-\infty, \sqrt[3]{\alpha})$ with $\alpha = t_0^3 + 3/u_0$, and the solution blows up if $t \uparrow \sqrt[3]{\alpha}$. (Here $\sqrt[3]{\alpha}$ is taken real, negative if $\alpha < 0$.) Likewise, if $u_0 < 0$ we get a unique solution on $(\sqrt[3]{\beta}, \infty)$, $\beta = t_0^3 + 3/u_0$, and if $u_0 = 0$ there is the solution $u(t) = 0$ for all $t \in \mathbb{R}$. From the local uniqueness property it follows that there are no other solutions. $\diamond$

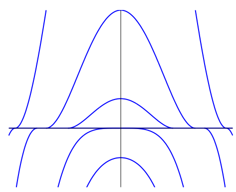
Example 2.8 Consider the initial value problem

$$u'(t) = -t\sqrt{|u(t)|}\,, \qquad u(t_0) = u_0\,,$$

with $t_0, u_0 \in \mathbb{R}$. The function $f(t, v) = -t\sqrt{|v|}$ is continuously differentiable around any point (t_0, u_0) with $u_0 \neq 0$. Therefore, according to Corollary 2.6, if $u_0 \neq 0$ we have locally a unique solution, and insight in its behaviour can be obtained by considering the sign of $u'(t)$. On the other hand, if $u_0 = 0$, then we do not know yet whether there is a unique solution.

This equation has again separable variables, and explicit expressions for solutions are easily derived. By considering $u > 0$, $u = 0$ and $u < 0$, the following solutions of the differential equation are found:

$$\begin{aligned} u(t) &= \tfrac{1}{16}(t^2 - \alpha)^2 && \text{if} \quad t^2 \leq \alpha\,, \\ u(t) &= 0 && \text{if} \quad t \in \mathbb{R}\,, \\ u(t) &= -\tfrac{1}{16}(t^2 - \beta)^2 && \text{if} \quad t^2 \geq \beta \end{aligned}$$



with integration constants $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$.

Let us consider the initial value problem with $u_0 > 0$. Then there is a unique solution $u(t) = \frac{1}{16}(t^2 - \alpha)^2$ on $[-\sqrt{\alpha}, \sqrt{\alpha}]$ with $\alpha = t_0^2 + 4\sqrt{u_0}$. The solution can be extended to the whole real line, but this extension is not unique. We can take $u(t) = 0$ on the intervals $[-\beta_1, -\alpha]$ and $[\alpha, \beta_2]$, with $\beta_1, \beta_2 \geq \alpha$, and then continue to the right with $u(t) = -\frac{1}{16}(t^2 - \beta_2)^2$ for $t \geq \sqrt{\beta_2}$ and to the left with $u(t) = -\frac{1}{16}(t^2 - \beta_1)^2$ for $t \leq -\sqrt{\beta_1}$ ◇

Remark 2.9 (Peano's theorem) In Corollary 2.6, the function f was assumed to be continuously differentiable on an open set $\mathcal{E}$ around (t_0, u_0) to ensure a local Lipschitz condition (2.6). In fact, even if f is merely continuous on $\mathcal{E}$ then local existence of a solution of the initial value problem is already guaranteed. This is known as *Peano's theorem.* This theorem is more difficult to prove than the above results; moreover, uniqueness does *not* follow, as illustrated by Example 2.8.

The proof of Peano's theorem can be found in the books listed in the preface. Also alternative proofs for above existence and uniqueness results with Lipschitz conditions can be found there. For example, in the book of Teschl (2012) existence and uniqueness is shown using the Banach fixed point theorem. For an older proof, due to Cauchy, based on the approximation method of Euler, we refer to the book of Hairer, Nørsett & Wanner (1993). ◇

2.4 A Perturbation Result

It was mentioned in Section 1 that we may want to know whether two models will give almost the same outcome, say a simple model that is easily analyzed and a more complicated model that gives an accurate description of the reality.

Suppose f is defined on $[t_0, T] \times \mathbb{R}^m$. Consider along with (2.1) also a solution $\tilde{u}$ of the initial value problem

$$\tilde{u}'(t) = \tilde{f}(t, \tilde{u}(t))\,, \qquad \tilde{u}(t_0) = \tilde{u}_0\,, \tag{2.12}$$

with $\tilde{f} : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\tilde{u}_0 \in \mathbb{R}^m$.

Theorem 2.10 Let u and $\tilde{u}$ be solutions on $[t_0, T]$ of (2.1) and (2.12), respectively. Assume f satisfies the Lipschitz condition (2.6) with $\mathcal{D} = \mathbb{R}^m$, and we have $\|f(t, v) - \tilde{f}(t, v)\| \le M$ for all $t \in [t_0, T]$ and $v \in \mathbb{R}^m$. Then

$$\|u(t) - \tilde{u}(t)\| \;\le\; e^{L(t-t_0)}\|u_0 - \tilde{u}_0\| + \frac{1}{L}\left(e^{L(t-t_0)} - 1\right) M \qquad (\text{for } t \in [t_0, T])\,.$$

Proof. Writing the initial value problems in integral form gives

$$\|u(t) - \tilde{u}(t)\| \;\le\; \|u_0 - \tilde{u}_0\| + \textstyle\int_{t_0}^{t} \|f(s, u(s)) - \tilde{f}(s, \tilde{u}(s))\|\, ds\,.$$

Furthermore, since $\|f(s, u) - \tilde{f}(s, \tilde{u})\| \le \|f(s, u) - f(s, \tilde{u})\| + \|f(s, \tilde{u}) - \tilde{f}(s, \tilde{u})\|$, it is seen that

$$\|f(s, u(s)) - \tilde{f}(s, \tilde{u}(s))\| \;\le\; L\,\|u(s) - \tilde{u}(s)\| + M\,.$$

Application of Lemma 2.2 with $\alpha(t) = \|u_0 - \tilde{u}_0\| + M(t - t_0)$ and $\beta = L$ provides the proof. $\square$

As in the previous subsection, this result with a global Lipschitz condition can be put into a local form with a bounded set $\mathcal{D}_0$, where we then only have to require that $\|f(t, v) - \tilde{f}(t, v)\| \le M$ for $t \in [t_0, T]$, $v \in \mathcal{D}_0$.

In applications, $\tilde{f}$ and $\tilde{u}_0$ are often viewed as perturbations of f and u_0. In particular, with $\tilde{f} = f$ we get $\|u(t) - \tilde{u}(t)\| \le e^{L(t-t_0)}\|u_0 - \tilde{u}_0\|$. This gives a bound on the sensitivity of solutions of our initial value problem with respect to perturbations on the initial value.

2.5 Exercises

Exercise 2.1. Determine whether a Lipschitz condition holds around $v = 0$ for the following scalar functions:

(a) $f(v) = \frac{1}{1-v^2}$, (b) $f(v) = |v|^{1/3}$, (c) $f(v) = v^2$.

For this last case, do we have a Lipschitz condition on the whole real line $\mathcal{D} = \mathbb{R}$?

Exercise 2.2.* Follow the derivation of Theorem 2.5 to find a lower bound for $\bar{T}$, which may depend on u_0, such that the initial value problem (2.1) has a unique solution on $[0, \bar{T}]$ with the following scalar differential equations:

(a) $u' = u^2$, (b) $u' = -|u|^{1/2}$, (c) $u' = \sin(\pi t)e^{-u^2}$.

Exercise 2.3. In the Examples 1.8 and 2.1, formulas are found for solutions of $u' = |u|^{\kappa+1}$ ($\kappa > 0$) and $u' = -3|u|^{2/3}$, respectively, with initial value $u(0) = 1$. Show that there are no other solutions.

Exercise 2.4. Consider the differential equation

$$u' = \frac{u^2}{1 + t^2}\,.$$

What can you say in advance about local existence and uniqueness of solutions passing through a point (t_0, u_0) in the (t, u)-plane? Solutions can be found by separation of variables. Make a sketch of the solutions. On what intervals do they exist?

Exercise 2.5. Show that the Picard iteration for the linear initial value problem $u'(t) = Au(t)$, $u(0) = u_0$ gives $v_k(t) = (I + tA + \ldots + \frac{1}{k!}(tA)^k)u_0$, where I stands for the identity matrix.

Exercise 2.6. Consider the initial value problem

$$w''(t) = g(t, w(t), w'(t)) \,, \qquad w(0) = w_0 \,, \quad w'(0) = w_0'$$

for a scalar second-order differential equation. Let $r_1, r_2 > 0$. Assume that $g(t, u_1, u_2)$ is continuous in t, continuously differentiable in u_1, u_2, such that

$$|\tfrac{\partial}{\partial u_j} g(t, u_1, u_2)| \le r_j \qquad (\text{for } j = 1, 2 \text{ and all } t \in [0, T],\ u_1, u_2 \in \mathbb{R}) \,.$$

Prove that this initial value problem has a unique solution on $[0, T]$. Hint: Consider (1.7) with the maximum norm in $\mathbb{R}^2$ and $L = \max\{1, r_1 + r_2\}$.

Exercise 2.7. Consider the autonomous problem $u'(t) = f(u(t))$, $u(t_0) = u_0$ on $\mathbb{R}^m$. Suppose f is continuously differentiable on $\mathbb{R}^m$. Show that: either $u(t)$ exists for all $t \ge t_0$, or there is a finite $t_1 > t_0$ such that $u(t)$ exists for $t \in [t_0, t_1)$ and $\lim_{t \uparrow t_1} \|u(t)\| = \infty$.

Exercise 2.8.* For (2.9) it was claimed that $\|h(v) - h(w)\| \le 2\|v - w\|$.
(a) Show that this is valid. Take for convenience $v_0 = 0$, $R_0 = 1$. Hint: write $h(v) = g(\|v\|)v$ with $g(s) = \min(1, \frac{1}{s})$ for $s \ge 0$, and use $|g(s) - g(t)| \le \frac{|s-t|}{st}$ to demonstrate that $\|(g(\|v\|) - g(\|w\|)w\| \le \|v - w\|$ if $\|v\| > 1$.
(b) One might think that $\|h(v) - h(w)\| \le \|v - w\|$ always holds. Find a counter example with the maximum norm on $\mathbb{R}^2$ with $v = (1, 1)^T$ and w near v.

3 Linear Systems

In this section we will study the solutions of initial value problems for systems of the type

$$u'(t) = Au(t) + g(t)\,, \qquad u(t_0) = u_0\,, \tag{3.1}$$

with a matrix $A \in \mathbb{R}^{m\times m}$ and $g : \mathbb{R} \to \mathbb{R}^m$ continuous. This differential equation is called *linear* with *constant coefficients*, and if $g = 0$ it is called *homogeneous*. Later we will also consider general linear equations where A may depend on t.

We will need in this section some concepts from linear algebra, such as norms on $\mathbb{R}^m$ and $\mathbb{C}^m$, induced matrix norms and the Jordan normal forms of matrices. These concepts can be found in the standard text-books on linear algebra (for instance: Horn & Johnson, *Matrix Analysis*, 1990) and will not be discussed here in full detail.

Even though we will be primarily interested in real valued problems, with $A \in \mathbb{R}^{m\times m}$, it is convenient to consider complex matrices $A \in \mathbb{C}^{m\times m}$. This is due to the fact that even if A is real, its eigenvalues and eigenvectors are complex in general. The results obtained thus far for real valued systems carry over directly to the complex case because we can always rewrite a differential equation in $\mathbb{C}^n$ as an equivalent system in $\mathbb{R}^{2n}$ by taking real and complex parts.

For a given norm $\|\cdot\|$ on $\mathbb{C}^m$, we define the induced matrix norm of a matrix $A \in \mathbb{C}^{m\times m}$ by

$$\|A\| = \max_{v\in\mathbb{C}^m, v\neq 0} \frac{\|Av\|}{\|v\|}\,. \tag{3.2}$$

Justification of this definition is given in Exercise 3.1. This matrix norm $\|A\|$ can be characterized as follows: it is the smallest nonnegative number α such that $\|Av\| \leq \alpha\|v\|$ for all $v \in \mathbb{C}^m$.

Furthermore, note that $\mathbb{C}^{m\times m}$ itself can be viewed as a linear vector space. The induced matrix norm (3.2) provides a norm on this space, so we can discuss convergence of sequences or series of matrices. Along with the triangle inequality $\|A + B\| \leq \|A\| + \|B\|$, it is also easy to see that $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathbb{C}^{m\times m}$, and in particular $\|A^k\| \leq \|A\|^k$ for any power $k \in \mathbb{N}$.

Example 3.1 As on $\mathbb{R}^m$, the most common norms on $\mathbb{C}^m$ are the Euclidian norm (also called l_2-norm) and the maximum norm (also known as the l_∞-norm):

$$\|v\|_2 = \left(\textstyle\sum_{j=1}^m |v_j|^2\right)^{1/2}\,, \qquad \|v\|_\infty = \max_{1\leq j\leq m} |v_j|$$

for vectors $v = (v_j) \in \mathbb{C}^m$ (short notation for $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{C}^m$). The corresponding induced matrix norms for $A = (a_{jk}) \in \mathbb{C}^{m\times m}$ are given by

$$\|A\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ eigenvalue of } A^*A\}\,, \qquad \|A\|_\infty = \max_{1\leq j\leq m} \textstyle\sum_{k=1}^m |a_{jk}|\,,$$

where $A^* = (\overline{a_{kj}})$ is the (Hermitian) adjoint of A. The derivation of these expressions is left as an exercise. $\diamond$

3.1 Matrix Exponentials

It was seen in Section 1 that for the scalar case an explicit solution for inhomogeneous equations (1.12) could be obtained by first deriving expressions for the homogeneous case. For systems we proceed similarly.

So first, let us consider homogeneous problems

$$u'(t) = Au(t)\,, \qquad u(0) = u_0\,, \tag{3.3}$$

with $A \in \mathbb{C}^{m\times m}$ and $t \in \mathbb{R}$. The starting time is taken for the moment as $t_0 = 0$ for notational convenience. The function $f(t,v) = Av$ satisfies a global Lipschitz condition with constant $L = \|A\|$. We therefore know, by Picard's theorem, that (3.3) has a unique solution on any time interval $[0,T]$. We can also consider $t \le 0$, as in Exercise 1.2, and extend the solution to $[-T,T]$ with arbitrary $T > 0$.

As we will see, the solution is given by $u(t) = \exp(tA)u_0$, where the exponent of the matrix is defined as

$$\exp(tA) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\,(t\,A)^k\,. \tag{3.4}$$

Here, in the first term we take by convention $(tA)^0 = I$, the identity matrix. Instead of $\exp(tA)$ we will usually write $e^{tA} = I + tA + \frac{1}{2}(t\,A)^2 + \cdots$. For this matrix exponential we have

$$\frac{d}{dt}e^{tA} = Ae^{tA}\,. \tag{3.5}$$

Theorem 3.2 The homogeneous problem (3.3) has unique solution $u(t) = e^{tA}u_0$, where $e^{tA} = \exp(tA)$ is defined by (3.4). For this matrix exponential, property (3.5) is valid.

Proof. The Picard iterates (2.5) for (3.3) are given by $v_n(t) = \sum_{k=0}^{n} \frac{1}{k!}(tA)^k u_0$ (see Exercise 2.5). We saw in the proof of Theorem 2.3 that these iterates converge to $u(t)$ for arbitrary u_0. Therefore $u(t) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}(tA)^k u_0 = e^{tA}u_0$, and we have $\frac{d}{dt}e^{tA}u_0 = Ae^{tA}u_0$ for any $u_0 \in \mathbb{C}^m$. □

An important property of the matrix exponential is the following: if $A, B \in \mathbb{C}^{m\times m}$ are commuting matrices ($AB = BA$), then

$$e^{t(A+B)} = e^{tA}e^{tB} \tag{3.6}$$

for all $t \in \mathbb{R}$. To prove this result we can mimic a proof of the scalar case with manipulation of the power series. A more elegant proof can be obtained by using uniqueness of solutions for linear initial value problems; see Exercise 3.4. In particular we see form (3.6) that for any $s, t \in \mathbb{R}$

$$e^{(t+s)A} = e^{tA}e^{sA}\,, \qquad (e^{tA})^{-1} = e^{-tA}\,. \tag{3.7}$$

Finally we mention that, similar as for the scalar case,

$$e^{tA} = \lim_{n\to\infty} (I + \tfrac{t}{n}A)^n \,. \tag{3.8}$$

The proof of this relation is a little technical; it is treated in Exercise 3.12.

Variation of constants formula. For inhomogeneous problems (3.1), with arbitrary starting time $t_0 \in \mathbb{R}$, we can derive a formula similar to (1.15) for the scalar case by considering $u(t) = e^{tA}c(t)$ with $c(t) \in \mathbb{R}^m$. An other way to derive it is to use the idea of integrating factors. Multiplying (3.3) by e^{-tA} we get

$$\tfrac{d}{dt}(e^{-tA}u(t)) \;=\; e^{-tA}u'(t) - Ae^{-tA}u(t) \;=\; e^{-tA}(u'(t) - Au(t)) = e^{-tA}g(t) \,.$$

Integration from t_0 to t gives $e^{-tA}u(t) - e^{-t_0A}u_0 = \int_{t_0}^t e^{-sA}g(s)\,ds$, and therefore

$$u(t) \;=\; e^{(t-t_0)A}\,u_0 \;+\; \int_{t_0}^t e^{(t-s)A}\,g(s)\,ds \,. \tag{3.9}$$

3.2 Computing Matrix Exponentials

To find formulas for matrix exponentials, we first observe that if $B, V \in \mathbb{C}^{m\times m}$ with V nonsingular, then $(VBV^{-1})^k = VB^kV^{-1}$ for any power. It therefore follows directly from (3.4) that

$$A = VBV^{-1} \qquad \Longrightarrow \qquad e^{tA} = Ve^{tB}V^{-1}.$$

If B is diagonal, $B = \mathrm{diag}(\beta_1, \dots, \beta_m)$, then $e^{tB} = \mathrm{diag}(e^{t\beta_j}, \dots, e^{t\beta_m})$. In the same way, if B is block-diagonal with blocks $B_1, \dots, B_n$ on the diagonal (which we denote as $B = \mathrm{Diag}(B_1, \dots, B_n)$ with capital D) then $e^{tB} = \mathrm{Diag}(e^{tB_1}, \dots, e^{tB_n})$.

If the matrix $A \in \mathbb{C}^{m\times m}$ has a complete set of m independent eigenvectors, then it is diagonalizable, $A = V\Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$. The λ_j are the eigenvalues of A and the j-th column of V is the corresponding eigenvector. Furthermore, $e^{t\Lambda}$ is the diagonal matrix with entries $e^{t\lambda_j}$ on the diagonal. So, for a diagonalizable matrix we can compute its exponent as

$$e^{tA} = Ve^{t\Lambda}V^{-1}\,, \qquad e^{t\Lambda} = \mathrm{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_m})\,. \tag{3.10}$$

Unfortunately, not all matrices are diagonalizable. However it is known (linear algebra) that we do always have a Jordan decomposition $A = VJV^{-1}$ where J is a block-diagonal matrix of the form

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{pmatrix}, \qquad J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{m_k\times m_k},$$

with $m_1 + m_2 + \cdots + m_l = m$. The same eigenvalue may appear in several Jordan blocks J_k. If an eigenvalue appears in a Jordan block of dimension larger than one, it is called *defective.* If all blocks have dimension $m_k = 1$ we are back in the diagonal case.

Since any power J^n is again block-diagonal, with blocks J_k^n, we see that

$$e^{tA} = V e^{tJ} V^{-1}\,, \quad e^{tJ} = \mathrm{Diag}(e^{tJ_1}, \dots, e^{tJ_l})\,. \tag{3.11}$$

It remains to compute the exponential for a single Jordan block.

For this, we write $J_k = \lambda_k I + E$, where

$$E = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Thus tE is the matrix with only entries t on the first upper diagonal, $(tE)^2$ has entries t^2 on the second upper diagonal and entries 0 elsewhere, etc., and finally $(tE)^{m_k} = O$, the zero matrix. Hence e^{tE} is the upper triangular Toeplitz matrix with entries $\frac{1}{j!}t^j$ on the j-th upper diagonal, $j = 1, \dots, m_k - 1$. Further we have, according to (3.6), $e^{tJ_k} = e^{t\lambda_k I} e^{tE} = e^{t\lambda_k} e^{tE}$, and therefore

$$\begin{aligned} e^{tJ_k} \;&=\; e^{t\lambda_k}\Big(I + tE + \tfrac{1}{2!}t^2E^2 + \cdots + \tfrac{1}{(m_k-1)!}t^{m_k-1}E^{m_k-1}\Big) \\ &=\; e^{t\lambda_k} \begin{pmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(m_k-1)!}t^{m_k-1} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{1}{2!}t^2 \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}. \end{aligned} \tag{3.12}$$

Consequently, we can now compute in principle the exponential e^{tA} for any matrix A. This means that for any homogeneous equation $u' = Au$ we have an explicit expression for the solutions, and by the variation of constants formula the same holds for the inhomogeneous problem (3.1).

For actual computations, it is in general much easier to let the exponential stand in its decomposed form (3.10) or (3.11). This holds in particular when dealing with an inhomogeneous term and the variation of constants formula (3.9); see e.g. Exercise 3.9.

Finally we mention that (3.11) implies

$$\frac{1}{C}\,\|e^{tJ}\| \;\le\; \|e^{tA}\| \;\le\; C\,\|e^{tJ}\| \qquad \text{with} \quad C = \|V\|\cdot\|V^{-1}\|\,. \tag{3.13a}$$

Here the first inequality follows by writing $e^{tJ} = V^{-1}e^{tA}V$. So, we will have $\sup_{t\ge 0}\|e^{tA}\| < \infty$ or $\lim_{t\to\infty}\|e^{tA}\| = 0$ iff the same properties hold for $\|e^{tJ}\|$. In the maximum norm, we further have the following simple expression:

$$\|e^{tJ}\|_\infty \;=\; \max_{1\le k\le l}\|e^{tJ_k}\|_\infty \;=\; \max_{1\le k\le l}\,|\,e^{t\lambda_k}|\,\sum_{j=0}^{m_k-1}\frac{|\,t\,|^j}{j!}\,. \tag{3.13b}$$

3.3 Two-Dimensional Problems and Phase Planes

Let us consider some examples for the simple case with real matrices $A \in \mathbb{R}^{2\times 2}$ and solutions $u(t) \in \mathbb{R}^2$ of the homogeneous problem (3.3).

Diagonalizable case. First assume A is diagonalizable, $A = V\Lambda V^{-1}$,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix},$$

where the columns $\underline{v}_1, \underline{v}_2 \in \mathbb{C}^2$ of V are the eigenvectors of A; these $\underline{v}_j$ are underlined to make it clear that they are vectors themselves, instead of components. Then the general solution of $u' = Au$ is

$$u(t) \;=\; c_1\, e^{\lambda_1 t}\, \underline{v}_1 \;+\; c_2\, e^{\lambda_2 t}\, \underline{v}_2\,. \tag{3.14}$$

The constants c_1, c_2 are determined by the initial condition, $u_0 = c_1\underline{v}_1 + c_2\underline{v}_2$, which is just $(c_1, c_2)^T = V^{-1}u_0$.

This follows from the general formula (3.10) for the matrix exponential. Instead of using that formula we can also get an equivalent, but more direct derivation by introducing $w(t) = (w_1(t), w_2(t))^T = V^{-1}u(t)$. Then $w'(t) = \Lambda w(t)$, that is,

$$w_j'(t) = \lambda_j w_j(t) \qquad (j = 1, 2)\,.$$

Hence $w_j(t) = c_j e^{\lambda_j t}$ with $c_j = w_j(0)$, and we obtain (3.14) from $u(t) = Vw(t)$.

Diagonalizable case with complex eigenvalues. If the eigenvalues are complex, then also the eigenvectors are complex. Even though formula (3.14) is still correct, it is then not very transparent. It can be rewritten by using $\lambda_{1,2} = \xi \pm i\eta$, since complex eigenvalues of a real 2×2 matrix will be complex conjugate. Using $e^{\lambda_{1,2}t} = e^{\xi t}(\cos(\eta t) \pm i \sin(\eta t))$, we obtain from (3.14) a formula

$$u(t) \;=\; e^{\xi t}\cos(\eta t)\, \underline{d}_1 + e^{\xi t}\sin(\eta t)\, \underline{d}_2 \tag{3.15}$$

with real vectors $\underline{d}_j \in \mathbb{R}^2$ (they must be real, because $u(t)$ is real for all t). These two vectors are not related anymore to the eigenvectors. Instead, by considering $u(0) = u_0$, $u'(0) = Au_0$, it is seen that $\underline{d}_1 = u_0$ and $\underline{d}_2 = \frac{1}{\eta}(A - \xi I)u_0$.

Defective case. For the case of a single, defective eigenvalue, we have the Jordan decomposition $A = VJV^{-1}$ with

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad V = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix},$$

where now only $\underline{v}_1$ is an eigenvector of A. For $\underline{v}_2$, which is called a generalized eigenvector, we have $(A - \lambda I)\underline{v}_2 = \underline{v}_1$. From (3.12) we now obtain

$$u(t) \;=\; (c_1 + c_2 t)\, e^{\lambda t}\, \underline{v}_1 \;+\; c_2\, e^{\lambda t}\, \underline{v}_2\,. \tag{3.16}$$

Again, the initial condition specifies the constants c_1, c_2 by $u_0 = c_1\underline{v}_1 + c_2\underline{v}_2$.

Instead of using the general formula (3.12), equation (3.16) can also be derived directly, which may give some more insight in the appearance of the $te^{\lambda t}$ term. If we set $w(t) = V^{-1}u(t)$, then $w'(t) = Jw(t)$, that is,

$$w_1'(t) = \lambda w_1(t) + w_2(t)\,, \qquad w_2'(t) = \lambda w_2(t)\,.$$

The second equation gives $w_2(t) = e^{\lambda t} w_2(0)$, of course. But then it is seen by the scalar variation of constants formula (1.15) that $w_1(t) = e^{\lambda t} w_1(0) + te^{\lambda t} w_2(0)$. By the back-transformation $u(t) = Vw(t)$ we arrive again at (3.16).

Phase portraits. Already for the simple 2×2 case there are some interesting features. To get insight in the behaviour of solutions, we could try to compute and plot the components $u_1(t)$ and $u_2(t)$ versus time t for a number of initial conditions. However, it is much more clear what is happening by considering trajectories, which will be discussed here.

For a real system in two dimensions, let us call $x(t) = u_1(t)$ and $y(t) = u_2(t)$. Then the solution between two time points, say $t = t_a$ and $t = t_b$, gives a curve $\{(x(t), y(t)) : t \in [t_a, t_b]\}$ in the xy-plane. This curve is called a *trajectory* or *orbit*. The xy-plane itself is usually called the *phase plane*. If we draw a number of trajectories, with different initial positions and $t \in \mathbb{R}$, we obtain a so-called *phase portrait*.

Some phase portraits are presented in Figure 3.1. Each plot corresponds to solutions with a certain matrix $A \in \mathbb{R}^{2\times 2}$. We see that there are a number of different cases that can be distinguished. For the following discussion, first observe that the origin always corresponds to a stationary solution. If all other trajectories stay bounded for $t \in [0, \infty)$ the origin is called a *stable* stationary solution. Otherwise, if some trajectories tend to infinity, we call it *unstable*. This behaviour, and also how the solutions tend to 0 or diverge from the origin (the shape of the trajectories), is determined by the eigenvalues.

Let us first suppose that the eigenvalues of the matrix A are not defective, so formula (3.14) applies. If (*i*) the eigenvalues are real with the same sign, all solutions will converge (negative sign) to the origin or diverge from it (positive sign). The origin is then called a stable or unstable *node*. The curvature of the orbits is primarily determined by the ratio $r = \lambda_1/\lambda_2$ of the eigenvalues (as can be seen by considering $w = V^{-1}u$, for which we get $w_2^{\lambda_1} = c \cdot w_1^{\lambda_2}$).

In case the eigenvalues are complex, they must be complex conjugate, $\lambda_{1,2} = \xi \pm i\eta$, and we can use formula (3.15). If (*ii*) $\xi \neq 0$ then the solutions will spiral towards the origin ($\xi < 0$) or away from it ($\xi > 0$), and the origin is then called a *focus* or *spiral point*. If (*iii*) $\xi = 0$ then the solutions become periodic.

We can also have (*iv*) two real nonzero eigenvalues of opposite sign. Then the origin is called a *saddle point* for the differential equation. Only solutions that start at a multiple of the eigenvector corresponding to the negative eigenvalue will tend to the origin. All other solutions will eventually tend to infinity.

Another phase portrait is obtained if (*v*) one eigenvalue is zero and the other

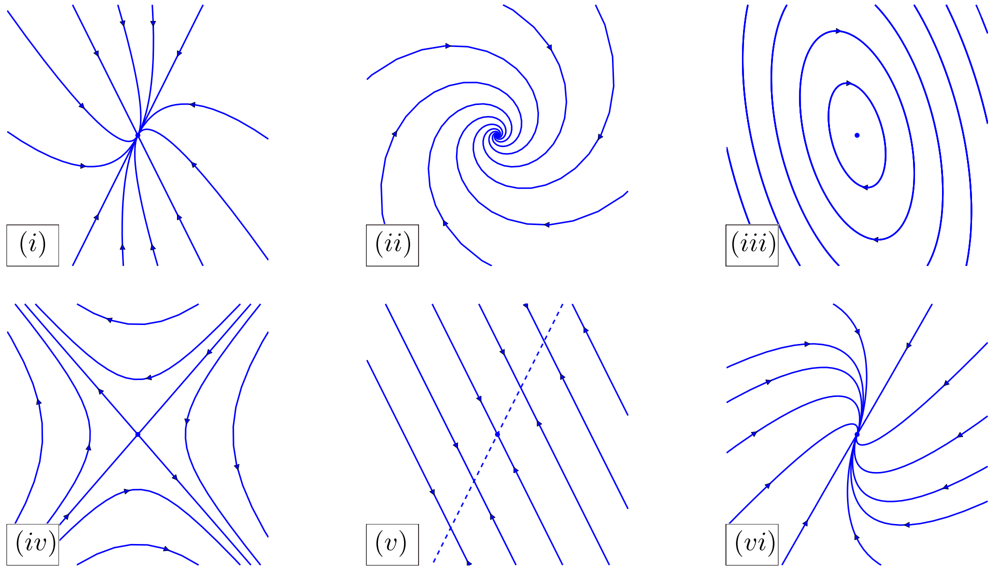


Figure 3.1: Phase portraits for the various cases: (*i*) negative, real eigenvalues; (*ii*) imaginary eigenvalues with negative real part; (*iii*) purely imaginary eigenvalues; (*iv*) real eigenvalues of opposite sign; (*v*) negative and zero eigenvalue; (*vi*) negative, defective eigenvalue.

one is not. Then all initial values u_0 that are a multiple of the eigenvector corresponding to the zero eigenvalue give stationary solutions.

Finally (*vi*), for the defective case with a single eigenvalue and only one eigenvector $\underline{v}_1$, formula (3.16) applies. For large t the term $c_2 t e^{\lambda t} \underline{v}_1$ will dominate. Again the sign of λ determined stability or instability. If $\lambda = 0$ we get a rather special situation: the origin is unstable but the growth of solutions is only linearly, instead of exponential.

Example 3.3 (Damped oscillator) The scalar linear second-order equation

$$x''(t) + 2\alpha\, x'(t) + \beta\, x(t) = 0, \tag{3.17}$$

arises in many applications. The initial values are $x(0) = x_0$, $x'(0) = y_0$. Setting $y(t) = x'(t)$ we get the first-order system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -2\alpha \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The eigenvalues and eigenvectors of this matrix are

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta}\,, \qquad \underline{v}_j = \begin{pmatrix} 1 \\ \lambda_j \end{pmatrix} \quad (j = 1, 2)\,.$$

For the origin to be a stable stationary point we therefore need $\alpha \geq 0$ and $\beta \geq 0$. For such α, β, three cases can distinguished.

Over-damping: if $\alpha^2 > \beta$, both eigenvalues are negative and the solutions are

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

with constants $c_1, c_2 \in \mathbb{R}$ determined by the initial values x_0, y_0. The phase portrait will be as in Figure 3.1 (i).

Critical damping: if $\alpha^2 = \beta$ we have only one eigenvalue and it is defective. Therefore we get the solutions

$$x(t) = (c_1 + c_2 t) e^{-\alpha t} .$$

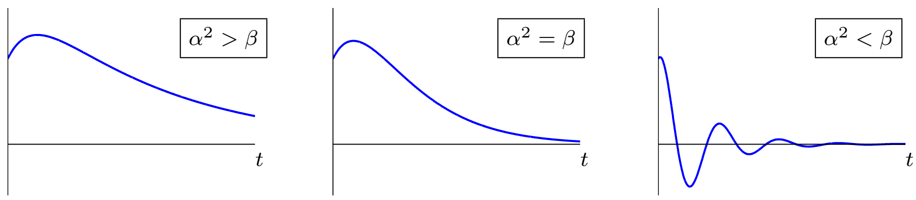
Here the phase portrait corresponds to Figure 3.1 (vi).

Damped oscillation: if $\alpha^2 < \beta$ the eigenvalues are complex conjugate, and we get the solutions

$$x(t) = c_1 e^{-\alpha t} \cos(\omega t) + c_2 e^{-\alpha t} \sin(\omega t) , \qquad \omega = \sqrt{\beta - \alpha^2} .$$

This can also be written as $x(t) = c e^{-\alpha t} \cos(\omega t - \vartheta)$ with $c = \sqrt{c_1^2 + c_2^2}$ and $\vartheta = \arctan(c_2/c_1)$. The phase portrait will now be as in Figure 3.1 (ii).

From the phase portraits we cannot see how fast the convergence to the steady state will be. For the three cases we plot the solution versus time (time axis horizontal) with the same α and $x_0 = y_0$, but varying β. It can now be observed that that the fastest decay of $x(t)$ without oscillations is achieved with critical damping, not with over-damping.



The solutions for $\alpha^2 \neq \beta$ are also easily found by simply trying $x(t) = e^{\lambda t}$. Inserting this in the differential equation, it is directly seen that this will indeed give a solution if $\lambda = \lambda_{1,2}$. Furthermore, linear combinations of solutions will again give a solution, leading to the constants c_1 and c_2. Guessing the general solution for the defective case $\alpha^2 = \beta$ is less obvious. $\diamond$

3.4 Linear Systems with Variable Coefficients

Solving linear systems of differential equations with a time-dependent matrix $A(t)$ is very much harder than for constant coefficients. In fact, explicit expressions for solutions are then only found for some special cases.

Suppose that $A(t) = (a_{ij}(t)) \in \mathbb{C}^{m \times m}$ is continuous in t, that is, all entries $a_{ij}(t)$ are continuous. We consider

$$u'(t) = A(t)\, u(t) , \qquad u(t_0) = u_0 \tag{3.18}$$

with $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{C}^m$. Because A is time-dependent, it is here more convenient to allow arbitrary starting points t_0, and we will consider also $t < t_0$. As before, we could have restricted ourselves to real valued matrices and initial vectors.

With a constant matrix A, the solution is $u(t) = \exp((t - t_0)A)u_0$. In view of formula (1.13) for the scalar case, one might think that the solutions of (3.18) will be given by $u(t) = \exp(\int_{t_0}^{t} A(s)\,ds)u_0$. However, this is *not correct* in general. In fact, it will only be valid if the $A(t)$ commute with each other, $A(t)A(s) = A(s)A(t)$ for all t, s, and in applications this rarely the case.

We do know that the problem (3.18) will have a unique solution on any bounded interval $\mathcal{I}$ containing t_0, because $L = \max_{t\in\mathcal{I}} \|A(t)\|$ will be a global Lipschitz constant on $\mathcal{I} \times \mathbb{C}^m$ for $f(t, v) = A(t)v$. Furthermore, it is easy to see that linear combinations solutions of the differential equation are again solutions: if $\underline{w}_j'(t) = A(t)\underline{w}_j(t)$ for $j = 1, \ldots, m$ then $u(t) = \sum_j c_j \underline{w}_j(t)$ also solves $u'(t) = A(t)u(t)$. This is often called the *superposition principle.* (The $\underline{w}_j$ are underlined to make it clear that they are vectors themselves, rather than components of a vector w.) If the vectors $\underline{w}_j(t_0)$ are linearly independent, we can find coefficients c_j such that $u_0 = \sum_{j=1}^{m} c_j \underline{w}_j(t_0)$, which will then provide a solution of our initial value problem (3.18).

This can be written in matrix form by letting $W(t) = (\underline{w}_1(t)\ \underline{w}_2(t) \ldots \underline{w}_m(t))$ be the $m \times m$ matrix with columns $\underline{w}_j(t)$. This $W(t)$ is called a *fundamental matrix* or *fundamental matrix solution* if

$$W'(t) \;=\; A(t)\,W(t)\,, \qquad W(t_0) = W_0 \tag{3.19}$$

with $W_0 \in \mathbb{C}^{m\times m}$ nonsingular. Then $u(t) = W(t)W_0^{-1}u_0$ is the solution of (3.18). In other words, if we define

$$S(t, t_0) \;=\; W(t)\,W(t_0)^{-1}\,, \tag{3.20}$$

then the solution of the initial value problem (3.18) is given by

$$u(t) \;=\; S(t, t_0)\,u_0\,. \tag{3.21}$$

Note that $S(t, t_0)$ does not depend on the choice of $W_0 = W(t_0)$. Actually, $V(t) = S(t, t_0)$ is again a fundamental matrix solution, but now with $V(t_0) = S(t_0, t_0) = I$, the identity matrix. This $S(t, t_0)$ is the generalization of the solution operator $\exp((t - t_0)A)$ of the constant-coefficient case. Although we cannot find explicit expressions in general, there are some interesting properties that can be demonstrated.

Since we have uniqueness of solutions of the initial value problem (3.18) with arbitrary starting points t_0, the solutions of $u'(t) = A(t)u(t)$ with initial value u_0 at t_0 and with $u_1 = S(t_1, t_0)u_0$ at t_1 must coincide. Consequently we have

$$S(t_2, t_0) \;=\; S(t_2, t_1)\,S(t_1, t_0)\,, \tag{3.22a}$$

for any $t_0, t_1, t_2 \in \mathbb{R}$. Taking $t_2 = t_0$, it is also seen that

$$S(t_1, t_0) \;=\; S(t_0, t_1)^{-1}\,. \tag{3.22b}$$

Apparently, $S(t_1,t_0)$ is invertible for arbitrary $t_0,t_1 \in \mathbb{R}$. In fact, the time evolution of the determinant of a fundamental matrix solution is precisely know. We have the following result, where $\det(W)$ is the determinant of W and $\mathrm{tr}(A)$ is the trace of A, the sum of the diagonal elements.

Theorem 3.4 Let $W(t)$ be a fundamental matrix solution (3.19). Then

$$\det(W(t)) \;=\; \exp\Big(\textstyle\int_{t_0}^t \mathrm{tr}(A(s))\,ds\Big)\cdot\det(W(t_0))\,. \tag{3.23}$$

Proof. Let $\mu(t)=\det(W(t))$. We have, for $h\to 0$,

$$W(t+h) \;=\; W(t)+hW'(t)+\mathcal{O}(h^2) \;=\; (I+hA(t))W(t) \;+\; \mathcal{O}(h^2)\,,$$

and therefore

$$\mu(t+h) \;=\; \det(I+hA(t))\,\mu(t) \;+\; \mathcal{O}(h^2)\,.$$

It is known from linear algebra that the determinant of a matrix is the product of its eigenvalues and the trace is the sum of the eigenvalues. If $\lambda_1,\ldots,\lambda_m$ are the eigenvalues of $A(t)$, then $\det(I+hA(t))=\prod_j(1+h\lambda_j)=1+h\sum_j\lambda_j+\mathcal{O}(h^2)$,

$$\det(I+hA(t)) \;=\; 1+h\,\mathrm{tr}(A(t))+\mathcal{O}(h^2)\,.$$

It follows that $\frac{1}{h}(\mu(t+h)-\mu(t))=\mathrm{tr}(A(t))\mu(t)+\mathcal{O}(h)$. Hence

$$\mu'(t) \;=\; \mathrm{tr}(A(t))\,\mu(t)\,,$$

from which the result follows. □

Relation (3.23) is known as *Liouville's formula*. It generalizes *Abel's identity* for differential equations which deals with the special systems obtained from linear, scalar second-order equations.

Example 3.5 Let x_1 and x_2 be two solutions of the second-order equation

$$x''(t)=p(t)x'(t)+q(t)x(t)\,,$$

with continuous $p(t)$ and $q(t)$. Writing this in the usual way as a first-order system with

$$u(t)=\begin{pmatrix} x(t)\\ x'(t)\end{pmatrix},\qquad A(t)=\begin{pmatrix} 0 & 1\\ q(t) & p(t)\end{pmatrix},$$

we see from (3.23) that $\mu(t)=x_1(t)x_2'(t)-x_1'(t)x_2(t)$ satisfies

$$\mu(t)=\exp\Big(\textstyle\int_{t_0}^t p(s)\,ds\Big)\cdot\mu(t_0)\,. \tag{3.24}$$

which is called Abel's identity.

If we have somehow found one solution $x_1\neq 0$, say by a lucky guess, then the second solution x_2 is obtained by solving the scalar equation $x_2'(t)=a(t)x_2(t)+b(t)$ with $a(t)=x_1'(t)/x_1(t)$ and $b(t)=\mu(t))/x_1(t)$ in an interval where x_1 is not zero. The general solution is then given by $x(t)=c_1x_1(t)+c_2x_2(t)$. ◇

Variation of constants formula. We now consider the inhomogeneous system

$$u'(t) = A(t)u(t) + g(t)\,, \qquad u(t_0) = u_0\,. \tag{3.25}$$

To find the solution, we can again make the "variation of constants" ansatz, $u(t) = S(t,t_0)c(t)$ with $c(t_0) = u_0$. Differentiation gives $u'(t) = A(t)u(t) + S(t,t_0)c'(t)$, and comparison with (3.25) shows that $c'(t) = S(t_0,t)g(t)$. Integration thus gives $c(t) = u_0 + \int_{t_0}^{t} S(s,t_0)g(s)\,ds$. This leads to the following expression for the solution:

$$u(t) = S(t,t_0)u_0 + \int_{t_0}^{t} S(t,s)g(s)\,ds\,. \tag{3.26}$$

Volumes in the phase space. Consider the homogeneous differential equation $u'(t) = A(t)u(t)$ with real matrix $A(t) \in \mathbb{R}^{m\times m}$. Let $\mathcal{D}_0$ be a set in $\mathbb{R}^m$ with volume $\mathrm{Vol}(\mathcal{D}_0) = \int_{\mathcal{D}_0} dv$. We can now define the set of points in $\mathbb{R}^m$ obtained from solutions at time t of the differential equation with $u(t_0) \in \mathcal{D}_0$,

$$\mathcal{D}_t = \{v \in \mathbb{R}^m : v = u(t),\ u \text{ is solution of } u'(s) = A(s)u(s),\ u(t_0) \in \mathcal{D}_0\}\,.$$

For any continuously differentiable function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ which is injective (one-to-one), we know, by the formula for substitution of variables in multiple integrals,

$$\int_{\varphi(\mathcal{D}_0)} dv \;=\; \int_{\mathcal{D}_0} |\det(\varphi'(v))|\,dv\,.$$

We have $\mathcal{D}_t = S(t,t_0)\mathcal{D}_0$ and the determinant of $S(t,t_0)$ is known by the Liouville formula. Taking $\varphi(v) = S(t,t_0)v$ with t, t_0 fixed, gives $\varphi'(v) = S(t,t_0)$ and

$$\mathrm{Vol}(\mathcal{D}_t) \;=\; \exp\left(\int_{t_0}^{t} \mathrm{tr}(A(s))\,ds\right)\cdot \mathrm{Vol}(\mathcal{D}_0)\,. \tag{3.27}$$

In particular, if $\mathrm{tr}(A(s)) = 0$ for all s, then the volume of $\mathcal{D}_0$ will be preserved in time.

In fact, the same formulas remain valid if we consider solutions of the inhomogeneous differential equations in (3.25) with arbitrary source terms $g(t)$. This is not surprising, because for fixed t and t_0 the variation of constants formula with $u_0 = v$ gives $u(t) = S(t,t_0)v + r(t,t_0) = \varphi(v)$, with $r(t,t_0)$ the result of the inhomogeneous term, which is independent of v. So, an inhomogeneous term will lead to a shift of $\mathcal{D}_t$ but not to a deformation.

Remark 3.6 Similar results for volumes are known for nonlinear differential equations, usually considered in autonomous form, $u' = f(u)$ with a continuously differentiable function f. The *flow* of the differential equation is the function $\varphi_t : \mathbb{R}^m \to \mathbb{R}^m$ that maps, for a given t, the initial value $v = u(0)$ to the solution value $u(t)$ at time t. Similar as above, it can then be shown that for $\mathcal{D}_t = \varphi_t(\mathcal{D}_0)$ we have

$$\frac{d}{dt}\mathrm{Vol}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \mathrm{div} f(v)\,dv\,,$$

where the divergence $\text{div} f(v) = \sum_{i=1}^{m} \frac{\partial}{\partial v_i} f_i(v)$ is the trace of the Jacobian matrix $f'(v)$. In particular, we have again preservation of volumes under the flow if $\text{div} f(v) = 0$ for all $v \in \mathbb{R}^m$. The difficult point in the proof of this result is to show that $\varphi_t(v)$ will be continuously differentiable w.r.t. v. This is true if f itself is continuously differentiable, but the proof is rather lengthy.

As an example, consider a Hamiltonian system

$$p_i' = -\frac{\partial}{\partial q_i} H(p,q) \,, \quad q_i' = \frac{\partial}{\partial p_i} H(p,q) \qquad (i = 1, 2, \ldots, n) \,,$$

where $p = (p_i) \in \mathbb{R}^n$ and $q = (q_i) \in \mathbb{R}^n$ are general momenta and positions of a mechanical system, and $H : \mathbb{R}^{2n} \to \mathbb{R}$ is called a Hamiltonian. This fits in the form $u'(t) = f(u(t))$ in $\mathbb{R}^m$, $m = 2n$, with

$$u = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad f(u) = \begin{pmatrix} -\frac{\partial}{\partial q} H(p,q) \\ \frac{\partial}{\partial p} H(p,q) \end{pmatrix}.$$

If H is twice differentiable, then the divergence of f is zero, and therefore the flow will be volume preserving. In mechanics this is known as Liouville's theorem. $\diamond$

3.5 Exercises

Exercise 3.1. The induced matrix norm is given by (3.2), but it is not obvious that this expression is well-defined. Show that (3.2) is equivalent to

$$\|A\| = \max\{\|Av\| : v \in \mathbb{C}^m,\ \|v\| = 1\}.$$

Note: it is allowed to put here 'max' instead of 'sup', because $\varphi(v) = \|Av\|$ defines a continuous function $\varphi : \mathbb{C}^m \to \mathbb{R}$, and the set $\{v \in \mathbb{C}^m : \|v\| = 1\}$ is compact.

Exercise 3.2. For $A = (a_{jk}) \in \mathbb{C}^{m\times m}$, let $\alpha = \max_{1\le j\le m} \sum_{k=1}^{m} |a_{jk}|$. Show that $\|Av\|_\infty \le \alpha \|v\|_\infty$ for any $v \in \mathbb{C}^m$. Then show that there is a $v \in \mathbb{C}^m$ for which equality holds, and conclude that $\|A\|_\infty = \alpha$. Hint: to show that equality can hold, consider a vector v all of whose components are one in modulus.

Exercise 3.3. Recall from linear algebra that a matrix $A \in \mathbb{C}^{m\times m}$ is called *Hermitian* if $A^* = A$, and it is called *unitary* if $A^*A = I$. If A is Hermitian, then $A = U\Lambda U^{-1}$ with diagonal Λ and unitary U. Furthermore, if U is unitary, then $\|Uv\|_2 = \|v\|_2$ for any $v \in \mathbb{C}^m$. (For real matrices the terms symmetric and orthogonal are used instead of Hermitian and unitary.)

(a) Assume A is Hermitian, and let $\lambda_1, \ldots \lambda_m$ be its eigenvalues. Show that $\|A\|_2 = \max_i |\lambda_i|$.

(b) Show that for an arbitrary matrix $A \in \mathbb{C}^{m\times m}$ we have $\|A\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ eigenvalue of } A^*A\}$. Hint: consider $\|Av\|_2^2/\|v\|_2^2$ and use the fact that A^*A is Hermitian.

Exercise 3.4. To prove (3.6) for commuting matrices, without manipulation of the power series, we can proceed as follows. First show that $A^k B = BA^k$ and $e^{tA}B = Be^{tA}$. Then show that $\frac{d}{dt}(e^{tA}e^{tB}) = (A+B)e^{tA}e^{tB}$. Finally, use uniqueness of solutions of $u' = (A+B)u$, $u(0) = u_0$ to show that (3.6) is valid.

Exercise 3.5. Construct an example with $A, B \in \mathbb{R}^{2\times 2}$ for which $e^{t(A+B)} \neq e^{tA}e^{tB}$. Hint: you can take any noncommuting pair A, B. Pick a simple pair with some zero columns.

Exercise 3.6. Compute e^{tA} for the following matrices A:

$$\begin{pmatrix} -1 & 3 \\ 6 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}.$$

The eigenvalues for the last matrix are complex, but the exponent of tA should be real. Computation by hand directly from (3.10) is already somewhat complicated. It is easier to proceed as in (3.15).

Exercise 3.7. Suppose that $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ has complex roots $\lambda_{1,2} = \xi \pm i\eta$ with $\xi < 0$. We know that all solutions will spiral towards the origin. Determine the orientation of the spiral directly from the sign of a_{21} (or a_{12}). Hint: consider in the phase plane the direction of $u'(0)$ starting with $u(0) = (1\ 0)^T$ (or $u(0) = (0\ 1)^T$).

Exercise 3.8. Let $D = a_{11}a_{22} - a_{12}a_{21}$ and $T = a_{11} + a_{22}$ be the determinant and trace of a matrix $A \in \mathbb{R}^{2\times 2}$. Determine in the T-D plane the regions in which the various possibilities (i)–(vi) occur and distinguish the stable/unstable cases.

Exercise 3.9. Consider the inhomogeneous problem $u'(t) = Au(t) + e^{\mu t}b$, with vector b and μ a real or complex number. Assume $A = V\mathrm{diag}(\lambda_j)V^{-1}$. Show that the solution is given by

$$u(t) \;=\; e^{tA}u(0) \;+\; r(t)$$

with $r(t) = V\mathrm{diag}(\rho_j(t))V^{-1}b$ and

$$\rho_j(t) = \begin{cases} \frac{1}{\mu-\lambda_j}(e^{\mu t} - e^{\lambda_j t}) & \text{if } \mu \neq \lambda_j\,, \\ t e^{\lambda_j t} & \text{if } \mu = \lambda_j\,. \end{cases}$$

Note: if λ_j is purely imaginary, $\mathrm{Re}(\lambda_j) = 0$, and $\mu = \lambda_j$, then $|\rho_j(t)|$ becomes very large after some time. This is known as the *resonance* effect.

Exercise 3.10. Let $a_0, a_1 \in \mathbb{R}$. The differential equation

$$x'' + \frac{a_1}{t}\,x' + \frac{a_0}{t^2}\,x \;=\; 0 \qquad (t > 0)$$

is called the Cauchy-Euler equation. Assume $a_0 < \frac{1}{4}(1-a_1)^2$. Solve the equation by introducing $s = \log(t)$ as independent variable. What is the behaviour for large t?

Exercise 3.11. For the linear system of differential equations with variable coefficients

$$u'(t) = \begin{pmatrix} -1 & 2e^{3t} \\ 0 & -2 \end{pmatrix} u(t)$$

explicit expressions for the solutions can be found quite easily, because we can first solve the second equation to get $u_2(t)$ and then solve the first equation to get the component $u_1(t)$. Show that

$$W(t) = \begin{pmatrix} e^{-t} & e^{t} - e^{-t} \\ 0 & e^{-2t} \end{pmatrix}$$

is a fundamental matrix solution and $\det(W(t)) = e^{-3t}$. [Note that the eigenvalues of the matrix $A(t)$ in this example are -1 and -2, but the fundamental matrix solution $W(t)$ is not bounded for $t \to \infty$.]

Exercise 3.12.* To prove relation(3.8) we first derive, in part (a), a small lemma.
(a) Consider a polynomial of degree n with coefficients p_j ($p_n \neq 0$) and roots θ_j,

$$p_0 + p_1 z + \cdots + p_n z^n \;=\; p_n\,(z-\theta_1)(z-\theta_2)\ldots(z-\theta_n)\,.$$

Prove, by induction to n, that $p_0 I + p_1 A + \cdots + p_n A^n = p_n\,(A-\theta_1 I)\ldots(A-\theta_n I)$.
(b) For any $z \in \mathbb{C}$, the binomial formula gives $\big(1+\frac{z}{n}\big)^n = \sum_{k=0}^{n} \frac{1}{k!}\beta_{k,n} z^k$ with $\beta_{0,n} = \beta_{1,n} = 1$ and $\beta_{k,n} = (1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}) \in (0,1)$ for $k = 2,\ldots,n$. Now show that

$$e^{tA} - (I + \tfrac{t}{n}A)^n \;=\; \textstyle\sum_{k=0}^{n} \frac{1}{k!}(1-\beta_{k,n})(tA)^k + \sum_{k>n} \frac{1}{k!}(tA)^k\,,$$

$$\|e^{tA} - (I + \tfrac{t}{n}A)^n\| \;\leq\; e^{t\|A\|} - (1 + \tfrac{t}{n}\|A\|)^n \quad \to\; 0 \quad \text{as } n \to \infty\,.$$

4 Stability and Linearization

4.1 Stationary Points

In this section and the following ones we will mainly look at *autonomous* systems of differential equations

$$u'(t) = f(u(t)) \tag{4.1}$$

with $f : \mathbb{R}^m \to \mathbb{R}^m$ continuously differentiable. Even though explicit expressions for the solutions cannot be found in general, we may be able to obtain a good qualitative description of solutions. For this, we first study the behaviour of solutions near stationary points.

Any $u_* \in \mathbb{R}^m$ which is a zero of the function f corresponds to a stationary solution of the differential equation, $u(t) = u_*$ for all $t \in \mathbb{R}$. Therefore, u_* is often called a *stationary point* or *equilibrium point* for the differential equation (4.1).

Definition 4.1 The stationary point u_* is said to be *stable* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that any solution of (4.1) with $\|u(0) - u_*\| < \delta$ exists for $t \geq 0$ and satisfies $\|u(t) - u_*\| < \varepsilon$ for all $t \geq 0$. If u_* is stable and there is a $\gamma > 0$ such that $\|u(t) - u_*\| \to 0$ (as $t \to \infty$) whenever $\|u(0) - u_*\| < \gamma$, then the stationary point u_* is called *asymptotically stable*. On the other hand, if u_* is not stable, we call it *unstable*.

It is important to note that stability, as defined here, is a local property. It roughly means that solutions that start sufficiently close to u_* remain close. The behaviour of solutions that do not start close to u_* is not involved in the definition. Some of those solutions may drift off to infinity.

For example, for the scalar equation (1.3), where $f(v) = \lambda v(1 - \mu v)$ with $\lambda, \mu > 0$, we already saw that there are two stationary points: the unstable point $u_* = 0$ and the asymptotically stable stationary point $u_* = 1/\mu$. Only solutions with $u(0) = u_0 > 0$ will tend to $u_* = 1/\mu$, whereas any solution that starts with $u(0) = u_0 < 0$ will diverge towards $-\infty$.

Remark 4.2 In these concepts of stability and asymptotic stability a norm $\|\cdot\|$ on $\mathbb{R}^m$ is involved, and we did not specify which norm this is. In fact, it does not matter because it is known (from linear algebra) that all norms on $\mathbb{R}^m$ are *equivalent*, in the sense that if $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are any two norms on $\mathbb{R}^m$, then there are positive constants $\underline{C}$ and $\overline{C}$ such that

$$\underline{C}\,\|v\|_\alpha \;\leq\; \|v\|_\beta \;\leq\; \overline{C}\,\|v\|_\alpha \qquad (\text{for all } v \in \mathbb{R}^m)\,.$$

For instance, with the Euclidian norm and the maximum norm on $\mathbb{R}^m$ we have $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{m}\|v\|_\infty$. Therefore, boundedness or convergence to zero are properties that are the same in any norm, and consequently stability and asymptotic stability are not influenced by the choice of the norm on $\mathbb{R}^m$. $\diamond$

4.2 Stability for Linear Systems

To investigate stability for general autonomous equations, we begin with the simple case of a linear homogeneous system of differential equations in $\mathbb{R}^m$,

$$u'(t) = Au(t)\,. \tag{4.2}$$

Clearly $u_* = 0$ is then a stationary point, and if the matrix $A \in \mathbb{R}^{m\times m}$ is not singular it is also the only stationary point. Since the solutions are $u(t) = e^{tA}u(0)$, stability means that there is a $K \geq 1$ such that $\|e^{tA}\| \leq K$ (for all $t \geq 0$); see Exercise 4.1. For asymptotic stability we need $\|e^{tA}\| \to 0$ (as $t \to \infty$). Observe that for this linear case stability and asymptotic stability are global properties, describing the behaviour of solutions that are not necessarily close to $u_* = 0$, due to the fact that if u is a solution, then so is $c \cdot u$ for any $c \in \mathbb{R}$.

Theorem 4.3 The stationary point $u_* = 0$ is stable for (4.2) if and only if $\mathrm{Re}\,\lambda \leq 0$ for any eigenvalue λ of A, and eigenvalues with $\mathrm{Re}\,\lambda = 0$ are not defective. The point is asymptotically stable if and only if $\mathrm{Re}\,\lambda < 0$ for all eigenvalues.

Proof. Let $\alpha = \max_k \mathrm{Re}\,\lambda_k$, where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the eigenvalues of A with corresponding dimensions $m_1, m_2, \ldots, m_l$ of the Jordan blocks.

According to the formulas (3.13), we have $\sup_{t\geq 0} \|e^{tA}\| < \infty$ if and only if $\alpha \leq 0$ and $m_k = 1$ for any eigenvalue λ_k with $\mathrm{Re}\,\lambda_k = 0$. Moreover, $\lim_{t\to\infty} \|e^{tA}\| = 0$ is seen to be equivalent to $\alpha < 0$. $\square$

For the asymptotically stable case, where all $\mathrm{Re}\,\lambda_k < 0$, we also have

$$\|e^{tA}\| \;\leq\; Ke^{-at} \qquad \text{for all } t \geq 0\,, \tag{4.3}$$

with a constant $a > 0$ such that $\max_k \mathrm{Re}\,\lambda_k < -a < 0$, and $K \geq 1$. This can again be shown from (3.13), but it also follows by the following argument: for $\tilde{A} = aI + A$, we get $\|e^{tA}\| = e^{-ta}\|e^{t\tilde{A}}\|$, and since $\tilde{A}$ has eigenvalues $\tilde{\lambda}_k = a + \lambda_k < 0$, we know that $\|e^{t\tilde{A}}\| \leq K$ (for $t \geq 0$) with some $K \geq 1$.

4.3 Stability for Nonlinear Systems

As a next step towards stability for nonlinear differential equations, we consider differential equations of the following type:

$$u'(t) \;=\; Au(t) \;+\; g(u(t))\,, \tag{4.4}$$

with a nonlinear term $g : \mathbb{R}^m \to \mathbb{R}^m$ that satisfies

$$\lim_{v\to 0} \frac{\|g(v)\|}{\|v\|} \;=\; 0\,. \tag{4.5}$$

This implies that $g(0) = 0$, so $u_* = 0$ is still a stationary point of (4.4).

Theorem 4.4 Suppose g is continuously differentiable and satisfies (4.5). Then the following statements about stability of $u_* = 0$ hold.
(a) If $\operatorname{Re}\lambda < 0$ for all eigenvalues λ of A, then $u_* = 0$ is an asymptotically stable stationary point of (4.4).
(b) If $\operatorname{Re}\lambda > 0$ for some eigenvalue of A, then $u_* = 0$ is an unstable stationary point of (4.4).

Proof of (a). We will only give a proof of statement (a). For systems, the proof of (b) is more technical; see Exercise 4.2 for the scalar case ($m = 1$).

Assume $\operatorname{Re}\lambda < 0$ for all eigenvalues λ of A. For a solution of the differential equation with initial value $u(0) = u_0$, we have, by the variation of constants formula (3.9),

$$u(t) \;=\; e^{tA}u_0 \;+\; \int_0^t e^{(t-s)A} g(u(s))\,ds\,.$$

Let $K \geq 1$ and $a > 0$ be as in (4.3). From (4.5) we see that there is an $\varepsilon > 0$ such that

$$\|g(v)\| \;\leq\; \frac{a}{2K}\,\|v\| \qquad \text{whenever } \|v\| \leq \varepsilon\,.$$

We will show that for any initial value $\|u_0\| \leq \frac{\varepsilon}{K}$, the solution tends to 0 as $t \to \infty$.

Suppose that $\|u_0\| \leq \delta < \varepsilon$. By the local existence and uniqueness result of Theorem 2.5, we know there is a time interval $[0, \bar{T}]$, $\bar{T} > 0$, for which the solution exists, and $\|u(t)\| \leq \varepsilon$ for $t \in [0, \bar{T}]$. On this interval we therefore have

$$\|u(t)\| \;\leq\; e^{-at}K\,\|u_0\| \;+\; \tfrac{1}{2}\,a \int_0^t e^{-a(t-s)}\|u(s)\|\,ds\,,$$

and consequently

$$e^{at}\|u(t)\| \;\leq\; K\,\|u_0\| \;+\; \tfrac{1}{2}\,a \int_0^t e^{as}\|u(s)\|\,ds\,.$$

Setting $\mu(t) = e^{at}\|u(t)\|$ we can apply Lemma 2.2 (Gronwall) with $\alpha = K\|u_0\|$ and $\beta = \frac{1}{2}a$. This gives $\mu(t) \leq e^{at/2}K\|u_0\|$, and therefore

$$\|u(t)\| \;\leq\; e^{-\frac{1}{2}at}K\,\|u_0\|\,.$$

Consequently, if $\|u_0\| \leq \frac{\varepsilon}{K}$ then $\|u(t)\| \leq e^{-at/2}\varepsilon < \varepsilon$ for $t \in (0, \bar{T}]$. But then it follows from Theorem 2.5 that the solution can be continued for larger t , still having $\|u(t)\| \leq e^{-at/2}\varepsilon$ by the above argument. We can therefore conclude that any solution with $\|u_0\| \leq \frac{\varepsilon}{K}$ will remain at distance less than ε from the origin for all time $t \geq 0$, and $\lim_{t\to\infty} \|u(t)\| = 0$. □

There is no statement in this theorem if the eigenvalues λ of A are such that $\max \operatorname{Re}\lambda = 0$. In this case stability or instability depends critically on the nonlinearity g. Examples can be easily found for $m = 1$ (with $A = 0$). Some examples that are more in line with the theorem are considered in Exercise 4.4.

Linearization. At first sight, it seems that the result of Theorem 4.4 only applies to special nonlinear systems. However, as we will see shortly, the result can be applied to any autonomous system (4.1) with a nonlinear function f which is continuously differentiable. To see this we will consider *linearization* around a stationary point u_*.

Recall from vector calculus that the function $f : \mathbb{R}^m \to \mathbb{R}^m$ is said to differentiable in $v \in \mathbb{R}^m$ if there exists a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ such that

$$\lim_{w\to 0} \frac{\|f(v+w) \, - \, f(v) \, - \, Aw\|}{\|w\|} \, = \, 0 \, .$$

Moreover, if all partial derivatives of f exist and are continuous around v, then A will be equal to the Jacobian matrix $f'(v)$ of partial derivatives, $a_{ij} = \frac{\partial}{\partial v_j} f_i(v)$. For such a given point $v \in \mathbb{R}^m$, we can define $g(w) = f(v+w) - f(v) - f'(v)w$ for $w \in \mathbb{R}^m$. This function $g(w)$ is continuously differentiable (with derivative $g'(w) = f'(v+w) - f'(v)$) and we have $\|g(w)\|/\|w\| \to 0$ for $w \to 0$.

This will be applied to our differential equation $u'(t) = f(u(t))$ with $v = u_*$. Consider a solution u and let

$$w(t) \, = \, u(t) - u_* \, .$$

Then $w'(t) = f(u_* + w(t)) = f(u_*) + f'(u_*)w(t) + g(w(t))$ with $\|g(w)\|/\|w\| \to 0$ for $w \to 0$. Since $f(u_*) = 0$, we obtain

$$w'(t) = Aw(t) + g(w(t)) \tag{4.6}$$

with $A = f'(u_*)$, and $w_* = 0$ is a stationary point of this differential equation. We can therefore apply Theorem 4.4 provided that the maximum of the real parts of the eigenvalues λ of A is not zero. Stability or instability of $w_* = 0$ for (4.6) can be directly translated to the same property for the stationary point u_* of the differential equation (4.1). This gives the following result:

Corollary 4.5 Suppose f is continuously differentiable. Let u_* be a stationairy point of the differential equation $u'(t) = f(u(t))$, and $A = f'(u_*)$. If $\mathrm{Re}\,\lambda < 0$ for all eigenvalues of A, then the stationary point u_* is asymptotically stable. On the other hand, if there is an eigenvalue of A with positive real part, then u_* is unstable. □

The differential equation $v'(t) = Av(t)$ with $A = f'(u_*)$ is called the *linearized equation* of $u'(t) = f(u(t))$ near the stationary point u_*. From this linearized equation we can often determine the stability of the stationairy point u_* of our nonlinear system (4.1).

Example 4.6 The system $u' = f(u)$ in $\mathbb{R}^2$ with

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad f(u) = \begin{pmatrix} -u_1 - u_2 + u_2^2 \\ u_1(1+u_2^2) \end{pmatrix}$$

has two stationary points: $u_* = (0,0)^T$ and $u_* = (0,1)^T$.

Near the origin the system has the form

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_2^2 \\ u_1 u_2^2 \end{pmatrix}.$$

The eigenvalues of the linearized system are $-\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$, so the origin is asymptotically stable.

Near the other stationary point, $u_* = (0,1)^T$, we consider $w_1 = u_1$ and $w_2 = u_2 - 1$. This gives $w_1' = -w_1 + w_2 + w_2^2$ and $w_2' = w_1(2 + 2w_2 + w_2^2)$, that is,

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} w_2^2 \\ 2w_1 w_2 + w_2^2 \end{pmatrix}.$$

Here the eigenvalues of the linearized system are $-\frac{1}{2} \pm \frac{3}{2}$. Since one eigenvalue is positive, this stationary point is unstable. (For this example, the nonlinear term $g(w)$ is written down explicitly, but that is not necessary.) $\diamond$

Remark 4.7 If the Jacobian matrix $A = f'(u_*)$ has no eigenvalues with real part zero, then not only stability of the linearized equation $v' = Av$ is the same as for $u' = f(u)$ near u_*, but also topological structure of the phase portrait (with nodes, spirals or saddle points) will be the same locally, including the orientation of the trajectories. This result is known as the *Grobman-Hartman theorem.* $\diamond$

4.4 Periodic Solutions and Limit Cycles

In the previous sections we often took the initial time t_0 to be zero. This was without loss of generality because we can always use $t - t_0$ as a new independent variable. However, for the following discussion it is convenient to allow arbitrary starting time points $t_0 \in \mathbb{R}$, and we also consider $t < t_0$.

For the autonomous system $u'(t) = f(u(t))$ with initial value $u(t_0) = u_0$ and with $f : \mathbb{R}^m \to \mathbb{R}^m$ continuously differentiable, local existence and uniqueness is guaranteed by Theorem 2.5. It may happen that a solution u blows up in finite time, $\lim_{t\uparrow t_+} \|u(t)\| = \infty$ with a $t_+ > t_0$. If not, then the solution will exist for all $t > t_0$. Likewise we may follow the solutions backwards in time, $t < t_0$, and either the solution will exist for all $t < t_0$ or $\lim_{t\downarrow t_-} \|u(t)\| = \infty$ at some $t_- < t_0$.

Let (t_-, t_+) be the maximal interval for which a solution of the initial value problem exists, where we allow $t_+ = \infty$ (and likewise $t_- = -\infty$) if there is no blow-up in finite time. Consider the trajectory $\mathcal{U} = \{u(t) : t_- < t < t_+\}$. Suppose $\tilde{u}$ is another solution of the differential equation, $\tilde{u}'(t) = f(\tilde{u}(t))$, but now with a different starting value and possibly a different starting time, $\tilde{u}(\tilde{t}_0) = \tilde{u}_0$. This gives a second trajectory $\tilde{\mathcal{U}} = \{\tilde{u}(t) : \tilde{t}_- < t < \tilde{t}_+\}$ with maximal interval of existence $(\tilde{t}_-, \tilde{t}_+)$.

Theorem 4.8 Assume $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable. Then two trajectories $\mathcal{U}$ and $\tilde{\mathcal{U}}$ either coincide ($\tilde{\mathcal{U}} = \mathcal{U}$), or they have no point in common ($\tilde{\mathcal{U}} \cap \mathcal{U} = \emptyset$). Consequently, for every point u_0 in the phase space $\mathbb{R}^m$, there is exactly one trajectory passing through that point.

Proof. Suppose $u_1 \in \tilde{\mathcal{U}} \cap \mathcal{U}$. Then there are $t_1 \in (t_-, t_+)$ and $\tilde{t}_1 \in (\tilde{t}_-, \tilde{t}_+)$ with $\tilde{u}(\tilde{t}_1) = u(t_1) = u_1$. Consider $v(t) = \tilde{u}(t - t_1 + \tilde{t}_1)$. This v satisfies again the differential equation $v'(t) = f(v(t))$, and we have $v(t_1) = u(t_1) = u_1$. So, u and v are solutions of the same initial value problem, with start time t_1. By uniqueness for the initial value problem it follows that $u(t) = v(t)$ for all $t \in (t_-, t_+)$. But then $u(t) = \tilde{u}(t - t_1 + \tilde{t}_1)$ for all $t \in (t_-, t_+)$, which shows that $\mathcal{U} \subset \tilde{\mathcal{U}}$. In the same way it follows that $\tilde{\mathcal{U}} \subset \mathcal{U}$. $\square$

The above theorem tells us that the trajectories of different solutions cannot intersect. In the same way it is seen that the trajectory of a solution may not intersect itself in one point; see Exercise 4.6. A solution may, however, catch-up with itself. This happens if the solution is *periodic*,

$$u(t+T) = u(t) \qquad \text{for all } t \in \mathbb{R}\,. \tag{4.7}$$

Here the smallest $T > 0$ for which this holds is called the period of the solution.

The trajectory of a periodic solution is a closed curve in the phase space. On the other hand, if $\mathcal{V} \subset R^m$ is a closed curve that does not contain stationary points, and u is a solution with $u(t) \in \mathcal{V}$ for all t, then there is a $c > 0$ such that $\|u'(t\| \geq c$ for all t, that is, the speed by which u moves along its trajectory $\mathcal{U}$ is strictly positive, cf. also Exercise 4.7. It follows that u is periodic and $\mathcal{U} = \mathcal{V}$.

The trajectory of a periodic solution may be surrounded by the trajectories of other periodic solutions; see for example Figure 3.1 (*iii*). For nonlinear systems it may also happen that a periodic solution attracts or repels nearby other solutions. In that case the trajectory of the periodic solution is called a *limit cycle*.

Some examples. In the examples for $u' = f(u)$ in $\mathbb{R}^2$ we will usually denote the components of u by $x = u_1$ and $y = u_2$. Further we will often suppress in the notation the explicit dependence of of x, y on the time t. With a slight abuse of notation, these x, y will also occasionally denote independent variables.

The following two examples are based on second-order differential equations of the form $x'' + a(x)x' + b(x)x = 0$, with a or b not constant.

Example 4.9 (Duffing equation: periodic solutions) Consider the system

$$\left\{ \begin{aligned} x' &= y\,, \\ y' &= x - x^3 - \alpha y\,, \end{aligned} \right. \tag{4.8}$$

with parameter $\alpha \geq 0$. It is known as the Duffing equation (without forcing).

There are three stationary points: $(0,0)$, $(\pm 1, 0)$. The Jacobi matrix $A = f'(u)$ is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -\alpha \end{pmatrix}. \tag{4.9}$$

The eigenvalues for $x = 0$ are $-\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 + 4}$. Hence the origin is an unstable stationary point (with one negative and one positive eigenvalue). For $x = \pm 1$ we

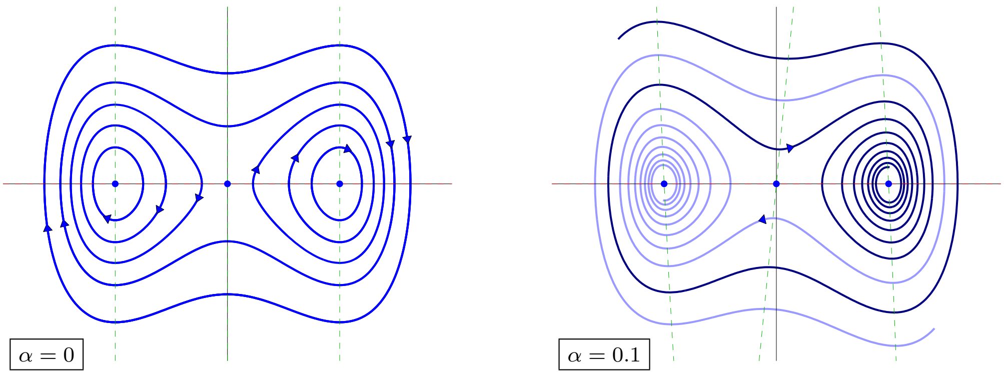


Figure 4.1: Trajectories for Duffing's equation with $\alpha = 0$ and $\alpha = 0.1$.

find the eigenvalues $-\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 8}$. If $\alpha > 0$ both eigenvalues have negative real part, so the stationary points $(1, 0)$ and $(-1, 0)$ are then asymptotically stable. If $\alpha = 0$ the eigenvalues are purely imaginary and Theorem 4.4 does not apply.

In fact, as will be seen shortly, for $\alpha = 0$ we get periodic solutions. Even though we do not have explicit expressions for these solutions, we can compute an explicit expression for the orbits. From (4.8) with $\alpha = 0$ we obtain $(x - x^3)\, x' = y\, y'$, and integration shows that $\frac{d}{dt}E(x, y) = 0$, where

$$E(x, y) \;=\; y^2 - x^2 + \tfrac{1}{2}x^4 \,.$$

The trajectories are therefore level curves $E(x, y) = c$ with c an integration constant.[2] These are closed curves, and the corresponding solutions are therefore periodic, except for the level curve $E(x, y) = 0$ that contains the unstable stationary point at the origin. A number of these orbits in the phase plane are plotted in the left panel of Figure 4.1.

The right panel of that figure contains a plot of two solutions for the case $\alpha > 0$. We see that these solutions converge for $t \to \infty$ to one of the two stable stationary points, so it appears that there are no periodic solutions anymore. This can be seen by differentiating $\varphi(t) = E(x(t), y(t))$ with respect to t, to give

$$\varphi'(t) \;=\; \frac{d}{dt}E(x, y) \;=\; (-2x + 2x^3)x' \,+\, 2\,y\,y' \;=\; -2\alpha\, y^2 \,.$$

Therefore, $\varphi(t) = E(x(t), y(t))$ will be monotonically decreasing for any solution. For a periodic solution, with period T, we would have $\alpha \int_0^T y(t)^2\, dt = 0$, that is $y(t) = 0$ for all $t \in [0, T]$. It follows that there are no periodic solutions. (It can also be shown that any solution will tend to one of the stationary points.) $\diamond$

[2] Often the expression of the orbits is derived by simply dividing the second equation of (4.8) by the first equation and setting $\frac{y'}{x'} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$, to obtain the separable scalar equation $\frac{dy}{dx} = \frac{x - x^3}{y}$. Here x is now viewed as an independent variable and y a dependent variable (depending on x).

In this example a differentiable function $E : \mathbb{R}^2 \to \mathbb{R}$ was found such that $\frac{d}{dt}E(x(t), y(t)) \le 0$ for any solution of the differential equation. For mechanical systems this E may represent the energy of the system, which can decrease in time because of friction.

Further it is noted that in Figure 4.1 some green and red dashed lines are drawn. The green line indicates that $y' = 0$ and the red line corresponds to $x' = 0$. In this example the dashed red line coincides with the x-axis. Such lines are often convenient since they divide the phase plane in regions where we know that the solutions will move along a trajectory in an upward-right, upward-left, downward-right or downward-left direction. This can give already a rough indication how the trajectories will look like.

An example with limit cycles is found in Exercise 4.5. The following example is more difficult, but also more interesting. We will not fully analyze it, so it should be merely considered as an illustration.

Example 4.10 (van der Pol equation: limit cycle) An interesting equation with a limit cycle in $\mathbb{R}^2$ is given by the van der Pol equation

$$(4.10) \qquad \begin{cases} x' = y\,, \\ y' = -x + \beta(1 - x^2)y\,, \end{cases}$$

with $\beta > 0$. Here we have only one stationary point: $u_* = 0$. The eigenvalues of $A = f'(0)$ are $\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4}$. Hence the eigenvalues have positive real part, and therefore $u_* = 0$ is an unstable stationary point. It can be also be shown (not too difficult) that all solutions are bounded for $t \ge 0$.

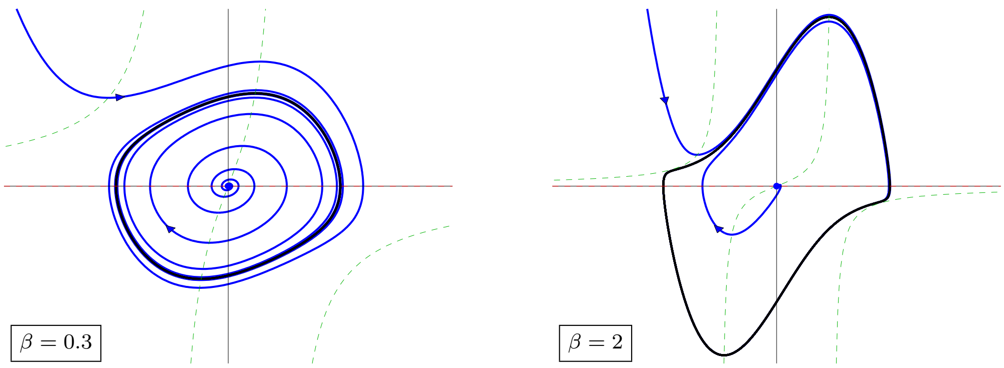


Figure 4.2: Trajectories for the van der Pol equation.

Furthermore, it can be shown (much more difficult) that there is a unique periodic solution, and this periodic solution acts as a limit cycle. All solutions spiral clockwise towards this limit cycle, either from the inside or from the outside, as shown in Figure 4.2 for two values of β. If β gets larger, the convergence to the limit cycle becomes faster. $\diamond$

4.5 Exercises

Exercise 4.1. Consider the linear system $u'(t) = Au(t)$ with stationary point $u_* = 0$. Show that the formal ε, δ-definition of stability for $u_* = 0$ is equivalent to the existence of a $K \geq 1$ such that $\|e^{tA}\| \leq K$ (for all $t \geq 0$).

Exercise 4.2. For the scalar case ($m = 1$) the proof of Theorem 4.4 is much easier (by taking into account the sign of u'). Prove statements (a) and (b) of the theorem for $m = 1$ by considering $u' = a\,u + g(u)$ with constant $a < 0$ or $a > 0$, respectively, and with $|g(v)| \leq \frac{1}{2}|a||v|$ for $|v| \leq \varepsilon$, $\varepsilon > 0$ small.

Exercise 4.3. Determine the stationary points for the system

$$\begin{cases} x' = (3-y)x\,, \\ y' = (1+x-y)y\,, \end{cases}$$

and discuss the stability of these points (x_*, y_*).

Exercise 4.4. Consider the following systems of differential equations

$$(a)\ \begin{cases} x' = y - \mu x(x^2+y^2)\,, \\ y' = -x - \mu y(x^2+y^2)\,, \end{cases} \qquad (b)\ \begin{cases} x' = xy - \mu x(x^2+y^2)\,, \\ y' = -x^2 - \mu y(x^2+y^2)\,, \end{cases}$$

with $\mu = \pm 1$. Explain why Theorem 4.4 is not applicable. Demonstrate stability or instability of the origin, by introducing $E(x,y) = x^2 + y^2$.

Exercise 4.5. Determine the limit cycles and stationary points of the following two systems:

$$(a)\ \begin{cases} x' = x - y - x\sqrt{x^2+y^2}\,, \\ y' = x + y - y\sqrt{x^2+y^2}\,, \end{cases} \qquad (b)\ \begin{cases} x' = -y + x\cos(x^2+y^2)\,, \\ y' = x + y\cos(x^2+y^2)\,. \end{cases}$$

Again, study the behaviour of $E(x,y) = x^2 + y^2$.

Exercise 4.6. Suppose u is a solution of the autonomous differential equation (4.1) with $f : \mathbb{R}^m \to \mathbb{R}^m$ continuously differentiable. Let $T > 0$. Show that if $u(t_0 + T) = u(t_0)$ for some t_0, then $u(t+T) = u(t)$ for all t.

Exercise 4.7.* Let $\mathcal{D} \subset \mathbb{R}^m$, $a, b > 0$, and suppose $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable and $\|f(v)\| \geq a$, $\|f'(v)f(v)\| \leq b$ for all $v \in \mathcal{D}$. Further assume u is a solution of the autonomous problem $u'(t) = f(u(t))$, $u(0) = u_0$ such that $u(t) \in \mathcal{D}$ for all $t \geq 0$. Show that

$$\|u(t+h) - u(t)\| \geq \tfrac{1}{2} h\, a \quad \text{if } t \geq 0,\ h \in (0, a/b)\,.$$

Hint: to derive this inequality, you may use the formula

$$u(t+h) - u(t) = hu'(t) + h^2 \textstyle\int_0^1 (1-\theta) u''(t+\theta h)\, d\theta\,,$$

which can be derived by partial integration of the integral term.

5 Some Models in $\mathbb{R}^2$ and $\mathbb{R}^3$

5.1 Population Models with Two Species

In this section we will study some simple, but interesting, population models with two species. Recall that for one species, the population density $u(t)$ can be described in first instance by $u' = a\,u$ where $a > 0$ is the natural growth rate (birth rate minus death rate). If the population increases this is no longer a realistic model, and a term $-bu^2$ with $b > 0$ should be added to describe internal competition (e.g. over food). The resulting differential equation

$$u' = a\,u - b\,u^2 \tag{5.1}$$

is called the *logistic equation.* In population dynamics it is also known as the *Verhulst model.*

5.1.1 Predator-Prey Model

We now consider two species: a prey population with density $x(t)$ and a predator population with density $y(t)$. Assume that for the prey population food is abundantly available, so the population is only held in check by predation. The number of contacts per unit time between predators and prey is proportional to $x\,y$, leading to the differential equation $x' = \alpha_1 x - \gamma_1 xy$ with $\alpha_1, \gamma_1 > 0$. In the same way it can be argued that the predators will have a natural rate of decline if there is no prey, but this predator population will increase at a rate proportional to $x\,y$. This gives $y' = -\alpha_2 y + \gamma_2 xy$ with $\alpha_2, \gamma_2 > 0$.

The resulting system of differential equations

$$\begin{cases} x' \,=\, \alpha_1\, x \,-\, \gamma_1\, x\, y\,, \\ y' \,=\, -\alpha_2\, y \,+\, \gamma_2\, x\, y\,, \end{cases} \tag{5.2}$$

with parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$, is known as the *predator-prey model* or the *Lotka-Volterra model.* Obviously, only solutions with $x, y \geq 0$ are relevant. Although we cannot solve the equations explicitly, there are interesting properties of the solutions that can be derived.

Assume that (x_0, y_0) is an initial value at time $t_0 = 0$. First, observe that if $x_0 = 0$, then we get the solution $x(t) = 0$, $y(t) = e^{-\alpha_2 t} y_0$. Therefore the positive y-axis in the phase plane is a trajectory. In the same way it is seen that the positive x-axis is a trajectory, corresponding to $y_0 = 0$. Since trajectories cannot intersect – they can only meet in stationary points – it can already be concluded that $x(t), y(t) > 0$ for all t whenever $x_0, y_0 > 0$.

The system has two stationary points: the origin, which is unstable, and the more interesting point $(x_*, y_*) = (\alpha_2/\gamma_2, \alpha_1/\gamma_1)$. The eigenvalues for the linearized problem at this stationary point are purely imaginary, which does not give much information about stability, but it is a first indication that the solutions might be periodic.

Proposition 5.1 Let $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$. Then all trajectories in the positive quadrant of the phase plane are closed curves, corresponding to periodic solutions.

Proof. Assume $x_0, y_0 > 0$. Note that $x'/x = \alpha_1 - \gamma_1 y$ only depends on y, and $y'/y = \alpha_2 - \gamma_2 x$ only depends on x. Hence $(x'/x)(y'/y)$ can be written in two ways:

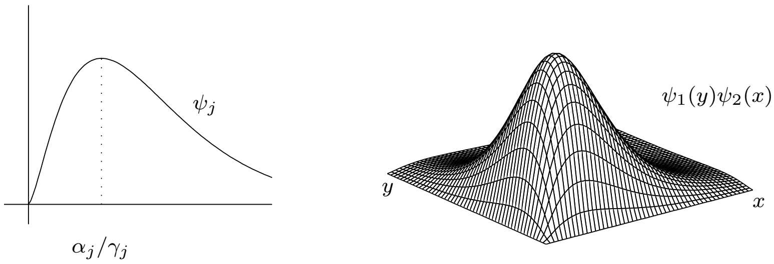
$$\Big(\frac{-\alpha_2}{x} + \gamma_2\Big)\, x' \;=\; \Big(\frac{\alpha_1}{y} - \gamma_1\Big)\, y' \,.$$

Integration shows that the trajectories are given by $E(x, y) = c$ with integration constant c and

$$E(x, y) \;=\; \alpha_1 \log(y) \;-\; \gamma_1 y \;+\; \alpha_2 \log(x) \;-\; \gamma_2 x \,.$$

The equation $E(x, y) = c$ is equivalent to

$$\psi_1(y) \cdot \psi_2(x) \;=\; e^c \qquad \text{with} \quad \psi_1(y) = y^{\alpha_1}/e^{\gamma_1 y}\,, \;\; \psi_2(x) = x^{\alpha_2}/e^{\gamma_2 x}\,.$$



The function $\psi_j(z) = z^{\alpha_j}/e^{\gamma_j z}$, defined for $z \geq 0$, has one maximum in $z_* = \alpha_j/\gamma_j$, and we have $\psi_j(0) = \lim_{z\to\infty} \psi_j(z) = 0$. The product function $\psi_1(y) \cdot \psi_2(x)$ (for $x, y \geq 0$) therefore has a single maximum, attained in the stationairy point $(x_*, y_*) = (\alpha_2/\gamma_2, \alpha_1/\gamma_1)$, and the contour lines $\psi_1(y)\psi_2(x) = e^c$ are closed curves in the first quadrant around this stationairy point. Since there are no stationary points on these curves, they correspond to trajectories of periodic solutions. $\square$

A number of these trajectories are plotted in the left panel of Figure 5.1. Each trajectory corresponds to a periodic solution, with some period T. Even though the solution itself and its period are unknown, we can compute the average values over one period.

Proposition 5.2 Let $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$. Then the average values

$$\overline{x} \;=\; \tfrac{1}{T}\int_0^T x(t)\, dt\,, \qquad \overline{y} \;=\; \tfrac{1}{T}\int_0^T y(t)\, dt\,,$$

are given by $\overline{x} = \alpha_2/\gamma_2$ and $\overline{y} = \alpha_1/\gamma_1$.

Proof. From the first equation in (5.2) it is seen that $x'/x = \alpha_1 - \gamma_1 y$. Hence

$$\tfrac{1}{T}\int_0^T (x'(t)/x(t))\, dt \;=\; \tfrac{1}{T}\int_0^T (\alpha_1 - \gamma_1 y(t))\, dt \;=\; \alpha_1 - \gamma_1\, \overline{y}\,.$$

Now, $\int_0^T (x'(t)/x(t))\, dt \;=\; \log(x(T)) - \log(u(0)) \;=\; 0$ since $x(T) = x(0)$. Consequently $\overline{y} = \alpha_1/\gamma_1$. The value for $\overline{x}$ is found in the same way. $\square$

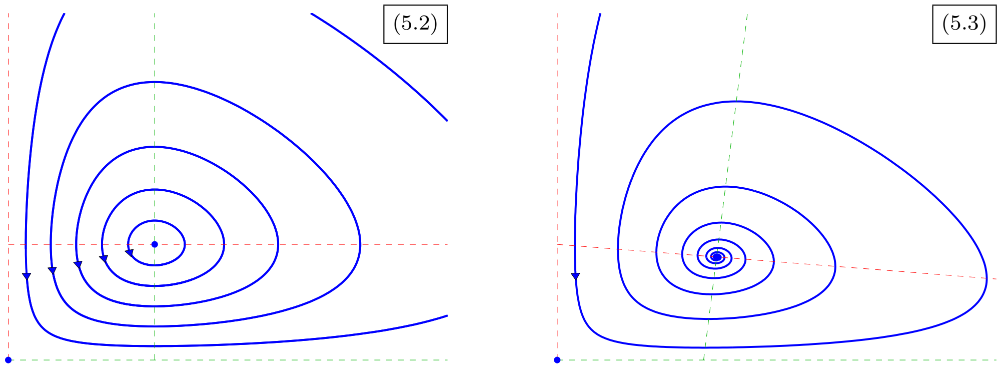


Figure 5.1: Trajectories for the Lotka-Volterra model. Left panel: equation (5.2). Right panel: equation (5.3) with small 'damping' factors $\beta_1, \beta_2 > 0$. The dashed lines indicate $x' = 0$ or $y' = 0$; these lines cover the x- and y-axis.

Volterra's principle. Proposition 5.2 has some important consequences for practical situations, for instance with insecticide treatment. Suppose that in a greenhouse there is an insect population x (e.g. aphids) feeding on the plants, and there is a predator insect population y (e.g. ladybird beetles) for which x is prey, and assume these populations will evolve according to equation (5.2). Now suppose that to decrease the population x some insecticide is sprayed. This insecticide will have an effect on x: the constant α_1 will be lowered to $\tilde{\alpha}_1 = \alpha_1 - \Delta\alpha_1$. However, the insecticide will also have a (possibly smaller) effect on y: the value α_2 will increase to $\tilde{\alpha}_2 = \alpha_2 + \Delta\alpha_2$. The interaction coefficients γ_1, γ_2 may be altered slightly, but let us assume they will not change at all and $\tilde{\alpha}_1$ is still positive. As a results the new average value of x will *increase* to $\overline{x} = (\alpha_2 + \Delta\alpha_2)/\gamma_2$, which is of course contrary to the intention. This remarkable effect is known as *Volterra's principle.*

Originally, Volterra studied the model to explain the observation by fisherman in the Mediterranean Sea that during the First World War the percentage of predatory fish showed a large increase (from 10% to 35%) compared to food fish. It seemed obvious that the greatly reduced level of fishing during this period should be responsible. But it was not clear why this would affect the predators and prey in a different way. We now see that since reduction of fishing will increase α_1 and decrease α_2, this leads to an increase of the ratio $\overline{y}/\overline{x}$, in agreement with the observations.

These conclusions have been criticized because they are based on the simple model (5.2). Biologists do observe oscillations for predator-prey ecosystems, but these oscillations tend to damp out. This calls for an improved model.

Competition within species. To improve the model, competition within the two populations over available resources may be taken into account. This leads to the model

$$\text{(5.3)} \qquad \begin{cases} x' = \alpha_1 x - \beta_1 x^2 - \gamma_1 x y \,, \\ y' = -\alpha_2 y - \beta_2 y^2 + \gamma_2 x y \,, \end{cases}$$

with constants $\beta_1, \beta_2 > 0$ as in the Verhulst model. As we will see shortly, they will act as 'damping' parameters.

If β_1, β_2 are small compared to the other parameters, then the solutions will slowly spiral towards the stationary point (x_*, y_*) in the positive quadrant given by $x_* = \kappa(\alpha_1\beta_2 + \alpha_2\gamma_1)$, $y_* = \kappa(\alpha_1\gamma_2 - \alpha_2\beta_1)$ with $\kappa = (\beta_1\beta_2 + \gamma_1\gamma_2)^{-1}$, and this point has become asymptotically stable. Apart from the origin, which is still unstable, a third stationary point $(\alpha_1/\beta_1\,, 0)$ has appeared, which is also unstable for small β_1. A typical trajectory is displayed in the right panel of Figure 5.1.

If we increase β_1, there is a transition around $\beta_1 = \alpha_1\gamma_2/\alpha_2$. If $\beta_1 > \alpha_1\gamma_2/\alpha_2$, then the point $(\alpha_1/\beta_1\,, 0)$ becomes asymptotically stable, whereas the other stationary point has moved out of the first quadrant (and has become unstable). This is considered in more detail in Exercise 5.1.

5.1.2 Competitive Species Model

The next model describes the struggle for survival between two species competing for the same limited food supply. Following the same modelling guidelines as before, we now arrive at the system

$$\text{(5.4)} \qquad \begin{cases} x' = \alpha_1 x - \beta_1 x^2 - \gamma_1 x y \,, \\ y' = \alpha_2 y - \beta_2 y^2 - \gamma_2 x y \,, \end{cases}$$

with parameters $\alpha_1, \alpha_2 > 0$ describing natural growth, $\beta_1, \beta_2 > 0$ giving competition within each species, and $\gamma_1, \gamma_2 > 0$ describing competitive interaction between the species. Depending on these parameters, the two species may coexist, or one species will drive the other one to extinction.

As for the previous models, the positive x- and y-axis are covered by trajectories, connected by stationary points. So, again we know that any initial value (x_0, y_0) with $x_0, y_0 > 0$ will lead to a solution that remains in the positive quadrant. To get a qualitative picture of the trajectories, the lines

$$\ell_1 = \{(x,y) : \alpha_1 - \beta_1 x - \gamma_1 y = 0\}\,, \quad \ell_2 = \{(x,y) : \alpha_2 - \beta_2 y - \gamma_2 x = 0\}$$

are important. On ℓ_1 we have $x' = 0$, and on ℓ_2 we have $y' = 0$. In the figures below, ℓ_1 is drawn as red dashed, ℓ_2 as green dashed. Likewise, on the x-axis (green dashed) we have $y' = 0$, whereas $x' = 0$ on the y-axis (red dashed). The stationary points are located on the intersections of the green and red lines.

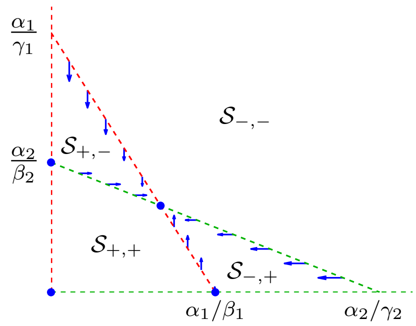
Coexistence of two species. Let us first study the case where

$$\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\gamma_2}\,, \qquad \frac{\alpha_2}{\beta_2} < \frac{\alpha_1}{\gamma_1}\,. \tag{5.5}$$

As we will see shortly, the two species can then eventually coexist.[3] In fact, for the other cases, one of the species will die out in the long run.

The lines ℓ_1, ℓ_2 divide the first quadrant of the phase plane into four regions:

$$\begin{aligned}
\mathcal{S}_{+,+} &= \{(x,y) : x' > 0, y' > 0\}\,,\\
\mathcal{S}_{+,-} &= \{(x,y) : x' > 0, y' < 0\}\,,\\
\mathcal{S}_{-,+} &= \{(x,y) : x' < 0, y' > 0\}\,,\\
\mathcal{S}_{-,-} &= \{(x,y) : x' < 0, y' < 0\}\,,
\end{aligned}$$



In $\mathcal{S}_{+,+}$ we know that solutions will move along a trajectory in upward-right direction, in $\mathcal{S}_{-,+}$ it is upward-left, and so on. This gives already a rough indication how the trajectories will look like. The intersections of the green and red dashed lines are the stationary points.

For this case (5.5), there are four stationary points in the first quadrant. Stability can be investigated with Theorem 4.4, provided that the eigenvalues of the matrix $A = f'(u_*)$ do not have real part equal to zero. By computing these matrices and their eigenvalues we see that the origin is unstable (two positive eigenvalues). The stationary points $(\alpha_1/\beta_1, 0)$ and $(0, \alpha_2/\beta_2)$ are also unstable, with one positive and one negative eigenvalue (saddle point). The remaining stationary point (x_*, y_*), on the intersection of ℓ_1 and ℓ_2 has two negative eigenvalues, so this point is asymptotically stable. (Computation by hand is here already a little cumbersome.)

All solutions with initial value (x_0, y_0) in the positive quadrant, $x_0, y_0 > 0$, will eventually tend to this stationary point (x_*, y_*).

To see this, let us first suppose that $(x_0, y_0) \in \mathcal{S}_{+,+}$. Since x' and y' are positive on $\mathcal{S}_{+,+}$, it is seen that $\frac{d}{dt}(x+y)$ is strictly positive on this region away from the stationairy points, and it follows that the solution must either tend to (x_*, y_*), or it will cross the lines ℓ_1 or ℓ_2, entering $\mathcal{S}_{+,-}$ or $\mathcal{S}_{-,+}$. For $(x_0, y_0) \in \mathcal{S}_{-,-}$ it is similar.

On the region $\mathcal{S}_{-,+}$, away from the stationairy points, $\frac{d}{dt}(-x+y)$ is strictly positive (solutions are 'swept' in upward-left direction). In fact, if $(x_0, y_0) \in \mathcal{S}_{-,+}$ the solution will stay in this region and it will ultimately approach the stationary point (x_*, y_*). This is intuitively clear by considering the direction of the flow in this region. To prove the statement, first note that on this region $y' > 0$, so the solution will not approach the x-axis. Now suppose that the solution reaches ℓ_1 at some given time t_1. Then $x'(t_1) = 0$ and by differentiation of the first equation

[3] The constants $\tilde{\beta}_j = \beta_j/\alpha_j$ and $\tilde{\gamma}_j = \gamma_j/\alpha_j$ measure the relative internal and external competition. Case (5.5) corresponds to $\tilde{\beta}_1 > \tilde{\gamma}_2$ and $\tilde{\beta}_2 > \tilde{\gamma}_1$, which means that for both species the relative internal competition is larger than the relative external competition in the other species.

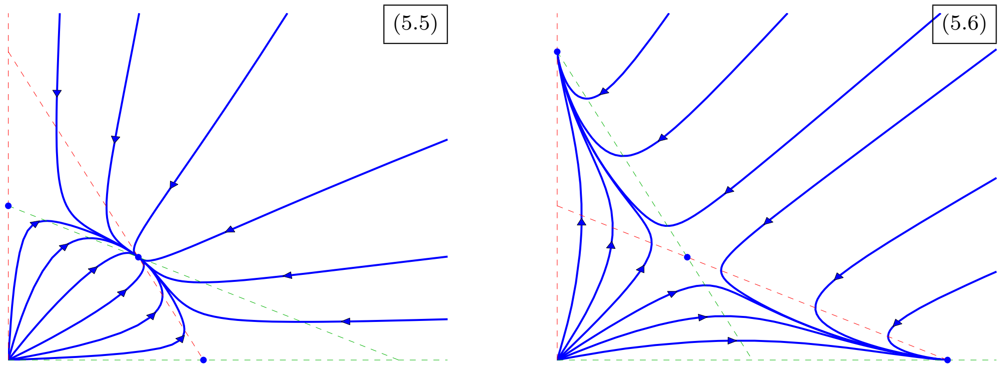


Figure 5.2: Trajectories competitive species model with parameters (5.5) or (5.6).

in (5.4) we see that $x''(t_1) = -\gamma_1 x(t_1) y'(t_1) < 0$. But this would mean that $x(t)$ has a maximum at $t = t_1$, which gives a contradiction with the fact that $x'(t) < 0$ for $t < t_1$. In the same way it is seen that the solution cannot reach ℓ_2.

An illustration with some trajectories is presented in Figure 5.2 (left panel).

Extinction of one species. As a second case we consider the model (5.4) with

$$\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\gamma_2}\,, \qquad \frac{\alpha_2}{\beta_2} > \frac{\alpha_1}{\gamma_1}\,. \tag{5.6}$$

There are again four stationary points in the first quadrant, with the origin being unstable. Now it is seen by some calculations that the stationary points $(\alpha_1/\beta_1, 0)$ and $(0, \alpha_2/\beta_2)$ are asymptotically stable, whereas the stationary points (x_*, y_*) with $x_*, y_* > 0$ has become an unstable saddle point.

By considering the regions $\mathcal{S}_{\pm,\pm}$ as in the previous case, it can be shown that any solution that starts with (x_0, y_0), $x_0, y_0 > 0$, will eventually tend to one of the two stable stationary points, except for initial values that are precisely on the trajectories that connect the saddle point (x_*, y_*) with the origin or infinity. For all other solutions one of the two species will eventually become extinct. An illustration with some trajectories is presented in Figure 5.2 (right panel).

An example for the case $\alpha_1/\beta_1 > \alpha_2/\gamma_2$, $\alpha_2/\beta_2 < \alpha_1/\gamma_1$ is treated in Exercise 5.2. For such parameters, species x will ultimately survive while y becomes extinct, irrespective of the initial state.

5.2 A Chaotic System in $\mathbb{R}^3$

For autonomous systems in $\mathbb{R}^2$ we have seen that solutions may tend to infinity, either in finite time or as $t \to \infty$. Bounded solutions can be stationary, tend to a stationary point, or they can be periodic or tend to a limit cycle, for example.

In fact, there is a famous result, called the *Poincaré-Bendixson theorem*, that states the following : Suppose a solution u of $u'(t) = f(u(t))$, with $f \in C^1(\mathbb{R}^2)$, stays for $t \geq t_0$ in a closed bounded set $\mathcal{D} \subset \mathbb{R}^2$ which contains no stationary points. Then the solution must be periodic or it tends to a limit cycle as $t \to \infty$.

In $\mathbb{R}^3$ this is no longer true, as was known already before Poincaré-Bendixson. Examples can be found in mechanical systems without friction, such as the spherical pendulum, with precessing orbits. Nevertheless, it was thought for a long time that the behaviour of solutions of autonomous systems in $\mathbb{R}^3$ would not be fundamentally different from systems in $\mathbb{R}^2$.

It was a big surprise when Lorenz introduced in the 1960's a simple system of differential equations in $\mathbb{R}^3$ with a totally different behaviour. This system is given by

$$\begin{cases} x' \;=\; \sigma\,(y-x)\,, \\ y' \;=\; -x\,z + r\,x - y\,, \\ z' \;=\; x\,y - b\,z \end{cases} \tag{5.7}$$

with positive constants σ, r and b. This system was obtained as simplified meteorological model with thermodynamic quantities $x(t)$, $y(t)$ and $z(t)$ at time t.

Proposition 5.3 Suppose $\sigma, r, b > 0$, and let $\mathcal{B}_R$ denote the closed ball around the point $(0, 0, \sigma + r)$ with radius R. Then there is an $R > 0$ such that for any solution $u = (x, y, z)$ of (5.7) we have :

(a) if $u(t_0) \in \mathcal{B}_R$, then $u(t) \in \mathcal{B}_R$ for all $t \geq t_0$;

(b) if $u(t_0) \notin \mathcal{B}_R$, then $u(t_1) \in \mathcal{B}_R$ for some $t_1 > t_0$.

Proof. Setting $c = \sigma + r$ and

$$E(x,y,z) \;=\; x^2 + y^2 + (z-c)^2\,,$$

it follows by some calculations that $\frac{d}{dt}E(x,y,z) = F(x,y,z)$ with

$$F(x,y,z) \;=\; -2\left(\sigma\,x^2 + y^2 + b(z-\tfrac{1}{2}c)^2\right) \;+\; \tfrac{1}{4}b\,c^2\,.$$

The set of points (x, y, z) in the phase space for which $F(x, y, z) = 0$ is an ellipsoid. Let $R > 0$ be such that this ellipsoid is contained in $\mathcal{B}_{R/2}$. Then there is a $d > 0$ such that $F(x, y, z) < -d$ for all (x, y, z) outside $\mathcal{B}_R$. Hence any solution starting outside $\mathcal{B}_R$ will enter this sphere in finite time, and once inside it cannot get out again. $\square$

It is obvious that the origin is a stationary point. It follows by some calculations that this point is asymptotically stable if $r < 1$, and this does not give very interesting solutions. If $r > 1$ the origin becomes unstable and two additional stationary points appear:

$$x_* \;=\; y_* \;=\; \pm\sqrt{b(r-1)}\,, \quad z_* \;=\; r-1\,.$$

It can be shown (cumbersome calculation when done by hand) that both these points are are asymptotically stable for r slightly larger rhan 1, but they become

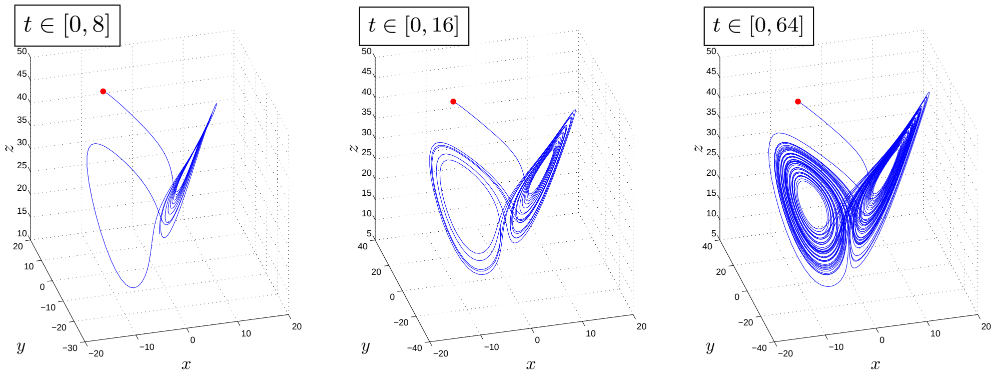


Figure 5.3: Orbits near the Lorenz attractor with time intervals $[0, T]$, $T = 8, 16, 64$. The initial values are $x_0 = -10$, $y_0 = 0$, $z_0 = 50$.

unstable if $\sigma > b + 1$ and

$$r > r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}.$$

There are no other stationary points, so now the question is what happens to these bounded solutions.

Let us consider $b = \frac{8}{3}$, $\sigma = 10$, giving $r_c \approx 24.7$, and $r = 28 > r_c$. For these parameter values, Lorenz found by numerical simulations that solutions do not approach a limit cycle. Accurate numerical trajectories are plotted in Figure 5.3, and a same behaviour is observed for any initial value. Solutions are attracted to the 'wings' of a set $\mathcal{V} \subset \mathbb{R}^3$ known as the Lorenz attractor (or the 'Lorenz butterfly'), they rotate for a while near one wing, and then suddenly jump to the other wing, where this process continues.

The precise time when such a jump occurs is very unpredictable; see Figure 5.4. Repeating the simulation with a slightly perturbed initial value will show the same behaviour but with different jumping times after a while. Therefore the two solutions will differ substantially after some time, but they both come closer and closer to the attractor $\mathcal{V}$ (which happens to be a fractal set, forming the two 'wings' and filaments connecting them).

Remark 5.4 We saw in Proposition 5.3 that there is a ball $\mathcal{B}$ such that any solution starting in this ball will stay in it. There is more we can say about the behaviour of solutions inside this ball. Writing (5.7) as $u' = f(u)$ with $u = (x, y, z)^T$, it is easily seen that the the trace of $f'(u)$ equals $-(\sigma + 1 + b)$, which is constant and negative. Using this, it can be shown (c.f. Remark 3.6) that there are sets $\mathcal{D}_0 = \mathcal{B} \supset \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots$ with exponentially decreasing volumes, $\mathrm{Vol}(\mathcal{D}_k) = e^{-k(\sigma+1+b)}\mathrm{Vol}(\mathcal{D}_0)$, such that $u(t) \in \mathcal{D}_k$ for all $t \geq k$. So any solution gets trapped in smaller and smaller volumes, and eventually it will tend to the attractor $\mathcal{V} = \mathcal{D}_0 \cap \mathcal{D}_1 \cap \mathcal{D}_2 \cap \cdots$ which has volume zero. $\diamond$

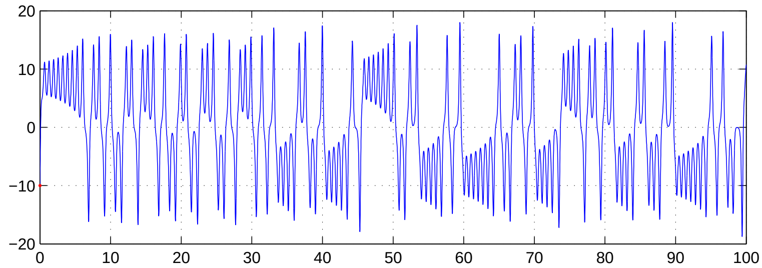


Figure 5.4: Plot of $x(t)$ for $t \in [0, 100]$. Initial values as in Figure 5.3.

In view of the unpredictability, systems like (5.7) are often called *chaotic*. Since Lorenz's discovery much work has been done on such systems, under the name of 'chaos theory'. It has been proven, among other things, that the Lorenz attractor has a fractal structure, with volume zero. It has helped to understand why long term weather prediction is so difficult, often phrased as 'a butterfly flapping its wings over Brazil, can cause a tornado over Texas two weeks later'. In spite of some initial hype, many interesting results and concepts have emerged.

5.3 Exercises

Exercise 5.1. Consider the Lotka-Volterra model (5.3) with $\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 1$ and with $\beta_2 = 0$. We keep $\beta_1 > 0$ as a free parameter.
(a) Determine stationary points and stability properties for $\beta_1 \in (0, 1)$ and $\beta_1 > 1$.
(b) Determine the regions $\mathcal{S}_{\pm,\pm}$ to get an indication about possible trajectories.
(c) Show that the predator population y will eventually become extinct if $\beta_1 > 1$. What will happen with the prey population x ?

Exercise 5.2. Consider model (5.4) with $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$, and $\alpha_1 > \alpha_2$.
(a) Determine the stationary points and their stability properties.
(b) Determine the regions $\mathcal{S}_{\pm,\pm}$.
(c) Show that the species y will eventually become extinct if $x(0) > 0$.

6 Quantitative Stability Estimates

The stability results presented in the previous sections are qualitative results, without concern of the constants involved. This is often not adequate, in particular for large systems. For this reason we will consider some quantitative stability estimates, with a given norm $\|\cdot\|$ on $\mathbb{R}^m$.

In this section we will consider a solution u on $[t_0,T]$ of the initial value problem in $\mathbb{R}^m$

$$u'(t) = f(t,u(t))\,, \qquad u(t_0) = u_0\,, \tag{6.1a}$$

together with a solution $\tilde{u}$ of the perturbed problem

$$\tilde{u}'(t) = \tilde{f}(t,\tilde{u}(t))\,, \qquad \tilde{u}(t_0) = \tilde{u}_0\,, \tag{6.1b}$$

where $f,\tilde{f} : [t_0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ and $u_0,\tilde{u}_0 \in \mathbb{R}^m$. Let $\mathcal{D} \subset \mathbb{R}^m$ be a convex set such that $u(t),\tilde{u}(t) \in \mathcal{D}$ for $t \in [t_0,T]$. It will be assumed that

$$\|f(t,v) - \tilde{f}(t,v)\| \,\le\, M \qquad (\text{for all } t \in [t_0,T],\ v \in \mathcal{D})\,. \tag{6.1c}$$

The aim in this section is to find useful upper bounds for $\|u(t)-\tilde{u}(t)\|$ under suitable additional assumptions on f. For this it will be convenient to introduce differential inequalities with generalized derivatives.

6.1 Differential Inequalities

For a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ we consider

$$D^+\varphi(t) = \limsup_{h\downarrow 0} \frac{\varphi(t+h)-\varphi(t)}{h}\,, \tag{6.2}$$

which is a so-called *Dini derivative*. Of course, if φ is differentiable in t then $D^+\varphi(t) = \varphi'(t)$. An inequality of the type $D^+\varphi(t) \le g(t,\varphi(t))$ for $t \in [t_0,T]$ is called a *differential inequality.*

Lemma 6.1 Let $\varphi,\psi : \mathbb{R}\to\mathbb{R}$ be continuous, and $g : [t_0,T]\times\mathbb{R}\to\mathbb{R}$. Assume that for $t \in [t_0,T]$

$$D^+\varphi(t) \,\le\, g(t,\varphi(t))\,, \qquad D^+\psi(t) \,>\, g(t,\psi(t))\,, \qquad \varphi(t_0) \,\le\, \psi(t_0)\,.$$

Then $\varphi(t) \,\le\, \psi(t)$ for all $t \in [t_0,T]$.

Proof. Suppose $\varphi(t_2) > \psi(t_2)$ for some $t_2 \in [t_0,T]$. Let $t_1 \ge t_0$ be the first point to the left of t_2 such that $\varphi(t_1) = \psi(t_1)$. Then, for $h>0$ small,

$$\varphi(t_1+h) - \varphi(t_1) \,>\, \psi(t_1+h) - \psi(t_1)\,,$$

and therefore $D^+\varphi(t_1) \ge D^+\psi(t_1)$. This gives a contradiction, because

$$D^+\varphi(t_1) \,\le\, g(t_1,\varphi(t_1)) = g(t_1,\psi(t_1)) < D^+\psi(t_1)\,. \qquad \square$$

Corollary 6.2 Suppose $D^+\varphi(t) \le \omega\,\varphi(t) + \rho$ on $[t_0, T]$ with $\omega, \rho \in \mathbb{R}$. Then

$$\varphi(t) \;\le\; e^{\omega(t-t_0)}\varphi(t_0) + \frac{1}{\omega}\left(e^{\omega(t-t_0)} - 1\right)\rho \qquad (\text{for } t \in [t_0, T])\,.$$

Here, by convention, $\frac{1}{\omega}(e^{\omega(t-t_0)} - 1) = (t - t_0)$ in case $\omega = 0$.

Proof. For arbitrary $\Delta\rho > 0$, let $\psi'(t) = \omega\psi(t) + (\rho + \Delta\rho)$ with $\psi(t_0) = \varphi(t_0)$. Application of Lemma 6.1 with $g(t, \varphi) = \omega\,\varphi + \rho$ shows that $\varphi(t) \le \psi(t)$ on $[t_0, T]$, and the inequality for $\varphi(t)$ now follows by letting $\Delta\rho \to 0$. □

We will mostly apply this with φ being the norm of a vector valued function, $\varphi(t) = \|w(t)\|$. Even if w is differentiable, its norm may not be so, but the Dini derivative will exist. Since $\|w(t+h)\| - \|w(t)\| \le \|w(t+h) - w(t)\|$ by the triangle inequality, it follows that

$$D^+\|w(t)\| \;\le\; \|w'(t)\|\,. \tag{6.3}$$

Estimates with Lipschitz Constants. As a typical application of differential inequalities, we first present an alternative proof of Theorem 2.10. This serves to refresh the memory, but it will also make the generalization in the next subsection more clear. So, we consider (6.1) with a constant $M \ge 0$, and assume that f satisfies the Lipschitz condition

$$\|f(t, v) - f(t, \tilde{v})\| \;\le\; L\|v - \tilde{v}\| \qquad (\text{for all } t \in [t_0, T],\ v, \tilde{v} \in \mathcal{D})\,. \tag{6.4}$$

Let $\varphi(t) = \|u(t) - \tilde{u}(t)\|$. Then

$$D^+\varphi(t) \;\le\; \|u'(t) - \tilde{u}'(t)\| \;\le\; L\varphi(t) + M\,,$$

and Corollary 6.2 now gives on $[t_0, T]$ the upper bound

$$\|u(t) - \tilde{u}(t)\| \;\le\; e^{L(t-t_0)}\|u_0 - \tilde{u}_0\| + \frac{1}{L}\left(e^{L(t-t_0)} - 1\right) M\,. \tag{6.5}$$

As we saw before, the usual way to establish the Lipschitz condition (6.4) for a continuously differentiable f is to require

$$\left\|\frac{\partial}{\partial v} f(t, v)\right\| \;\le\; L \qquad \text{for all } t \in [t_0, T] \text{ and } v \in \mathcal{D}\,, \tag{6.6}$$

where $\frac{\partial}{\partial v} f(t, v)$ denotes the Jacobi matrix with partial derivatives w.r.t. $v \in \mathbb{R}^m$.

6.2 Estimates with Logarithmic Matrix Norms

In many applications the Lipschitz constant L is large, and then an estimate like (6.5) may not be very useful. For example, for linear autonomous systems $u'(t) = Au(t)$, a stability estimate with Lipschitz constant $L = \|A\|$ essentially amounts to

$$\|e^{tA}\| \;\le\; e^{t\|A\|} \qquad \text{for } t \ge 0\,. \tag{6.7}$$

Such an estimate will often be a crude over-estimation, as is already seen with $m = 1$, $a \ll 0$, where $|e^{t\,a}| \ll e^{t|a|}$ for $t > 0$. We will improve this estimate (6.7), replacing $\|A\|$ by a quantity $\mu(A)$ which is often much smaller than the norm of A.

The *logarithmic norm* of a matrix $A \in \mathbb{R}^{m\times m}$ is defined as

$$\text{(6.8)} \qquad \mu(A) \;=\; \lim_{h\downarrow 0} \mu_h(A)\,, \qquad \mu_h(A) \;=\; \frac{\|I + hA\| - 1}{h}\,.$$

Properties of this logarithmic norm are discussed later, but it should be noted here already that it is not a norm: it can be negative. For example, if $m = 1$, $a < 0$, then $\mu(a) = a$.

First we consider an application where, instead (6.6), it is assumed that

$$\text{(6.9)} \qquad \mu\Big(\tfrac{\partial}{\partial v} f(t,v)\Big) \;\le\; \omega \qquad \text{for all } t \in [t_0,T] \text{ and } v \in \mathcal{D}\,,$$

Theorem 6.3 Consider (6.1) with constant $M \ge 0$ and $f, \tilde f$ continuously differentiable on $[t_0,T]\times\mathcal{D}$, and assume (6.9) is valid. Then, for all $t \in [t_0,T]$,

$$\|u(t) - \tilde u(t)\| \;\le\; e^{\omega(t-t_0)}\|u_0 - \tilde u_0\| + \frac{1}{\omega}\left(e^{\omega(t-t_0)} - 1\right) M\,.$$

Proof. The proof will only be given for linear equations, $f(t,v) = A(t)v + g(t)$ with $A(t) \in \mathbb{R}^{m\times m}$ such that $\mu(A(t)) \le \omega$ for $t \in [t_0,T]$. For the general nonlinear case the proof is a bit longer and more technical.[4]

Let $\varphi(t) = \|u(t) - \tilde u(t)\|$. Both u and $\tilde u$ are twice continuously differentiable. For the difference $w(t) = u(t) - \tilde u(t)$ we therefore have $w(t{+}h) = w(t) + hw'(t) + h^2\int_0^1 (1-\theta) w''(t+\theta h)\,d\theta$, and consequently $\|w(t{+}h)\| \le \|w(t) + hw'(t)\| + h^2 K$ with $K = \frac{1}{2}\max_{s\in[t_0,T]}\|w''(s)\|$. Hence

$$\begin{aligned}
\varphi(t+h) \;&\le\; \|u(t) + hf(t,u(t)) - \tilde u(t) - h\tilde f(t,\tilde u(t))\| \;+\; h^2 K \\
&\le\; \|u(t) - \tilde u(t) + h\left(f(t,u(t)) - f(t,\tilde u(t))\right)\| \;+\; hM \;+\; h^2 K \\
&\le\; \|I + hA(t)\|\,\varphi(t) \;+\; hM \;+\; h^2 K\,,
\end{aligned}$$

which gives

$$\frac{\varphi(t{+}h) - \varphi(t)}{h} \;\le\; \frac{\|I{+}hA(t)\| - 1}{h}\cdot\varphi(t) \;+\; M \;+\; h\,K\,.$$

Letting $h \downarrow 0$ it follows that $D^+\varphi(t) \le \omega\,\varphi(t) + M$. The stability estimate is now obtained from Corollary 6.2. □

Properties of logarithmic norms. To discuss logarithmic norms, we should first verify that the definition (6.8) makes sense. For this, observe that for any

[4] For the general case one can introduce $A(t) = \int_0^1 \frac{\partial}{\partial v} f(t, \tilde u(t) + \theta(u(t) - \tilde u(t))\,d\theta$. Then $f(t,u(t)) - f(t,\tilde u(t)) = A(t)\cdot(u(t) - \tilde u(t))$. Using property (6.10c), it can be shown that $\mu(A(t)) \le \omega$ by writing the integral as a limit of a Riemann sum.

$h > 0$ we have $-\|A\| \le \mu_h(A) \le \|A\|$. Moreover, $\mu_h(A)$ is monotonically non-increasing in h: if $0 < \theta < 1$ then

$$\mu_{\theta h}(A) \;\le\; \frac{1}{\theta h}\big(\|\theta I + \theta h A\| + \|(1-\theta)I\| - 1\big) \;=\; \mu_h(A)\,.$$

Hence the limit in (6.8) exists, and $\mu(A) \le \|A\|$.

The importance of logarithmic norms lies in the following result, which tells us that $\mu(A)$ is the smallest number ω such that $\|e^{tA}\| \le e^{t\omega}$ for all $t \ge 0$.

Lemma 6.4 For $A \in \mathbb{R}^{m\times m}$ we have

$$\mu(A) \;\le\; \omega \qquad \Longleftrightarrow \qquad \|e^{tA}\| \;\le\; e^{t\omega} \quad (\text{for all } \; t \ge 0)\,.$$

Proof. To prove the implication from right to left, note that $I+hA = e^{hA}+\mathcal{O}(h^2)$. Therefore, if $\|e^{tA}\| \le e^{t\omega}$ (for $t \ge 0$), then $\|I + hA\| \le 1 + h\omega + \mathcal{O}(h^2)$ (for $h \downarrow 0$), and hence $\mu(A) \le \omega$.

The estimate $\|e^{tA}\| \le e^{t\,\mu(A)}$ for $t \ge 0$ follows from Theorem 6.3, with $M = 0$. For this linear case with constant matrix A, a more direct proof is possible by using formula (3.8), written as

$$e^{tA} \;=\; \lim_{h\downarrow 0}\,(I + hA)^n \qquad (t = nh \text{ fixed, } n \to \infty)\,.$$

If $\mu(A) \le \omega$, then $\|I + hA\| \le 1 + \omega h + o(h)$ for $h \downarrow 0$. Hence

$$\|(I + hA)^n\| \;\le\; \big(1 + \omega h + o(h)\big)^n \;\to\; e^{t\omega} \quad (t = nh \text{ fixed, } n \to \infty)\,,$$

from which it is seen that $\|e^{tA}\| \le e^{t\omega}$. $\square$

Some properties of logarithmic norm are:

(6.10a) $$\mu(cI + A) \;=\; c + \mu(A) \qquad \text{if } c \in \mathbb{R}\,,$$

(6.10b) $$\mu(cA) \;=\; c\,\mu(A) \qquad \text{if } c \ge 0\,,$$

(6.10c) $$\mu(A + B) \;\le\; \mu(A) + \mu(B)$$

for arbitrary $A, B \in \mathbb{R}^{m\times m}$. Proof of these properties is straightforward.

For some common vector norms, the corresponding logarithmic norms are easy to compute.

Example 6.5 Let $A = (a_{ij}) \in \mathbb{R}^{m\times m}$. For the maximum norm on $\mathbb{R}^m$, it follows from a direct calculation, using the expression for the induced matrix norm, that

$$\mu_\infty(A) \;=\; \max_i \Big(a_{ii} + \textstyle\sum_{j\ne i} |a_{ij}|\Big)\,,$$

see Exercise 6.4. For the Euclidian norm we have

$$\mu_2(A) \;=\; \max\big\{\lambda : \lambda \text{ eigenvalue of } \tfrac{1}{2}(A + A^T)\big\}\,.$$

Again this can be shown, by some calculations, from the expression for the induced matrix norm. See also Exercise 6.5 for an alternative proof. $\diamond$

For applications it is important to notice that the inequality $\|e^{tA}\| \le e^{t\mu(A)}$ is in general only sharp for $t \downarrow 0$. The extent to which the inequality will be adequate for larger t may depend crucially on the choice of a suitable norm.

6.3 Applications to Large Systems

Systems of ordinary differential equations (ODEs) with large dimension m arise in many applications, for instance with large electrical circuits. Historically, the first systems that were studied described problems in elasticity and heat conduction, leading to partial differential equations (PDEs). Here we consider an example which is related to the problem of heat conduction, as introduced by Fourier.

Consider the system of ODEs

$$u_j'(t) \;=\; \tfrac{\kappa}{h^2}\Big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\Big) \;+\; r(u_j(t))\,, \tag{6.11}$$

with component index $j = 1, 2, \ldots, m$, $h = \frac{1}{m+1}$, and $u_0(t) = u_{m+1}(t) = 0$. Initial values $u_j(0)$ are assumed to be given for all components, and $\kappa > 0$, $r : \mathbb{R} \to \mathbb{R}$.

This system is related to the partial differential equation

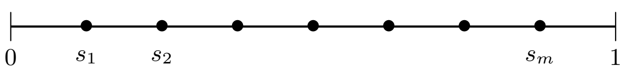
$$\frac{\partial}{\partial t} v(s,t) \;=\; \kappa \frac{\partial^2}{\partial s^2} v(s,t) \;+\; r(v(s,t)) \tag{6.12}$$

with spatial variable $s \in [0,1]$. Here $v(s,t)$ may stand for a temperature in a rod of length 1, or it may denote a concentration of a biological species that varies over space and time. For example, if $r(v) = av - bv^2$ with $a, b > 0$, then (6.12) combines the simple Verhulst model for population growth with spatial migration by diffusion. We consider equation (6.12) for $t \in [0,T]$ and $0 < s < 1$. Together with the initial condition $v(s,t) = v_0(s)$ we also have boundary conditions $v(0,t) = 0$, $v(1,t) = 0$. This is called an initial-boundary value problem for a PDE.

If we impose a spatial grid $s_j = jh$, $j = 1, \ldots, m$, with h the mesh-width in space, and use the approximations

$$\frac{\partial^2}{\partial s^2} v(s,t) \;=\; \frac{1}{h^2}\Big(v(s-h,t) - 2v(s,t) + v(s+h,t)\Big) + \mathcal{O}(h^2) \tag{6.13}$$

at the grid points, omitting the $\mathcal{O}(h^2)$ remainder term, then we obtain the ODE system (6.11) where the components u_j approximate the PDE solution at the grid points, $u_j(t) \approx v(s_j,t)$. The resulting ODE system is often called a *semi-discrete system* because space has been discretized but time is still continuous.



On the other hand, if we start with (6.11) and then let $m \to \infty$ the partial differential equation (6.12) can be obtained. This is how (6.12) was derived by Fourier. Actually, Fourier considered the heat distribution in a rod without source term r. Equation (6.12) with $r = 0$ is known as the *heat equation*.

We can write the system (6.11) in vector form as

$$u'(t) \;=\; f(u(t)) \;=\; A\,u(t) \;+\; g(u(t))\,, \tag{6.14}$$

with matrix $A \in \mathbb{R}^{m\times m}$ and with g obtained by component-wise application of r,

$$A \;=\; \frac{\kappa}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}, \qquad g(u) \;=\; \begin{pmatrix} r(u_1) \\ r(u_2) \\ \vdots \\ r(u_m) \end{pmatrix} \tag{6.15}$$

for $u = (u_j) \in \mathbb{R}^m$. We are of course mainly interested in the behaviour for small $h > 0$, that is, large m. In the following we consider the maximum norm.

Stability. It is directly seen that $\mu_\infty(A) \le 0$, and hence

$$\|e^{tA}\|_\infty \;\le\; 1\,,$$

Note that $\|A\|_\infty = \frac{4\kappa}{h^2}$ and therefore the estimate $\|e^{tA}\|_\infty \le e^{t\|A\|_\infty}$ is not very useful for small h.

If r satisfies a Lipschitz condition, say $|r(v) - r(\tilde v)| \le \ell|v - \tilde v|$ on $\mathbb{R}$, we have $\mu_\infty(f'(u)) \le \ell$ on $\mathbb{R}^m$. In fact, if r is differentiable and $r'(v) \le \omega$ on $\mathbb{R}$ we get, by using property (6.10c), the sharper estimate

$$\mu_\infty(f'(u)) \;\le\; \mu_\infty(A) + \mu_\infty(g'(u)) \;\le\; \omega \qquad \text{for all } u \in \mathbb{R}^m. \tag{6.16}$$

Remark 6.6 Similar results are valid the Euclidian norm; see Exercise 6.5. For this, note that A is symmetric, and from $\|e^{tA}\|_\infty \le 1$ (for $t \ge 0$) we know that all eigenvalues are nonpositive. $\diamond$

Convergence. If the PDE solution v is four times continuously differentiable w.r.t. the spatial variable s, then it follows by Taylor expansion that (6.13) is valid with a remainder term bounded by $K\,h^2$ for small $h > 0$, with $K = \frac{1}{12}\max_{s,t}|\frac{\partial^4}{\partial s^4}v(s,t)|$. If $\mu_\infty(f'(u)) \le \omega$ on $\mathbb{R}^m$ we can apply Theorem 6.3 with $\tilde u_j(t) = v(s_j, t)$, $\|f(u) - \tilde f(u)\| \le Kh^2$, to obtain

$$\max_{1\le j\le m} |u_j(t) - v(s_j,t)| \;\le\; \frac{1}{\omega}\left(e^{\omega t} - 1\right)\, K\,h^2\,. \tag{6.17}$$

In case $\omega = 0$ the right-hand side reads $t\,K\,h^2$. Hence for any given time interval $[0, T]$ we have for $h \to 0$ convergence – in the sense of (6.17) – of the ODE solution towards the PDE solution.

Properties of the ODE system can then be transferred to the PDE solution. For example, for the heat equation ($r = 0$), the combination of (6.17) with $\|e^{tA}\|_\infty \le 1$ for $t \ge 0$ gives

$$\max_{0\le s\le 1} |v(s,t)| \;\le\; \max_{0\le s\le 1} |v(s,0)| \qquad (\text{for all } t \ge 0)\,. \tag{6.18}$$

6.4 Exercises

Exercise 6.1. Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and $g : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume

$$(*) \qquad D^+\varphi(t) \,\le\, g(t, \varphi(t)) \,, \qquad D^+\psi(t) \,\ge\, g(t, \psi(t)) \,, \qquad \varphi(t_0) \,\le\, \psi(t_0) \,,$$

for $t \in [t_0, T]$. In general, this is *not* a sufficient condition to have

$$(**) \qquad \varphi(t) \,\le\, \psi(t) \quad (\text{for all } t \in [t_0, T]) \,.$$

(a) Demonstrate this with $g(t, \varphi) = \sqrt{|\varphi|}$. Hint: use non-uniqueness of solutions of $\varphi' = g(t, \varphi)$, $\varphi(t_0) = 0$.
(b) Show that if $g(t, \varphi)$ satisfies a Lipschitz condition w.r.t. φ, then assumption $(*)$ is sufficient for $(**)$. Hint: consider ψ_n satisfying $\psi_n'(t) = g(t, \psi_n(t)) + \frac{1}{n}$.

Exercise 6.2. Consider the initial value problem

$$u'(t) = t^2 + u(t)^2 \,, \qquad u(0) = 1 \,.$$

Show that: $\dfrac{1}{1-t} \le u(t) \le \tan(t + \frac{1}{4}\pi)$ for $0 \le t < \frac{1}{4}\pi$.

Exercise 6.3. Let $\varepsilon > 0$ small, and

$$A \,=\, \begin{pmatrix} -2 & 1 \\ \varepsilon & -2 \end{pmatrix} \,=\, V \Lambda V^{-1} \,.$$

(a) Compute the eigenvalues ($\Lambda = \mathrm{diag}(\lambda_j)$) and eigenvectors ($V$), and determine upper bounds for $\|e^{tA}\|_\infty$ in the maximum norm. What happens if $\varepsilon \to 0$?
(b) Compare this with the following estimates: $\|e^{tA}\|_\infty \le \|V\|_\infty \|V^{-1}\|_\infty \|e^{t\Lambda}\|_\infty$, $\|e^{tA}\|_\infty \le e^{t\|A\|_\infty}$ and $\|e^{tA}\|_\infty \le e^{t\mu_\infty(A)}$.
(c) Consider the norm $\|w\| = \|V^{-1}w\|_\infty$ on $\mathbb{R}^2$. Determine $\mu(A)$ in this new norm. (Long calculations can be avoided; consider $\|I + hA\|$.)

Exercise 6.4. Let $A = (a_{ij}) \in \mathbb{R}^{m \times m}$ and $\alpha = \max_i \left(a_{ii} + \sum_{j \ne i} |a_{ij}| \right)$. Show that $\|I + hA\|_\infty = 1 + h \cdot \alpha$ if $h > 0$ is sufficiently small. Consequently $\mu_\infty(A) = \alpha$.

Exercise 6.5. Let $A \in \mathbb{R}^{m \times m}$. Consider the Euclidian norm $\|v\|_2 = \sqrt{v^T v}$ on $\mathbb{R}^m$.
(a) Let $B = \frac{1}{2}(A + A^T)$. Show that

$$\|e^{tA}\|_2 \le 1 \quad (\text{for all } t \ge 0) \quad \Longleftrightarrow \quad v^T B\, v \le 0 \quad (\text{for all } v \in \mathbb{R}^m).$$

Hint: consider $\frac{d}{dt}\|u(t)\|_2^2$ for a solution of $u'(t) = Au(t)$, $u(0) = u_0$.
(b) Since $B = \frac{1}{2}(A + A^T)$ is symmetric, we have $B = U \Lambda U^{-1}$ with U orthogonal ($U^T = U^{-1}$) and real $\Lambda = \mathrm{diag}(\lambda_j)$. Show that

$$v^T B\, v \le 0 \quad (\text{for all } v \in \mathbb{R}^m) \quad \Longleftrightarrow \quad \max_j \lambda_j \le 0 \quad (1 \le j \le m).$$

(c) For general $A \in \mathbb{R}^{m \times m}$, show that $\mu_2(A)$ is the largest eigenvalue of $\frac{1}{2}(A + A^T)$. Hint: from property (6.10a) we know that $\mu_2(A) \le \omega$ iff $\mu_2(A - \omega I) \le 0$.

7 Boundary Value Problems

In this section we will study boundary value problems for ordinary differential equations, also known as two-point boundary value problems. In such problems the independent variable is often a space coordinate, and it will therefore be denoted by s. This will also be convenient when boundary value problems are discussed in connection with partial differential equations.

The general form of a two-point boundary value problem on an interval $[a, b]$ is

$$u'(s) = f\big(s, u(s)\big)\,, \qquad h\big(u(a), u(b)\big) = 0\,, \tag{7.1}$$

with $f : [a, b] \times \mathbb{R}^m \to \mathbb{R}^m$ and $h : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. General statements about existence and uniqueness of solutions are much more difficult than for initial value problems. In many applications boundary value problems do appear in the special form of a scalar second-order differential equation, and such forms are considered in the following. Also, for convenience of notation, we take $[a, b] = [0, 1]$.

In this section we will mainly restrict ourselves to two-point boundary value problems of the form

$$w''(s) \;=\; g\big(s, w(s), w'(s)\big)\,, \tag{7.2a}$$

$$w(0) = \alpha\,, \quad w(1) = \beta\,, \tag{7.2b}$$

with $g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$. The next example shows that, even for smooth functions g, this problem may not have a unique solution.

Example 7.1 Consider the problem

$$w'' \;=\; -e^w\,, \qquad w(0) = 0\,, \quad w(1) = \beta\,. \tag{7.3}$$

Instead of the right boundary condition, we first consider the initial conditions $w(0) = 0$, $w'(0) = \xi$, and denote the solution as $w(s, \xi)$. Figure 7.1 shows numerical approximations for $\xi = 2, 4, \ldots, 16$. It appears that no matter how large ξ is chosen, we cannot get $w(1, \xi)$ larger than some critical value β_c.

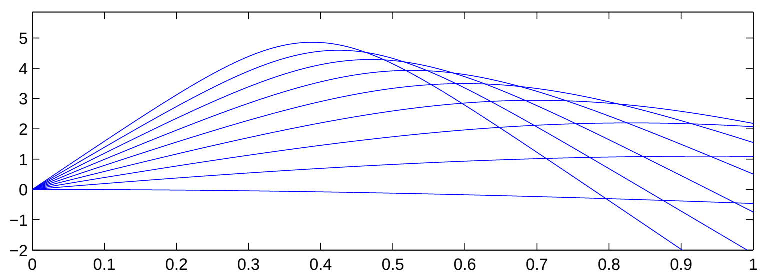


Figure 7.1: Solutions $w(s, \xi)$ versus $s \in [0, 1]$, for $\xi = 0, 2, 4, \ldots, 16$.

It follows that the boundary value problem (7.3) does not have a solution if $\beta > \beta_c$. On the other hand, if $\beta < \beta_c$ the solution is not unique. In Exercise 7.6 this will be analyzed. $\diamond$

7.1 Existence, Uniqueness and Shooting

It is clear from the above example, that to guarantee unique solvability of the boundary value problem (7.2), it is not sufficient that g is differentiable. Also a Lipschitz condition is not sufficient. We do have the following result:

Theorem 7.2 Assume $g(s, u_1, u_2)$ is continuous in s, continuously differentiable in u_1, u_2, and there are $q_0, q_1, p_0, p_1 \in \mathbb{R}$ such that

$$0 \le q_0 \le \frac{\partial}{\partial u_1} g(s, u_1, u_2) \le q_1 \,, \qquad p_0 \le \frac{\partial}{\partial u_2} g(s, u_1, u_2) \le p_1 \,,$$

for all $s \in [0, 1]$ and $u_1, u_2 \in \mathbb{R}$. Then the boundary value problem (7.2) has a unique solution.

To prove this theorem, we will first derive some intermediate results, which are of interest on their own. As in Example 7.1, we consider along with the boundary value problem (7.2) also the initial value problem consisting of the differential equation (7.2a) with initial condition

$$w(0) = \alpha \,, \quad w'(0) = \xi \tag{7.4}$$

where $\xi \in \mathbb{R}$. Let us denote the solution of this initial value problem as $w(s, \xi)$, and introduce

$$F(\xi) = w(1, \xi) - \beta \,. \tag{7.5}$$

It will be shown that F has precisely one root ξ_*, and $w(s, \xi_*)$ is then the solution of our boundary value problem. This approach, where ξ is determined such that the boundary values (7.2b) are satisfied is called *shooting*. Here ξ is considered as the 'shooting angle', aiming at $w(1, \xi) = \beta$.

Lemma 7.3 Let g satisfy the assumptions of Theorem 7.2. Then the function F is defined and continuous on $\mathbb{R}$.

Proof. We can formulate the differential equation (7.2a) in a first-order form $u' = f(s, u)$ in $\mathbb{R}^2$ with

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix}, \qquad f(s, u) = \begin{pmatrix} u_2 \\ g(s, u_1, u_2) \end{pmatrix}.$$

This function f satisfies a global Lipschitz condition; see Exercise 2.6. We therefore know that the initial value problem (7.2a), (7.4) has a unique solution for any $\xi \in \mathbb{R}$. Moreover, from Theorem 2.10 with $M = 0$ it follows that $w(1, \xi)$ depends continuously on ξ, and the same thus holds for $F(\xi)$. $\square$

Lemma 7.4 Assume $p, q : [0,1] \to \mathbb{R}$ are continuous, and $0 \le q_0 \le q(s)$, $p_0 \le p(s)$ for all $s \in [0,1]$. Let u be the solution of

$$u'' = p(s)\,u' + q(s)\,u\,, \qquad u(0) = 0\,, \quad u'(0) = 1\,.$$

Then $u(s) \ge \int_0^s e^{p_0 t}\,dt$ for all $s \in [0,1]$.

Proof. Since $u(0) = 0$, $u'(0) = 1$, there is an $s_1 \in (0,1]$ such that $u(s) > 0$ for $s \in (0, s_1)$. On this interval we have $u'' - p(s)\,u' \ge 0$. Multiplication by an integrating factor $\exp(-\int_0^s p(t)\,dt)$ shows that

$$\frac{d}{ds}\Big(\exp\big(-\textstyle\int_0^s p(t)\,dt\big) \cdot u'(s) \Big) \ \ge\ 0\,.$$

Therefore $\exp\big(-\int_0^s p(t)\,dt\big) \cdot u'(s) - u'(0) \ge 0$, which gives

$$u'(s) \ \ge\ \exp\big(\textstyle\int_0^s p(t)\,dt\big) \ \ge\ e^{p_0 s} \qquad (0 \le s \le s_1)\,.$$

Using $u(s) = \int_0^s u'(t)\,dt$, it follows that $u(s) \ge \int_0^s e^{p_0 t}\,dt$ for all $s \in [0, s_1]$.

It is now clear that $u(s_1) > 0$. Consequently $u(s) > 0$ on the entire interval $(0,1]$, and $u(s) \ge \int_0^s e^{p_0 t}\,dt$ for all $s \in [0,1]$. $\square$

Lemma 7.5 Let g satisfy the assumptions of Theorem 7.2. Then there is a $\gamma > 0$ such that $F(\xi) - F(\tilde\xi) \ge \gamma \cdot (\xi - \tilde\xi)$ for all $\xi > \tilde\xi$.

Proof. For given $\xi > \tilde\xi$, denote $w(s) = w(s,\xi)$, $\tilde w(s) = w(s,\tilde\xi)$ and let $v(s) = w(s) - \tilde w(s)$. Then $F(\xi) - F(\tilde\xi) = v(1)$.

Denoting $h_j(s,u_1,u_2) = \frac{\partial}{\partial u_j} g(s,u_1,u_2)$, $j = 1,2$, we have

$$\begin{aligned} v'' &= g(s,w,w') - g(s,\tilde w,w') + g(s,\tilde w,w') - g(s,\tilde w,\tilde w') \\ &= \big(\textstyle\int_0^1 h_1(s,\tilde w + \theta v, w')\,d\theta\big)\,v + \big(\textstyle\int_0^1 h_2(s,\tilde w,\tilde w' + \theta v')\,d\theta\big)\,v' \end{aligned}$$

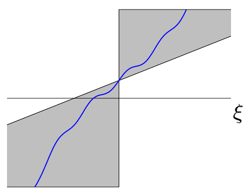
Hence v satisfies a linear initial value problem

$$v'' = p(s)\,v' + q(s)\,v\,, \qquad v(0) = 0\,, \quad v'(0) = \xi - \tilde\xi\,,$$

where $0 \le q_0 \le q(s)$ and $p_0 \le p(s)$ for all $s \in [0.1]$. Application of Lemma 7.4 to $u(s) = v(s)/(\xi - \tilde\xi)$ shows that $v(1) \ge \gamma(\xi - \tilde\xi)$ with $\gamma = \int_0^1 e^{p_0 t}\,dt$. $\square$

Proof of Theorem 7.2. The proof of the theorem is now simple. We know that

$$\begin{cases} F(\xi) \ \ge\ F(0) + \gamma\xi & \text{if } \xi \ge 0\,, \\ F(\xi) \ \le\ F(0) + \gamma\xi & \text{if } \xi \le 0\,. \end{cases}$$



Since F is also continuous and strictly monotonically increasing, it follows that it has a unique root ξ_*. $\square$

Alternative results for linear problems. The most restrictive assumption in Theorem 7.2 is $q_0 \geq 0$, that is, nonnegativity of $\frac{\partial}{\partial u_1} g(s, u_1, u_2)$. It will be seen in the next subsection, for the simple equation $w'' = qw$, that this assumption is not always necessary, but it cannot be omitted in the theorem.

There are other results available in the literature, in particular for linear problems. As a typical case we consider

$$w'' = p(s)w' + q(s)w + r(s)\,, \qquad w(0) = \alpha\,, \;\; w(1) = \beta\,, \tag{7.6}$$

together with the corresponding homogeneous problem

$$v'' = p(s)v' + q(s)v\,, \qquad v(0) = 0\,, \;\; v(1) = 0\,. \tag{7.7}$$

The next theorem gives a link between existence and uniqueness for (7.6) and the simpler problem (7.7). Moreover, we will see in the proof how to construct solutions of the boundary value problem (7.6) from solutions of two initial value problems.

Theorem 7.6 Let p, q, r be continuous on $[0, 1]$. Then the problem (7.6) has a unique solution for arbitrary $\alpha, \beta \in \mathbb{R}$ if and only if the homogeneous problem (7.7) only has the trivial solution $v = 0$.

Proof. Denote $L\,u = u'' - p\,u' - q\,u$. Let u_1, u_2 be defined by

$$\begin{aligned} L\,u_1 &= r\,, \quad u_1(0) = \alpha\,, \;\; u_1'(0) = 0\,, \\ L\,u_2 &= 0\,, \quad u_2(0) = 0\,, \;\; u_2'(0) = 1\,. \end{aligned}$$

These u_1, u_2 are well defined because the linear initial value problems have a unique solution. Further we consider the linear combinations $u = u_1 + c\,u_2$ with $c \in \mathbb{R}$. We have

$$L\,u = r\,, \quad u(0) = \alpha\,.$$

This will provide a solution to (7.6) if $u(1) = u_1(1) + c\,u_2(1) = \beta$, which can be achieved with some $c \in \mathbb{R}$ if $u_2(1) \neq 0$.

Suppose that (7.7) only has the trivial solution. Then $u_2(1) \neq 0$ and therefore (7.6) has a solution. Moreover, this solution is unique, because if w_1, w_2 are two solutions then $v = w_1 - w_2$ solves (7.7).

On the other hand, if (7.7) has a non-trivial solution v, then with any solution w of (7.6) we get other solutions $w + c\,v$. □

Remark 7.7 The above theorems can also be formulated for other boundary conditions. The conditions in (7.2), (7.6), where the value of w is specified at the boundaries, are known as *Dirichlet conditions.* We can also specify the derivative w' at a boundary point, and that is known as a *Neumann condition.* Also combinations are possible, such as

$$a_0 w(0) + a_1 w'(0) = \alpha\,, \quad b_0 w(1) + b_1 w'(1) = \beta\,,$$

which still fits in the general form (7.1). ◇

Remark 7.8 Linear differential equations in boundary value problems often appear in the form

$$(k(s)w'(s))' - l(s)w(s) = f(s)$$

with differentiable, positive $k(s)$. At first sight, this form of the differential equation seems different from (7.6), but division by $k(s)$ returns (7.6) with $q(s) = -k'(s)/k(s)$, $p(s) = l(s)/k(s)$ and $r(s) = f(s)/k(s)$. $\diamond$

7.2 Eigenvalue Problems

There are interesting applications where one is actually interested in cases where the boundary value problem does *not* have a unique solution. As a simple, but important example we consider

$$w''(s) = \lambda\, w(s)\,, \qquad w(0) = w(1) = 0\,. \tag{7.8}$$

We want to find $\lambda \in \mathbb{R}$ such that this problem has a solution w not identically equal to zero. As we will see, this is only possible for certain values of λ. Of course, $w = 0$ is always a solution.

Problem (7.8) has the form $Lw = \lambda w$, with linear operator L, and this problem is therefore called an *eigenvalue problem*, with *eigenvector* λ and *eigenfunction* w.

We see from Theorem 7.2 that if $\lambda \geq 0$, then we only have the trivial solution $w = 0$. This can also be seen more directly: the general solutions of the differential equation is $w(s) = c_1 + c_2 s$ for $\lambda = 0$ and $w(s) = c_1 e^{\sqrt{\lambda}\, s} + c_2 e^{-\sqrt{\lambda}\, s}$ for $\lambda > 0$, and from the boundary conditions $w(0) = w(1) = 0$ it follows that we must have $c_1 = c_2 = 0$.

On the other hand, if $\lambda < 0$, then the general solution of the differential equation was found in Example 3.3 to be

$$w(s) = c_1 \cos(\sqrt{|\lambda|}\, s) + c_2 \sin(\sqrt{|\lambda|}\, s)\,.$$

Here the homogeneous boundary conditions $w(0) = w(1) = 0$ imply $c_1 = 0$ and $c_2 \sin(\sqrt{|\lambda|}) = 0$. Therefore, if $\sqrt{|\lambda|}$ is a multiple of π we can find a solution $w \neq 0$. This gives the following result:

Proposition 7.9 Problem (7.8) has the eigenvalues

$$\lambda_j = -j^2\pi^2 \qquad (j = 1, 2, \ldots)\,.$$

The corresponding eigenfunctions are given by $w_j(s) = \sin(\pi j s)$ for $s \in [0, 1]$. $\Box$

Remark 7.10 Important generalizations of the above result for (7.8) are known for the Sturm-Liouville eigenvalue problems $(k(s)w'(s))' - l(s)w(s) = \lambda w(s)$, $w(0) = w(1) = 0$, with k, l given smooth functions and $0 < k_0 \leq k(s) \leq k_1$, $l_0 \leq l(s) \leq l_1$ on $[0, 1]$. It is known, among other things, that such a problem possesses an infinite sequence of eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ with $-\lambda_j \in [l_0 + k_0\, j^2\pi^2,\, l_1 + k_1\, j^2\pi^2]$, $j = 1, 2, \ldots$, with corresponding eigenfunctions w_j that satisfy the orthogonality property $\int_0^1 w_i(s) w_j(s)\, ds = 0$ if $i \neq j$. $\diamond$

The heat equation. Eigenvalue problems arise in a natural way from the study of partial differential equations. Let us consider the heat equation

$$\frac{\partial}{\partial t}v(s,t) \;=\; \frac{\partial^2}{\partial s^2}v(s,t) \tag{7.9}$$

with space variable $s \in [0,1]$ and time variable $t \geq 0$, together with the boundary conditions $v(0,t) = v(1,t) = 0$. Now we can try to find solutions by the ansatz (educated guess)

$$v(s,t) = e^{\lambda t} w(s)\,, \tag{7.10}$$

with separated variables, where we are interested in having w not identically equal to zero. Inserting this expression into (7.9) we see that w should be a non-trivial solution of the boundary value problem (7.8).

With the eigenvalues and eigenvectors found above, we thus obtain solutions $v_j(s,t) = e^{-\pi^2 j^2 t}\sin(\pi j s)$ ($j \in \mathbb{N}$) for the heat equation. These solutions are often called fundamental solutions. By taking linear combinations we then also obtain solutions of the form

$$v(s,t) \;=\; \sum_{j\geq 1} a_j e^{-\pi^2 j^2 t}\sin(\pi j s)\,, \tag{7.11}$$

with coefficients $a_j \in \mathbb{R}$, $j = 1, 2, \ldots$

Solutions of the heat equation (7.9) are specified by an initial condition $v(s,0) = \varphi(s)$. Trying to match the coefficients a_j to the initial profile, Fourier was led to the representation

$$\varphi(s) \;=\; \sum_{j\geq 1} a_j \sin(\pi j s)\,,$$

which we nowadays call a *Fourier series* representation of φ.[5] Since the integral $\int_0^1 \sin(\pi j s)\sin(\pi k s)\,ds$ equals 0 if $j \neq k$ and $\frac{1}{2}$ if $j = k$, we find that the Fourier coefficients are given by

$$a_j = 2\int_0^1 \sin(\pi j s)\varphi(s)\,ds\,.$$

For a proper mathematical justification of these equalities with infinite series we refer to the course 'Fourier Theory' or text-books on Fourier series.

The wave equation. The propagation of sound in air or vibrations in an elastic medium are described by the so-called *wave equation*

$$\frac{\partial^2}{\partial t^2}v(s,t) \;=\; \frac{\partial^2}{\partial s^2}v(s,t)\,. \tag{7.12}$$

[5] Nowadays it is known that the Fourier series converges for any φ in the function space $L_2[0,1]$, consisting of square integrable functions with identification of functions that differ only in isolated points (or sets of measure zero). In Fourier's time that was not clear, and he had great trouble getting his results published; see for instance M. Kline, Mathematical Thought from Ancient to Modern Times, Vol. 2, 1990.

The study of this equation – by Johann Bernoulli, d'Alembert and others – preceded Fourier's study of the heat equation. For this wave equation with homogeneous boundary conditions $v(0,t) = v(1,t) = 0$ we can obtain solutions by making the ansatz $v(s,t) = \cos(\mu t)w(s)$ or $v(s,t) = \sin(\mu t)w(s)$. Similar as above, this leads to an eigenvalue problem

$$w'' = -\mu^2 w\,, \qquad w(0) = w(1) = 0\,, \tag{7.13}$$

and we now find nontrivial solutions for the values $\mu_j = \pi\, j \quad (j = 1, 2, \ldots)$ with corresponding eigenfunctions $w_j(s) = \sin(\pi j s)$.

This gives solutions for the wave equations by the series

$$v(s,t) \;=\; \sum_{j\geq 1} \big(a_j \cos(\pi j t) + b_j \sin(\pi j t)\big)\, \sin(\pi j s)\,, \tag{7.14}$$

with coefficients $a_j, b_j \in \mathbb{R}$ determined by the initial conditions, $v(s,0) = \varphi(s)$, $\frac{\partial}{\partial t}v(s,0) = \psi(s)$ for $s \in [0.1]$. Note that since the wave equation is a second-order equation in t, both $v(s,0)$ and $\frac{\partial}{\partial t}v(s,0)$ are to be specified.

7.3 Exercises

Exercise 7.1. The form of a hanging cable, held fixed at the two ends, is described by

$$w''(s) = \kappa\sqrt{1 + w'(s)^2}\,, \qquad w(0) = \alpha\,, \quad w(1) = \beta\,,$$

where s is the horizontal space component, $w(s)$ is the vertical height, and $\kappa > 0$ is the weight of the cable per unit length. Show that this problem has a unique solution and find this solution. Hint: $\frac{d}{du}\log(u + \sqrt{1+u^2}) = (1+u^2)^{-1/2}$. Fitting the general solution of the differential equation to the boundary conditions requires some calculations.

Exercise 7.2. A nontrivial solution of the eigenvalue problem (7.8) will satisfy $w'(0) \neq 0$ (why?), and we may therefore require $w'(0) = 1$ (scaling). Show that (7.8) fits in the general formulation (7.1) with $u(s) = (w(s), w'(s), \lambda)^T \in \mathbb{R}^3$.

Exercise 7.3. Let $p, q \in \mathbb{R}$. Determine those constants p, q for which the linear problem $w'' = p\,w' + q\,w + r(s)$, $w(0) = \alpha$, $w(1) = \beta$, has a unique solution for all $\alpha, \beta \in \mathbb{R}$ and any continuous $r : [0,1] \to \mathbb{R}$.

Exercise 7.4. Compute the eigenvalues and eigenfunctions of the problem

$$w'' = \lambda w\,, \qquad w(0) = 0\,, \quad w'(1) = 0\,,$$

with a Dirichlet condition at the left boundary and a Neumann condition on the right.

Exercise 7.5.* Consider the partial differential equation

$$\frac{\partial}{\partial t} v(s,t) = \frac{\partial}{\partial s}\Big(k(s)\frac{\partial}{\partial s} v(s,t)\Big) - l(s)\,v(s,t)$$

with boundary conditions $v(0,t) = v(1,t) = 0$. Find relations for u and v such that $v(s,t) = u(t)w(s)$ is a solution of this partial differential equation. Which eigenvalue problem do we get for w, and how is u related to an eigenvalue?

Exercise 7.6.* To understand the numerical observations in Example 7.1, we will try to analyze the behaviour of the solutions of $w'' = -e^w$, $w(0) = 0$, $w'(0) = \xi$ for varying ξ, and in particular of $H(\xi) = w(1)$.
(a) Let $v = w'$. Show that $w = \log(-v')$ and

$$v' = \tfrac{1}{2}v^2 - c$$

with integration constant $c = 1 + \frac{1}{2}\xi^2$. Hint: multiply the differential equation by v. Observe that $v'(s) < 0$ on $[0,1]$. Note: the relation $\frac{1}{2}v^2 = -e^w + c$ can be used to draw trajectories in the phase plane, but to determine $w(1)$ we have to do more work.
(b) Determine α, β, γ as function of ξ such that

$$v(s) = -\gamma\,\frac{\beta e^{\alpha s} - 1}{\beta e^{\alpha s} + 1}\,.$$

Remark: this guess is motivated by the fact that $u(s) = \coth(s)$ solves $u' = 1 - u^2$.
(c) Show that $w(1) \to -\infty$ if $\xi \to \pm\infty$ (this involves some calculation). Conclude that $\beta_c = \sup_{\xi\in\mathbb{R}} H(\xi) < \infty$, and show that the boundary value problem (7.3) has no solution if $\beta > \beta_c$, and it has multiple solutions if $\beta < \beta_c$. Remark: the actual value of β_c is difficult to establish analytically; numerically we find $\beta_c \approx 2.24$.