Forbidden minor characterizations for low-rank optimal solutions to semidefinite programs over the elliptope

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Abstract

We study a new geometric graph parameter $\text{egd}(G)$, defined as the smallest integer $r \geq 1$ for which any partial symmetric matrix which is completable to a correlation matrix and whose entries are specified at the positions of the edges of G, can be completed to a matrix in the convex hull of correlation matrices of rank at most r. This graph parameter is motivated by its relevance to the problem of finding low rank solutions to semidefinite programs over the elliptope, and also by its relevance to the bounded rank Grothendieck constant. Indeed, $\text{egd}(G) \leq r$ if and only if the rank-r Grothendieck constant of G is equal to 1. We show that the parameter $\text{egd}(G)$ is minor monotone, we identify several classes of forbidden minors for $\text{egd}(G) \leq r$ and we give the full characterization for the case $r = 2$. We also show an upper bound for $\text{egd}(G)$ in terms of a new tree-width-like parameter $\text{la}_{\boxtimes}(G)$, defined as the smallest r for which G is a minor of the strong product of a tree and K_r. We show that, for any 2-connected graph $G \neq K_{3,3}$ on at least 6 nodes, $\text{egd}(G) \leq 2$ if and only if $\text{la}_{\boxtimes}(G) \leq 2$.

Keywords: matrix completion, semidefinite programming, correlation matrix, Gram representation, graph minor, tree-width, Grothendieck constant.

1. Introduction

1.1. Semidefinite programs

A semidefinite program (SDP) is a convex program defined as the minimization of a linear function over an affine section of the cone of positive semidefinite (psd) matrices. Semidefinite programming is a far reaching generalization of linear programing with a wide range of applications in a number of different areas

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such as approximation algorithms [11], control theory [30], polynomial optimization [20] and quantum information theory [5]. A semidefinite program in canonical primal form looks as follows:

$$\begin{aligned} \inf\ & \langle A_0, X\rangle \\ \text{subject to } & \langle A_k, X\rangle = b_k, \qquad k = 1,\dots,m \\ & X \succeq 0, \end{aligned} \tag{P}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Frobenius inner product of matrices and where A_k $(0 \le k \le m)$ are n-by-n symmetric matrices, called the *coefficient* matrices of the SDP. The generalized inequality $X \succeq 0$ means that X is positive semidefinite, i.e., all its eigenvalues are nonnegative.

The field of semidefinite programming has grown enormously in recent years. This success can be attributed to the fact that SDP's have significant modeling power, exhibit a powerful duality theory and there exist efficient algorithms, both in theory and in practice, for solving them.

The first landmark application of semidefinite programming is the work of Lovász [27] on approximating the Shannon capacity of graphs with the theta number, which gives rise to the only known polynomial time algorithm for calculating these parameters in perfect graphs (see [14]). Starting with the seminal work of Goemans and Williamson on the max-cut problem [12], SDP's have also proven to be an invaluable tool in the design of approximation algorithms for hard combinatorial optimization problems. This success is vividly illustrated by the fact that many SDP-based approximation algorithms are essentially optimal for a number of problems, assuming the validity of the Unique Games Conjecture (see e.g. [16, 29]).

In this paper we consider the problem of identifying conditions that guarantee the existence of low-rank optimal solutions for a certain class of SDP's. Results of this type are important for approximation algorithms. Indeed, SDP's are widely used as convex tractable relaxations for hard combinatorial problems. Then, rank-one solutions typically correspond to optimal solutions of the initial discrete problem and low-rank optimal solutions can decrease the error of the rounding methods and lead to improved performance guarantees.

An illustrative example is the max-cut problem where we are given as input an edge-weighted graph and the goal is to find a cut of maximum weight. It is known that, using the Goemans and Williamson semidefinite programming relaxation, the max-cut problem can be approximated in polynomial time within a factor of 0.878 [12]. Furthermore, assuming that this SDP relaxation for max-cut has an optimal solution of rank 2 (resp., 3), this approximation ratio can be improved to 0.8844 (resp., 0.8818) [2].

Low-rank solutions to SDP's are also relevant to the study of geometric representations of graphs. In this setting we consider representations obtained by assigning vectors to the vertices of a graph, where we impose restrictions on the vectors labeling adjacent vertices (e.g. orthogonality, or unit distance conditions). Then, questions related to the existence of low-dimensional representations can be reformulated as the problem of deciding the existence of a low-rank solution to an appropriate SDP, and they are connected to interesting graph properties (see [25] for an overview).

1.2. The Gram and extreme Gram dimension parameters

Our main goal in this paper is to identify combinatorial conditions guaranteeing the existence of low-rank optimal solutions to SDP's. This question has been raised by Lovász in [26]. Quoting Lovász [26, Problem 8.1] it is important to "find combinatorial conditions that guarantee that the semidefinite relaxation has a solution of rank 1". Furthermore, the version of this problem "with low rank instead of rank 1, also seems very interesting".

Our focus lies on combinatorial conditions that capture the sparsity of the coefficient matrices of a semidefinite program. To encode this information, with any semidefinite program of the form (P) we associate a graph $\mathcal{A}_P = (V_P, E_P)$, called the *aggregate sparsity pattern* of (P), where $V_P = \{1, \ldots, n\}$ and $ij \in E_P$ if and only if there exists $k \in \{0, 1, \ldots, m\}$ such that $(\mathcal{A}_k)_{ij} \neq 0$.

The structure of the aggregate sparsity pattern of a semidefinite program can be used to prove the existence of low-rank optimal solutions. This statement can be formalized by using the following graph parameter, introduced in [22].

Definition 1.1. *[22] The* Gram dimension *of a graph G is defined as the smallest integer $r \geq 1$ with the following property: Any semidefinite program that attains its optimum and whose aggregate sparsity pattern is a subgraph of G has an optimal solution of rank at most r.*

It is shown in [22] that the graph parameter $\mathrm{gd}(G)$ is minor monotone. Consequently, by the celebrated graph minor theorem of Robertson and Seymour [31], for any fixed integer $r \geq 1$, the graphs satisfying $\mathrm{gd}(G) \leq r$ can be characterized by a finite list of minimal forbidden minors. The forbidden minors for the graphs with $\mathrm{gd}(G) \leq r$ for the values $r = 2, 3$ and 4 were identified in [22].

Theorem 1.2. *[22] For any graph G we have that*

(i) $\mathrm{gd}(G) \leq 2$ *if and only if G has no K_3-minor,*

(ii) $\mathrm{gd}(G) \leq 3$ *if and only if G has no K_4-minor,*

(iii) $\mathrm{gd}(G) \leq 4$ *if and only if G has no K_5 and $K_{2,2,2}$-minors.*

Moreover it is shown in [22] that there are close connections between the Gram dimension and results concerning Euclidean graph realizations of Belk and Connelly [3, 4] and with linear algebraic properties of positive semidefinite matrices, whose zero pattern is prescribed by a fixed graph [33].

In this paper we restrict our attention to SDP's involving only constraints on the diagonal entries, namely, requiring that every feasible matrix has all its diagonal entries equal to 1. Specifically, for a graph $G = ([n], E)$ and a vector of edge-weights $w \in \mathbb{R}^E$, we consider SDP's of the following form:

$$\mathrm{sdp}(G, w) = \max \sum_{ij \in E} w_{ij} X_{ij} \quad \text{s.t.} \quad X_{ii} = 1 \ (i \in [n]), \ X \succeq 0. \qquad (P_G^w)$$

Semidefinite programs of this form arise naturally in the context of approximation algorithms. As an example, the Goemans-Williamson SDP relaxation of the max-cut problem (when formulated as a linear program in ± 1 variables) fits into this framework.

Clearly, for any $w \in \mathbb{R}^E$, the optimal value of (P_G^w) is attained since the objective function is linear and the feasible region is a compact set. Moreover, the aggregate sparsity pattern of (P_G^w) is a subgraph of G. Consequently, the semidefinite program (P_G^w) has an optimal solution of rank at most $\mathrm{gd}(G)$ (recall Definition 1.1). Our objective in this paper is to strengthen the upper bound $\mathrm{gd}(G)$. For this we introduce the following graph parameter.

Definition 1.3. *The* extreme Gram dimension *of a graph $G = ([n], E)$, denoted $\mathrm{egd}(G)$, is defined as the smallest integer $r \geq 1$ such that, for any $w \in \mathbb{R}^E$, the program (P_G^w) has an optimal solution of rank at most r.*

It follows from the definitions that $\mathrm{egd}(G)$ is upper bounded by $\mathrm{gd}(G)$, i.e.,

$$\mathrm{egd}(G) \leq \mathrm{gd}(G).$$

Moreover, this inequality is strict, for instance, for the complete graph K_n. Indeed, as we will see in Section 3.2,

$$\mathrm{egd}(K_n) = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor < n = \mathrm{gd}(K_n), \text{ for } n \geq 2.$$

We will show that the graph parameter $\mathrm{egd}(\cdot)$ is minor monotone (Lemma 3.1). Consequently, by the celebrated graph minor theorem of Robertson and Seymour, for any fixed integer $r \geq 1$, the class of graphs satisfying $\mathrm{egd}(G) \leq r$ can be characterized by a finite list of minimal forbidden minors. It is known that a graph G has $\mathrm{egd}(G) \leq 1$ if and only if G has no K_3-minor [19]. Our main result in this paper is the characterization of the graphs satisfying $\mathrm{egd}(G) \leq 2$ in terms of two forbidden minors (cf. Theorem 5.1).

Next, we give a series of reformulations for the extreme Gram dimension that will be useful throughout the paper. We start by introducing some necessary definitions and relevant notation. Throughout, $\mathcal{S}^n$ denotes the set of $n \times n$ symmetric matrices, $\mathcal{S}^n_+$ is the cone of positive semidefinite (psd) matrices and $\mathcal{S}^n_{++}$ is the cone of positive definite matrices. A psd matrix whose diagonal entries are all equal to one is called a *correlation matrix*. The set

$$\mathcal{E}_n = \{X \in \mathcal{S}^n_+ \ : \ X_{ii} = 1 \ (i \in [n])\}$$

of all $n \times n$ correlation matrices is known as the *elliptope*. For an integer $r \geq 1$, we define also the (in general non-convex) *bounded rank elliptope*

$$\mathcal{E}_{n,r} = \{X \in \mathcal{E}_n \ : \ \mathrm{rank}\, X \leq r\}.$$

Given a graph $G = ([n], E)$, π_E denotes the projection from $\mathcal{S}^n$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of G, i.e.,

$$\pi_E : \mathcal{S}^n \to \mathbb{R}^E \qquad X \mapsto (X_{ij})_{ij \in E}.$$

Lastly, the *elliptope* of the graph $G = ([n], E)$ is defined as the projection of the elliptope $\mathcal{E}_n$ onto the subspace indexed by the edge set of G:

$$\mathcal{E}(G) = \pi_E(\mathcal{E}_n).$$

The study of the elliptope is motivated by its relevance to the positive semidefinite matrix completion problem. Indeed, the elements of $\mathcal{E}(G)$ can be seen as the G-partial matrices that admit a completion to a full correlation matrix. A G-partial matrix is a matrix whose entries are specified only at the off-diagonal positions corresponding to edges of G and at the diagonal positions, with all diagonal entries being equal to 1. Consequently, deciding whether such a G-partial matrix admits a positive semidefinite completion is equivalent to deciding membership in the elliptope $\mathcal{E}(G)$.

We now give the first reformulation for the extreme Gram dimension. For a graph $G = ([n], E)$ and $w \in \mathbb{R}^E$, consider the rank-constrained SDP:

$$\mathrm{sdp}_r(G, w) = \max \sum_{ij \in E} w_{ij} X_{ij} \quad \text{s.t.} \quad X \in \mathcal{E}_{n,r}. \tag{1}$$

Then, it is clear that $\mathrm{egd}(G)$ can be equivalently defined as the smallest integer $r \geq 1$ for which equality holds:

$$\mathrm{sdp}(G, w) = \mathrm{sdp}_r(G, w), \quad \text{for all } w \in \mathbb{R}^E. \tag{2}$$

For the second reformulation of the parameter $\mathrm{egd}(\cdot)$, we first observe that program (P_G^w) is equivalent to

$$\mathrm{sdp}(G, w) = \max \ w^\mathsf{T} x \quad \text{s.t.} \quad x \in \mathcal{E}(G), \tag{3}$$

and thus it corresponds to optimization over the projected elliptope $\mathcal{E}(G)$. On the other hand, as its objective function is linear, the (non-convex rank constrained) program (1) can be equivalently reformulated as optimization over the convex set $\pi_E(\mathrm{conv}(\mathcal{E}_{n,r}))$. That is,

$$\mathrm{sdp}_r(G, w) = \max \ w^\mathsf{T} x \quad \text{s.t.} \quad x \in \pi_E(\mathrm{conv}(\mathcal{E}_{n,r})). \tag{4}$$

Then, in view of (2), we arrive at the following geometric reformulation for $\mathrm{egd}(\cdot)$.

Lemma 1.4. *The* extreme Gram dimension *of a graph $G = ([n], E)$ is equal to the smallest integer $r \geq 1$ for which*

$$\mathcal{E}(G) = \pi_E(\mathrm{conv}(\mathcal{E}_{n,r})). \tag{5}$$

Using this geometric reformulation for the parameter $\mathrm{egd}(\cdot)$, we are now in a position to explain why we have chosen to name it as the *extreme* Gram dimension. Since the inclusion $\pi_E(\mathrm{conv}(\mathcal{E}_{n,r})) \subseteq \mathcal{E}(G)$ is always valid, it follows from Lemma 1.4 that $\mathrm{egd}(G)$ is equal to the smallest $r \geq 1$ for which the reverse inclusion $\mathcal{E}(G) \subseteq \pi_E(\mathrm{conv}(\mathcal{E}_{n,r}))$ holds. Moreover, since $\mathcal{E}(G)$ is a compact convex subset of $\mathbb{R}^E$, by the Krein–Milman theorem, $\mathcal{E}(G)$ is equal to the convex hull of its set of extreme points. With $\operatorname{ext}\mathcal{E}(G)$ denoting the set of extreme points of $\mathcal{E}(G)$, it follows that

$$\mathcal{E}(G) \subseteq \pi_E(\operatorname{conv}(\mathcal{E}_{n,r})) \Longleftrightarrow \operatorname{ext}\mathcal{E}(G) \subseteq \pi_E(\mathcal{E}_{n,r}).$$

Summarizing, the parameter $\operatorname{egd}(G)$ can be reformulated as the smallest integer $r \geq 1$ for which:

$$\operatorname{ext}\mathcal{E}(G) \subseteq \pi_E(\mathcal{E}_{n,r}). \tag{6}$$

In other words, $\operatorname{egd}(G)$ is equal to the smallest $r \geq 1$ for which every extreme point of $\mathcal{E}(G)$ has a positive semidefinite completion of rank at most r.

1.3. Relation with the Gram dimension

As we will now see, both $\operatorname{gd}(\cdot)$ and $\operatorname{egd}(\cdot)$ can be phrased within the common framework of Gram representations introduced below. This reformulation will allow us to clarify the relationship between these two parameters.

Definition 1.5. *Given a graph $G = ([n], E)$ and a vector $x \in \mathbb{R}^E$, a* Gram representation *of x in $\mathbb{R}^r$ is a set of unit vectors $p_1, \ldots, p_n \in \mathbb{R}^r$ such that*

$$x_{ij} = p_i^{\mathsf{T}} p_j \quad \forall \{i,j\} \in E.$$

The Gram dimension *of $x \in \mathcal{E}(G)$, denoted by $\operatorname{gd}(G, x)$, is the smallest integer $r \geq 1$ for which x has such a Gram representation in $\mathbb{R}^r$.*

Recall that, for a matrix $X \in \mathcal{S}^n$, $X \in \mathcal{E}_n$ if and only if there exists a family of unit vectors $p_1, \ldots, p_n$ such that $X_{ij} = p_i^{\mathsf{T}} p_j$ for all $i, j \in [n]$. Hence, for a vector $x \in \mathcal{E}(G)$, it is easy to see that $\operatorname{gd}(G, x)$ is equal to the smallest rank of a completion for x to a full correlation matrix.

Using this notion of Gram representations, we find the following equivalent definition for the Gram dimension of a graph, as originally introduced in [22]. We sketch a proof of this fact for clarity.

Lemma 1.6. *For any graph $G = (V, E)$,*

$$\operatorname{gd}(G) = \max_{x \in \mathcal{E}(G)} \operatorname{gd}(G, x). \tag{7}$$

Proof. Let $x \in \mathcal{E}(G)$. Then $\operatorname{gd}(G, x)$ is the smallest r for which the SDP:

$$\min\ 0 \ \text{ s.t. } X \in \mathcal{E}_n,\ X_{ij} = x_{ij}\ \forall ij \in E$$

has an optimal solution of rank at most r. As the aggregate sparsity pattern of this SDP is equal to G, it follows that $\operatorname{gd}(G) \geq \operatorname{gd}(G, x)$. This shows the inequality $\operatorname{gd}(G) \geq \max_{x \in \mathcal{E}(G)} \operatorname{gd}(G, x)$. We now show the reverse inequality: $\operatorname{gd}(G) \leq \max_{x \in \mathcal{E}(G)} \operatorname{gd}(G, x) =: r$. For this consider an SDP of the form (P) whose aggregate sparsity pattern is a subgraph of G. Let X be an optimal solution of (P); we construct another optimal solution X' with rank at most

r. For simplicity let us assume that all diagonal entries of X are positive (if not, just work with the principal submatrix of X with only positive diagonal entries). With D denoting the diagonal matrix with diagonal entries $\sqrt{X_{ii}}$, we can rescale X so that $Y := D^{-1}XD^{-1}$ belongs to $\mathcal{E}_n$. Then, the projection $y = \pi_E(Y)$ belongs to $\mathcal{E}(G)$. Hence there exists a matrix $Y' \in \mathcal{E}_n$ of rank at most r such that $\pi_E(Y') = y$. Scaling back, the matrix $X' = DY'D$ is a psd completion of x and thus it is also an optimal solution of (P). Moreover, X' has rank at most r, which concludes the proof. $\square$

Moreover, we have the following analogous reformulation for the extreme Gram dimension.

Lemma 1.7. *For any graph G,*

$$\mathrm{egd}(G) = \max_{x \in \mathrm{ext}\, \mathcal{E}(G)} \mathrm{gd}(G, x). \tag{8}$$

Proof. Let $x \in \mathrm{ext}\, \mathcal{E}(G)$. By (6), x has a psd completion of rank at most $\mathrm{egd}(G)$ and thus $\mathrm{gd}(G,x) \leq \mathrm{egd}(G)$. This shows $\max_{x \in \mathrm{ext}\, \mathcal{E}(G)} \mathrm{gd}(G,x) \leq \mathrm{egd}(G)$. The reverse inequality follows directly from (6). $\square$

Combining (7) and (8) we find again the inequality: $\mathrm{egd}(G) \leq \mathrm{gd}(G)$.

1.4. Relation with the bounded rank Grothendieck constant

The study of programs of the form (1) is of significant practical interest, the main motivation coming from statistical mechanics and in particular from the *r-vector model* introduced by Stanley [32]. This model consists of an interaction graph $G = (V, E)$, where vertices correspond to particles and edges indicate whether there is interaction (ferromagnetic or antiferromagnetic) between the corresponding pair of particles. Additionally, there is a potential function $A: V \times V \to \mathbb{R}$ satisfying $A_{ij} = 0$ if $ij \notin E$, $A_{ij} > 0$ if there is ferromagnetic interaction between i and j and $A_{ij} < 0$ if there is antiferromagnetic interaction between i and j. Additionally, particles possess a vector valued *spin* given by a function $f : V \to \mathbb{S}^{r-1}$, where $\mathbb{S}^{r-1}$ denotes the unit sphere in $\mathbb{R}^r$. Assuming that there is no external field acting on the system, its total energy is given by the *Hamiltonian* defined as

$$H(f) = -\sum_{ij \in E} A_{ij} f(i)^{\mathsf{T}} f(j).$$

A *ground state* is a configuration of spins that minimizes the Hamiltonian. The case $r = 1$ corresponds to the *Ising model*, the case $r = 2$ corresponds to the *XY model* and the case $r = 3$ to the *Heisenberg model*. Consequently, calculating the Hamiltonian and computing ground states in any of these models amounts to solving a rank-constrained semidefinite program of the form (1).

As the rank function is non-convex and non-differentiable, such problems are computationally challenging. Indeed, problem (1) (or its reformulation (4)) is hard. In the case $r = 1$, the feasible region of (4) is equal to the cut polytope of the graph G (in ± 1 variables) and thus (4) is NP-hard. It is believed that (1) is also NP-hard for any fixed integer $r \geq 2$ (cf., e.g., the quote of Lovász [28, p. 61]). For any $r \geq 2$, it is shown in [10] that membership in $\pi_E(\text{conv}(\mathcal{E}_{n,r}))$ is NP-hard. This motivates the need for identifying tractable instances for (1).

Clearly (P_G^w) is a semidefinite programming relaxation for program (1) obtained by removing the rank constraint. The quality of this relaxation is measured by its integrality gap defined below.

Definition 1.8. *The* rank-r Grothendieck constant *of a graph G, denoted as $\kappa(r, G)$, is defined as*

$$\kappa(r, G) = \sup_{w \in \mathbb{R}^E} \frac{\text{sdp}(G, w)}{\text{sdp}_r(G, w)}. \tag{9}$$

For $r = 1$, the special case where G is a complete bipartite graph was studied by A. Grothendieck [13], although in a quite different language, and for general graphs by Alon et al. [1]. The general case $r \geq 2$ is studied by Briët et al. [6], their main motivation being the polynomial-time approximation of ground states of spin glasses.

The extreme Gram dimension of a graph is closely related to the rank-r Grothendieck constant of a graph as we now point out. Indeed it follows directly from the definitions that a graph has extreme Gram dimension at most r if and only if its rank-r Grothendieck constant is 1, i.e.,

$$\text{egd}(G) \leq r \Longleftrightarrow \kappa(r, G) = 1.$$

Hence, for any graph G with $\text{egd}(G) \leq r$, we have that $\text{sdp}(G, w) = \text{sdp}_r(G, w)$ for all $w \in \mathbb{R}^E$, and thus the value of program (1) can be approximated within arbitrary precision in polynomial time.

1.5. Contributions and outline of the paper

We now briefly summarize the main contributions of the paper. Our first result is to show that the new graph parameter $\text{egd}(G)$ is minor monotone. As a consequence the class $\mathcal{G}_r$ consisting of all graphs G with $\text{egd}(G) \leq r$ can be characterized by finitely many minimal forbidden minors. It is known that for the case $r = 1$ the only forbidden minor is K_3, i.e., the class $\mathcal{G}_1$ consists of all forests [19]. One of the main contributions of this paper is a complete characterization of the class $\mathcal{G}_2$ (Theorem 5.2).

Furthermore, we identify three families of graphs F_r, G_r, H_r which are forbidden minors for the class $\mathcal{G}_{r-1}$. This gives all the minimal forbidden minors for $r \leq 2$. The graphs G_r were already considered in [7, 18].

On the other hand we show an upper bound for the extreme Gram dimension in terms of a tree-width-like parameter. This graph parameter, which we denote as $\text{la}_{\boxtimes}(G)$, is defined as the smallest integer r for which G is a minor of the strong product $T \boxtimes K_r$ of a tree T and the complete graph K_r. We call it the *strong largeur d'arborescence* of G, in analogy with the *largeur d'arborescence* $\text{la}_{\square}(G)$ introduced by Colin de Verdière [7], defined similarly by replacing the strong product with the Cartesian product of graphs. Another main contribution is to show the upper bound: $\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G)$.

Our main result is that, for a graph $G \neq K_{3,3}$ which is 2-connected and has at least 6 nodes, $\mathrm{egd}(G) \leq 2$ if and only if $\mathrm{la}_{\boxtimes}(G) \leq 2$ if and only if G does not have F_3 or H_3 as a minor. We also characterize the graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$ and recover the characterization of [17] for the graphs with $\mathrm{la}_{\square}(G) \leq 2$.

The results and techniques in the paper come in two flavours: in Section 4 they rely mostly on the geometry of faces of the elliptope and linear algebraic tools to construct suitable extreme points of the projected elliptope and, in Section 5, they are purely graph theoretic.

The paper is organized as follows. Section 2 contains preliminaries about graphs and basic facts about the geometry of the faces of the elliptope. In Section 3 we study properties of the new graph parameter $\mathrm{egd}(G)$. In particular, in Section 3.1 we show minor-monotonicity and investigate the behaviour under the clique-sum graph operation and in Section 3.2 we show some bounds on $\mathrm{egd}(G)$. In Section 3.3 we introduce the strong largeur d'arborescence parameter $\mathrm{la}_{\boxtimes}(G)$ and we show that it upper bounds the extreme Gram dimension, i.e., that $\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G)$. In Sections 4.1-4.3 we compute the extreme Gram dimension of the three graph classes F_r, G_r and H_r and in Section 4.4 we compute the extreme Gram dimension of the graphs K_5 and $K_{3,3}$, which play a special role within the class $\mathcal{G}_2$. Section 5 is devoted to identifying the forbidden minors for the class $\mathcal{G}_2$. In Section 5.1 we characterize the chordal graphs in $\mathcal{G}_2$ (Theorem 5.4). In Section 5.2 we show that any graph with no minor F_3 or K_4 admits a chordal extension avoiding these two minors (Theorem 5.7) and in Section 5.3 we show the analogous result for graphs with no F_3 and H_3 minor (Theorem 5.12). Finally in Section 6 we characterize the graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$, we explain the links to results about $\mathrm{la}_{\square}(G)$, and we point out connections with the graph parameter $\nu(G)$ of Colin de Verdière [7].

2. Preliminaries

2.1. Preliminaries about graphs

We recall some definitions about graphs. Let $G = (V, E)$ be a graph, we also denote its node set by $V(G)$ and its edge set by $E(G)$. A *component* is a maximal connected subgraph of G. A *cutset* is a set $U \subseteq V$ for which $G\backslash U$ (deleting the nodes in U) has more connected components than G, U is a *cut node* if $|U| = 1$, and G is *2-connected* if it is connected and has no cut node. For $W \subseteq V$, $G[W]$ is the subgraph induced by W. Given $\{u, v\} \notin E(G)$, $G + \{u, v\}$ is the graph obtained by adding the edge $\{u, v\}$ to G.

Given an edge $e = \{u, v\} \in E$, $G\backslash e = (V, E \setminus \{e\})$ is the graph obtained from G by *deleting* the edge e and G/e is obtained by *contracting* the edge e: Replace the two nodes u and v by a new node, adjacent to all the neighbors of u and v. A graph M is a *minor* of G, denoted as $M \preceq G$, if M can be obtained from G by a series of edge deletions and contractions and node deletions. Equivalently, M is a minor of a connected graph G if there is a partition of $V(G)$ into nonempty subsets $\{V_i : i \in V(M)\}$ where each $G[V_i]$ is connected and, for each edge $\{i,j\} \in E(M)$, there exists at least one edge in G between V_i and V_j. The collection $\{V_i : i \in V(M)\}$ is called an *M-partition* of G and the V_i's are the *classes* of the partition.

Given a finite list $\mathcal{M}$ of graphs, $\mathcal{F}(\mathcal{M})$ denotes the collection of all graphs that do not admit any graph in $\mathcal{M}$ as a minor. By the celebrated graph minor theorem of Robertson and Seymour [31], any family of graphs which is closed under the operation of taking minors is of the form $\mathcal{F}(\mathcal{M})$ for some finite set $\mathcal{M}$ of graphs. In this setting, closed means that every minor of a graph in the family is also contained in the family.

A *graph parameter* is any function from the set of graphs (up to isomorphism) to the natural numbers. A graph parameter $f(\cdot)$ is called *minor monotone* if

$$f(G\backslash e) \leq f(G) \text{ and } f(G/e) \leq f(G),$$

for any graph G and any edge e of G. Given a minor monotone graph parameter $f(\cdot)$ and a fixed integer $k \geq 1$, the family of graphs G satisfying $f(G) \leq k$ is closed under taking minors. Then, for any fixed integer $k \geq 1$ there exists a forbidden minor characterization for the family of graphs satisfying $f(G) \leq k$.

A *homeomorph* (or subdivision) of a graph M is obtained by replacing its edges by paths. When M has maximum degree at most 3, G admits M as a minor if and only if it contains a homeomorph of M as a subgraph.

A *clique* in G is a set of pairwise adjacent nodes and $\omega(G)$ denotes the maximum cardinality of a clique in G. A k-clique is a clique of cardinality k.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs, where $V_1 \cap V_2$ is a clique in both G_1 and G_2. Their *clique sum* is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$, also called their *clique k-sum* when $k = |V_1 \cap V_2|$.

If C is a circuit in G, a *chord* of C is an edge $\{u,v\} \in E$ where u and v are two nodes of C that are not consecutive on C. G is said to be *chordal* if every circuit of length at least 4 has a chord. As is well known, a graph G is chordal if and only if G is a clique sum of cliques.

The *Cartesian product* of two graphs $G = (V,E)$ and $G' = (V', E')$, denoted by $G \Box G'$, is the graph with node set $V \times V'$, where distinct nodes $(i,i'), (j,j') \in V \times V'$ are adjacent in $G \Box G'$ when $i = j$ and $(i',j') \in G'$, or $(i,j) \in G$ and $i' = j'$.

The *strong product* of two graphs $G = (V,E)$ and $G' = (V', E')$, denoted by $G \boxtimes G'$, is the graph with node set $V \times V'$, where distinct nodes $(i,i'), (j,j') \in V \times V'$ are adjacent in $G \boxtimes G'$ when $i = j$ or $(i,j) \in E$, and $i' = j'$ or $(i',j') \in E'$.

The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the smallest integer $k \geq 1$ such that G is contained in a clique sum of copies of K_{k+1}. This parameter was introduced by Robertson and Seymour in their fundamental work on graph minors [31] and is commonly used in the parameterized complexity analysis of graph algorithms. It is known that $\mathrm{tw}(\cdot)$ is a minor-monotone graph parameter and that $\mathrm{tw}(K_n) = n - 1$ (see e.g. [9]). It follows from the above definition that, if G is obtained as the clique sum of G_1 and G_2 then

$$\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}. \tag{10}$$

Colin de Verdière [7] introduced the following treewidth-like parameter: The *largeur d'arborescence* of a graph G, denoted by $\mathrm{la}_{\Box}(G)$, is the smallest integer $r \geq 1$ for which G is a minor of $T\Box K_r$ for some tree T. Then,

$$\mathrm{tw}(G) \leq \mathrm{la}_{\Box}(G) \leq \mathrm{tw}(G) + 1,$$

where the upper bound is shown in [7] and the lower bound in [33].

The *strong largeur d'arborescence* of a graph G, denoted by $\mathrm{la}_{\boxtimes}(G)$, is the smallest integer $r \geq 1$ for which G is a minor of $T \boxtimes K_r$ for some tree T. In Section 3.3 we will come back to this parameter and we will show that it is an upper bound for the extreme Gram dimension.

2.2. Preliminaries about positive semidefinite matrices

Throughout we set $[n] = \{1, \ldots, n\}$. For $U \subseteq [n]$ and $X \in \mathcal{S}^n$, $X[U]$ denotes the principal submatrix of X with row and column indices in U and, for $j \in [n]$, $X[\cdot, j]$ denotes the j-th column of X. For a set $A \subseteq \mathbb{R}^n$, $\langle A\rangle$ denotes the vector space spanned by A and $\mathrm{conv}A$ denotes the convex hull of A.

We group here some basic properties of positive semidefinite matrices that we will use throughout. Given vectors $p_1, \ldots, p_n \in \mathbb{R}^k$ ($k \geq 1$), we let the matrix $\mathrm{Gram}(p_1, \ldots, p_n) = (p_i^{\mathsf{T}} p_j)_{i,j=1}^n$ denote their *Gram matrix*. Then, the rank of $\mathrm{Gram}(p_1, \ldots, p_n)$ is equal to $\dim\langle p_1, \ldots, p_n\rangle$. If $X = \mathrm{Gram}(p_1, \ldots, p_n)$ we also say that the p_i's form a *Gram representation* of X. As is well known, a matrix $X \in \mathcal{S}^n$ is positive semidefinite if and only if

$$\exists p_1, \ldots, p_n \in \mathbb{R}^k \text{ for some } k \geq 1 \text{ such that } X = \mathrm{Gram}(p_1, \ldots, p_n). \tag{11}$$

For $X \in \mathcal{S}^n$, $\mathrm{Ker}\, X$ is the kernel of X, consisting of the vectors $u \in \mathbb{R}^n$ such that $Xu = 0$. When $X \succeq 0$, $u \in \mathrm{Ker}\, X$ (i.e., $Xu = 0$) if and only if $u^T X u = 0$. Given a matrix $X \in \mathcal{S}^n$ in block-form

$$X = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

the following holds:

$$\text{if } X \succeq 0, \text{ then } \mathrm{Ker}\, A \subseteq \mathrm{Ker}\, B. \tag{12}$$

2.3. The geometry of the elliptope

In this section we group some geometric properties of the elliptope, which we will need in the paper. First we recall some basic definitions and facts about faces of convex sets.

Given a convex set K, a set $F \subseteq K$ is a *face* of K if, for all $x \in F$, the condition $x = ty + (1-t)z$ with $y, z \in K$ and $t \in (0, 1)$ implies $y, z \in F$. For $x \in K$ the smallest face $F(x)$ of K containing x is well defined, it is the unique face of K containing x in its relative interior. A point $x \in K$ is an *extreme point* of K if $F(x) = \{x\}$. We denote the set of extreme points of a convex set K by $\mathrm{ext}\, K$. A useful property of extreme points is that if F is a face of the convex set K then

$$\mathrm{ext}\, F \subseteq \mathrm{ext}\, K. \tag{13}$$

Moreover, z is said to be a *perturbation* of $x \in K$ if $x \pm \epsilon z \in K$ for some $\epsilon > 0$, then the segment $[x - \epsilon z, x + \epsilon z]$ is contained in $F(x)$ and the dimension of $F(x)$ is equal to the dimension of the linear space $\mathcal{P}(x)$ of perturbations of x.

We now recall some facts about the faces of the elliptope that we need here. We refer e.g. to [21] for details. For a matrix $X \in \mathcal{E}_n$, the smallest face $F(X)$ of $\mathcal{E}_n$ containing X is given by

$$F(X) = \{Y \in \mathcal{E}_n : \operatorname{Ker} X \subseteq \operatorname{Ker} Y\}. \tag{14}$$

Therefore, two matrices in the relative interior of a face F of $\mathcal{E}_n$ have the same rank, while $\operatorname{rank} X > \operatorname{rank} Y$ if X is in the relative interior of F and Y lies on the boundary of F. Here is the explicit description of the space $\mathcal{P}(X)$ of perturbations of a matrix $X \in \mathcal{E}_n$.

Proposition 2.1. *([24], see also [8, §31.5]) Let $X \in \mathcal{E}_n$ with rank r. Let $u_1, \ldots, u_n \in \mathbb{R}^r$ be a Gram representation of X, let U be the $r \times n$ matrix with columns $u_1, \ldots, u_n$ and set $\mathcal{U}_V = \langle u_1 u_1^{\mathsf{T}}, \ldots, u_n u_n^{\mathsf{T}} \rangle \subseteq \mathcal{S}^r$. The space of perturbations $\mathcal{P}(X)$ at X is given by*

$$\mathcal{P}(X) = U^{\mathsf{T}} \mathcal{U}_V^{\perp} U = \{U^{\mathsf{T}} R U : R \in \mathcal{S}^r, \langle R, u_i u_i^{\mathsf{T}} \rangle = 0 \ \forall i \in [n]\} \tag{15}$$

and the dimension of the smallest face $F(X)$ of $\mathcal{E}_n$ containing X is

$$\dim F(X) = \dim \mathcal{P}(X) = \binom{r+1}{2} - \dim\ \mathcal{U}_V. \tag{16}$$

In particular, X is an extreme point of $\mathcal{E}_n$ if and only if

$$\binom{r+1}{2} = \dim\ \mathcal{U}_V. \tag{17}$$

Hence, if $X \in \operatorname{ext} \mathcal{E}_n$ with $\operatorname{rank} X = r$ then

$$\binom{r+1}{2} \leq n. \tag{18}$$

An application for the previous proposition is the following example:

Example 2.1. *Let $e_1, \ldots, e_r \in \mathbb{R}^r$ be the standard unit vectors. The matrix with Gram representation $\{e_i : i \in [r]\} \cup \{(e_i + e_j)/\sqrt{2} : 1 \leq i < j \leq r\}$ is an extreme point of $\mathcal{E}_n$, since $\mathcal{U}_V$ is full dimensional in $\mathcal{S}^r$, where $n = \binom{r+1}{2}$.*

The next theorem shows that every number in the range prescribed in (18) corresponds to an extremal element of $\mathcal{E}_n$.

Theorem 2.2. *[24] For any natural number r satisfying $\binom{r+1}{2} \leq n$ there exists a matrix $X \in \mathcal{E}_n$ which is an extreme point of $\mathcal{E}_n$ and has rank equal to r.*

Next we establish some tools which will be useful to study the extreme points of the projected elliptope $\mathcal{E}(G)$.

Lemma 2.3. *Consider a partial matrix $x \in \mathcal{E}(G)$ and let $X \in \mathcal{E}_n$ be a rank r completion of x with Gram representation $\{u_1, \ldots, u_n\}$ in $\mathbb{R}^r$. Moreover, let U be the $r \times n$ matrix with columns $u_1, \ldots, u_n$. Set*

$$U_{ij} = \frac{u_i u_j^{\mathsf{T}} + u_j u_i^{\mathsf{T}}}{2},\ \mathcal{U}_V = \langle U_{ii} : i \in V \rangle,\ \mathcal{U}_E = \langle U_{ij} : \{i,j\} \in E \rangle \subseteq \mathcal{S}^r. \quad (19)$$

If x is an extreme point of $\mathcal{E}(G)$, then $\mathcal{U}_E \subseteq \mathcal{U}_V$.

Proof. Assume that $\mathcal{U}_E \not\subseteq \mathcal{U}_V$. Then there exists a matrix $R \in \mathcal{U}_V^{\perp} \setminus \mathcal{U}_E^{\perp}$. As $R \in \mathcal{U}_V^{\perp}$, the matrix $Z = U^{\mathsf{T}} R U = (\langle R, U_{ij} \rangle)_{i,j=1}^n \in \mathcal{S}^n$ is a perturbation of X (recall (15) and (19)). As $R \notin \mathcal{U}_E^{\perp}$, $Z_{ij} \neq 0$ for some edge $\{i,j\} \in E$. Now, $X \pm \epsilon Z \in \mathcal{E}_n$ for some $\epsilon > 0$. Hence, x can be written as the convex combination $(\pi_E(X + \epsilon Z) + \pi_E(X - \epsilon Z))/2$, where $\pi_E(X \pm \epsilon Z)$ are distinct points of $\mathcal{E}(G)$. This contradicts the assumption that x is an extreme point of $\mathcal{E}(G)$. □

Given $x \in \mathcal{E}(G)$, its *fiber* is the set of all psd completions of x in $\mathcal{E}_n$, i.e.,

$$\mathrm{fib}(x) = \{X \in \mathcal{E}_n : \pi_E(X) = x\}.$$

We close this section with a simple but useful lemma about extreme points of projected elliptopes.

Lemma 2.4. *For a vector $x \in \mathcal{E}(G)$ we have that*

(i) $x \in \mathrm{ext}\,\mathcal{E}(G)$ *if and only if* $\mathrm{fib}(x)$ *is a face of* $\mathcal{E}_n$.

(ii) If $x \in \mathrm{ext}\,\mathcal{E}(G)$ *then* $\mathrm{ext}\,\mathrm{fib}(x) \subseteq \mathrm{ext}\,\mathcal{E}_n$.

Proof. (i) Say $x \in \mathrm{ext}\,\mathcal{E}(G)$ and let $\lambda A + (1-\lambda)B \in \mathrm{fib}(x)$, where $A, B \in \mathcal{E}_n$ and $\lambda \in (0,1)$. Then $x = \lambda \pi_E(A) + (1-\lambda)\pi_E(B) \in \mathcal{E}(G)$ and since $x \in \mathrm{ext}\,\mathcal{E}(G)$ this implies that $A, B \in \mathrm{fib}(x)$. The other direction is similar.

(ii) The assumption combined with (i) imply that $\mathrm{fib}(x)$ is a face of $\mathcal{E}_n$ and using (13) the claim follows. □

3. Properties of the extreme Gram dimension

3.1. Minor-monotonicity and clique sums

In this section we investigate the behavior of the graph parameter $\mathrm{egd}(\cdot)$ under the graph operations of taking minors and clique sums.

Lemma 3.1. *The parameter* $\mathrm{egd}(\cdot)$ *is minor monotone. That is, for any edge e of G,*

$$\mathrm{egd}(G\backslash e) \leq \mathrm{egd}(G) \quad \textit{and} \quad \mathrm{egd}(G/e) \leq \mathrm{egd}(G).$$

Proof. Consider a graph $G = ([n], E)$ and an edge $e \in E$, and set $r = \mathrm{egd}(G)$. We begin by showing that $\mathrm{egd}(G \setminus e) \leq r$. Using Lemma 1.4, it suffices to show that $\mathcal{E}(G \setminus e) \subseteq \pi_{E\setminus e}(\mathrm{conv}(\mathcal{E}_{n,r}))$. For this, let $x \in \mathcal{E}(G \setminus e)$ and choose a scalar $x_e \in [-1,1]$ such that $(x, x_e) \in \mathcal{E}(G)$. Since $\mathrm{egd}(G) = r$ it follows that $(x, x_e) \in \pi_E(\mathrm{conv}(\mathcal{E}_{n,r}))$ and thus $x \in \pi_{E\setminus e}(\mathrm{conv}(\mathcal{E}_{n,r}))$.

We now show that $\mathrm{egd}(G/e) \leq r$. Say, e is the edge $(n-1, n)$ and set $G/e = ([n-1], E')$. By Lemma 1.4, it suffices to show $\mathcal{E}(G/e) \subseteq \pi_{E'}(\mathrm{conv}(\mathcal{E}_{n-1,r}))$. For this, let $x \in \mathcal{E}(G/e)$; we show that $x \in \pi_{E'}(\mathrm{conv}(\mathcal{E}_{n-1,r}))$. As x belongs to $\mathcal{E}(G/e)$, it follows that $x = \pi_{E'}(X)$ for some matrix $X \in \mathcal{E}_{n-1}$. Say, $X = \mathrm{Gram}(p_1, \ldots, p_{n-1})$ for some vectors $p_1, \ldots, p_{n-1}$. Let $X[\cdot, n-1]$ denote the last column of X and define the new matrix

$$Y = \begin{pmatrix} X & X[\cdot, n-1] \\ X[\cdot, n-1]^{\mathsf{T}} & 1 \end{pmatrix} \in \mathcal{S}^n.$$

Then, $Y_{n-1,n} = X_{n-1,n-1} = 1$ holds. Moreover, $Y = \mathrm{Gram}(p_1, \ldots, p_{n-1}, p_{n-1})$, which shows that $Y \in \mathcal{E}_n$. Therefore, the projected vector $y = \pi_E(Y)$ belongs to $\mathcal{E}(G)$ and its $(n-1,n)$-coordinate satisfies: $y_{n-1,n} = Y_{n-1,n} = 1$. As $y \in \mathcal{E}(G)$ with $\mathrm{egd}(G) = r$, it follows from Lemma 1.4 that there exist matrices $Y_1, \ldots, Y_m \in \mathcal{E}_{n,r}$ and scalars $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ satisfying

$$y = \pi_E(\sum_{i=1}^m \lambda_i Y_i). \tag{20}$$

Combining with $y_{n-1,n} = 1$, this implies:

$$1 = \sum_{i=1}^m \lambda_i (Y_i)_{n-1,n}. \tag{21}$$

Since the matrices Y_i ($i \in [m]$) are psd with diagonal entries equal to 1, all their entries are bounded in absolute value by 1. Moreover, as $\lambda_i \in (0,1]$ for every $i \in [m]$, (21) implies that $(Y_i)_{n-1,n} = 1$ and thus

$$Y_i[\{n-1, n\}] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the vector $(1,-1)$ lies in the kernel of $Y_i[\{n-1,n\}]$. Using (12), we can conclude that the last two columns of Y_i indexed by $n-1$ and by n are equal.

For $i \in [m]$, let X_i be the matrix obtained from Y_i by removing its n-th row and its n-th column. Since X_i is a submatrix of Y_i we have that $\mathrm{rank}\, X_i \leq \mathrm{rank}\, Y_i \leq r$. Set $X = \sum_{i=1}^m \lambda_i X_i$ and notice that, by construction, it belongs to $\mathrm{conv}(\mathcal{E}_{n-1,r})$. Moreover, since $Y_i[\cdot, n-1] = Y_i[\cdot, n]$ for all $i \in [m]$, it follows that $x = \pi_{E'}(X)$. Lastly, since $X \in \mathrm{conv}(\mathcal{E}_{n-1,r})$, it follows that $x \in \pi_{E'}(\mathrm{conv}(\mathcal{E}_{n-1,r}))$. This concludes the proof that $\mathrm{egd}(G/e) \leq r$. □

We now recall a well known useful fact concerning completions of psd matrices. We include a short proof for completeness.

Lemma 3.2. *Consider two psd matrices X_i indexed respectively by V_i for $i = 1, 2$. Assume that $X_1[V_1 \cap V_2] = X_2[V_1 \cap V_2]$. Then X_1 and X_2 admit a common psd completion X indexed by $V_1 \cup V_2$ with* $\mathrm{rank}\, X = \max\{\mathrm{rank}\,(X_1), \mathrm{rank}\,(X_2)\}$.

Proof. Set $r = \max\{\text{rank}\,(X_1), \text{rank}\,(X_2)\}$. Let $u_j^{(i)}$ $(j \in V_i)$ be a Gram representation of X_i (for $i = 1, 2$) and assume without loss of generality that the two families of vectors lie in the same space $\mathbb{R}^r$. Then, there exists an orthogonal $r \times r$ matrix Q mapping $u_j^{(1)}$ to $u_j^{(2)}$ for $j \in V_1 \cap V_2$. Clearly, the Gram matrix of the vectors of $Qu_j^{(1)}$ $(j \in V_1)$ together with $u_j^{(2)}$ $(j \in V_2 \setminus V_1)$ is a common psd completion with rank at most r. □

As a direct application of the above lemma, we obtain the followin result of [22]: if G is the clique sum of G_1 and G_2, then its Gram dimension satisfies: $\text{gd}(G) = \max\{\text{gd}(G_1), \text{gd}(G_2)\}$. For the extreme Gram dimension, the analogous result holds only for clique k-sums with $k \leq 1$.

Lemma 3.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. If $|V_1 \cap V_2| \leq 1$ then the clique sum G of G_1, G_2 satisfies* $\text{egd}(G) = \max\{\text{egd}(G_1), \text{egd}(G_2)\}$.

Proof. Let $x \in \mathcal{E}(G)$ and set $r = \max\{\text{egd}(G_1), \text{egd}(G_2)\}$. We will show that $x \in \pi_E(\text{conv}(\mathcal{E}_{n,r}))$. For $i = 1, 2$, the vector $x_i = \pi_{E_i}(x)$ belongs to $\pi_{E_i}(\text{conv}(\mathcal{E}_{|V_i|,r}))$. Hence, $x_i = \pi_{E_i}(\sum_{j=1}^{m_i} \lambda_{i,j} X^{i,j})$ for some $X^{i,j} \in \mathcal{E}_{|V_i|,r}$ and $\lambda_{i,j} \geq 0$ with $\sum_j \lambda_{i,j} = 1$. As $|V_1 \cap V_2| \leq 1$, any two matrices $X^{1,j}$ and $X^{2,k}$ share at most one diagonal entry, equal to 1 in both matrices. By Lemma 3.2, $X^{1,j}$ and $X^{2,k}$ have a common completion $Y^{j,k} \in \mathcal{E}_{n,r}$. This implies that $x = \pi_E(\sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \lambda_{1,j}\lambda_{2,k} Y^{jk})$, which shows $x \in \pi_E(\text{conv}(\mathcal{E}_{n,r}))$. □

Throughout this paper we denote by $\mathcal{G}_r$ the class of graphs having extreme Gram dimension at most r. By Lemma 3.1 and Lemma 3.3 the class $\mathcal{G}_r$ is closed under taking disjoint unions and clique 1-sums of graphs. Nevertheless, it is *not* closed under clique k-sums when $k \geq 2$. For example, the graph F_3 seen in Figure 1 is a clique 2-sum of triangles, however $\text{egd}(F_3) = 3$ (Theorem 4.2) while triangles have extreme Gram dimension 2 (Lemma 3.4).

3.2. Upper and lower bounds

From Lemma 1.7 we know that, for any graph G,

$$\text{egd}(G) = \max_{x \in \text{ext}\,\mathcal{E}(G)} \text{gd}(G, x).$$

According to this characterization, in order to show that $\text{egd}(G) \leq r$, it suffices to show that every partial matrix $x \in \text{ext}\,\mathcal{E}(G)$ has a psd completion of rank at most r. Using results from Section 2.3 we obtain the following:

Lemma 3.4. *The extreme Gram dimension of the complete graph K_n is*

$$\text{egd}(K_n) = \max\left\{r \in \mathbb{Z}_+ : \binom{r+1}{2} \leq n\right\} = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor. \tag{22}$$

Hence, for any graph G on n nodes we have that:

$$\text{egd}(G) \leq \max\left\{r \in \mathbb{Z}_+ : \binom{r+1}{2} \leq n\right\} = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor. \tag{23}$$

Proof. Notice that $\mathcal{E}(K_n)$ is the bijective image of $\mathcal{E}_n$ in $\mathbb{R}^{\binom{n}{2}}$, obtained by considering only the upper triangular part of matrices in $\mathcal{E}_n$. Then, for any $X \in \operatorname{ext}\mathcal{E}_n$ with $\operatorname{rank} X = r$ we know that $\binom{r+1}{2} \leq n$ (recall (18)). Moreover, from Theorem 2.2 we know that for any natural number r satisfying $\binom{r+1}{2} \leq n$ there exists an extreme point of $\mathcal{E}_n$ with rank equal to r. The second part follows from (22) using the fact that $\operatorname{egd}(\cdot)$ is minor monotone (Lemma 3.1). $\square$

The following lemma is a direct consequence of (23).

Lemma 3.5. *Consider a graph G with $|V(G)| = \binom{r+1}{2}$. Then $\operatorname{egd}(G) \leq r$.*

On the other hand, in order to obtain a lower bound $\operatorname{egd}(G) \geq r$, we need an extreme point of $\mathcal{E}(G)$, all of whose positive semidefinite completions have rank at least r. This poses two difficulties: how to construct a suitable extreme point of $\mathcal{E}(G)$ and then, given an extreme point of $\mathcal{E}(G)$, how to verify that *all* its positive semidefinite completions have rank at least r. We resolve this by using the construction of extreme points from Lemma 2.4. Indeed, if we can find a point $x \in \mathcal{E}(G)$ admitting a *unique* completion $X \in \mathcal{E}_n$ which is an extreme point of $\mathcal{E}_n$ and has rank r, then we can conclude that x is an extreme point of $\mathcal{E}(G)$ which has Gram dimension r, thus showing that $\operatorname{egd}(G) \geq r$. We summarize this in the following lemma for further reference.

Lemma 3.6. *Assume that there exists $x \in \mathcal{E}(G)$ which has a unique completion $X \in \mathcal{E}_n$. Assume moreover that X is an extreme point of $\mathcal{E}_n$ and that X has rank r. Then, $\operatorname{egd}(G) \geq r$.*

Proof. As $\operatorname{fib}(x) = \{X\}$ and $X \in \operatorname{ext}\mathcal{E}_n$, it follows that $\operatorname{fib}(x)$ is a face of $\mathcal{E}_n$ and then Lemma 2.4 implies that $x \in \operatorname{ext}\mathcal{E}(G)$. $\square$

3.3. The strong largeur d'arborescence

In this section we introduce a new treewidth-like parameter that will serve as an upper bound for the extreme Gram dimension.

Definition 3.7. *The* strong largeur d'arborescence *of a graph G, denoted by $\operatorname{la}_{\boxtimes}(G)$, is the smallest integer $k \geq 1$ for which G is a minor of $T \boxtimes K_k$ for some tree T.*

Notice the analogy with the largeur d'arborescence where the Cartesian product has been substituted with the strong graph product. It is clear from its definition that the parameter $\operatorname{la}_{\boxtimes}(\cdot)$ is minor monotone. Moreover,

Lemma 3.8. *For any graph G we have that*

$$\frac{\operatorname{tw}(G)+1}{2} \leq \operatorname{la}_{\boxtimes}(G) \leq \operatorname{la}_{\square}(G).$$

Proof. The rightmost inequality follows directly from the definitions. For the leftmost inequality assume that $\operatorname{la}_{\boxtimes}(G) = k$, i.e., G is minor of $T \boxtimes K_k$ for some tree T. Notice that the graph $T \boxtimes K_k$ can be obtained by taking clique k-sums of copies of the graph $K_2 \boxtimes K_k$. By (10), $\mathrm{tw}(T \boxtimes K_k) = \mathrm{tw}(K_2 \boxtimes K_k) = 2k-1$. Combining this with the fact that the treewidth is minor-monotone, we obtain that $\mathrm{tw}(G) \leq \mathrm{tw}(T \boxtimes K_k) = 2k-1$ and the claim follows. $\square$

Our main goal in this section is to show that the extreme Gram dimension is upper bounded by the strong largeur d'arborescence: $\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G)$ for any graph G. As we will see in later sections, this property will play a crucial role in characterizing graphs with extreme Gram dimension at most 2. We start with a technical lemma which we need for the proof of Theorem 3.11 below.

Lemma 3.9. *Let $\{u_1, \ldots, u_{2r}\}$ be a set of vectors, denote its rank by ρ. Let $\mathcal{U}$ denote the linear span of the matrices $U_{ij} = (u_i u_j^{\mathsf{T}} + u_j u_i^{\mathsf{T}})/2$ for all $i, j \in \{1, \ldots, r\}$ and all $i, j \in \{r+1, \ldots, 2r\}$. If $\rho \geq r+1$ then $\dim \mathcal{U} < \binom{\rho+1}{2}$.*

Proof. Let $I \subseteq \{1, \ldots, r\}$ for which $\{u_i : i \in I\}$ is a maximum linearly independent subset of $\{u_1, \ldots, u_r\}$ and let $J \subseteq \{r+1, \ldots, 2r\}$ such that the set $\{u_i : i \in I \cup J\}$ is maximum linearly independent; thus $|I| + |J| = \rho$. Set $K = \{1, \ldots, r\} \setminus I$, $L = \{r+1, \ldots, 2r\} \setminus J$, and $J' = J \setminus \{k\}$, where k is some given (fixed) element of J. For any $l \in L$, there exists scalars $a_{l,i} \in \mathbb{R}$ such that

$$u_l = \sum_{i \in I \cup J'} a_{l,i} u_i + a_{l,k} u_k. \tag{24}$$

Set

$$A_l = \sum_{i \in I \cup J'} a_{l,i} U_{ik} \quad \text{for} \quad l \in L.$$

Then, define the set $\mathcal{W}$ consisting of the matrices U_{ii} for $i \in I \cup J$, U_{ij} for all $i \neq j$ in $I \cup J'$, U_{kj} for all $j \in J'$, and A_l for all $l \in L$. Then, $|\mathcal{W}| = \rho + \binom{\rho-1}{2} + r - 1 = \binom{\rho}{2} + r = \binom{\rho+1}{2} + r - \rho \leq \binom{\rho+1}{2} - 1$. In order to conclude the proof it suffices to show that $\mathcal{W}$ spans the space $\mathcal{U}$.

Clearly, $\mathcal{W}$ spans all matrices U_{ij} with $i, j \in \{1, \ldots, r\}$. Moreover, by its definition $\mathcal{W}$ contains all matrices U_{ij} for $i, j \in J$. Consequently, it remains to show that $U_{kl} \in \mathcal{W}$ for all $l \in L$, $U_{lj} \in \mathcal{W}$ for all $l \in L$ and $j \in J'$ and that $U_{ll'} \in \mathcal{W}$ for all $l, l' \in L$. Fix $l \in L$. Using (24) we obtain that $U_{lk} = A_l + a_{l,k} U_{kk}$ lies in the span of $\mathcal{W}$. Moreover, for $j \in J'$, $U_{lj} = \sum_{i \in I \cup J'} a_{l,i} U_{ij} + a_{l,k} U_{kj}$ also lies in the span of $\mathcal{W}$. Finally, for $l' \in L$, $U_{ll'} = \sum_{i,j \in I \cup J'} a_{l,i} a_{l',j} U_{ij} + a_{l',k} A_l + a_{l,k} A_{l'} + a_{l,k} a_{l',k} U_{kk}$ is also spanned by $\mathcal{W}$. This concludes the proof. $\square$

Lemma 3.10. *Let $v_1, \ldots, v_n$ be a family of linearly independent vectors in $\mathbb{R}^n$. Then the matrices $(v_i v_j^{\mathsf{T}} + v_j v_i^{\mathsf{T}})$ for $1 \leq i \leq j \leq n$ span $\mathcal{S}^n$.*

Proof. Consider a matrix $Z \in \mathcal{S}^n$ such that $\langle Z, (v_i v_j^{\mathsf{T}} + v_j v_i^{\mathsf{T}})/2 \rangle = 0$ for all $i, j \in [n]$; we show that Z is the zero matrix. For any vector $x \in \mathbb{R}^n$, we can write $x = \sum_{i=1}^n \lambda_i v_i$ for some scalars λ_i ($i \in [n]$) and thus $x^{\mathsf{T}} Z x = 0$. This implies $Z = 0$. $\square$

We can now show the main result of this section.

Theorem 3.11. *For any tree T, we have that* $\operatorname{egd}(T \boxtimes K_r) \leq r$.

Proof. Let $G = T \boxtimes K_r$, where T is a tree on $[t]$. Say, $G = (V, E)$ with $|V| = n$. So the node set of G is $V = \cup_{i=1}^t V_i$, where the V_i's are pairwise disjoint sets, each of cardinality r. By definition of the strong product, for any edge $\{i, j\}$ of T, the set $V_i \cup V_j$ induces a clique in G, denoted as C_{ij}. Then, G is the union of the cliques C_{ij} over all edges $\{i, j\}$ of T. We show that $\operatorname{egd}(G) \leq r$. For this, pick an element $x \in \operatorname{ext} \mathcal{E}(G)$. Then $x = \pi_E(X)$ for some $X \in \mathcal{E}_n$. As C_{ij} is a clique in G, the principal submatrix $X^{ij} := X[C_{ij}]$ is fully determined from x. To show that x has a psd completion of rank at most r, it suffices to show that $\operatorname{rank} X^{ij} \leq r$ for all edges $\{i, j\}$ of T. Indeed, by applying Lemma 3.2, we can then conclude the existence of a common psd completion of the X^{ij} of rank at most r.

Pick an edge $\{i, j\}$ of T and set $\rho = \operatorname{rank} X^{ij}$. Assume that $\rho \geq r + 1$; we show below that there exists a nonzero perturbation Z of X^{ij} such that

$$\begin{aligned} &Z_{hk} = 0 \text{ for all } (h, k) \in (V_i \times V_i) \cup (V_j \times V_j), \\ &Z_{hk} \neq 0 \text{ for some } (h, k) \in V_i \times V_j. \end{aligned} \tag{25}$$

This permits to reach a contradiction: As Z is a perturbation of X^{ij}, there exists $\epsilon > 0$ for which $X^{ij} + \epsilon Z,\ X^{ij} - \epsilon Z \succeq 0$. By construction, C_{ij} is the only maximal clique of G containing the edges $\{h, k\}$ of G with $h \in V_i$ and $k \in V_j$. Hence, one can find a psd completion X' (resp., X'') of the matrix $X^{ij} + \epsilon Z$ (resp., $X^{ij} - \epsilon Z$) and the matrices $X^{i'j'}$ for all edges $\{i', j'\} \neq \{i, j\}$ of T. Now, $x = \frac{1}{2}(\pi_E(X') + \pi_E(X''))$, where $\pi_E(X'), \pi_E(X'')$ are distinct elements of $\mathcal{E}(G)$, contradicting the fact that x is an extreme point of $\mathcal{E}(G)$.

We now construct the desired perturbation Z of X^{ij} satisfying (25). For this let u_h ($h \in V_i \cup V_j$) be a Gram representation of X^{ij} in $\mathbb{R}^\rho$ and let $\mathcal{U} \subseteq \mathcal{S}^\rho$ denote the linear span of the matrices $U_{hk} = (u_h u_k^{\mathsf{T}} + u_k u_h^{\mathsf{T}})/2$ for all $h, k \in V_i$ and all $h, k \in V_j$. Applying Lemma 3.9, as $\rho \geq r + 1$, we deduce that $\dim \mathcal{U} < \binom{\rho+1}{2}$. Hence, by Lemma 3.10, there exists a nonzero matrix $R \in \mathcal{S}^\rho$ lying in $\mathcal{U}^\perp$ for which the matrix $Z \in \mathcal{S}^{2r}$ defined by $Z_{hk} = \langle R, U_{hk} \rangle$ for all $h, k \in V_i \cup V_j$, is nonzero. By construction, Z is a perturbation of X^{ij} (recall Proposition 2.1) and it satisfies $Z_{hk} = 0$ whenever the pair (h, k) is contained in V_i or in V_j. Moreover, as $Z \neq 0$, we have $Z_{hk} \neq 0$ for some $h \in V_i$ and $k \in V_j$. Thus (25) holds and the proof is completed. □

Corollary 3.12. *For any graph G,* $\operatorname{egd}(G) \leq \operatorname{la}_\boxtimes(G)$.

Proof. If $\operatorname{la}_\boxtimes(G) = k$, then G is a minor of $T \boxtimes K_k$ for some tree T and thus $\operatorname{egd}(G) \leq \operatorname{egd}(T \boxtimes K_k) \leq k$, by Lemma 3.1 and Theorem 3.11. □

4. The extreme Gram dimension of some graph classes

In this section we construct three classes of graphs F_r, G_r, H_r, whose extreme Gram dimension is equal to r. Therefore, they are forbidden minors for the class

$\mathcal{G}_{r-1}$ of graphs with extreme Gram dimension at most $r-1$. As we will see in the next section, this gives all the forbidden minors for the class $\mathcal{G}_2$.

The graphs G_r were already considered by Colin de Verdière [7] in relation to the graph parameter $\nu(\cdot)$, to which we will come back in Section 6. Each of the graphs $G = F_r, G_r, H_r$ has $\binom{r+1}{2}$ nodes and thus their extreme Gram dimension is at most r (recall Lemma 3.5). Moreover, they satisfy: $\text{egd}(G/e) \leq r-1$ after contracting any edge e. In order to show equality $\text{egd}(G) = r$, we will rely on Lemma 3.6.

To use Lemma 3.6 we need tools permitting to show existence of a *unique* completion for a partial matrix $x \in \mathcal{E}(G)$. We introduce below such a tool: 'forcing a non-edge with a singular clique'. This is based on the following property, which is a special case of relation (12):

$$\begin{pmatrix} A & b \\ b^{\mathsf{T}} & \alpha \end{pmatrix} \succeq 0 \Longrightarrow b^{\mathsf{T}} u = 0 \quad \forall u \in \operatorname{Ker} A. \tag{26}$$

Lemma 4.1. *Let $x \in \mathcal{E}(G)$, let $C \subseteq V$ be a clique of G and let $\{i,j\} \notin E(G)$ with $i \notin C$, $j \in C$. Set $x[C] = (x_{ij})_{i,j \in C} \in \mathcal{E}_{|C|}$ (setting $x_{ii} = 1$ for all i). Assume that i is adjacent to all nodes of $C \setminus \{j\}$, $x[C]$ is singular and $x[C \setminus \{j\}]$ is nonsingular. Then, for any psd completion X of x, the (i,j)-th entry X_{ij} is uniquely determined.*

Proof. Let X be a psd completion of x. The principal submatrix $X[C \cup \{i\}]$ has the block form shown in (26) (with A being indexed by C), where all entries are specified (from x) except the entry $b_j = X_{ij}$ which is unspecified since $\{i,j\} \notin E(G)$. As $x[C]$ is singular there exists a nonzero vector u in the kernel of $x[C]$. Moreover, since $x[C \setminus \{j\}]$ is nonsingular if follows that $u_j \neq 0$. Hence the condition $b^{\mathsf{T}} u = 0$ permits to derive the value of X_{ij} from x. □

When applying Lemma 4.1 we will say that "the clique C forces the pair $\{i,j\}$". The lemma will be used in an iterative manner: Once a non-edge $\{i,j\}$ has been forced, we know the value X_{ij} in any psd completion X and thus we can replace G by $G + \{i,j\}$ and search for a new forced pair in the extended graph $G + \{i,j\}$.

We note in passing that a general framework for constructing partial psd matrices with a unique psd completion has been developed in [23]. Following that approach, most of the constructions described below can be easily recovered.

4.1. The class F_r

For $r \geq 2$ the graph F_r has $r + \binom{r}{2} = \binom{r+1}{2}$ nodes, denoted as v_i (for $i \in [r]$) and v_{ij} (for $1 \leq i < j \leq r$); it consists of a clique K_r on the nodes $\{v_1, \ldots, v_r\}$ together with the cliques C_{ij} on $\{v_i, v_j, v_{ij}\}$ for all $1 \leq i < j \leq r$. The graphs F_3 and F_4 are illustrated in Figure 1.

For $r = 2$, $F_2 = K_3$ has extreme Gram dimension 2. More generally:

Theorem 4.2. *For $r \geq 2$, $\text{egd}(F_r) = r$. Moreover, F_r is a minimal forbidden minor for the class $\mathcal{G}_{r-1}$.*

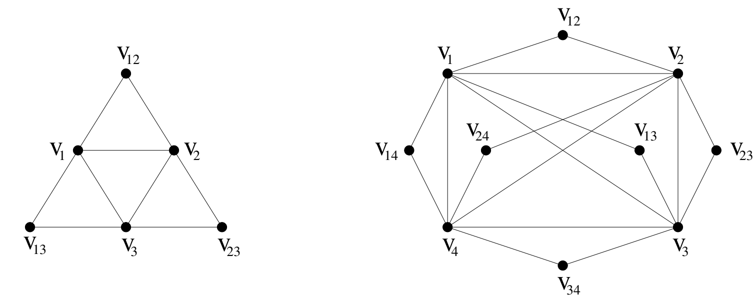

Figure 1: The graphs F_3 and F_4.

Proof. Since F_r has $\binom{r+1}{2}$ nodes it follows from Lemma 3.5 that $\text{egd}(F_r) \le r$. We now show that $\text{egd}(F_r) \ge r$. For this we label the nodes $v_1, \ldots, v_r$ by the standard unit vectors $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by the vector $(e_i + e_j)/\sqrt{2}$. Consider the Gram matrix X of these $n = \binom{r+1}{2}$ vectors and its projection $x = \pi_{E(F_r)}(X) \in \mathcal{E}(F_r)$. Using (17) it follows directly that X is an extreme point of $\mathcal{E}_n$. We now show that X is the only psd completion of x which, in view of Lemma 3.6, implies that $\text{egd}(F_r) \ge r$. For this we use Lemma 4.1. Observe that, for each $1 \le i < j \le r$, the matrix $x[C_{ij}]$ is singular. First, for any $k \in [r] \setminus \{i, j\}$, the clique C_{ij} forces the non-edge $\{v_k, v_{ij}\}$ and then, for any other $1 \le i' < j' \le r$, the clique C_{ij} forces the non-edge $\{v_{ij}, v_{i'j'}\}$. Hence, in any psd completion of x, all the entries indexed by non-edges are uniquely determined, i.e., $\text{fib}(x) = \{X\}$.

Next, we show minimality. Let e be an edge of F_r, we show that $\text{egd}(H) \le r-1$ where $H = F_r \backslash e$. If e is an edge of the form $\{v_i, v_{ij}\}$, then H is the clique 1-sum of an edge and a graph on $\binom{r+1}{2} - 1$ nodes and thus $\text{egd}(H) \le r-1$ follows using Lemmas 3.3 and 3.4. Suppose now that e is contained in the central clique K_r, say $e = \{v_1, v_2\}$. We show that H is contained in a graph of the form $T \boxtimes K_{r-1}$ for some tree T. We choose T to be the star $K_{1,r-1}$ and we give a suitable partition of the nodes of F_r into sets $V_0 \cup V_1 \cup \ldots \cup V_{r-1}$, where each V_i has cardinality at most $r-1$, V_0 is assigned to the center node of the star $K_{1,r-1}$ and $V_1, \ldots, V_{r-1}$ are assigned to the $r-1$ leaves of $K_{1,r-1}$. Namely, set $V_0 = \{v_{12}, v_3, \ldots, v_r\}$, $V_1 = \{v_1, v_{13}, \ldots, v_{1r}\}$, $V_2 = \{v_2, v_{23}, \ldots, v_{2r}\}$ and, for $k \in \{3, \ldots, r-1\}$, $V_k = \{v_{kj} : k+1 \le j \le r\}$. Then, in the graph H, each edge is contained in one of the sets $V_0 \cup V_k$ for $1 \le k \le r-1$. This shows that H is a subgraph of $K_{1,r-1} \boxtimes K_{r-1}$ and thus $\text{egd}(H) \le r-1$ (by Theorem 3.11). $\square$

As an application of Theorem 4.2 we get:

Corollary 4.3. *If the tree T has a node of degree at least $(r-1)/2$ then*

$$\text{egd}(T \boxtimes K_r) = r.$$

Proof. Directly from Theorem 4.2, as $T \boxtimes K_r$ contains a subgraph F_r. $\square$

4.2. The class G_r

Consider an equilateral triangle and subdivide each side into $r-1$ equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into $(r-1)^2$ congruent equilateral triangles. The graph G_r corresponds to the edge graph of this triangulation. The graph G_5 is illustrated in Figure 2.

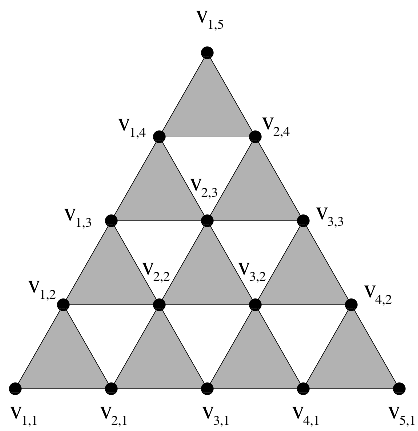

Figure 2: The graph G_5.

The graph G_r has $\binom{r+1}{2}$ vertices, denoted as $v_{i,l}$ for $l \in [r]$ and $i \in [r-l+1]$ (with $v_{1,l}, \ldots, v_{r-l+1,l}$ at level l, see Figure 2). Note that $G_2 = K_3 = F_2$, $G_3 = F_3$, but $G_r \neq F_r$ for $r \geq 4$. Using the following lemma we can construct some points of $\mathcal{E}(G_r)$ with a unique completion.

Lemma 4.4. *Consider a labeling of the nodes of G_r by vectors $w_{i,l}$ satisfying the following property (P_r): For each triangle $C_{i,l} = \{v_{i,l}, v_{i+1,l}, v_{i,l+1}\}$ of G_r, the set $\{w_{i,l}, w_{i+1,l}, w_{i,l+1}\}$ is minimally linearly dependent. (These triangles are shaded in Figure 2). Let X be the Gram matrix of the vectors $w_{i,l}$ and let $x = \pi_{E(G_r)}(X)$ be its projection. Then X is the unique completion of x.*

Proof. For $r = 2$, $G_2 = K_3$ and there is nothing to prove. Let $r \geq 3$ and assume that the claim holds for $r-1$. Consider a labeling $w_{i,l}$ of G_r satisfying (P_r) and the corresponding vector $x \in \mathcal{E}(G_r)$. We show, using Lemma 4.1, that the entries Y_{uv} of a psd completion Y of x are uniquely determined for all $\{u,v\} \notin E(G_r)$. For this, denote by H, R, L the sets of nodes lying on the 'horizontal' side, the 'right' side and the 'left' side of G_r, respectively (refer to the drawing of G_r of Figure 2). Observe that each of $G_r \backslash H$, $G_r \backslash R$, $G_r \backslash L$ is a copy of G_{r-1}. As the induced vector labelings on each of these graphs satisfies the property (P_{r-1}), we can conclude using the induction assumption that the entry Y_{uv} is uniquely determined whenever the pair $\{u,v\}$ is contained in the vertex set of one of $G_r \backslash H$, $G_r \backslash R$, or $G_r \backslash L$. The only non-edges $\{u,v\}$ that are not yet covered arise when u is a corner of G_r and v lies on the opposite side, say

$u = v_{1,1}$ and $v = v_{r-l+1,l} \in R$. If $l \neq 1, r$ then the clique $C_{1,1} = \{v_{1,1}, v_{2,1}, v_{1,2}\}$ forces the pair $\{u, v\}$ (since $\{v, v_{1,2}\} \subseteq E(G_r \backslash H)$ and $\{v, v_{2,1}\} \subseteq E(G_r \backslash L)$). If $l = r$ then the clique $C_{1,r-1} = \{v_{1,r-1}, v_{2,r-1}, v_{1,r}\}$ forces the pair $\{u, v\}$ (since $\{u, v_{1,r-1}\} \subseteq E(G_r \backslash R)$ and the value at the pair $\{u, v_{2,r-1}\}$ has just been specified). Analogously for the case $l = 1$. This concludes the proof. $\square$

Theorem 4.5. *We have that* $\text{egd}(G_r) = r$ *for all* $r \geq 2$. *Moreover,* G_r *is a minimal forbidden minor for the class* $\mathcal{G}_{r-1}$.

Proof. Since G_r has $\binom{r+1}{2}$ nodes it follows from Lemma 3.5 that $\text{egd}(G_r) \leq r$. We now show that $\text{egd}(G_r) \geq r$. For this, choose a vector labeling of the nodes of G_r satisfying the conditions of Lemma 4.4: Label the nodes $v_{1,1}, \ldots, v_{r,1}$ at level $l = 1$ by the standard unit vectors $w_{1,1} = e_1, \ldots, w_{r,1} = e_r$ in $\mathbb{R}^r$ and define inductively $w_{i,l+1} = \frac{w_{i,l} + w_{i+1,l}}{\|w_{i,l} + w_{i+1,l}\|}$ for $l = 1, \ldots, r-1$. By Lemma 4.4 their Gram matrix X is the unique completion of its projection $x = \pi_{E(G_r)}(X) \in \mathcal{E}(G_r)$. Moreover, X is extreme in $\mathcal{E}_n$ since $\mathcal{U}_V$ is full-dimensional in $\mathcal{S}^r$. This shows $\text{egd}(G_r) \geq r$, by Lemma 3.6.

We now show that $\text{egd}(G_r \backslash e) \leq r-1$. For this use the following inequalities: $\text{egd}(G_r \backslash e) \leq \text{la}_{\boxtimes}(G_r \backslash e) \leq \text{la}_{\square}(G_r \backslash e) \leq r-1$, where the leftmost inequality follows from Corollary 3.12 and the rightmost one is shown in [17]. $\square$

We conclude with two immediate corollaries.

Corollary 4.6. *The graph parameter* $\text{egd}(G)$ *is unbounded for the class of planar graphs.*

Corollary 4.7. *Let* T *be a tree which contains a path with* $2r-2$ *nodes. Then,* $\text{egd}(T \boxtimes K_r) = r$.

Proof. It is shown in [7] that G_r is a minor of the Cartesian product of two paths P_r and P_{2r-2} (with, respectively, r and $2r-2$ nodes). Hence, $G_r \preceq P_{2r-2} \square P_r \preceq T \boxtimes K_r$ and thus $r = \text{egd}(G_r) \leq \text{egd}(T \boxtimes K_r)$. $\square$

4.3. The class H_r

In this section we consider a third class of graphs H_r for every $r \geq 3$. In order to explain the general definition we first describe the base case $r = 3$.

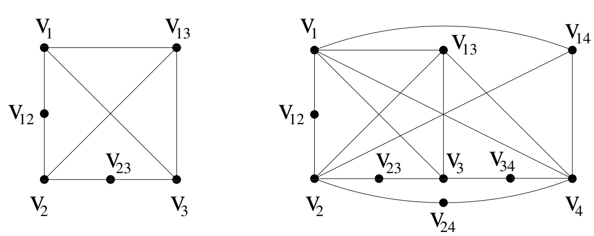

Figure 3: The graphs H_3 and H_4.

The graph H_3 is shown in Figure 3. It is obtained by taking a complete graph K_4, with vertices v_1, v_2, v_3 and v_{13}, and subdividing two adjacent edges: here we insert node v_{12} between v_1 and v_2 and node v_{23} between nodes v_2 and v_3.

Lemma 4.8. $\mathrm{egd}(H_3) = 3$ *and* H_3 *is a minimal forbidden minor for* $\mathcal{G}_2$.

Proof. As H_3 has 6 nodes, $\mathrm{egd}(H_3) \le 3$. To show equality, we use the following vector labeling for the nodes of H_3: Label the nodes v_1, v_2, v_3 by the standard unit vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ and v_{ij} by $(e_i + e_j)/\sqrt{2}$ for $1 \le i < j \le 3$. Let $X \in \mathcal{E}_6$ be their Gram matrix and set $x = \pi_{E(H_3)}(X) \in \mathcal{E}(H_3)$. Then X has rank 3 and X is an extreme point of $\mathcal{E}_6$. We now show that X is the unique completion of x in $\mathcal{E}_6$. For this let $Y \in \mathrm{fib}(x)$. Consider its principal submatrices Z, Z' indexed by $\{v_1, v_2, v_3, v_{13}\}$ and $\{v_1, v_2, v_{12}\}$, of the form:

$$Z = \begin{pmatrix} 1 & a & 0 & \sqrt{2}/2 \\ a & 1 & b & 0 \\ 0 & b & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 1 \end{pmatrix} \qquad Z' = \begin{pmatrix} 1 & a & \sqrt{2}/2 \\ a & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Then, $\det(Z) = -(a+b)^2/2$ implies $a + b = 0$, and $\det(Z') = a(1-a)$ implies $a \ge 0$. Similarly, $b \ge 0$ using the principal submatrix of Y indexed by $\{v_2, v_3, v_{23}\}$. This shows $a = b = 0$ and thus the entries of Y at the positions $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are uniquely specified. Remains to show that the entries are uniquely specified at the non-edges containing v_{12} or v_{23}. For this we use Lemma 4.1: First the clique $\{v_2, v_3, v_{23}\}$ forces the pairs $\{v_1, v_{23}\}$ and $\{v_{13}, v_{23}\}$ and then the clique $\{v_1, v_2, v_{12}\}$ forces the pairs $\{v_{23}, v_{12}\}$, $\{v_{13}, v_{12}\}$, and $\{v_3, v_{12}\}$. Thus we have shown $Y = X$, which concludes the proof that $\mathrm{egd}(H_3) = 3$.

We now verify that it is a minimal forbidden minor. Contracting any edge results in a graph on 5 nodes and we are done by (23). Lastly, we verify that $\mathrm{egd}(H_3 \backslash e) \le 2$ for each edge $e \in E(H_3)$. If deleting the edge e creates a cut node, then the result follows using Lemma 3.3. Otherwise, $H_3 \backslash e$ is contained in $T \boxtimes K_2$, where T is a path (for $e = \{v_2, v_{13}\}$) or a claw $K_{1,3}$ (for $e = \{v_1, v_{13}\}$ or $\{v_3, v_{13}\}$), and the result follows from Theorem 3.11. $\square$

We now describe the graph H_r, or rather a class $\mathcal{H}_r$ of such graphs. Any graph $H_r \in \mathcal{H}_r$ is constructed in the following way. Its node set is $V = V_0 \cup V_3 \cup \ldots \cup V_r$, where $V_0 = \{v_{ij} : 3 \le i < j \le r\}$ and, for $i \in \{3, \ldots, r\}$, $V_i = \{v_1, v_2, v_{12}, v_i, v_{1i}, v_{2i}\}$. So H_r has $n = \binom{r+1}{2}$ nodes. Its edge set is defined as follows: On each set V_i we put a copy of H_3 (with index i playing the role of index 3 in the description of H_3 above) and, for each $3 \le i < j \le r$, we have the edges $\{v_i, v_{ij}\}$ and $\{v_j, v_{ij}\}$ as well as exactly one edge, call it e_{ij}, from the set

$$F_{ij} = \{\{v_i, v_j\}, \{v_i, v_{1j}\}, \{v_j, v_{1i}\}, \{v_{1i}, v_{1j}\}\}. \tag{27}$$

Figure 3 shows the graph H_4 for the choice $e_{34} = \{v_4, v_{13}\}$.

Theorem 4.9. *For any graph* $H_r \in \mathcal{H}_r$ *(*$r \ge 3$*),* $\mathrm{egd}(H_r) = r$.

Proof. We label the nodes $v_1, \ldots, v_r$ by $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by $(e_i + e_j)/\sqrt{2}$. Let $X \in \mathcal{E}_n$ be their Gram matrix and $x = \pi_{E(H_r)}(X) \in \mathcal{E}(H_r)$. Then X is an extreme point of $\mathcal{E}_n$, we show that $\mathrm{fib}(x) = \{X\}$. For this let $Y \in \mathrm{fib}(x)$. We already know that $Y[V_i] = X[V_i]$ for each $i \in \{3, \ldots, r\}$. Indeed, as the subgraph of H_r induced by V_i is H_3, this follows from the way we have chosen the labeling and from the proof of Lemma 4.8. Hence we may now assume that we have a complete graph on each V_i and it remains to show that the entries of Y are uniquely specified at the non-edges that are not contained in some set V_i ($3 \le i \le r$). For this note that the vectors labeling the set $C_{ij} = \{v_i, v_j, v_{ij}\}$ are minimally linearly dependent. Using Lemma 4.1, one can verify that all remaining non-edges are forced using these sets C_{ij} and thus $Y = X$. This shows that $\mathrm{egd}(H_r) \ge r$. $\square$

In contrast to the graphs F_r and G_r, we do not know whether $H_r \in \mathcal{H}_r$ is a *minimal* forbidden minor for $\mathcal{G}_{r-1}$ for $r \ge 4$.

4.4. Two special graphs: $K_{3,3}$ and K_5

In this section we consider the graphs $K_{3,3}$ and K_5 which will play a special role in the characterization of the class $\mathcal{G}_2$. First we compute the extreme Gram dimension of $K_{3,3}$. Note that its Gram dimension is $\mathrm{gd}(K_{3,3}) = 4$ as $K_{3,3}$ contains a K_4-minor but it contains no K_5 and $K_{2,2,2}$-minor; cf. Theorem 1.2.

Our main goal in this section is to show that the extreme Gram dimension of the graph $K_{3,3}$ is equal to 2, i.e., for any $x \in \mathrm{ext}\,\mathcal{E}(K_{3,3})$ there exists a psd completion of rank at most 2. We start by showing that any completion of an element of $\mathrm{ext}\,\mathcal{E}(K_{3,3})$ has rank at most 3.

Lemma 4.10. *For $x \in \mathrm{ext}\,\mathcal{E}(K_{3,3})$, any $X \in \mathrm{fib}(x)$ has rank at most 3.*

Proof. Let $x \in \mathrm{ext}\,\mathcal{E}(K_{3,3})$ and let $X \in \mathrm{fib}(x)$ with $\mathrm{rank}\, X \ge 4$. Let $u_1, \ldots, u_6$ be a Gram representation of X and choose a subset $\{u_i : i \in I\}$ of linearly independent vectors with $|I| = 4$. Let E_I denote the set of edges of $K_{3,3}$ induced by I and set

$$\mathcal{U}_I = \{U_{ii} : i \in I\} \cup \{U_{ij} : \{i,j\} \in E_I\}.$$

Then $\mathcal{U}_I$ consists of linearly independent elements; cf. Lemma 3.10. By Lemma 2.3, $\mathcal{U}_I$ is contained in $\{\mathcal{U}_{ii} : i \in [6]\}$ and thus it has dimension at most 6. On the other hand, as any four nodes induce at least three edges in $K_{3,3}$, we have that $|\mathcal{U}_I| \ge 4 + 3 = 7$ and thus the dimension of $\mathcal{U}_I$ is at least 7, a contradiction. $\square$

The proof of the main theorem relies on the following two lemmas.

Lemma 4.11. *Let $X, Z \in \mathcal{S}^n$ with $X \succeq 0$ and satisfying:*

$$Xz = 0 \Longrightarrow z^{\mathsf{T}} Z z \ge 0, \quad Xz = 0, z^{\mathsf{T}} Z z = 0 \Longrightarrow Zz = 0. \tag{28}$$

Then $X + tZ \succeq 0$ for some $t > 0$.

Proof. Up to an orthogonal transformation we may assume $X = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where D is a diagonal matrix with positive diagonal entries. Correspondingly, write Z in block form: $Z = \begin{pmatrix} A & B \\ B^\mathsf{T} & C \end{pmatrix}$. The conditions (28) show that $C \succeq 0$ and that the kernel of C is contained in the kernel of B. This implies that $X + tZ \succeq 0$ for some $t > 0$. $\square$

Lemma 4.12. *Let $x \in \operatorname{ext} \mathcal{E}(K_{3,3})$, let $X \in \operatorname{ext} \operatorname{fib}(x)$ with $\operatorname{rank} X = 3$ and with Gram representation $\{u_1, \ldots, u_6\} \subseteq \mathbb{R}^3$. Let $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, 6\}$ be the bipartition of the node set of $K_{3,3}$. Then, there exist matrices $Y_1, Y_2 \in \mathcal{S}^3$ such that $Y_1 + Y_2 \succ 0$ and*

$$\langle Y_k, U_{ii} \rangle = 0 \ \forall i \in V_k \ \forall k \in \{1, 2\} \quad \textit{and} \quad \exists k \in \{1, 2\} \ \exists i, j \in V_k \ \langle Y_k, U_{ij} \rangle \neq 0.$$

Proof. Define $\mathcal{U}_k = \langle U_{ii} : i \in V_k \rangle \subseteq \mathcal{W}_k = \langle U_{ij} : i, j \in V_k \rangle \subseteq \mathcal{S}^3$ for $k = 1, 2$. With this notation we are looking for two matrices Y_1, Y_2 such that $Y_1 + Y_2 \succ 0, Y_1 \in \mathcal{U}_1^\perp, Y_2 \in \mathcal{U}_2^\perp$ and either $Y_1 \notin \mathcal{W}_1^\perp$ or $Y_2 \notin \mathcal{W}_2^\perp$.

Since $x \in \operatorname{ext} \mathcal{E}(K_{3,3})$ by Lemma 2.4 it follows that $\operatorname{fib}(x)$ is a face of $\mathcal{E}_6$ and by (13) we have that $X \in \operatorname{ext} \mathcal{E}_6$. Then (17) implies that $\dim \langle U_{ii} : i \in [6] \rangle = 6$ and thus $\dim \mathcal{U}_1 = \dim \mathcal{U}_2 = 3$. This implies that $\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$ and thus $\mathcal{U}_1^\perp \cup \mathcal{U}_2^\perp = \mathcal{S}^3$. Moreover, as $\dim \mathcal{U}_1^\perp = \dim \mathcal{U}_2^\perp = 3$ it follows that $\mathcal{U}_1^\perp \cap \mathcal{U}_2^\perp = \{0\}$ and thus $\mathcal{S}^3 = \mathcal{U}_1^\perp \oplus \mathcal{U}_2^\perp$. Lastly, we have that $\mathcal{W}_k^\perp \subseteq \mathcal{U}_k^\perp$ $(k = 1, 2)$ and thus $\mathcal{W}_1^\perp \cap \mathcal{W}_2^\perp \subseteq \mathcal{U}_1^\perp \cap \mathcal{U}_2^\perp = \{0\}$.

Assume for contradiction that $\mathcal{S}^3_{++}$ is contained in $\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. This implies that

$$\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp = \mathcal{S}^3 = \mathcal{U}_1^\perp \oplus \mathcal{U}_2^\perp \tag{29}$$

and thus $\mathcal{W}_k = \mathcal{U}_k$ $(k = 1, 2)$. Indeed, (29) implies that $\dim \mathcal{W}_1^\perp + \dim \mathcal{W}_2^\perp = \dim \mathcal{U}_1^\perp + \dim \mathcal{U}_2^\perp$ which combined with the fact that $\mathcal{W}_k^\perp \subseteq \mathcal{U}_k^\perp$ $(k = 1, 2)$ gives that $\dim \mathcal{W}_k^\perp = \dim \mathcal{U}_k^\perp$ $(k = 1, 2)$. Lastly, using the fact that $\mathcal{U}_k \subseteq W_k$ $(k = 1, 2)$ the claim follows. In turn this implies that $\mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$.

As $\dim \mathcal{U}_k = 3$, we have $\dim \langle u_i : i \in V_k \rangle \geq 2$ for $k = 1, 2$. Say, $\{u_1, u_2\}$ and $\{u_4, u_5\}$ are linearly independent. As $\dim \langle u_i : i \in [6] \rangle = 3$, there exists a nonzero vector $\lambda \in \mathbb{R}^4$ such that $0 \neq w = \lambda_1 u_1 + \lambda_2 u_2 = \lambda_3 u_4 + \lambda_4 u_5$. Notice that the scalar w is nonzero for otherwise the vectors u_1, u_2 would be dependent. Hence we obtain that $ww^\mathsf{T} \in \mathcal{W}_1 \cap \mathcal{W}_2$, contradicting the fact that $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$.

Hence we have shown that $\mathcal{S}^3_{++} \not\subseteq \mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. So there exists a positive definite matrix Y which does not belong to $\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. Write $Y = Y_1 + Y_2$, where $Y_k \in \mathcal{U}_k^\perp$ for $k = 1, 2$. We may assume, say, that $Y_1 \notin \mathcal{W}_1^\perp$. Thus Y_1, Y_2 satisfy the lemma. $\square$

We can now show the main result of this section.

Theorem 4.13. *For the graph $K_{3,3}$ we have that $\operatorname{egd}(K_{3,3}) = 2$, i.e., for any partial matrix $x \in \operatorname{ext} \mathcal{E}(K_{3,3})$ there exists a completion $X \in \operatorname{fib}(x)$ with $\operatorname{rank} X \leq 2$.*

Proof. Let $x \in \operatorname{ext} \mathcal{E}(K_{3,3})$ and let $X \in \operatorname{fib}(x)$ be an extreme point of $\mathcal{E}_6$ (which exists by Lemma 2.4 (ii)). We assume that $\operatorname{rank} X = 3$ (else we are done). Let $\{u_1, \ldots, u_6\} \subseteq \mathbb{R}^3$ be a Gram representation of X and let Y_1 and Y_2 be the matrices provided by Lemma 4.12. Moreover, define the matrix $Z \in \mathcal{S}^6$ by $Z_{ij} = \langle Y_k, U_{ij} \rangle$ for $i, j \in V_k$, $k \in \{1, 2\}$, and $Z_{ij} = 0$ for $i \in V_1$, $j \in V_2$. By Lemma 4.12, Z is a nonzero matrix with zero diagonal entries and with zeros at the positions corresponding to the edges of $K_{3,3}$.

Next we show that $X + tZ \succeq 0$ for some $t > 0$, using Lemma 4.11. For this it is enough to verify that (28) holds. Assume $Xz = 0$, i.e., $a := \sum_{i \in V_1} z_i u_i = -\sum_{j \in V_2} z_j u_j$. Then,

$$z^{\mathsf{T}} Z z = \sum_{k=1,2} \sum_{i,j \in V_k} z_i z_j \langle Y_k, U_{ij} \rangle = \langle Y_1 + Y_2, aa^{\mathsf{T}} \rangle \geq 0,$$

since $Y_1 + Y_2 \succ 0$. Moreover, $z^{\mathsf{T}} Z z = 0$ implies $a = 0$ and thus $Zz = 0$ since, for $i \in V_k$, $(Zz)_i = \sum_{j \in V_k} \langle Y_k, U_{ij} \rangle z_j = \pm \langle Y_k, (u_i a^{\mathsf{T}} + a u_i^{\mathsf{T}})/2 \rangle$. Hence, the matrix $X + tZ$ also belongs to the fiber of x. This shows that $|\operatorname{fib}(x)| \geq 2$. As, by Lemma 4.10, all matrices in $\operatorname{fib}(x)$ have rank at most 3, it follows that $\operatorname{fib}(x)$ contains a matrix of rank at most 2 (indeed, any matrix in the relative interior of $\operatorname{fib}(x)$ has rank 3 and any matrix on the boundary has rank at most 2). $\square$

We now know that both graphs $K_{3,3}$ and K_5 belong to the class $\mathcal{G}_2$. We next show that they are in some sense maximal for this property.

Lemma 4.14. *Let G be a 2-connected graph that contains K_5 or $K_{3,3}$ as a proper subgraph. Then, G contains H_3 as a minor and thus* $\operatorname{egd}(G) \geq 3$.

Proof. The proof is based on the following observations. If G is a 2-connected graph containing K_5 or $K_{3,3}$ as a proper subgraph, then G has a minor H which is one of the following graphs: (a) H is K_5 with one more node adjacent to two nodes of K_5, (b) H is $K_{3,3}$ with one more edge added, (c) H is $K_{3,3}$ with one more node adjacent to two adjacent nodes of $K_{3,3}$. Then H contains a H_3 subgraph in cases (a) and (b), and a H_3 minor in case (c) (easy verification). Hence, $\operatorname{egd}(G) \geq \operatorname{egd}(H_3) = 3$. $\square$

We conclude this section with a lemma that will be used in the proof of Theorem 5.4.

Lemma 4.15. *Let G be a 2-connected graph with $n \geq 6$ nodes. Then,*

(i) If G has no F_3-minor then $\omega(G) \leq 4$.

(ii) If G is chordal and has no F_3-subgraph then $\omega(G) \leq 4$.

Proof. Assume for contradiction that $\omega(G) \geq 5$ and let $U \subseteq V$ with $G[U] = K_5$. Since G is 2-connected and $n \geq 6$, there exists a node $u \notin U$ which is connected by two vertex disjoint paths to two distinct nodes $v, w \in U$; let P_{uv} and P_{uw} be such shortest paths. In case (i), contract the paths P_{uv} and P_{uw} to get a node adjacent to both v and w. Then, we can easily see that G has an F_3-minor, a contradiction. In case *(ii)*, let $v' \in P_{uv}$ and $w' \in P_{uw}$ with $(v, v'), (w, w') \in E(G)$. Since G is chordal and the paths are the shortest possible, at least one of the edges (v, w') or (w, v') will be present in G. This implies that G contains an F_3-subgraph, a contradiction. □

5. Graphs with extreme Gram dimension at most 2

In this section we characterize the class $\mathcal{G}_2$ of graphs with extreme Gram dimension at most 2. Our main result is the following:

Theorem 5.1. *For any graph G,*

$$\mathrm{egd}(G) \leq 2 \textit{ if and only if } G \textit{ has no minors } F_3 \textit{ or } H_3.$$

The graphs F_3 and H_3 are illustrated in Figure 4 below.

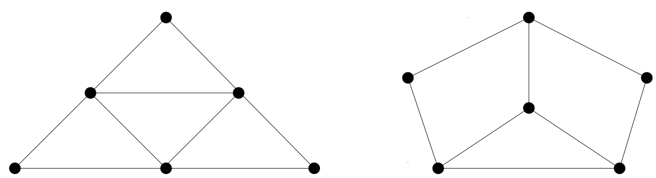

Figure 4: The graphs F_3 and H_3

In the previous sections we established that the graphs F_3 and H_3 are minimal forbidden minors for the class of graphs satisfying $\mathrm{egd}(G) \leq 2$. In order to prove Theorem 5.1 it remains to show that a graph G having no F_3 and H_3 minors satisfies $\mathrm{egd}(G) \leq 2$.

By Lemma 3.3 we may assume that G is 2-connected. Moreover, we may assume that $|V(G)| \geq 6$, since for graphs on at most five nodes we know that $\mathrm{egd}(G) \leq \mathrm{egd}(K_5) = 2$ (recall (23)). Additionally, since $\mathrm{egd}(K_{3,3}) = 2$ (recall Theorem 4.13) we may also assume that $G \neq K_{3,3}$.

Consequently, it suffices to consider 2-connected graphs with at least 6 nodes that are different from $K_{3,3}$. Then, Theorem 5.1 follows from the equivalence of the first two items in the next theorem.

Theorem 5.2. *Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. Then, the following assertions are equivalent.*

(i) $\mathrm{egd}(G) \leq 2$.

(ii) *G has no minors F_3 or H_3.*

(iii) $\mathrm{la}_{\boxtimes}(G) \leq 2$, *i.e., G is a minor of $T \boxtimes K_2$ for some tree T.*

The implication *(i)* $\Longrightarrow$ *(ii)* follows from Theorem 4.2 and Theorem 4.8. Moreover, the implication *(iii)* $\Longrightarrow$ *(i)* follows from Theorem 3.11. The rest of this chapter is dedicated to proving the implication *(ii)* $\Longrightarrow$ *(iii)*. The proof consists of two steps. First we consider the chordal case and show:

(1) **The chordal case:** Let G be a 2-connected chordal graph with $n \geq 6$ nodes. Then, G has no F_3 or H_3-minors if and only if G is a contraction minor of $T \boxtimes K_2$, for some tree T (Section 5.1, Theorem 5.4).

Then, we reduce the general case to the chordal case and show:

(2) **Reduction to the chordal case:** Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. If G has no F_3 or H_3-minors then G is a subgraph of a chordal graph with no F_3 or H_3-minors.

Notice that, in case (1), G is by assumption chordal and thus the case $G = K_{3,3}$ is automatically excluded. For case (2), we first need to exclude K_4 instead of H_3 (Section 5.2, Theorem 5.7) and then we derive from this special case the general result (Section 5.3, Theorem 5.12).

5.1. The chordal case

Our goal in this section is to characterize the 2-connected chordal graphs G with $\text{egd}(G) \leq 2$. By Lemma 4.14, if $G \neq K_5$ has $\text{egd}(G) \leq 2$, then $\omega(G) \leq 4$. Throughout this section we denote by $\mathcal{C}$ the family of all 2-connected chordal graphs with $\omega(G) \leq 4$. Any graph $G \in \mathcal{C}$ is a clique 2- or 3-sum of K_3's and K_4's. Note that F_3 belongs to $\mathcal{C}$ and has $\text{egd}(F_3) = 3$. On the other hand, any graph $G = T \boxtimes K_2$ where T is a tree, belongs to $\mathcal{C}$ and has $\text{egd}(G) = 2$. These graphs are "special clique 2-sums" of K_4's, as they satisfy the following property: every 4-clique has at most two edges which are cutsets and these two edges are not adjacent. This motivates the following definitions, useful in the proof of Theorem 5.4 below.

Definition 5.3. *Let G be a 2-connected chordal graph with $\omega(G) \leq 4$.*

(i) An edge of G is called free *if it belongs to exactly one maximal clique and* non-free *otherwise.*

(ii) A 3-clique in G is called free *if it contains at least one free edge.*

(iii) A 4-clique in G is called free *if it does not have two adjacent non-free edges. A free 4-clique can be partitioned as $\{a, b\} \cup \{c, d\}$, called its* two sides*, where only $\{a, b\}$ and $\{c, d\}$ can be non-free.*

(iv) G is called free if all its maximal cliques are free.

For instance, F_3, $K_5 \setminus e$ (the clique 3-sum of two K_4's) are not free. Hence any free graph in $\mathcal{C}$ is a clique 2-sum of free K_3's and free K_4's. Note also that $\text{la}_{\boxtimes}(K_5 \setminus e) = 3$. We now show that for a graph $G \in \mathcal{C}$ the property of being free is equivalent to having $\text{la}_{\boxtimes}(G) \leq 2$ and also to having $\text{egd}(G) \leq 2$.

Theorem 5.4. *Let G be a 2-connected chordal graph with $n \geq 6$ nodes. The following assertions are equivalent:*

(i) G has no minors F_3 or H_3.

(ii) *G does not contain F_3 as a subgraph.*

(iii) $\omega(G) \leq 4$ *and G is free.*

(iv) *G is a contraction minor of $T \boxtimes K_2$ for some tree T.*

(v) $\mathrm{la}_{\boxtimes}(G) \leq 2$.

(vi) $\mathrm{egd}(G) \leq 2$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear and the implications $(iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$ follow from earlier results.
$(ii) \Rightarrow (iii)$: Assume that (ii) holds. By Lemma 4.15 (ii) it follows that $\omega(G) \leq 4$. Our first goal is to show that G does not contain clique 3-sums of K_4's, i.e., it does not contain a $K_5 \setminus e$ subgraph. For this, assume that $G[U] = K_5 \setminus e$ for some $U \subseteq V(G)$. As $|V(G)| \geq 6$ and G is 2-connected chordal, there exists a node $u \notin U$ which is adjacent to two adjacent nodes of U. Then, one can find a F_3 subgraph in G, a contradiction. Therefore, G is a clique 2-sum of K_3's and K_4's. We now show that each of them is free.

Suppose first that $C = \{a, b, c\}$ is a maximal 3-clique which is not free. Then, there exist nodes $u, v, w \notin C$ such that $\{a, b, u\}$, $\{a, c, v\}$, $\{b, c, w\}$ are cliques in G. Moreover, u, v, w are pairwise distinct (if $u = v$ then $C \cup \{u\}$ is a clique, contradicting maximality of C) and we find a F_3 subgraph in G.

Suppose now that $C = \{a, b, c, d\}$ is a 4-clique which is not free and, say, both edges $\{a, b\}$ and $\{a, c\}$ are non-free. Then, there exist nodes $u, v \notin C$ such that $\{a, b, u\}$ and $\{a, c, v\}$ are cliques. Moreover, $u \neq v$ (else we find a $K_5 \setminus e$ subgraph) and thus we find a F_3 subgraph in G. Thus (iii) holds.
$(iii) \Rightarrow (iv)$: Assume that G is free, $G \neq K_4, K_3$ (else we are done). When all maximal cliques are 4-cliques, it is easy to show using induction on $|V(G)|$ that $G = T \boxtimes K_2$, where T is a tree and each side of a 4-clique of G corresponds to a node of T.

Assume now that G has a maximal 3-clique $C = \{a, b, c\}$. Say, $\{b, c\}$ is free and $\{a, b\}$ is a cutset. Write $V(G) = V' \cup V'' \cup \{a, b\}$, where V'' is the (vertex set of the) component of $G \setminus \{a, b\}$ containing c, and V' is the union of the other components. Now replace node a by two new nodes a', a'' and replace C by the 4-clique $C' = \{a', a'', b, c\}$. Moreover, replace each edge $\{u, a\}$ by $\{u, a'\}$ if $u \in V'$ and by $\{u, a''\}$ if $u \in V''$. Let G' be the graph obtained in this way. Then $G' \in \mathcal{C}$ is free, G' has one less maximal 3-clique than G, and $G = G'/\{a', a''\}$. Iterating, we obtain a graph $\widehat{G}$ which is a clique 2-sum of free K_4's and contains G as a contraction minor. By the above, $\widehat{G} = T \boxtimes K_2$ and thus G is a contraction minor of $T \boxtimes K_2$. □

5.2. Structure of the graphs with no F_3 and K_4-minor

In this section we investigate the structure of the graphs with no F_3 or K_4-minors. We start with two technical lemmas.

Lemma 5.5. *Let G and M be two 2-connected graphs, let $\{x, y\} \notin E(G)$ be a cutset in G, and let $r \geq 2$ be the number of components of $G \setminus \{x, y\}$.*

(i) Assume that $G \in \mathcal{F}(M)$, but the graph $G + \{x, y\}$ has a M-minor with M-partition $\{V_i : i \in V(M)\}$. If $x \in V_i$ and $y \in V_j$, then $M \setminus \{i, j\}$ has at least $r \geq 2$ components (and thus $i \neq j$).

(ii) Assume that M does not have two adjacent nodes forming a cutset in M. If $G \in \mathcal{F}(M)$, then $G + \{x, y\} \in \mathcal{F}(M)$.

Proof. (i) Let $C_1, \ldots, C_r \subseteq V(G)$ be the node sets of the components of $G \setminus \{x, y\}$. As G is 2-connected, there is an $x - y$ path P_s in $G[C_s \cup \{x, y\}]$ for each $s \in [r]$. Notice that $P_s \neq \{x, y\}$ since P_s is a path in G. Our first goal is to show that every component of $M \setminus \{i, j\}$ corresponds to exactly one component of $G \setminus \{x, y\}$. For this, let U be a component of $M \setminus \{i, j\}$. By the definition of the M-partition, the graph $G[\bigcup_{k \in U} V_k]$ is connected. As $x, y \notin \bigcup_{k \in U} V_k$, we deduce that $\bigcup_{k \in U} V_k \subseteq C_s$ for some $s \in [r]$. We can now conclude the proof. Assume for contradiction that $M \setminus \{i, j\}$ has less than r components. Then there is at least one component C_s which does not correspond to any component of $M \setminus \{i, j\}$ which means that $(\bigcup_{k \neq i,j} V_k) \cap C_s = \emptyset$. Indeed, if $V_k \cap C_s$ for some $k \neq i, j$ then since C_s is a connected component of $G \setminus \{x, y\}$ it follows that $\cup_{\lambda \in U} V_\lambda \subseteq C_s$, where U is the component of $M \setminus \{i, j\}$ that contains k. Summarizing we know that $C_s \subseteq V_i \cup V_j$. Hence the path P_s is contained in $G[V_i \cup V_j]$, thus $\{V_i : i \in V(M)\}$ remains an M-partition of G (recall that $P_s \neq \{x, y\}$) and we find a M-minor in G, a contradiction. Therefore, $M \setminus \{i, j\}$ has at least $r \geq 2$ components. This implies that $\{i, j\}$ is a cutset of M since M is 2-connected it follows that $i \neq j$.

(ii) Assume $G + \{x, y\}$ has a M-minor, with corresponding M-partition $\{V_i : i \in V(M)\}$. By (i), the nodes x and y belong to two distinct classes V_i and V_j and $\{i, j\}$ is a cutset in M. By the hypothesis, this implies that $\{i, j\} \notin E(M)$ and thus M is a minor of G, a contradiction. □

We continue with a lemma that will be essential for the next theorem.

Lemma 5.6. *Let $G \in \mathcal{F}(K_4)$ be a 2-connected graph and let $\{x, y\} \notin E(G)$. If there are at least three (internally vertex) disjoint paths from x to y, then $\{x, y\}$ is a cutset and $G \backslash \{x, y\}$ has at least 3 components.*

Proof. Let P_1, P_2, P_3 be distinct vertex disjoint paths from x to y. Then $P_1 \setminus \{x, y\}$, $P_2 \setminus \{x, y\}$ and $P_3 \setminus \{x, y\}$ lie in distinct components of $G \setminus \{x, y\}$, for otherwise G would contain a homeomorph of K_4. □

We now arrive at the main result of this section.

Theorem 5.7. *Let $G \in \mathcal{F}(F_3, K_4)$ be a 2-connected graph on $n \geq 6$ nodes. Then, there exists a chordal graph $Q \in \mathcal{F}(F_3, K_4)$ containing G as a subgraph.*

Proof. Let G be a 2-connected graph in $\mathcal{F}(F_3, K_4)$. As a first step, consider $\{x, y\} \notin E(G)$ such that there exist at least three disjoint paths in G from x to y. Then, Lemma 5.6 implies that $\{x, y\}$ is a cutset of G and $G \setminus \{x, y\}$ has at least three components. As a first step we show that we can add the edge $\{x, y\}$ without creating a K_4 or F_3-minor, i.e., $G + \{x, y\} \in \mathcal{F}(F_3, K_4)$.

As $\{x, y\}$ is a cutset, Lemma 5.5 (ii) applied for $M = K_4$ gives that $G+\{x, y\}$ does not have a K_4 minor. Assume for contradiction that $G + \{x, y\}$ has an F_3 minor. Again, Lemma 5.5 (i) applied for $M = F_3$ implies that $x \in V_i, y \in V_j$, where $F_3 \setminus \{i, j\}$ has at least 3 components. Clearly there is no such pair of vertices in F_3 so we arrived at a contradiction. Consequently, we can add edges iteratively until we obtain a graph $\widehat{G} \in \mathcal{F}(F_3, K_4)$ containing G as a subgraph and satisfying:

$$\forall \{x, y\} \notin E(\widehat{G}) \text{ there are at most two disjoint } x - y \text{ paths in } \widehat{G}. \qquad (30)$$

If $\widehat{G}$ is chordal we are done. So consider a chordless circuit C in $\widehat{G}$. Note that any circuit C' distinct from C, which meets C in at least two nodes, meets C in exactly two nodes (if they meet in at least 3 nodes then we can find a F_3 minor) that are adjacent (if they are not adjacent then there exist three internally vertex disjoint paths between them, contradicting (30)). Call an edge of C *busy* if it is contained in some circuit $C' \neq C$. If $e_1 \neq e_2$ are two busy edges of C and $C_i \neq C$ is a circuit containing e_i, then C_1, C_2 are (internally) disjoint (use (30)). Hence C can have at most two busy edges, for otherwise one would find a F_3-minor in $\widehat{G}$.

We now show how to triangulate C without creating a K_4 or F_3-minor: If C has two busy edges denoted, say, $\{v_1, v_2\}$ and $\{v_k, v_{k+1}\}$ (possibly $k = 2$), then we add the edges $\{v_1, v_i\}$ for $i \in \{3, \ldots, k\}$ and the edges $\{v_k, v_i\}$ for $i \in \{k+2, \ldots, |C|\}$, see Figure 5 a). If C has only one busy edge $\{v_1, v_2\}$, add the edges $\{v_1, v_i\}$ for $i \in \{3, \ldots, |C|-1\}$, see Figure 5 b). (If C has no busy edge then $G = C$, triangulate from any node and we are done).

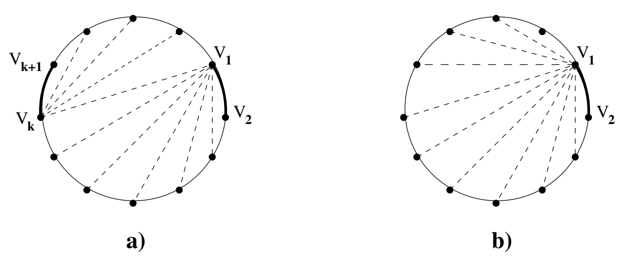

Figure 5: Triangulating a chordless circuit with a) two or b) one busy edge.

Let Q denote the graph obtained from $\widehat{G}$ by triangulating all its chordless circuits in this way. Hence Q is a chordal extension of $\widehat{G}$ (and thus of G). We show that $Q \in \mathcal{F}(F_3, K_4)$. First we see that $Q \in \mathcal{F}(K_4)$ by applying iteratively Lemma 5.5 (ii) (for $M = K_4$): For each $i \in \{3, \ldots, k\}$, $\{v_1, v_i\}$ is a cutset of $\widehat{G}$ and of $\widehat{G} + \{\{v_1, v_j\} : j \in \{3, \ldots, i-1\}\}$ (and analogously for the other added edges $\{v_k, v_i\}$). Hence Q is a clique 2-sum of triangles. We now verify that each triangle is free which will conclude the proof, using Theorem 5.4.

For this let $\{a, b, c\}$ be a triangle in Q. First note that if (say) $\{a, b\} \in E(Q) \setminus E(\widehat{G})$, then a, b, c lie on a common chordless circuit C of $\widehat{G}$. Indeed, let C be a chordless circuit of $\widehat{G}$ containing a, b and assume $c \notin C$. By (30), $\widehat{G} \setminus \{a, b\}$ has at most two components and thus there is a path from c to one of the two paths composing $C \setminus \{a, b\}$. Together with the triangle $\{a, b, c\}$ this gives a homeomorph of K_4 in Q, contradicting $Q \in \mathcal{F}(K_4)$, just shown above. Hence the triangle $\{a, b, c\}$ lies in C and thus has a free edge.

Suppose now that $\{a, b, c\}$ is a triangle contained in $\widehat{G}$. If it is not free then there is a F_3 on $\{a, b, c, x, y, z\}$ where x (resp., y, and z) is adjacent to a, b (resp., a, c, and b, c). Say $\{x, a\} \in E(Q) \setminus E(\widehat{G})$ (as there is no F_3 in $\widehat{G}$). Then x, a, b lie on a chordless circuit C of $\widehat{G}$ and $\{x, b\} \in E(\widehat{G})$ (since $\{a, b\}$ is a busy edge). Moreover, $c, y, z \notin C$ for otherwise we get a K_4-minor in Q. Then delete the edge $\{x, a\}$ and replace it by the $\{x, a\}$-path along C. Do the same for any other edge of $E(Q) \setminus E(\widehat{G})$ connecting y and z to $\{a, b, c\}$. After that we get a F_3-minor in $\widehat{G}$, a contradiction. $\square$

5.3. Structure of the graphs with no F_3 and H_3-minor

Here we investigate the graphs G in the class $\mathcal{F}(F_3, H_3)$. By the results in Section 5.2 we may assume that G contains some homeomorph of K_4. Figure 6 shows a homeomorph of K_4, where the original nodes are denoted as 1,2,3,4 and called its *corners*, and the wiggled lines correspond to subdivided edges (i.e., to paths P_{ij} between the corners $i \neq j \in [4]$).

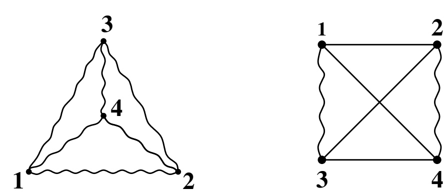

Figure 6: A homeomorph of K_4 and its two sides (cf. Lemma 5.8)

To help the reader visualize F_3 and H_3 we use Figure 7. Notice the special role of node 5 in H_3 (denoted by a square) and of the (dashed) triangle $\{1, 2, 3\}$.

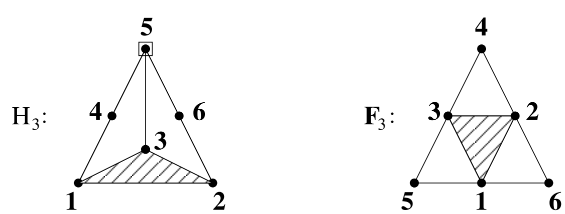

Figure 7: The graphs H_3 and F_3.

The starting point of the proof is to investigate the structure of homeomorphs of K_4 in a graph of $\mathcal{F}(H_3)$.

Lemma 5.8. *Let G be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there is a partition of the corner nodes of H into $\{1,3\}$ and $\{2,4\}$ for which the following holds.*

(i) Only the paths P_{13} and P_{24} can have more than two nodes.

(ii) Every component of $G \setminus H$ is connected to P_{13} or to P_{24}.

Then P_{13} and P_{24} are called the two sides *of H (cf. Figure 6).*

Proof. We use the graphs from Figure 8 which all contain a subgraph H_3.
Case 1: $H = K_4$. If $G \setminus H$ has a unique component C then $|C| \geq 2$ as $n \geq 6$. If C is connected to two nodes of H, then the conclusion of the lemma holds. Otherwise, C is connected to at least three nodes in H and then the graph from Figure 8 a) is a minor of G, a contradiction.

If there are at least two components in $G \setminus H$, then they cannot be connected to two adjacent edges of H for, otherwise, the graph of Figure 8 b) is a minor of G, a contradiction. Hence the lemma holds.
Case 2: $H \neq K_4$. Say, P_{13} has at least 3 nodes. Then the edges $\{1,i\}$, $\{3,i\}$ $(i = 2,4)$ cannot be subdivided (else H is a homeomorph of H_3). So (i) holds. We now show (ii). Indeed, if a component of $G \setminus H$ is connected to both P_{13} and P_{24}, then at least one of the graphs in Figure 8 c) and d) will be a minor of G, a contradiction. $\square$

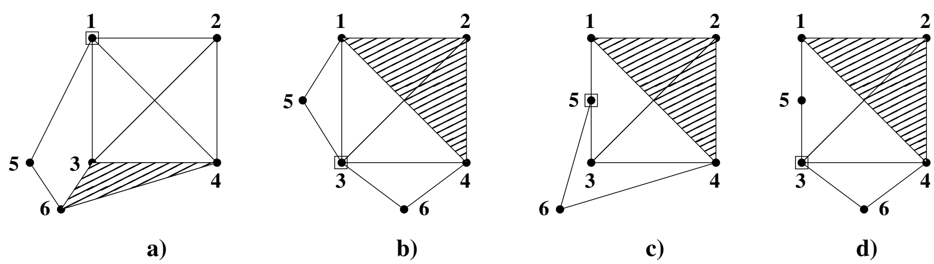

Figure 8: Bad subgraphs in the proof of Lemma 5.8.

Lemma 5.8 implies that there is no path with at least 3 nodes between the sides of a K_4-homeomorph. We now show that, moreover, there is no additional edge between the two sides. More precisely:

Lemma 5.9. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there exists no edge between the two sides of H except between their endpoints.*

Proof. Say, P_{13} and P_{24} are the two sides of H. Assume for a contradiction that $\{a,b\} \in E(G)$, where a lies on P_{13} and b on P_{24}.

Assume first that a is an internal node of P_{13} and b is an internal node of P_{24}. If $|V(H)| = 6$, then $H = K_{3,3}$ and Lemma 4.14 implies that G has a H_3 minor, a contradiction. Hence, $|V(H)| > 6$ and we can assume w.l.o.g. that the path from 1 to a within P_{13} has at least 3 nodes. Then G contains a homeomorph of K_4 with corner nodes $1, b, 4, a$, where the two paths from 1 to a and from 1 to b (via 2) have at least 3 nodes, giving a H_3 minor and thus a contradiction.

Assume now that only a is an internal node of P_{13} and, say $b = 2$. If $|V(H)| = 5$, then $G \setminus H$ has at least one component. By Lemma 5.8, this component connects either to the path P_{13} or to the edge $\{2,4\}$. In both cases, it is easy to verify that one of the graphs in Figure 9 will be a minor of G, a contradiction since all of them have a H_3 subgraph. On the other hand, if $|V(H)| \geq 6$, then one of the paths from 1 to a, from a to 3 (within P_{13}), or P_{24} is subdivided. This implies that G contains a homeomorph of K_4 with corner nodes $a, 1, 2, 4$ or $a, 2, 3, 4$, which thus contains two adjacent subdivided edges, giving a H_3 minor. $\square$

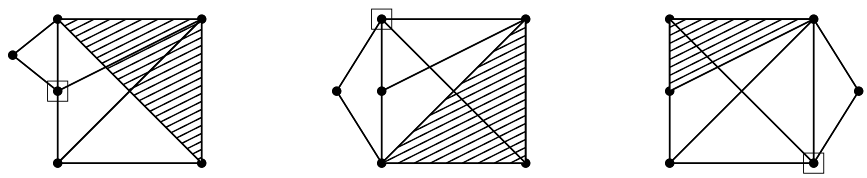

Figure 9: Bad subgraphs in the proof of Lemma 5.9.

Lemmas 5.8 and 5.9 imply directly:

Corollary 5.10. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then the endnodes of at least one of the two sides of H form a cutset in G. Moreover, if P_{13} is a side of H and its endnodes $\{1,3\}$ do not form a cutset, then $P_{13} = \{1,3\}$ and there is no component of $G \setminus H$ which is connected to P_{13}.*

We now show that one may add edges to G so that all minimal homeomorphs of K_4 are 4-cliques, without creating a F_3 or H_3 minor.

Lemma 5.11. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(F_3, H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. The graph obtained by adding to G the edges between the endpoints of the sides of H belongs to $\mathcal{F}(F_3, H_3)$.*

Proof. Say P_{13} and P_{24} are the sides of H. Assume $|V(P_{13})| \geq 3$ and $\{1,3\} \notin E(G)$. By Corollary 5.10, $\{1,3\}$ is a cutset in G. We show that $\widehat{G} = G + \{1,3\} \in \mathcal{F}(F_3, H_3)$. First, applying Lemma 5.5 (ii) with $M = H_3$ and $\{x,y\} = \{1,3\}$, we obtain that $\widehat{G} \in \mathcal{F}(H_3)$.

Next, assume for contradiction that $\widehat{G}$ has a F_3 minor, where the labelling of F_3 is given in Figure 10. Applying Lemma 5.5 (i) with $M = F_3$ and $\{x,y\} = \{1,3\}$, we see that the nodes 1 and 3 belong to distinct classes of the F_3-partition,

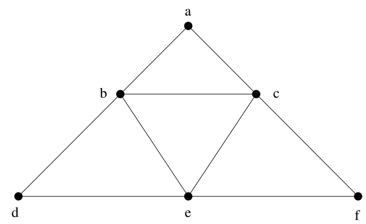

Figure 10: A labelling of F_3 used in Lemma 5.11.

which corresponds to a cutset of F_3. Say, $1 \in V_e$ and $3 \in V_c$. Then the nodes 2 and 4 do not lie in $V_e \cup V_c$ (for otherwise, one would have an F_3-partition in G). Next we show that the nodes 2 and 4 do not belong to the same class of the F_3-partition. Assume for contradiction that $2, 4 \in V_k$. If $\{2, 4\}$ is not a cutset in G then, by Corollary 5.10, $P_{24} = \{2, 4\}$ and no component of $G \setminus H$ connects to $\{2, 4\}$. Hence $V_k = \{2, 4\}$ and we can move node 2 to the class V_e, so that we obtain a F_3-partition of G, a contradiction. If $\{2, 4\}$ is a cutset of G, then every component of $G \setminus \{2, 4\}$ except the one containing 1 and 3 has to lie within V_k, so we can again move node 2 to V_e and obtain a F_3-partition of G.

Accordingly, the nodes $1, 2, 3$ and 4 belong to distinct classes and we can assume without loss of generality that $2 \notin V_f$. Observe that every $1-2$ path in G is either the edge $\{1, 2\}$ or meets the nodes 3 or 4. Similarly, every $2-3$ path in G is either the edge $\{2, 3\}$ or meets the nodes 1 or 4. An easy case analysis shows that whatever the position of nodes 2 and 4 in the F_3-partition we always find a $1-2$ or a $2-3$ path violating the above conditions. □

We are now ready to show the main result of this section.

Theorem 5.12. *Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. If $G \in \mathcal{F}(F_3, H_3)$ then there exists a chordal graph $Q \in \mathcal{F}(F_3, H_3)$ containing G as a subgraph.*

Proof. If $G \in \mathcal{F}(F_3, K_4)$ then we are done by Theorem 5.7. Otherwise, we augment the graph G by adding the edges between the endpoints of the sides of every homeomorph of K_4 contained in G. Let $\widehat{G}$ be the graph obtained in this way. By Lemma 5.11, we know that $\widehat{G} \in \mathcal{F}(F_3, H_3)$. Hence, for each K_4-homeomorph H in $\widehat{G}$, its corners form a 4-clique. Moreover, if C, C' are two distinct 4-cliques of $\widehat{G}$, then $C \cap C'$ is contained in a side of C and C'.

Consider a 4-clique $C = \{1, 2, 3, 4\}$ in $\widehat{G}$, say with sides $\{1, 3\}$, $\{2, 4\}$ (so each component of $\widehat{G} \setminus C$ connects to $\{1, 3\}$ or to $\{2, 4\}$, by Lemma 5.8). Pick an edge f between the two sides (i.e., $f = \{i, j\}$ with $i \in \{1, 3\}$, $j \in \{2, 4\}$) and delete this edge f from $\widehat{G}$. We repeat this process with every 4-clique in $\widehat{G}$ and obtain the graph $G_0 = \widehat{G} \setminus \{f_1, \ldots, f_k\}$, if $\widehat{G}$ has k 4-cliques.

By construction, G_0 belongs to $\mathcal{F}(F_3, K_4)$ and is 2-connected. Hence, we can apply Theorem 5.7 to G_0 and obtain a chordal graph $Q_0 \in \mathcal{F}(F_3, K_4)$ containing G_0 as a subgraph. Hence, Q_0 is a clique 2-sum of free triangles. It suffices now to show that the augmented graph $Q = Q_0 + \{f_i : i \in [k]\}$ is a clique 2-sum of free K_3's and K_4's. Then Q is a chordal graph in $\mathcal{F}(F_3, H_3)$ (by Theorem 5.4) containing $\widehat{G}$ and thus G, and the proof is completed.

For this, consider again a 4-clique $C = \{1, 2, 3, 4\}$ in $\widehat{G}$ with sides $\{1, 3\}$ and $\{2, 4\}$. Then, each component of $\widehat{G} \setminus C$ connects to $\{1, 3\}$ or $\{2, 4\}$. We claim that the same holds for each component of $Q_0 \setminus C$. Indeed, a component of $Q_0 \setminus C$ is a union of some components of $\widehat{G} \setminus C$. Thus it connects to two nodes (to 1,3, or to 2,4), or to at least three nodes of C. But the latter case cannot occur since we would then find a K_4 minor in Q_0.

Assume that the edge $f = \{1, 4\}$ was deleted from the 4-clique C when making the graph G_0. We now show that adding it back to Q_0 results in a free graph. Indeed, by adding the edge $\{1, 4\}$ we only replace the two maximal 3-cliques $\{1, 3, 4\}$ and $\{1, 2, 4\}$ by a new maximal 4-clique $\{1, 2, 3, 4\}$, which is free. We iterate this process for each of the edges $f_1, \ldots, f_k$ and obtain that $Q = Q_0 + \{f_i : i \in [k]\}$ is the clique 2-sum of free K_3's and K_4's. Summarizing, Q is a 2-connected chordal graph with $\omega(G) \leq 4$ which is free. Then, Theorem 5.4 (*iii*) implies that Q does not have F_3 or H_3 as minors. $\square$

6. Characterization of graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$

Recall that the *largeur d'arborescence* of a graph G, denoted by $\mathrm{la}_{\square}(G)$, is defined as the smallest integer $k \geq 1$ such that G is a minor of $T \square K_k$, for some tree T. Colin de Verdière [7] introduced the largeur d'arborescence as upper bound for his graph parameter $\nu(\cdot)$, which is defined as the maximum corank of a matrix $A \in \mathcal{S}^n_+$ satisfying the condition: $A_{ij} = 0$ if and only if $i \neq j$ and $\{i, j\} \notin E(G)$, as well as the following condition known as the *Strong Arnold Property*: if $AX = 0$ where $X \in \mathcal{S}^n$ satisfies $X_{ij} = 0$ for all $i = j$ and all $\{i, j\} \in E(G)$, then $X = 0$.

In [7] it was shown that $\nu(\cdot)$ is minor monotone and that for any graph G, $\nu(G) \leq \mathrm{la}_{\square}(G)$. Moreover, this holds with equality for the family of graphs G_r, i.e., $\nu(G_r) = \mathrm{la}_{\square}(G_r) = r$ for all $r \geq 2$ (recall Section 4). Furthermore,

$$\mathrm{la}_{\square}(G) \leq 1 \Longleftrightarrow \nu(G) \leq 1 \Longleftrightarrow G \text{ has no minor } K_3. \tag{31}$$

Lastly, Kotlov [17] shows:

$$\mathrm{la}_{\square}(G) \leq 2 \Longleftrightarrow \nu(G) \leq 2 \Longleftrightarrow G \text{ has no minors } F_3, K_4. \tag{32}$$

The most work in obtaining the characterization (32) is to show that $\mathrm{la}_{\square}(G) \leq 2$ if $G \in \mathcal{F}(K_4, F_3)$. In fact, this also follows from our characterization of the class $\mathcal{F}(K_4, F_3)$. Indeed, if $G \in \mathcal{F}(K_4, F_3)$ is 2-connected then we have shown that G is subgraph of G' which is a clique 2-sum of free triangles. Now our argument in the proof of Theorem 5.4 also shows that G' is a contraction minor of $T \square K_2$ for some tree T (as each triangle of G' arises as contraction of a 4-clique which can be replaced by a 4-circuit). In this sense our characterization is a refinement of Kotlov's result tailored to our needs.

We now characterize the graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$. The *wheel* W_5 is obtained from the circuit C_4 by adding a node adjacent to all nodes of C_4.

Theorem 6.1. *For a graph G, $\mathrm{la}_{\boxtimes}(G) \leq 2$ if and only if $G \in \mathcal{F}(F_3, H_3, W_5)$.*

Proof. We already know that $\mathrm{la}_{\boxtimes}(G) \geq \mathrm{egd}(G) = 3$ for $G = F_3, H_3$. Suppose for contradiction that $\mathrm{la}_{\boxtimes}(W_5) \leq 2$. Then $\mathrm{la}_{\boxtimes}(W_5) = \mathrm{la}_{\boxtimes}(H)$ where H is a chordal extension of W_5 and H is a contraction minor of some $T \boxtimes K_2$. As W_5 is not chordal, H contains W_5 with one added chord on its 4-circuit, i.e., H contains $K_5 \setminus e$ and thus $\mathrm{la}_{\boxtimes}(H) \geq \mathrm{la}_{\boxtimes}(K_5 \setminus e) = 3$. Therefore, F_3, H_3, W_5 are forbidden minors for the property $\mathrm{la}_{\boxtimes}(G) \leq 2$. Conversely, assume that $G \in \mathcal{F}(F_3, H_3, W_5)$ is 2-connected, we show that $\mathrm{la}_{\boxtimes}(G) \leq 2$. This is clear if G has $n \leq 4$ nodes, or if G has $n = 5$ nodes and it has a node of degree 2. If G has $n = 5$ nodes and each node has degree at least 3, then one can easily verify that G contains W_5. If G has $n \geq 6$ nodes then $\mathrm{la}_{\boxtimes}(G) \leq 2$ follows from Theorem 5.2 (since $G \neq K_{3,3}$ as $W_5 \preceq K_{3,3}$). $\square$

Summarizing, the following inequalities are known:

$$\nu(G) \leq \mathrm{la}_{\square}(G) \text{ and } \mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G) \leq \mathrm{la}_{\square}(G).$$

Moreover, by combining (31), (32) and Theorem 5.1 it follows that if G is a graph with $\nu(G) \leq 2$, then $\mathrm{egd}(G) = \nu(G)$. Furthermore, it is known that $\nu(K_n) = n - 1$ [7] and thus $\nu(K_n) > \mathrm{la}_{\boxtimes}(K_n) \geq \mathrm{egd}(K_n)$ if $n \geq 4$.

An interesting open question is whether the inequality $\mathrm{egd}(G) \leq \nu(G)$ holds in general. We point out that the analogous inequality $\nu^{=}(G) \leq \mathrm{gd}(G)$ was shown in [22]. Recall that the parameter $\nu^{=}(\cdot)$ is the analogue of $\nu(\cdot)$ studied by van der Holst [34] (same definition as $\nu(G)$, but now requiring only that $A_{ij} = 0$ for $\{i, j\} \in E(G)$ and allowing zero entries at positions on the diagonal and at edges), and $\nu^{=}(\cdot)$ satisfies: $\nu(G) \leq \nu^{=}(G)$.

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