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Continuity and computability of reachable sets

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ABSTRACT
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Continuity and Computability of Reachable Sets

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Abstract

The computation of reachable sets of nonlinear dynamic and control systems is an important problem of systems theory. In this paper we consider the computability of reachable sets using Turing machines to perform approximate computations. We use Weihrauch's type-two theory of effectivity for computable analysis and topology, which provides a natural setting for performing computations on sets and maps. The main result is that the reachable set is lower-computable, but is only outer-computable if it equals the chain-reachable set. In the course of the analysis, we extend the computable topology theory to locally-compact Hausdorff spaces and semicontinuous set-valued maps, and provide a framework for computing approximations.

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1 Introduction

The purpose of this paper is to study the computability of reachable sets for nonlinear dynamic and control systems, and to introduce the computable analysis and topology as a powerful tool for the study of nonlinear systems. The reachability problem is important in applications, since it can be viewed as a *nonlinear verification* problem, and used for the validation of safety properties of the system. Further, of all the important problems in nonlinear systems the reachability problem also seems to be the most amenable to study by the methods of computable analysis and topology, and hence forms a good starting point for the application of these techniques.

We use the framework of *type-two effectivity* developed by Weihrauch [23] and co-workers. In this theory, computations are performed by standard Turing machines with *input tapes*, which can only be sequentially read, and *output tapes*, which can only be sequentially written to, and *work tapes*. Unlike standard computability theory (type-one effectivity) in which inputs and outputs are *words* (elements of Σ^*), type-two machines can compute on *sequences* (elements of Σ^ω). This allows representations of, and computations on, the standard objects of analysis and topology, such as real numbers, open, closed and compact subsets of Euclidean space, continuous functions and semicontinuous multivalued functions. Type-two effectivity theory provides a standard representation for elements of a topological space, and the main result of the theory is that only continuous functions are computable in the standard representation.

The *reachable set* for a discrete-time system F with initial set X_0 is defined by $\mathrm{Reach}(F, X_0) := \bigcup_{i=0}^{\infty} F^i(X_0)$. There are already many software packages which compute approximations to the reachable set, such as d/dt for linear hybrid systems [2]. However, since general sets and functions cannot be represented exactly in a finite amount of data, there is always the question of what is it possible to compute. In particular, we wish to know whether it is possible to compute the standard representations of the reachable set (an infinite computation), and whether it is possible to compute approximations to the reachable set by a finite computation.

We show that given arbitrarily good lower approximations to the initial set and the system, we can compute arbitrarily good lower approximations to the reachable set. Unfortunately, it is not possible, in general, to compute arbitrarily good outer approximations. Instead, for uniformly bounded systems, we show that it is possible to compute outer approximations to the *chain reachable set*, $\mathrm{ChainReach}(F, X_0)$, which contains all points which can be reached by introducing an arbitrarily small amount of noise. (An introduction to ϵ-chains can be found in Conley [10].) Finally, we show that it is only possible to compute arbitrary-precision approximations to the reachable set if $\mathrm{cl}(\mathrm{Reach}(F, X_0)) = \mathrm{ChainReach}(F, X_0)$.

The main results of the paper are summarised in the following theorem.

Theorem 1.1. *It is possible to compute lower approximations to the reachable set of a lower-semicontinuous system, and outer approximations to the chain-reachable set of an upper-semicontinuous system. It is possible to compute arbitrary-precision approximations to the reachable set of a continuous system if, and only if, the closure of the reachable set equals the chain reachable set.*

We remark that the negative computability results here assume that the *only* information we have about sets and systems are lower and upper approximations. If more detailed information is available (e.g. a description in terms of polynomials with rational coefficients) then it may be possible to determine the reachable and chain reachable sets exactly, even if they differ. In other words, a lack of computability in the approximative sense used here does not imply a lack of computability in some other computational framework. On the other hand, there may be reachability questions which cannot be answered exactly but can be determined approximately.

The computational topology used here for the representation of sets and functions is based mostly on Chapters 5 and 6 of Weihrauch [23]. However, rather than restrict ourselves to Euclidean spaces or separable metric spaces, we generalise to second-countable, locally compact Hausdorff spaces. The resulting theory is essentially the same as that for Euclidean spaces, and provides the most general natural setting for our results. While we anticipate that the main application areas will be Euclidean spaces, the more general approach also includes, for example, computability on manifolds. For more detailed description of computability on subsets of metric spaces, see Brattka and Presser [8] and Brattka [7].

We also develop new *approximation representations* of sets and semicontinuous functions. These allow sets and functions to be represented be sequences of *denotable* sets and functions, which can be specified exactly. Denotable sets and functions are already used in packages for rigorous numerics such as GAIO [11], which allows outer approximations of Lipschitz continuous systems.

There are a number of other works in which set-valued methods, and approximations to reachable sets are considered. A number of applications of set-valued methods to control problems are given in Szolnoki [22]. There is a large body of literature on approximation methods in viability theory such as Aubin and Frankowska [5] and Cardaliaguet et. al. [9]. Approximation methods based on ellipsoidal techniques have been considered by Kurzhanski and Varaiya [16, 17]. The integration of differential inclusions has been studied by Puri, Varaiya and Borkar [21]. The relation between reachability and chain reachability has been considered by Asarin and Bouajjani [1]. Reachability for systems with piecewise-constant derivatives was shown to be undecidable for by Asarin, Maler and Pnueli [3]. For an approximation framework based on first-order logic over the reals, see Franzle [12, 13].

The paper is organised as follows. We first give a simple example system for which the reach set fails to be computable, in order to motivate the results of the rest of the paper. In Section 2, we give an introduction to the topological aspects of the computable analysis of Weihrauch, which form the core techniques. In Section 3 we develop computable topology for semicontinuous multivalued maps, which provide our basic model for control systems. In Section 4 we apply these techniques to solve reachability problems for (semi)continuous systems, and also discuss the subclass of *closure-interior systems* for which inner and outer approximations to the reachable set are possible. In Section 5, we relate the abstract representations of points and sets defined in [23] to approximations by denotable elements. Finally, we state some conclusions and give directions for future work in Section 6.

We have endeavoured to make the paper as self-contained as possible, and have hence included a brief, but comprehensive introduction to general topology, computable analysis and multivalued maps. The material in these sections can mostly be found in the books [14, 20, 23]. Although we give definitions and state theorems formally in terms of the language of type-two effectivity, we write proofs in the language of standard topology and analysis, since we feel that this is more transparent for the reader. The proofs of the results can therefore be viewed as "constructive topology". For a self-proclaimed work of constructivist propaganda, see Bishop and Bridges [6].

Example 1.2. We now give a simple example which illustrates the difficulties involved in computing reachable sets. Consider the maps $f_\epsilon : \mathbb{R} \to \mathbb{R}$ given by

$$f_\epsilon(x) := \epsilon + x + x^2 - 9x^4, \tag{1}$$

where ϵ is a small parameter.

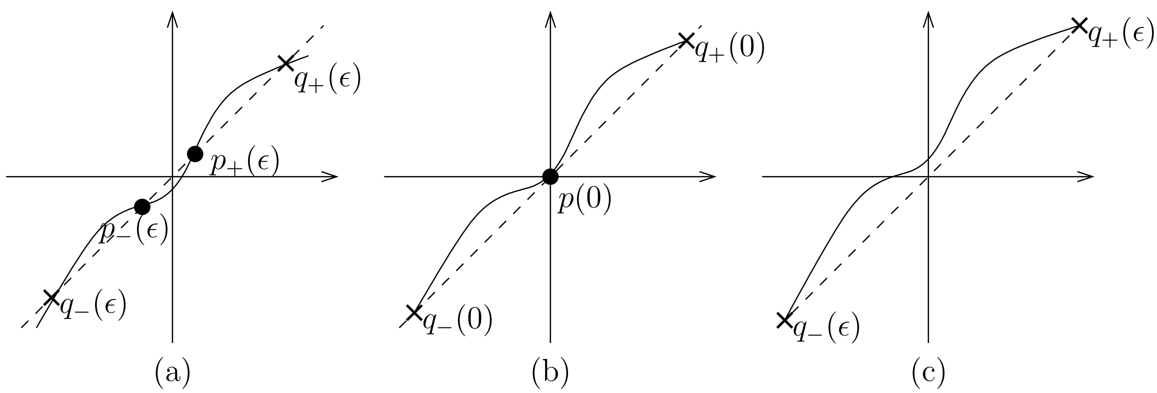


Figure 1: The map $f(x) := \epsilon + x + x^2 - 9x^4$ for (a) $\epsilon < 0$, (b) $\epsilon = 0$ and (c) $\epsilon > 0$.

For $\epsilon = 0$, there are fixed points at $p(0) = 0$, $q_-(0) = -1/3$ and $q_+(0) = +1/3$, as shown in Figure 1(b). Since $f_0'(-1/3) = 5/3$ and $f_0'(1/3) = 1/3$, the fixed points $q_-(0)$ and $q_+(0)$ are hyperbolic, and can be continued to give families of fixed points $q_-(\epsilon)$ and $q_+(\epsilon)$ for some neighbourhood of $\epsilon = 0$, as shown in Figure 1(a-c). The fixed point p at $x = 0$ can be continued to two branches of fixed points $p_-(\epsilon)$ and $p_+(\epsilon)$ for $\epsilon < 0$, as shown in Figure 1(a), but does not exist for $\epsilon > 0$, as shown in Figure 1(c).

Since $f_\epsilon'(x) = 1 + 2x - 36x^3$, we can show that $f_\epsilon'(x) > 0$ for $x \leqslant 5/14$, and hence f_ϵ is an increasing function. If $\epsilon > 0$ is sufficiently small, then $f_\epsilon(x) > x$ for all $x \in (q_-(\epsilon), q_+(\epsilon))$, and $f_\epsilon(x) \geqslant x + \epsilon$ if $x \in [-1/3, +1/3]$.

Consider an initial point $x_0 \in (-1/3, 0)$. For ϵ sufficiently close to 0, we have $x_0 > q_-(\epsilon)$ and $x_0 < p_-(\epsilon)$ if $\epsilon < 0$. Let $x_i = f_\epsilon^i(x_0)$ for $i \in \mathbb{Z}^+$. Then the reachable set of f_ϵ starting from x_0 is just the orbit $\{x_i : i \in \mathbb{Z}^+\}$.

If $\epsilon < 0$, then since $q_-(\epsilon) < x_0 < p_-(\epsilon)$, we have $f(q_-(\epsilon)) < f(x_0) < f(p_-(\epsilon))$ by monotonicity of f_ϵ, so $q_-(\epsilon) < x_1 < p_-(\epsilon)$. Hence $x_i \in (q_-(\epsilon), p_-(\epsilon))$ for all i. Further, since $f(x) > x$ for $x \in (q_-(\epsilon), p_-(\epsilon))$, the orbit (x_i) is an increasing sequence in $[x_0, p_-(\epsilon)]$. Indeed, we can show that $\lim_{i\to\infty} x_i = p_-(\epsilon)$. In particular, $\mathrm{Reach}(f_\epsilon, \{x_0\} \subset [x_0, p_-(\epsilon)]$. Similarly, if $\epsilon = 0$, we see that $\mathrm{Reach}(f_0, \{x_0\}) \subset [x_0, 0]$.

If $\epsilon > 0$, the situation is very different. Since $f_\epsilon(x) \geqslant x + \epsilon$ for $x \in (-1/3, +1/3)$, it must be the case that $x_i > 1/3$ for some i. In fact, for ϵ sufficiently small, we have $\lim_{i\to\infty} x_i = q_+(\epsilon)$. The reachable set is therefore not contained in a small neighbourhood of $[x_0, 0]$ for $\epsilon > 0$, even if $\epsilon \ll 1$, and in fact jumps discontinuously at $\epsilon = 0$.

Hence, to find a good approximation to the reachable set, it is necessary to determine whether $\epsilon > 0$. If ϵ is known precisely (e.g. ϵ is a given rational), then $\mathrm{Reach}(f_\epsilon, x_0)$ can be approximated to arbitrary precision. However, if ϵ is only known approximately, then it may be impossible to decide whether $\epsilon > 0$, and hence find a good approximation to $\mathrm{Reach}(f_\epsilon, x_0)$.

The above example shows that computability of system properties depends on the *class* of systems under consideration, and the *representation* of systems in that class. In the framework of computable analysis, a function is described approximately; even for a polynomial function with real coefficients, the coefficients are given by approximating sequences of rationals or rational intervals. In an algebraic framework, such as polynomial systems with rational coefficients, we can describe a system exactly, and more quantities may be computable. However, the class of systems we can deal with algebraically is restricted compared with that of computational analysis.

We could conceive of a reachability algorithm using special techniques for one-dimensional polynomial systems, and more general techniques for other systems. Unfortunately, the question of whether a one-dimensional continuous function (described in terms of computational analysis) is a polynomial with rational coefficients is undecidable. Hence a dual-method algorithm would need to be told whether its input was a polynomial description or an approximate description.

2 Computable analysis and topology

Computable analysis deals with real numbers, continuous functions on real and Euclidean spaces, and subsets of Euclidean spaces. We consider a more general computable topology dealing with continuous functions on Hausdorff spaces. In this section, we review the elements of the literature which we need. The material in Section 2.1 can be found in [20], and that of the other subsections in [23].

2.1 Topological spaces

We first recall the basic facts of general topology.

A *topological space* is a pair (M, τ) where M is a set and τ is a set of subsets of M (i.e. $\tau \subset \mathcal{P}(M)$) such that

1. $\emptyset \in \tau$ and $X \in \tau$,
2. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$, and
3. If $\mathcal{U} \subset \tau$, then $\bigcup \mathcal{U} \in \tau$.

The sets in τ are called *open* sets, and the complement of an open set is a *closed* set. A set B is a *neighbourhood* of a point x if there exists and open set $U \in \tau$ such that $x \in U \subset B$.

A topological space (M, τ) is T_0 or *Kolmogorov* if given any two disjoint points x, x', there is an open set U containing exactly one of x and x'. The space (M, τ) is T_2 or *Hausdorff* if given any two disjoint points x, x', there are disjoint open sets U, U' such that $x \in U$ and $x' \in U'$. The Hausdorff space (M, τ) is T_4 or *normal* if given any two disjoint closed sets A, A', there are disjoint open sets U, U' with $A \subset U$ and $A' \subset U'$.

An open cover of a set $B \subset M$ is a set $\mathcal{U} \subset \tau$ such that $B \subset \bigcup \mathcal{U}$. A set $C \subset M$ is *compact* if every open cover of C has a finite subcover. A set $B \subset M$ is *pre-compact* if $\mathrm{cl}(B)$ is compact. A topological space (M, τ) is *locally compact* if every point has a compact neighbourhood.

An open cover $\mathcal{U}$ of M is *locally finite* if for every compact $C \subset M$, $\{U \in \mathcal{U} : U \cap C \neq \emptyset\}$ is finite. We say an open cover $\mathcal{U}_2$ is a *refinement* of a cover $\mathcal{U}_1$, denoted $\mathcal{U}_2 \prec \mathcal{U}_1$, if for all $U_2 \in \mathcal{U}_2$, there exists $U_1 \in \mathcal{U}_1$ such that $U_2 \subset U_1$. We say a refinement $\mathcal{U}_2$ of $\mathcal{U}_1$ is a *strong refinement*, if for all $U_2 \in \mathcal{U}_2$, there exists $U_1 \in \mathcal{U}_1$ such that $\overline{U}_2 \subset U_1$, and a *proper refinement* if for all $U_1 \in \mathcal{U}_1$, $U_1 = \bigcup\{U_2 \in \mathcal{U}_2 : U_2 \subset U_1\}$.

A subset β of a topology τ on M is a *base* for τ if every element of τ is a union of elements of β. If β is a base of τ, then an element $U \in \beta$ is a *basic (open) set*, and $\mathrm{cl}(U)$ is a *basic closed set*. A subset σ

of a topology τ on M is a *subbase* or *generator* of τ if τ is the smallest topology containing σ. (i.e. τ is the smallest subset of $\mathcal{P}(M)$ which contains σ and satisfies the axioms for a topology.) A base for the topology generated by σ is given by all finite intersections of elements of σ.

A topological space is *second countable* if it has a countable base of open sets. In particular, if a topology τ has a countable generating set σ, then it has a countable basis (consisting of all finite intersections of elements of σ).

If (M, τ) is a T_0 topological space and σ is a generator for τ, then for any pair $x, x' \in M$ with $x \neq x'$, there is an element U of σ containing exactly one of x, x'. This means that every point x can be specified by giving the subset $\{U \in \sigma : x \in U\}$ of elements of σ containing x.

A sequence (x_n) *converges* to x_∞ if for every open set U containing x_∞, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geqslant N$. If (M, τ) is a Hausdorff space, then any convergent sequence has a unique limit, but otherwise limits need not be unique. (Unique limits for non-Hausdorff spaces can be defined using *convergent nets*.)

Where there is no confusion as to the topology on M, we denote the set of open subsets of a topological space M by $\mathcal{O}(M)$, the set of closed subsets by $\mathcal{A}(M)$, and the set of compact subsets by $\mathcal{K}(M)$.

2.2 Computability and naming systems

We consider computability in terms of words and sequences on a finite alphabet Σ. For digital computers, $\Sigma = \{0, 1\}$, words Σ^* can be thought of as files or data structures, and sequences Σ^ω can be thought of as infinite "data streams". The binary alphabet $\{0, 1\}$ can of course be used to represent any other alphabet, such as the ASCII character set. The alphabet Σ is frequently taken to contain a special *blank* symbol $\sqcup$, which can denote a space or the end of an input.

Computations are performed by Turing machines with n input tapes and a single output tape. Each input tape must be specified as either containing a word or a sequence. A partial function $f :\subset Y_1, \ldots, Y_n \to Y_0$ with $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for $i = 0, \ldots, n$ is *computable* if there is some Turing machine which computes $y_0 = f(y_1, \ldots y_k)$, where in the case $Y_0 = \Sigma^*$ the computation halts with y_0 on the output tape, and in the case $Y_0 = \Sigma^\omega$ the computation continuous forever, writing y_0 on the output tape.

The theory of computability on words and sequences is known as *type-two effectivity (TTE)*, as opposed to *type-one effectivity*, which can be considered as "ordinary" computation on words.

In order to formalise computability on more general sets, we consider *naming systems*.

Definition 2.1 (Naming systems).

1. A *notation* of a set M is a surjective partial function $\nu :\subset \Sigma^* \to M$.
2. A *representation* of a set M is a surjective partial function $\delta :\subset \Sigma^\omega \to M$.

A notation ν is *effective* if the set

$$\{(u, v) \in \Sigma^* \times \Sigma^* : u, v \in \mathrm{dom}(\nu) \text{ and } \nu(u) = \nu(v)\}$$

is recursively enumerable (r.e.).

Note that the domain of an effective notation is recursively enumerable. In most situations of interest, the equivalence problem $\nu(u) = \nu(v)$ will be recursive (decidable), or even trivial (i.e. $\nu(u) = \nu(v) \iff u = v$.)

Remark 2.2. We could also use functions $\nu :\subset \mathbb{N} \to M$ as notations, and functions $\delta : \mathbb{N}^\omega \to M$ as representations.. This is more in the language of recursive function theory, whereas our naming systems are in the language of Turing computability.

Definition 2.3 (Translation and equivalence). Given two naming systems $\gamma :\subset Y \to M$ and $\gamma' :\subset Y' \to M'$, where $Y, Y' \in \{\Sigma^*, \Sigma^\omega\}$ and $M \subset M'$, we say a computable function $f :\subset Y \to Y'$ *translates* γ to γ' if $\gamma(y) = \gamma'(f(y))$ for all $y \in \mathrm{dom}(\gamma)$. We write $\gamma \leqslant \gamma'$ if some computable function translates γ to γ'. We say γ and γ' are *equivalent*, denoted $\gamma \equiv \gamma'$, if $\gamma \leqslant \gamma'$ and $\gamma' \leqslant \gamma$.

Definition 2.4 (Realisation). Given a function $f : M \to M'$, and naming systems $\gamma :\subset Y \to M$ and $\gamma' :\subset Y' \to M'$, a function $g : Y \to Y'$ is a *realisation* of f if $\gamma'(g(y)) = f(\gamma(y))$ for all $y \in \mathrm{dom}(\gamma)$.

Given a notation ν of a set M we may wish to give a representation of tuples M^* and sequences M^ω. There are a number of methods for performing such a "tupling" operation:

1. If Σ contains a blank symbol $\sqcup$ which is not contained in any word in $\mathrm{dom}(\nu)$, we can construct a representation δ by
$$\delta(w_0 \sqcup w_1 \sqcup w_2 \sqcup \cdots) = (\nu(w_0), \nu(w_1), \nu(w_2), \ldots)\,.$$
2. If $\mathrm{dom}(\nu) \subset \Sigma^*$ is prefix free, then any sequence $p \in \Sigma^\omega$ parses uniquely into a sequence $p = w_0 w_1 w_2 \cdots$ with each $w_i \in \mathrm{dom}(\nu)$. We can take
$$\delta(w_0 w_1 w_2 \cdots) = (\nu(w_0), \nu(w_1), \nu(w_2), \ldots)\,.$$
3. We can construct a *wrapping function* $\imath : \Sigma^* \to \Sigma^*$ such that $\imath(\Sigma^*)$ is prefix-free. One particular choice for $\Sigma = \{0, 1\}$ is
$$\imath(a_1 a_2 \cdots a_n) = \mathtt{110}a_1\mathtt{0}a_2\mathtt{0}\cdots\mathtt{0}a_n\mathtt{011}.$$
A representation for M^ω is then given by
$$\delta(\imath(w_0)\imath(w_1)\imath(w_2)\cdots) = (\nu(w_0), \nu(w_1), \nu(w_2), \ldots).$$

Regardless of which "tupling" method is chosen, will write $\langle w_0, w_1, w_2, \ldots\rangle$ for the tupling of words $(w_0, w_1, w_2, \ldots)$, and write $w \lhd p$ if $p = \langle w_0, w_1, w_2, \ldots\rangle$ and $w = w_i$ for some $i \in \mathbb{N}$. We may also tuple finitely many words $\langle w_1, \ldots, w_k\rangle$, a word and a sequence $\langle w, p\rangle$, finitely many sequences $\langle p_1, \ldots, p_k\rangle$ or even infinitely many sequences $\langle p_1, p_2, \ldots\rangle$. The tupling of sequences may be effected by shuffling, e.g. $\langle p_1, p_2\rangle = \langle w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, \ldots\rangle$ where $p_i = \langle w_{i,1}, w_{i,2}, w_{i,3}, \ldots\rangle$ for $i = 1, 2$.

2.3 Computable topological spaces

The essence of a computable topological space is to perform all computations on a countable generator σ of τ. Computability properties may therefore depend on the generator chosen. To formally relate computability concepts to Turing computability, we need a naming system for elements of σ in terms of some finite alphabet Σ.

If (M, τ) is a T_0-topological space, then every point is specified by the set of open sets containing it. This property also holds for a generator σ of τ, so every point is specified by $\{U \in \sigma : x \in U\}$. This gives us a way of representing points in topological spaces in a way which respects the topology.

Definition 2.5 (Computable topological space). A *computable topological space* is a quadruple (M, τ, σ, ν) such that M is a non-empty set, $\tau \subset \mathcal{P}(M)$ is a topology on M, $\sigma \subset \tau$ is generator of τ, and $\nu : \Sigma^* \to \sigma$ is an effective notation for σ.

We denote the closures of the elements of σ by $\overline{\nu}(w) := \mathrm{cl}(\nu(w))$. We also consider all finite unions of elements of σ, with notation $\widetilde{\nu}\langle w_1, \ldots w_k\rangle := \bigcup_{i=1}^k \nu(w_i)$.

There is a canonical representation of elements of a computable topological space.

Definition 2.6 (Standard representation). The *standard representation* $\delta_{\mathbf{S}}$ of a computable topological space $\mathbf{S} = (M, \tau, \sigma, \nu)$ is the representation $\delta_{\mathbf{S}} :\subset \Sigma^\omega \to M$ given by
$$\delta_{\mathbf{S}}(p) = x \quad :\Longleftrightarrow \quad \{\nu(w) : w \lhd p\} = \{J \in \sigma : x \in J\}$$

Remark 2.7. Informally, we can think of the standard representation δ of (M, τ, σ, ν) as *encoding* a sequence $(J_i)_{i \in \mathbb{N}}$ containing all sets $J_i \in \sigma$ for which $x \in J_i$. When writing proofs, we shall usually consider the sequence encoded by the representation, and not the representation itself, to avoid obscuring the idea of the proof in technical notation.

Definition 2.8 (Admissible representation). Let (M, τ) be a second-countable T_0-space. A representation $\gamma :\subset \Sigma^\omega \to M$ is *admissible* with respect to τ if $\gamma \equiv \delta_{\mathbf{S}}$ for some computable topological space $\mathbf{S} = (M, \tau, \sigma, \nu)$.

We will want to consider sets and functions on Hausdorff spaces with a given base.

Definition 2.9 (Computable Hausdorff space). A computable topological space (X, τ, β, ν) is a *computable Hausdorff space* if (X, τ) is a locally-compact separable Hausdorff space, and β is a base for τ such that each $I \in \beta$ is pre-compact.

Following Brattka and Presser [8], we now define some important properties of a computable topological spaces.

Definition 2.10 (Effectivity properties). A computable topological space (X, τ, β, ν) has

1. the *effective intersection* property, if $\{(w_0, w_1) : \nu(w_0) \cap \nu(w_1) \neq \emptyset\}$ is r.e.,
2. the *effective disjointness* property, if $\{(v_0, v_1) : \overline{\nu}(v_0) \cap \overline{\nu}(v_1) = \emptyset\}$ is r.e.,
3. the *effective inclusion* property, if $\{(v, w) : \overline{\nu}(v) \subset \nu(w)\}$ is r.e., and
4. the *effective covering* property, if $\{(v, \langle w_0, \ldots, w_n\rangle) : \overline{\nu}(v) \subset \bigcup_{i=0}^{n} \nu(w_i)\}$ is r.e.

These sets are typically not recursive, since we can only verify *robust* properties in general. We note that the effective covering property implies the effective inclusion property. We will see later the the effective intersection property of a computable topological space is equivalent to lower-computability of all basic closed sets, and the effective disjointness property is equivalent to upper-computability, and the effective covering property is equivalent to upper-computability of basic closed sets considered as compact sets.

The main theorem of computable analysis is that only continuous functions are computable in the standard representation. We use the following form, which is Corollary 3.2.12 of [23].

Theorem 2.11 (Computable implies continuous). *For $i = 0, \ldots, k$ let $\mathbf{S}_i = (M_i, \tau_i, \sigma_i, \nu_i)$ be a computable topological space, and δ_i the standard representation of $\mathbf{S}_i$. Then every $(\delta_1, \ldots, \delta_k; \delta_0)$-computable function $f : M_1 \times \cdots \times M_k \to M_0$ is $(\tau_1, \ldots, \tau_n; \tau_0)$-continuous.*

2.4 Representations of real numbers

Let $\mathbb{R}$ be the set of real numbers, and τ the standard topology on $\mathbb{R}$. A base for τ is given by the set of all finite open rational intervals, $\beta := \{(a, b) : a, b \in Q, a < b\}$. Given a notation ν for β, we obtain a computable topological space $(\mathbb{R}, \tau, \beta, \nu)$. The standard representation ρ of a real number x encodes a list of all (a, b) with $a < x < b$.

However, it is more natural to consider other representations. In particular, instead of considering *all* intervals containing x, we need only take a sequence of intervals (a_n, b_n) such that $a_n < a_{n+1} < b_{n+1} < b_n$ for all n, and $\{x\} = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$. This gives the *interval representation* ρ_I, defined formally as

$$\rho_I \langle w_1, w_2, \ldots \rangle = x \quad :\Longleftrightarrow \quad \forall i \in \mathbb{N},\ \overline{\nu}(w_{i+1}) \subset \nu(w_i) \text{ and } \textstyle\bigcap_{i=1}^{\infty} \nu(w_i) = \{x\} \tag{2}$$

It is also possible to define weaker topologies $\tau_<$ and $\tau_>$ on $\mathbb{R}$, with bases $\beta_< := \{(a, \infty) : a \in \mathbb{Q}\}$ and $\beta_> := \{(-\infty, a) : a \in \mathbb{Q}\}$, respectively. The resulting standard representations are denoted $\rho_<$ and $\rho_>$, and give lower and upper bounds for x, respectively.

Euclidean space $(\mathbb{R}^n, \tau^n)$ has a base β^n consisting of all rational cubes,

$$\beta^n := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q},\ a_i < b_i \text{ for } i = 1, \ldots, n\} \tag{3}$$

and becomes a computable Hausdorff space by giving a notation ν^n for β^n. The resulting standard representation is ρ^n, which encodes a list of all open cubes containing a point x. An equivalent representation is to use a decreasing sequence of cubes (J_i) such that $\overline{J}_{i+1} \subset J_i$ and $\{x\} = \bigcap_{i=1}^{\infty} J_i$ as a name for x.

2.5 Representations of closed sets

We now consider topologies on the set of closed subsets of a second-countable locally compact Hausdorff space (X, τ). Let β be a base for τ on M, and define

$$\begin{aligned} \sigma^{\mathcal{A}}_{<} &:= \big\{\{A \in \mathcal{A}(X) : A \cap J \neq \emptyset\} : J \in \beta\big\} \\ \sigma^{\mathcal{A}}_{>} &:= \big\{\{A \in \mathcal{A}(X) : A \cap \overline{J} \neq \emptyset\} : J \in \beta\big\} \\ \sigma^{\mathcal{A}} &:= \sigma^{\mathcal{A}}_{<} \cup \sigma^{\mathcal{A}}_{>}. \end{aligned} \tag{4}$$

Let $\tau^{\mathcal{A}}_{<}$, $\tau^{\mathcal{A}}_{>}$ and $\tau^{\mathcal{A}}$ be the topologies generated, respectively, by $\sigma^{\mathcal{A}}_{<}$, $\sigma^{\mathcal{A}}_{>}$ and $\sigma^{\mathcal{A}}$. We denote the topological spaces $(\mathcal{A}(X), \tau_<)$, $(\mathcal{A}(X), \tau_>)$ and $(\mathcal{A}(X), \tau)$ by, respectively, $\mathcal{A}_<(X)$, $\mathcal{A}_>(X)$ and $\mathcal{A}_=(X)$.

We can give representations $\psi_<$, $\psi_>$ and ψ for the topologies which are equivalent to the standard representations as follows.

$$\begin{aligned} \psi_<(p) = A \quad &:\Longleftrightarrow \quad \{\nu(w) : w \lhd p\} = \{J \in \beta : A \cap J \neq \emptyset\} \\ \psi_>(p) = A \quad &:\Longleftrightarrow \quad \{\nu(w) : w \lhd p\} = \{J \in \beta : A \cap \overline{J} = \emptyset\} \\ \psi\langle p, q\rangle = A \quad &:\Longleftrightarrow \quad \psi_<(p) = A \text{ and } \psi_>(q) = A. \end{aligned} \tag{5}$$

The representation $\psi_<$ encodes a list of all basic open sets J such that $A \cap J \neq \emptyset$, and $\psi_>$ encodes a list of all basic closed sets $\overline{J}$ such that $A \cap \overline{J} = \emptyset$. The representations are *robust* in the sense that if $A \cap J \neq \emptyset$, then there exists I with $\overline{I} \subset J$ such that $A \cap I \neq \emptyset$, and if $A \cap \overline{J} = \emptyset$, then there exists I with $\overline{J} \subset I$ such that $A \cap \overline{I} = \emptyset$.

A closed subset A of X *recursively enumerable* if A is $\psi_<$-computable, *co-recursively enumerable* if it is $\psi_>$-computable, and *recursive* if it is ψ-computable. Note that membership of a recursive set need not be decidable.

The topologies $\tau^{\mathcal{A}}_{<}$ and $\tau^{\mathcal{A}}_{>}$ are T_0 topologies, since given two distinct closed sets A_0 and A_1, there is a point x_0 of $A_0 \setminus A_1$ (or $A_1 \setminus A_0$). Then since (X, τ) is normal there is a basic open set $J \ni x_0$ such that $J \cap A_1 = \emptyset$. Hence $A_0 \in \{A \in \mathcal{A} : A \cap J \neq \emptyset\}$ but $A_1 \notin \{A \in \mathcal{A} : A \cap J \neq \emptyset\}$. A similar argument shows that if A_0 and A_1 are distinct closed sets, then there is a basic closed set $\overline{J}$ such that $\overline{J}$ intersects exactly one of A_0 and A_1. The topology $\tau^{\mathcal{A}}$ is a normal Hausdorff topology.

The following result on intersection and union operations on closed sets is Theorem 4.1.13 of Weihrauch [23].

Theorem 2.12.

1. *Union $(A, B) \mapsto A \cup B$ on $\mathcal{A}$ is $(\psi_<, \psi_<; \psi_<)$-computable, $(\psi_>, \psi_>; \psi_>)$-computable and $(\psi, \psi; \psi)$-computable.*
2. *Intersection $(A, B) \mapsto A \cap B$ on $\mathcal{A}$ is $(\psi_>, \psi_>; \psi_>)$-computable.*
3. *The function $A \mapsto A \cap \{0\}$ is not $(\tau^{\mathcal{A}}; \tau^{\mathcal{A}}_{<})$-continuous, and so intersection $(A, B) \mapsto A \cap B$ on $\mathcal{A}$ is not $(\tau^{\mathcal{A}}, \tau^{\mathcal{A}}; \tau^{\mathcal{A}}_{<})$-continuous or $(\psi, \psi; \psi_<)$-computable.*

We note that, since $\tau^{\mathcal{A}}$ is a stronger topology than $\tau^{\mathcal{A}}_{<}$, it is immediate that intersection is not $(\tau^{\mathcal{A}}, \tau^{\mathcal{A}}; \tau^{\mathcal{A}})$-continuous. Similarly, intersection is not (ψ, ψ, ψ)-computable, or $(\psi_<, \psi_<; \psi_<)$-computable, since ψ translates to $\psi_<$.

2.6 Representations of open sets

Since an open set is the complement of a closed set, we can use the representations of closed sets to give representations of open sets. We let $\tau^{\mathcal{O}}_{<}$, $\tau^{\mathcal{O}}_{>}$ and $\tau^{\mathcal{O}}$ be the topologies generated, respectively, by $\sigma^{\mathcal{O}}_{<}$, $\sigma^{\mathcal{O}}_{>}$ and $\sigma^{\mathcal{O}}$ defined below:

$$\begin{aligned} \sigma^{\mathcal{O}}_{<} &:= \big\{\{U \in \mathcal{O}(X) : (X \setminus U) \cap \overline{J} = \emptyset\} : J \in \beta\big\} = \big\{\{U \in \mathcal{O}(X) : \overline{J} \subset U\} : J \in \beta\big\} \\ \sigma^{\mathcal{O}}_{>} &:= \big\{\{U \in \mathcal{O}(X) : (X \setminus U) \cap J \neq \emptyset\} : J \in \beta\big\} = \big\{\{U \in \mathcal{O}(X) : J \not\subset U\} : J \in \beta\big\} \\ \sigma^{\mathcal{O}} &:= \sigma^{\mathcal{O}}_{<} \cup \sigma^{\mathcal{O}}_{>}. \end{aligned} \tag{6}$$

The topologies $\tau^{\mathcal{O}}_{<}$ and $\tau^{\mathcal{O}}_{>}$ are T_0 topologies, and $\tau^{\mathcal{O}}$ is a Hausdorff topology. We can give representations $\theta_{<}$, $\theta_{>}$ and θ for the topologies which are equivalent to the standard representations as follows.

$$\begin{aligned} \theta_{<}(p) = U &\;:\Longleftrightarrow\; \{\nu(w) : w \lhd p\} = \{J \in \beta : \overline{J} \subset U\} \\ \theta_{>}(p) = U &\;:\Longleftrightarrow\; \{\nu(w) : w \lhd p\} = \{J \in \beta : J \not\subset U\} \\ \theta\langle p, q\rangle = U &\;:\Longleftrightarrow\; \theta_{<}(p) = U \text{ and } \theta_{>}(q) = U. \end{aligned} \tag{7}$$

The representation $\theta_{<}$ encodes a list of all basic closed sets $\overline{J}$ such that $\overline{J} \subset U$ (equivalently $(X \setminus U) \cap \overline{J} = \emptyset$) and $\theta_{>}$ encodes a list of all basic open sets J such that $J \not\subset U$ (equivalently $(X \setminus U) \cap J \neq \emptyset$).

2.7 Representations of compact sets

Let $\mathcal{K}(X)$ be the set of compact subsets of X. A subset of a locally compact Hausdorff space is compact if it is closed and bounded. We can specify a bound for a compact C as a finite open cover of C by basic open sets. The standard representations of compact sets are then given by

$$\begin{aligned} \kappa_{<}\langle u, p\rangle = C &\;:\Longleftrightarrow\; C \subset \widetilde{\nu}(u) \text{ and } \psi_{<}(p) = C \\ \kappa_{>}\langle u, p\rangle = C &\;:\Longleftrightarrow\; C \subset \widetilde{\nu}(u) \text{ and } \psi_{>}(p) = C \\ \kappa\langle u, p, q\rangle = C &\;:\Longleftrightarrow\; C \subset \widetilde{\nu}(u) \text{ and } \psi\langle p, q\rangle = C, \end{aligned} \tag{8}$$

where $u \in \Sigma^*$ and $p, q \in \Sigma^\omega$. Note that this differs slightly from that of [23], in which only a single basic open set can be used as a cover. (The representation here is more general, since we do not require that every compact set is contained in a single basic open set.)

We can define topologies on compact sets by using generators

$$\begin{aligned} \sigma^{\mathcal{K}}_{>} &:= \Big\{\{C \in \mathcal{K} : C \subset \textstyle\bigcup_{i=1}^{k} J_i\} : J_1, \ldots J_k \in \beta\Big\} \\ \sigma^{\mathcal{K}} &:= \sigma^{\mathcal{A}}_{<} \cup \sigma^{\mathcal{K}}_{>}, \end{aligned} \tag{9}$$

and taking $\tau^{\mathcal{K}}_{>}$ and $\tau^{\mathcal{K}}$ to be the topologies generated, respectively, by $\sigma^{\mathcal{K}}_{>}$ and $\sigma^{\mathcal{K}}$. The resulting topological spaces are $\mathcal{K}_{>}(X) := (\mathcal{K}(X), \tau^{\mathcal{K}}_{>})$ and $\mathcal{K}_{=}(X) := (\mathcal{K}(X), \tau^{\mathcal{K}})$.

The standard representations of the computable topological spaces give representations

$$\begin{aligned} \kappa^{\mathrm{cv}}_{>}(p) = C &\;:\Longleftrightarrow\; \{(\nu(w_1), \ldots, \nu(w_k)) : \langle w_1, \ldots, w_k\rangle \lhd p\} \\ &\qquad\qquad = \{(J_1, \ldots, J_k) \subset \beta : C \subset \textstyle\bigcup_{i=1}^{k} J_i\} \\ \kappa^{\mathrm{cv}}\langle p, q\rangle = C &\;:\Longleftrightarrow\; \psi_{<}(p) = C \text{ and } \kappa^{\mathrm{cv}}_{>}(q) = C. \end{aligned} \tag{10}$$

The representation $\kappa^{\mathrm{cv}}_{>}$ encodes a list of all tuples of basic open sets $(J_1, \ldots, J_k)$ such that $C \subset \bigcup_{i=1}^{k} J_i$. The representation is robust, since if $C \subset \bigcup_{i=1}^{k} J_i$, then there exists $(I_1, \ldots, I_k)$ with $\overline{I}_i \subset J_i$ for $i = 1, \ldots, k$ and $C \subset \bigcup_{i=1}^{k} I_i$.

By Lemma 5.2.5 of [23], we have $\kappa^{\mathrm{cv}}_{>} \equiv \kappa_{>}$ and $\kappa^{\mathrm{cv}} \equiv \kappa$. The equivalence of $\kappa_{>}$ and $\kappa^{\mathrm{cv}}_{>}$ implies that every open cover of C can be computed from a single open cover and a list of basic closed sets disjoint from C.

The situation for lower approximations is rather more complicated. We are not aware of (and conjecture that there does not exist) a topology on $\mathcal{K}$ for which $\kappa_<$ is an admissible representation. However, the topology $\tau_<^{\mathcal{A}}|_{\mathcal{K}}$ provides a topology on $\mathcal{K}$ for which many operations on compact sets are continuous. The representation $\kappa_<$ strengthens the representation $\psi_<|_{\mathcal{K}}$ by supplying a bound on the compact set. Hence, for lower approximations, we often consider properties of $\psi_<$ as well as $\kappa_<$, since $\psi_<$ is a more natural representation.

2.8 Representations of continuous functions

The natural topology for the space of continuous functions $f : X \to Y$ is the *compact-open topology*, $\tau^{\mathcal{C}}$. This topology is generated by the open sets

$$\sigma^{\mathcal{C}} := \big\{\{f \in C(X \to Y) : f(C) \subset U\} : C \in \mathcal{K}(X),\ U \in \mathcal{O}(Y)\big\}. \tag{11}$$

The *compact-open representation* is the standard representation of this topological space, and is given by

$$\delta^{\mathrm{co}}(p) = f \quad :\Longleftrightarrow \quad \{(\nu_X(w_1), \nu_Y(w_2)) : (w_1, w_2) \lhd p\} = \{(I, J) \in \beta_X \times \beta_Y : f(\overline{I}) \subset J\}. \tag{12}$$

The representation δ^{co} encodes a list of pairs $(\overline{I}, J)$ with $I \in \beta_X$ and $J \in \beta_Y$ for which $f(\overline{I}) \subset J$. Equivalent to $f(\overline{I}) \subset J$ is $\overline{I} \subset f^{-1}(J)$. The compact-open representation is *robust*, in the sense that if (I, J) is such that $f(\overline{I}) \subset J$, then there exist (K, L) with $\overline{I} \subset K$, $\overline{L} \subset J$ such that $f(\overline{K}) \subset L$.

As discussed in [23, Chapter 6.1], there are a number of equivalent representation for the space of continuous functions $C(X \to Y)$. In particular, there is a standard representation $\delta_\to$ under which the evaluation map $(f, x) \mapsto f(x)$ and the composition map $(g, f) \mapsto g \circ f$ are computable. The equivalence of the representation $\delta_\to$ and the compact-open representation δ^{co} is shown by [23, Lemma 6.1.7].

The compact-open representation has the following properties:

Theorem 2.13.

1. *The evaluation map $(f, x) \mapsto f(x)$ is $(\delta^{\mathrm{co}}, \rho; \rho)$-computable.*
2. *The composition map $(g, f) \mapsto g \circ f$ is $(\delta^{\mathrm{co}}, \delta^{\mathrm{co}}; \delta^{\mathrm{co}})$-computable.*
3. *The set-image map $(f, A) \mapsto \mathrm{cl}(f(A))$ for $A \in \mathcal{A}(X)$ is $(\delta^{\mathrm{co}}, \psi_<; \psi_<)$-computable.*
4. *The set-image map $(f, C) \mapsto f(C)$ for $C \in \mathcal{K}(X)$ is $(\delta^{\mathrm{co}}, \kappa_<; \kappa_<)$-computable, $(\delta^{\mathrm{co}}, \kappa_>; \kappa_>)$-computable and $(\delta^{\mathrm{co}}, \kappa; \kappa)$-computable.*

A representation for sequences is given by the compact-open representation of functions $\mathbb{N} \to X$.

The *graph* of a map $f : X \to Y$ is the set

$$\mathrm{Graph}(f) := \{(x, y) \in X \times Y : y = f(x)\}.$$

Since the graph of a continuous function $f : X \to Y$ is closed, we can consider the representations $\psi_<$, $\psi_>$ and ψ of this set. It turns out that the representation $\psi_>$ of $\mathcal{A}(X \times Y)$ gives a representation δ^{cc} of $C(X \to Y)$ which is equivalent to the standard representation if Y is compact.

2.9 Union and intersection of sets

We need to extend the results on unions and intersections to the case of infinite unions and intersections. For countable sequences, we use the topology of convergence on finite sequences. Countable unions and intersections have the following computability properties.

Theorem 2.14 (Countable unions and intersections).

1. *Countable closed union* $(A_1, A_2, \ldots) \mapsto \mathrm{cl}(\bigcup_{n\in\mathbb{N}} A_n)$ *on* $\mathcal{A}$ *is* $(\psi_<, \psi_<, \ldots; \psi_<)$*-computable.*
2. *Countable intersection* $(A_1, A_2, \ldots) \mapsto \bigcap_{n\in\mathbb{N}} A_n$ *on* $\mathcal{A}$ *is* $(\psi_>, \psi_>, \ldots; \psi_>)$*-computable.*
3. *Countable intersection* $(C_1, C_2, \ldots) \mapsto \bigcap_{n\in\mathbb{N}} C_n$ *on* $\mathcal{K}$ *is* $(\kappa_>, \kappa_>, \ldots; \kappa_>)$*-computable.*

Proposition 2.15.

1. *Countable closed union is neither* $(\tau^{\mathcal{A}}, \tau^{\mathcal{A}}, \ldots; \tau^{\mathcal{A}}_>)$*-continuous nor* $(\tau^{\mathcal{K}}, \tau^{\mathcal{K}}, \ldots; \tau^{\mathcal{K}}_>)$*-continuous.*
2. *Countable intersection is neither* $(\tau^{\mathcal{A}}, \tau^{\mathcal{A}}, \ldots; \tau^{\mathcal{A}}_<)$*-continuous nor* $(\tau^{\mathcal{K}}, \tau^{\mathcal{K}}, \ldots; \tau^{\mathcal{K}}_<)$*-continuous.*

The proofs are straightforward.

3 Multivalued maps

In system theory, it is useful to consider multivalued maps $F : X \rightrightarrows Y$, since these represent control systems $f : X \times U \to X$ as $F(x) = f(x, U)$.

We typically represent a multivalued map $F : X \rightrightarrows Y$ by a single-valued map $X \to \mathcal{P}(Y)$, but may also identify F with its graph, $\mathrm{Graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. If $A \in \mathcal{P}(X)$, then we define $F(A) := \{y \in Y : \exists\, x \in A,\ y \in F(x)\}$. Thus a multivalued map $F : X \rightrightarrows Y$ induces a single valued map $\mathcal{P}(X) \to \mathcal{P}(Y)$. If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$, the *composition* of F and G is $G \circ F : X \rightrightarrows Z$ given by $G \circ F\,(x) := G(F(x)) = \{z \in Z : \exists\, y \in Y,\ y \in F(x) \text{ and } z \in G(y)\}$. Note that $G \circ F\,(A) = G(F(A))$, and composition is associative.

There are two natural set-valued preimages of $F : X \rightrightarrows Y$, the *weak preimage* $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, and the *strong preimage*, $F^{\Leftarrow}(B) = \{x \in X : F(x) \subset B\}$. The graph of F^{-1} is the "transpose" of the graph of F; i.e. $(x, y) \in \mathrm{Graph}(F) \iff (y, x) \in \mathrm{Graph}(F^{-1})$. If $F : X \rightrightarrows X$, then an *orbit* of F is a sequence (x_i) such that $x_{i+1} \in F(x_i)$ for all i, so the reverse of an orbit of F is an orbit of F^{-1}.

We say F is *lower-semicontinuous* if $F^{-1}(U)$ is open whenever U is open, or equivalently, if $F^{\Leftarrow}(A)$ is closed whenever A is closed. F is *upper-semicontinuous* if $F^{-1}(A)$ is closed whenever A is closed, or equivalently, if $F^{\Leftarrow}(U)$ is open whenever U is open. A function F is *weakly upper-semicontinuous* if $F^{-1}(C)$ is closed whenever C is compact. A multivalued function is *continuous* if it is both lower-semicontinuous and upper-semicontinuous.

Henceforth, we restrict attention to functions with *closed values*, which means that $F(x)$ is closed for all x, denoted $F : X \to \mathcal{A}(Y)$. We also consider functions with *compact values*, which means $F(x)$ is compact for all x, denoted or $F : X \to \mathcal{K}(Y)$.

A closed-valued function $F : X \to \mathcal{A}(Y)$ is lower-semicontinuous if, and only if, it is $(\tau_X; \tau^{\mathcal{A}(Y)}_<)$-continuous, and weakly upper-semicontinuous if, and only if, it is $(\tau_X; \tau^{\mathcal{A}(Y)}_>)$-continuous. A compact-valued function $F : X \to \mathcal{K}(Y)$ is upper-semicontinuous if, and only if, it is $(\tau_X; \tau^{\mathcal{K}(Y)}_>)$-continuous.

If F is locally-bounded, then F is *(strongly) upper-semicontinuous* if $F : X \to \mathcal{K}_>(Y)$ is continuous. A multivalued function is *continuous* if it is both lower-semicontinuous and upper-semicontinuous.

We denote closed-valued lower-semicontinous functions by $LSC_{\mathcal{A}}$, closed-valued weakly upper-semicontinuous functions by $USC_{\mathcal{A}}$, and compact-valued upper semicontinous functions by $USC_{\mathcal{K}}$. We denote closed-valued weakly continuous functions by $C_{\mathcal{A}}$ and compact-valued continuous functions by $C_{\mathcal{K}}$.

If $F \in LSC_{\mathcal{A}}$, then $F(\mathrm{cl}(A)) \subset \mathrm{cl}(F(A))$ for any set A, and therefore $\mathrm{cl}(G \circ F(x)) = \mathrm{cl}(G(\mathrm{cl}(F(x))))$. If $F \in USC_{\mathcal{A}}$, then $F(C)$ is closed whenever C is compact, and $F \in USC_{\mathcal{K}}$, then $F(C)$ is compact whenever C is compact, but in both cases $F(A)$ need not be closed even if A is closed. If $F \in USC_{\mathcal{A}}$ if, and only if, $\mathrm{Graph}(F)$ is closed.

Upper-semicontinuity with compact values is preferable to weak upper-semicontinuity with closed values, since (strong) upper-semicontinuity is preserved under composition.

For a closed-valued lower-semicontinuous function F, the image $F(A)$ need not be closed even if A is closed. This means that the composition $(F, G) \mapsto F \circ G$ need not be closed-valued. We therefore take a *closed-valued composition* $(F, G) \mapsto \mathrm{cl}(F \circ G)$ defined by $\mathrm{cl}(F \circ G)\,(x) := \mathrm{cl}(F(G(x)))$.

For more information on multivalued functions, see Klein and Thompson [14].

3.1 Topology of multivalued semicontinuous functions

To define topologies on the spaces of closed-valued (semi)continuous maps, we identify $LSC_{\mathcal{A}}(X \rightrightarrows Y)$ with $C(X \to \mathcal{A}_<(Y))$, $USC_{\mathcal{A}}(X \rightrightarrows Y)$ with $C(X \to \mathcal{A}_>(Y))$ and $C_{\mathcal{A}}(X \rightrightarrows Y)$ with $C(X \to \mathcal{A}(Y))$, and use the compact-open topologies. Explicit generators for the topologies $\tau_<^{\mathcal{MA}}$ on $LSC_{\mathcal{A}}$ and $\tau_>^{\mathcal{MA}}$ on $LSC_{\mathcal{K}}$ are given by

$$\begin{aligned} \sigma_<^{\mathcal{MA}} &:= \big\{\{F \in LSC_{\mathcal{A}} : \overline{I} \subset F^{-1}(J)\} : I \in \beta_X,\ J \in \beta_Y\big\}, \\ \sigma_>^{\mathcal{MA}} &:= \big\{\{F \in USC_{\mathcal{A}} : \overline{I} \cap F^{-1}(\overline{J}) = \emptyset\} : I \in \beta_X,\ J \in \beta_Y\big\}. \end{aligned} \tag{13}$$

Note that $\overline{I} \subset F^{-1}(J) \iff \forall x \in \overline{I},\ F(x) \cap J \neq \emptyset$, and that $\overline{I} \cap F^{-1}(\overline{J}) = \emptyset \iff F(\overline{I}) \cap \overline{J} = \emptyset$.

The lower-semicontinuous functions $LSC_{\mathcal{K}}(X \rightrightarrows Y)$ are somewhat degenerate, and have no natural topology other than that induced from $LSC_{\mathcal{A}}(X \rightrightarrows Y)$. To define topologies on the spaces of compact-valued (semi)continuous maps, we identify $USC_{\mathcal{K}}(X \rightrightarrows Y)$ with $C(X \to \mathcal{K}_>(Y))$ and $C_{\mathcal{K}}(X \rightrightarrows Y)$ with $C(X \to \mathcal{K}(Y))$, and again use the compact-open topologies. An explicit generator for the topology $\tau_>^{\mathcal{MK}}$ on $USC_{\mathcal{K}}$ is

$$\sigma_>^{\mathcal{MK}} := \Big\{\{F \in USC_{\mathcal{K}} : \overline{I} \subset F^{\Leftarrow}(\textstyle\bigcup_{i=1}^k J_i)\} : I \in \beta_X,\ J_1, \ldots J_k \in \beta_Y\Big\}. \tag{14}$$

Note that $\overline{I} \subset F^{\Leftarrow}(\bigcup_{i=1}^k J_i) \iff F(\overline{I}) \subset \bigcup_{i=1}^k J_i$.

3.2 Representations of multivalued semicontinuous functions

We now define representations $\mu_<$ for lower-semicontinuous maps, $\mu_>^{\mathcal{A}}$ for weakly upper-semicontinuous maps, and $\mu_>^{\mathcal{K}}$ for upper-semicontinuous compact-valued maps.

Admissible representations for $\tau_<^{\mathcal{MA}}$, $\tau_>^{\mathcal{MA}}$ and $\tau^{\mathcal{MA}}$ are given by

$$\begin{aligned} \mu_<^{\mathcal{A}}(p) = F \in LSC_{\mathcal{A}} \quad &:\iff \quad \{(\nu_X(v), \nu_Y(w)) : \langle v, w\rangle \lhd p\} \\ &\qquad\qquad = \{(I, J) \in \beta_X \times \beta_Y : \overline{I} \subset F^{-1}(J)\}, \\ \mu_>^{\mathcal{A}}(p) = F \in USC_{\mathcal{A}} \quad &:\iff \quad \{(\nu_X(v), \nu_Y(w)) : \langle v, w\rangle \lhd p\} \\ &\qquad\qquad = \{(I, J) \in \beta_X \times \beta_Y : \overline{I} \cap F^{-1}(\overline{J}) = \emptyset\} \\ \mu^{\mathcal{A}}\langle p, q\rangle = F \in C_{\mathcal{A}} \quad &:\iff \quad \mu_<^{\mathcal{A}}(p) = \mu_>^{\mathcal{A}}(q) = F. \end{aligned} \tag{15}$$

Note that $\mu_<^{\mathcal{A}}$ encodes a list of all pairs $(\overline{I}, J)$ with $I \in \beta_X$, $J \in \beta_Y$ such that $\overline{I} \subset F^{-1}(J)$ (equivalently, $\forall x \in \overline{I},\ F(x) \cap J \neq \emptyset$), and $\mu_>^{\mathcal{A}}$ encodes a list of all pairs $(\overline{I}, \overline{J})$ with $I \in \beta_X$, $J \in \beta_Y$ such that $F(\overline{I}) \cap \overline{J} = \emptyset$.

An admissible representation for compact-valued upper-semicontinous functions is given by

$$\begin{aligned} \mu_>^{\mathcal{K}}(p) = F \in USC_{\mathcal{K}} \quad &:\iff \quad \{(\nu_X(v), \nu_Y(w_1), \ldots, \nu_Y(w_k)) : \langle v, w_1, \ldots, w_k\rangle \lhd p\} \\ &\qquad\qquad = \{(I, J_1, \ldots, J_k) : \overline{I} \subset F^{-1}(\textstyle\bigcup_{i=1}^k J_i)\}. \end{aligned} \tag{16}$$

Note that $\mu_>^{\mathcal{K}}$ encodes a list of all tuples $(\overline{I}, J_1, \ldots, J_k)$ such that $F(\overline{I}) \subset \bigcup_{i=1}^k J_i$.

The following result on representations is immediate from the definitions.

Lemma 3.1.

1. *The representations* $\mu_<^{\mathcal{A}}$ *of* $LSC_{\mathcal{A}}(X \rightrightarrows Y)$ *and* δ^{co} *of* $C(X \to \mathcal{A}_<(Y))$ *are equivalent.*
2. *The representations* $\mu_>^{\mathcal{A}}$ *of* $USC_{\mathcal{A}}(X \rightrightarrows Y)$*,* δ^{co} *of* $C(X \to \mathcal{A}_>(Y))$*, and* $\psi_>$ *of* Graph(F) *are equivalent.*
3. *The representations* $\mu_>^{\mathcal{K}}$ *of* $USC_{\mathcal{A}}(X \rightrightarrows Y)$ *and* δ^{co} *of* $C(X \to \mathcal{K}_>(Y))$ *are equivalent.*

For single-valued maps, the situation is simpler.

Lemma 3.2. *The representations* δ^{co}*,* $\mu_<^{\mathcal{A}}$*,* $\mu_>^{\mathcal{K}}$*, and* $\mu^{\mathcal{K}}$ *are equivalent representations for* $C(X \to Y)$*.*

Proof. The representations δ^{co} and $\mu_<^{\mathcal{A}}$ are trivially equivalent, since $f(C) \subset U$ if, and only if, $\forall x \in C$, $f(x) \cap U \neq \emptyset$. We need then only show that $\mu_<^{\mathcal{A}}$ and $\mu_>^{\mathcal{K}}$ are equivalent, since the other equivalences follow by definition.

$\mu_<^{\mathcal{A}} \leqslant \mu_>^{\mathcal{K}}$:
We need to compute a list of all $(\overline{K}, L_1, \ldots, L_l)$ with $f(\overline{K}) \subset \bigcup_{j=1}^{l} L_j$ from a list of all $(\overline{I}, J)$ with $f(\overline{I}) \subset J$. We claim that an algorithm which outputs $(\overline{K}, L_1, \ldots, L_l)$ if there exists a finite set $\{(I_i, J_i) : i = 1 \ldots k\}$ with $F(\overline{I}_i) \subset J_i$ for $i = 1, \ldots k$ such that $\overline{K} \subset \bigcup_{i=1}^{k} I_i$ and $\overline{J}_i \subset \bigcup_{j=1}^{l} L_j$ (covering) for all $i = 1, \ldots, k$ performs the calculation.

If $(\overline{K}, L_1, \ldots, L_l)$ is output, then $F(\overline{I}_i) \subset J_i$, $\overline{K} \subset \bigcup_{i=1}^{k} I_i$ and $\overline{J}_i \subset \bigcup_{j=1}^{l} L_j$, so $F(\overline{K}) \subset \bigcup_{j=1}^{l} L_j$.

If $F(\overline{K}) \subset \bigcup_{j=1}^{l} L_j$, then every $x \in \overline{K}$ has a neighbourhood I_x such that $F(\overline{I}_x) \subset J_x$ with $\overline{J}_x \subset \bigcup_{j=1}^{l} L_j$. Since $\overline{K}$ is compact, there is a finite subset $\{x_i : i = 1, \ldots, k\}$ with $\overline{K} \subset \bigcup_{i=1}^{k} I_{x_i}$. Hence $(\overline{K}, L_1, \ldots, L_l)$ is output.

$\mu_>^{\mathcal{K}} \leqslant \mu_<^{\mathcal{A}}$:
We need to compute a list of all $(\overline{I}, J)$ with $f(\overline{I}) \subset J$ from a list of all $(\overline{K}, L_1, \ldots, L_l)$ such that $f(\overline{K}) \subset \bigcup_{j=1}^{l} L_j$. To do this, we simply output $(\overline{I}, J) = (\overline{K}, L_1)$ if $F(\overline{K}) \subset \bigcup_{j=1}^{l} L_j$ with $l = 1$. □

3.3 Counterexamples for multivalued functions

We now give some examples illustrating counterexamples for multivalued semicontinuous functions.

The following example shows that a map $F : X \to \mathcal{A}_>(Y)$ may have $F(x)$ compact for all $x \in X$, but not be continuous as a map $F : X \to \mathcal{K}_>(Y)$.

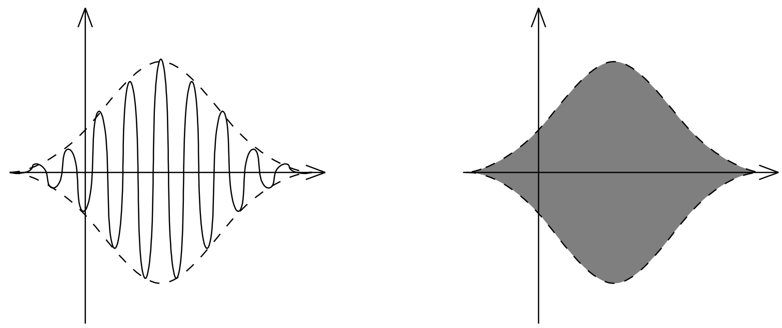
Example 3.3. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $F(x) = \{0\}$ if $x \leqslant 0$, and $F(x) = \{0, 1/x\}$ if $x > 0$. Then $F : \mathbb{R} \to \mathcal{A}_>(\mathbb{R})$ is continuous, but $F^{-1}(-1, 1) = (-\infty, 0] \cup (1, \infty)$ which is not open, and $F^{\Leftarrow}[1, \infty) = (0, 1]$ which is not closed, so is not upper-semicontinuous with compact values.

If $G(x) = \{0\}$ if $x < 1$, and $G(x) = \{0, 1\}$ for $x \geqslant 1$, then G is upper semicontinuous, but $G \circ F(x) = \{0\}$ if $x \in (-\infty, 0] \cup (1, \infty)$ and $G \circ F(x) = \{0, 1\}$ if $x \in (0, 1]$, so $G \circ F$ is not upper semicontinuous.

Rather than consider compact-valued maps, we could consider, with more generality, closed-valued maps. However, the composition of two closed-valued upper-semicontinuous maps need not be upper-semicontinuous, as Example 3.4 shows.

Example 3.4. Let $F(x) = \{0, 1/x\}$ for $x > 0$, and $F(0) = \{0\}$. Let $G(x) = \{0, 1\}$ if $x \geqslant 1$ and $G(x) = \{0\}$ if $x < 1$. Then $G \circ F(x) = \{0\}$ if $x = 0$ or $x \geqslant 1$, and $G \circ F(x) = \{0, 1\}$ if $0 < x \leqslant 1$. Hence $G \circ F$ is not upper-semicontinuous.

We could also consider the representation $\psi_<$ of Graph(F) on $\mathcal{A}(X \times Y)$ as a lower representation for $USC(X \rightrightarrows Y)$. It is straightforward to show that $\mu_< \leqslant \psi_<$ on $USC(X \rightrightarrows Y)$. However, a $\psi_<$ is strictly weaker than $\mu_<$, even for continuous functions, as the following example shows.



(a) F_n (b) F

Figure 2: The limit of a continuous multivalued map may exist in the graph topology but not the compact-open topology. (a) F_n, (b) the limit F.

Example 3.5. Let $g(x) : [-1,1] \to [-1,1]$ be continuous, and let $F_n(x) = \{g(x)\sin(nx)\}$. Then in the $\tau^{\mathcal{A}}$ topology on $\mathcal{A}(X \times Y)$, $\mathrm{Graph}(F_n) \to \{(x,y) : |y| \leqslant |g(x)|\} = \mathrm{Graph}(F)$, a continuous multivalued map, but F_n does not converge in the compact-open topology $\tau^{\mathcal{M}}_{<}$ on multivalued maps, since if C is compact and $U = (0,1)$, then (C,U) is a pair such that $\forall x \in C$, $F(x) \cap U \neq \emptyset$ and $U \subset (0,1)$, then for sufficiently large n, $\exists x_n \in C$ with $\sin(nx_n) < 0$, and then $F_n(x_n) \cap U = \emptyset$. Hence F_n does not converge to F.

3.4 Composition of multivalued maps

We now show that composition of multivalued maps, where continuous, is computable in the appropriate representation.

Theorem 3.6.

1. *The closed composition function* $(F,G) \mapsto \mathrm{cl}(F \circ G)$ *is* $(\mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<}; \mu^{\mathcal{A}}_{<})$*-computable.*
2. *The composition function* $(F,G) \mapsto F \circ G$ *is* $(\mu^{\mathcal{A}}_{>}, \mu^{\mathcal{K}}_{>}; \mu^{\mathcal{A}}_{>})$*-computable and* $(\mu^{\mathcal{A}}, \mu^{\mathcal{K}}; \mu^{\mathcal{A}})$*-computable.*
3. *The composition function* $(F,G) \mapsto F \circ G$ *is* $(\mu^{\mathcal{K}}_{>}, \mu^{\mathcal{K}}_{>}; \mu^{\mathcal{K}}_{>})$*-computable and* $(\mu^{\mathcal{K}}, \mu^{\mathcal{K}}; \mu^{\mathcal{K}})$*-computable.*

Proof. $(F,G) \mapsto \mathrm{cl}(F \circ G)$ is $(\mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<}; \mu^{\mathcal{A}}_{<})$-computable:
Output $(\overline{I}, K)$ if there exists a finite set $\{(I_i, J_i) : i = 1 \ldots k\}$ such that $I \subset \bigcup_{i=1}^{k} I_i$ (covering), $\overline{I}_i \subset G^{-1}(J_i)$, and $\overline{J}_i \subset F^{-1}(K)$.

If $(\overline{I}, K)$ is output, then $\forall x \in \overline{I}$, $\exists i$ with $x \in I_i$. Then $G(x) \cap J_i \neq \emptyset$, so $\exists y \in G(x) \cap J_i$, and $F(y) \cap K \neq \emptyset$ since $y \in \overline{J}_i$, so $F \circ G(x) \cap K \neq \emptyset$.

Conversely, if $\overline{I} \subset (F \circ G)^{-1}(K)$, then $\forall x \in \overline{I}$, $\exists y \in Y,\ z \in K$ with $y \in G(x)$ and $z \in F(y)$. Hence by lower-semicontinuity, $\exists J_x$ such that $y \in J_x$ and $z \in F(\overline{J}_x)$, so $\overline{J}_x \subset F^{-1}(K)$. Similarly, $\exists I_x$ such that $x \in I_x$ and $\overline{I}_x \subset G^{-1}(J_x)$. Since $\overline{I}$ is compact, there is a finite subset $\{x_i : i = 1, \ldots, k\}$ with $\overline{I} \subset \bigcup_{i=1}^{k} I_{x_i}$. Hence $(\overline{I}, K)$ is an output.

$(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{A}}_{>}, \mu^{\mathcal{K}}_{>}; \mu^{\mathcal{A}}_{>})$-computable:
Output $(\overline{I}, \overline{K})$ if there exists a finite set $\{J_1, \ldots, J_k\}$ such that $G(\overline{I}) \subset \bigcup_{i=1}^{k} J_i$ and $F(\overline{J}_i) \cap \overline{K} = \emptyset$ for $i = 1, \ldots, k$.

If $(\overline{I}, \overline{K})$ is output, then $G(F(\overline{I})) \subset G(\bigcup_{i=1}^{k}(J_i)) \subset \bigcup_{i=1}^{k} G(\overline{J}_i)$, so $G \circ F(\overline{I}) \cap \overline{K} = \emptyset$.

Conversely, suppose $F(G(\overline{I})) \cap \overline{K} = \emptyset$. Since F is weakly upper-semicontinuous, $F^{-1}(\overline{K})$ is closed, and so $V = F^{\Leftarrow}(Z \setminus \overline{K})$ is open. Hence $G(\overline{I}) \subset V$, and since G is compact-valued upper-semicontinuous, $G(\overline{I})$ is compact. Thus there exist $J_1, \ldots, J_k$ such that $G(\overline{I}) \subset \bigcup_{i=1}^{k} J_i$ and $\overline{J}_i \subset V$ for $i = 1, \ldots, k$. Hence $F(\overline{J}_i) \cap \overline{K} = \emptyset$ for $i = 1, \ldots, k$, and so $(\overline{I}, \overline{K})$ is output.

$(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{A}}, \mu^{\mathcal{K}}; \mu^{\mathcal{A}})$-computable:
Immediate since $(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<})$-computable and $(\mu^{\mathcal{A}}_{>}, \mu^{\mathcal{K}}_{>}, \mu^{\mathcal{A}}_{>})$-computable.

$(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{K}}_{>}, \mu^{\mathcal{K}}_{>}; \mu^{\mathcal{K}}_{>})$-computable:
Output $(\overline{I}, K_1, \ldots K_k)$ if $\exists\, J_1, \ldots J_m$ such that $G(\overline{I}) \subset \bigcup_{j=1}^{m} J_j$ and $F(\overline{J}_j) \subset \bigcup_{i=1}^{k} K_i$ for $j = 1, \ldots, m$.

If $(\overline{I}, K_1, \ldots, K_k)$ is output, then $F \circ G(\overline{I}) \subset F(\bigcup_{j=1}^{m} J_j)$ and $F(\overline{J}_j) \subset \bigcup_{i=1}^{k} K_i$ for all j, so $F \circ G(\overline{I}) \subset \bigcup_{i=1}^{k} K_i$.

Conversely, if $F \circ G(\overline{I}) \subset \bigcup_{i=1}^{k} K_i$, then $F(y) \subset \bigcup_{i=1}^{k} K_i$ for all $y \in G(\overline{I})$. By upper-semicontinuity, for each $y \in G(\overline{I})$, there exists J_y such that $F(\overline{J}_y) \subset \bigcup_{i=1}^{k} K_i$, and since $G(\overline{I})$ is compact, there is a finite subset $\{y_1, \ldots y_m\}$ of $G(\overline{I})$ such that $G(\overline{I}) \subset \bigcup_{j=1}^{m} J_{y_j}$. Then $G(\overline{I}) \subset \bigcup_{j=1}^{m} J_{y_j}$ and $F(\overline{J}_{y_j}) \subset \bigcup_{i=1}^{k} K_i$ for $j = 1, \ldots m$. Hence $(\overline{I}, K_1, \ldots, K_k)$ is output.

$(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{K}}, \mu^{\mathcal{K}}; \mu^{\mathcal{K}})$-computable:
Immediate since $(F,G) \mapsto F \circ G$ is $(\mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<}, \mu^{\mathcal{A}}_{<})$-computable and $(\mu^{\mathcal{K}}_{>}, \mu^{\mathcal{K}}_{>}, \mu^{\mathcal{K}}_{>})$-computable. □

A closed set A can be considered as a function from a one-point space $\underline{1}$ to A. Then the representations $\psi_{<}$, $\psi_{>}$ and ψ of $\mathcal{A}(X)$ are equivalent, respectively, to $\mu^{\mathcal{A}}_{<}$, $\mu^{\mathcal{A}}_{>}$ and $\mu^{\mathcal{A}}$ of $C(\underline{1} \rightrightarrows X)$. Similarly, the representations $\kappa_{<}$, $\kappa_{>}$ and κ of $\mathcal{K}(X)$ are equivalent, respectively, to $\mu^{\mathcal{K}}_{<}$, $\mu^{\mathcal{K}}_{>}$ and $\mu^{\mathcal{K}}$ of $C_{\mathcal{K}}(\underline{1} \rightrightarrows X)$. This gives the following

Corollary 3.7.

1. *The function $(F,A) \mapsto \mathrm{cl}(F(A))$ is $(\mu^{\mathcal{A}}_{<}, \psi_{<}; \psi_{<})$-computable.*
2. *The function $(F,C) \mapsto F(C)$ is $(\mu^{\mathcal{A}}_{>}, \kappa_{>}; \psi_{>})$-computable and $(\mu^{\mathcal{A}}, \kappa; \psi)$-computable.*
3. *The function $(F,C) \mapsto F(C)$ is $(\mu^{\mathcal{K}}_{>}, \kappa_{>}; \kappa_{>})$-computable and $(\mu^{\mathcal{K}}, \kappa; \kappa)$-computable.*

If F is an upper-semicontinuous map, then $F(A)$ need not be closed even if A is closed. We can consider the composition function $(F,A) \mapsto \mathrm{cl}(F(A))$ for $F \in \mathcal{USC}_{\mathcal{K}}$ and $A \in \mathcal{A}$, and attempt to compute a $\psi_{>}$-name of $\mathrm{cl}(F(A))$. However, the following result shows that this is impossible.

Theorem 3.8. *The function $(F,A) \mapsto \mathrm{cl}(F(A))$ is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{A}}; \tau^{\mathcal{A}}_{>})$-continuous.*

Proof. Let $A = [0, \infty)$, and $F_a(x) = \{0\}$ if $x \notin [a-1, a+1]$, $F_a(x) = [0, x-a+1]$ if $x \in [a-1, a]$ and $F_a(x) = [0, a+1-x]$ if $x \in [a, a+1]$. Then $F_a \to F$ given by $F(x) = \{0\}$ as $a \to \infty$, but $F_a(A) = [0,1]$ which does not converge to $F(A) = \{0\}$. Hence $(F,A) \mapsto \mathrm{cl}(F(A))$ is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{A}}; \tau^{\mathcal{A}}_{>})$-continuous. □

4 Reachability problems

We now apply the material developed in Section 3 to the study of the reachability problem for semicontinuous systems. We first define the reachable, closed-reachable and chain-reachable sets, and give an alternative formulation of the chain reachable set. We then prove some straightforward results on computability of countable unions and intersections, and use these to prove the main results on reachability. Finally, we discuss *closure-interior systems*, which have inner as well as outer approximations, and show that the computability results extend to these systems as well.

4.1 Reachable and chain reachable sets

Definition 4.1 (Reachability). Let $F : X \rightrightarrows X$ be a multivalued map, and $X_0 \subset X$. Then the *reachable set* of F from X_0 is

$$\begin{aligned}\mathrm{Reach}(F, X_0) \quad := \quad \{y \in X : \exists\, x_0, x_1, \ldots x_n \text{ such that} \\ x_0 \in X_0,\ (x_i, x_{i+1}) \in F \text{ for } i = 0, \ldots, n-1, \text{ and } x_n = y.\end{aligned} \tag{17}$$

The reachable set need not be closed, so we take its closure, and define the *closed reachable set* as

$$\overline{\mathrm{Reach}}(F, X_0) := \mathrm{cl}(\mathrm{Reach}(F, X_0)). \tag{18}$$

We now briefly recall the concepts of ϵ-chains as considered by Conley [10]. If (X, d) is a metric space and $F : X \rightrightarrows X$ is a multivalued map, then a sequence of points $x_0, x_1, \ldots, x_n$ is an *ϵ-chain* if there exist $y_1, \ldots, y_n \in X$ with $y_{i+1} \in F(x_i)$ and $d(y_{i+1}, x_{i+1}) < \epsilon$ for $i = 0, \ldots, n-1$. A point x is *ϵ-reachable* from a set X_0 if there is an ϵ-chain $x_0, x_1, \ldots, x_n$ with $x_0 \in X_0$ and $x_n = x$. A point x is *chain-reachable* from X_0 if there is an ϵ-chain from X_0 to x for all $\epsilon > 0$.

The concept of chains can be generalised to non-metric spaces as follows:

Definition 4.2 ($\mathcal{U}$-chain). Let $\mathcal{U}$ be an open cover, and $F : X \rightrightarrows X$. A sequence $x_0, \ldots, x_n$ is a *$\mathcal{U}$-chain* for F if there exist points $y_1, \ldots, y_n \in X$ and open sets $U_1, \ldots, U_n \in \mathcal{U}$ such that $y_{i+1} \in F(x_i)$ and $x_{i+1}, y_{i+1} \in U_{i+1}$ for $i = 0, \ldots, n-1$.

Equivalently, we can define the $\mathcal{U}$-neighbourhood of a set B by $N_{\mathcal{U}}(B) := \bigcup\{U \in \mathcal{U} : B \cap U \neq \emptyset\}$. Then a sequence $x_0, \ldots, x_n$ is a $\mathcal{U}$-chain for F if, and only if, $x_{i+1} \in N_{\mathcal{U}}(F(x_i))$ for $i = 0, \ldots, n-1$.

Definition 4.3 (Chain reachability). Let $F : X \rightrightarrows X$ be a multivalued map, and $X_0 \subset X$. Define

$$\mathrm{Reach}(F, X_0, \mathcal{U}) := \{x \in X : \exists\ \mathcal{U}\text{-chain } x_0, x_1, \ldots, x_n \text{ for } F \text{ such that } x_0 \in X_0 \text{ and } x_n = x\} \tag{19}$$

the set of points reachable from X_0 by a $\mathcal{U}$-chain. The *chain reachable set* of F from X_0 is

$$\mathrm{ChainReach}(F, X_0) := \textstyle\bigcap_{\mathcal{U}} \mathrm{Reach}(F, X_0, \mathcal{U}), \tag{20}$$

where $\mathcal{U}$ runs over all locally finite open covers of X.

It is straightforward to show [10] that $\mathrm{ChainReach}(F, X_0)$ is closed for any system F and any initial set X_0. An equivalent definition of the chain reachable set of an upper-semicontinuous closed-valued function can be given in terms of graphs.

$$\mathrm{ChainReach}(F, X_0) = \bigcap\{\mathrm{Reach}(G, X_0) : G \in LSC_{\mathcal{O}} \text{ and } \mathrm{Graph}(F) \subset \mathrm{Graph}(G)\} \tag{21}$$

We now give an alternative characterisation of the chain reachable set which will be useful when performing a computability analysis. We use the following lemma on compact chain-reachable sets.

Lemma 4.4. *If* $\mathrm{ChainReach}(F, C)$ *is compact, then for any open neighbourhood* U *of* $\mathrm{ChainReach}(F, C)$*, there exists an open cover* $\mathcal{U}$ *such that* $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U})) \subset U$*. In particular, there exists an open cover* $\mathcal{U}$ *such that* $\mathrm{Reach}(F, C, \mathcal{U})$ *is pre-compact.*

Proof. Suppose $\mathrm{ChainReach}(F, C)$ is compact, and let V be a pre-compact open neighbourhood of $\mathrm{ChainReach}(F, C)$ such that $\mathrm{cl}(V) \subset U$. Then since F is upper-semicontinuous and $F(\mathrm{ChainReach}(F, C)) \subset \mathrm{ChainReach}(F, C)$, we see that $F^{\Leftarrow}(V)$ is an open neighbourhood of $\mathrm{ChainReach}(F, C)$. Hence there is an open neighbourhood W of $\mathrm{ChainReach}(F, C)$ such that $\mathrm{cl}(W) \subset V$ and $F(\mathrm{cl}(W)) \subset V$. Choose an open cover $\mathcal{V}$ such that $N_{\mathcal{V}}(F(\mathrm{cl}(W))) \subset V$, and let $B = \mathrm{cl}(V) \setminus W$, a compact set. Now if $\mathcal{U}$ is any refinement of $\mathcal{V}$, then either $\mathrm{Reach}(F, C, \mathcal{U}) \subset W$, or there exists a $\mathcal{U}$-chain $x_0, x_1, \ldots, x_n$ with $x_0 \in C$, $x_i \in W$ for $i < n$ and $x_n \notin W$. Then $x_n \in N_{\mathcal{U}}(F(x_{n-1})) \subset N_{\mathcal{V}}(F(\mathrm{cl}(W))) \subset V$, so $x_n \in B$, and hence $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U})) \cap B \neq \emptyset$. Since $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U}))$ decreases on taking refinements, and converges to $\mathrm{ChainReach}(F, C, \mathcal{U})$, we must have $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U})) \cap B = \emptyset$ for some $\mathcal{U}$. Then $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U})) \subset W$, so $\mathrm{cl}(\mathrm{Reach}(F, C, \mathcal{U})) \subset U$ and $\mathrm{Reach}(F, C, \mathcal{U})$ is pre-compact. □

Theorem 4.5 (Characterisation of the chain-reachable set). *Let* $F \in USC_{\mathcal{K}}$ *and* C *a compact set. Suppose* $\mathrm{ChainReach}(F, C)$ *is compact. Then*

$$\mathrm{ChainReach}(F, C) = \bigcap\{U \in \mathcal{O}(X) : C \subset U \text{ and } F(\mathrm{cl}(U)) \subset U\}. \tag{22}$$

Proof. We first show that for any neighbourhood V of $\mathrm{ChainReach}(F,C)$, there exists $U \subset V$ with $C \subset U$ and $F(\mathrm{cl}(U)) \subset U$. For any open cover $\mathcal{U}$, we have $C \subset \mathrm{Reach}(F,C,\mathcal{U})$ and $\mathrm{cl}(F(\mathrm{Reach}(F,C,\mathcal{U}))) \subset \mathrm{Reach}(F,C,\mathcal{U})$. By Lemma 4.4, if V is any open neighbourhood of $\mathrm{ChainReach}(F,C)$, then there is an open cover $\mathcal{U}$ such that $\mathrm{cl}(\mathrm{Reach}(F,C,\mathcal{U})) \subset V$. Hence there is an open set U such that $C \cup \mathrm{cl}(F(\mathrm{Reach}(F,C,\mathcal{U}))) \subset U$ and $\mathrm{cl}(U) \subset \mathrm{Reach}(F,C,\mathcal{U})$. Then $U \subset V$, $C \subset U$ and $F(\mathrm{cl}(U)) \subset F(\mathrm{Reach}(F,C,\mathcal{U})) \subset \mathrm{cl}(F(\mathrm{Reach}(F,C,\mathcal{U}))) \subset U$ as required.

To complete the proof, we let U be such that $C \subset U$ and $F(\mathrm{cl}(U)) \subset U$, and need to show that $\mathrm{ChainReach}(F,C) \subset U$. We have $N_{\mathcal{U}}(F(\mathrm{cl}(U))) \subset U$ for some open cover $\mathcal{U}$. Defining sets X_n recursively by $X_{n+1} := N_{\mathcal{U}}(F(X_n))$, we see by induction that $X_n \subset U$ for all n, so $\mathrm{Reach}(F,C,\mathcal{U}) \subset U$ and hence $\mathrm{ChainReach}(F,C) \subset U$. □

To consider computability of the reachable and chain reachable sets, we reformulate the reachability conditions as operators. The closed reachability operator naturally operates on lower-semicontinuous maps, and the chain reachability operator on upper-semicontinuous maps.

Definition 4.6 (Reachability operators).

1. The *closed reachability operator* is the function $\overline{\mathrm{Reach}} : LSC_{\mathcal{A}}(X \rightrightarrows X) \times \mathcal{A}(X) \to \mathcal{A}(X)$ given by $\overline{\mathrm{Reach}}(F,A) := \mathrm{cl}(\mathrm{Reach}(F,A))$.
2. The *chain reachability operator* is the function $\mathrm{ChainReach} : USC_{\mathcal{K}}(X \rightrightarrows X) \times \mathcal{A}(X) \to \mathcal{A}(X)$ given by $\mathrm{ChainReach}(F,A) := \bigcap_{\mathcal{U}} \mathrm{Reach}(F,A,\mathcal{U})$, where $\mathcal{U}$ runs over all locally-finite open covers.

The following example shows that the chain reachability operator may be badly behaved if the chain-reachable set is not compact.

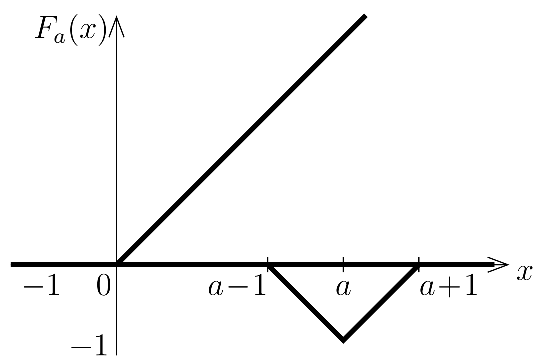


Figure 3: The map F_a of Example 4.7.

Example 4.7. Define continuous multivalued maps $F : \mathbb{R} \rightrightarrows \mathbb{R}$ and $F_a : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) := \begin{cases} \{0\} & \text{if } x \leqslant 0, \\ \{0,x\} & \text{if } x \geqslant 0 \end{cases} \qquad F_a(x) := \begin{cases} F(x) \cup \{a-1-x\} & \text{if } x \in [a-1,a], \\ F(x) \cup \{x-a-1\} & \text{if } x \in [a,a+1], \\ F(x) & \text{otherwise.} \end{cases} \tag{23}$$

The graph of F_a is shown in Figure 3. Note that $F_a \to F$ as $a \to \infty$ in $\tau^{\mathcal{MK}}$, since for any compact set C, $F_a|_C = F|_C$ for a sufficiently large, and that $F(x) \subset F_a(x)$ for all x.

Let $X_0 = \{0\}$, and consider chain reachable sets $\mathrm{ChainReach}(F,X_0)$. We have $\mathrm{ChainReach}(F,\{0\}) = [0,\infty)$, since we can reach any point in $[0,\infty)$ from 0 by an ϵ-chain (x_i) by taking $y_{i+1} = x_i$ as $x \in F(x)$ for all x, and $x_{i+1} > y_i$. Since $F_a(x) \supset F(x)$ for any x, we must have $\mathrm{ChainReach}(F_a,\{0\}) \supset \mathrm{ChainReach}(F,\{0\})$ for any a. Hence $[a-1,a+1] \subset \mathrm{ChainReach}(F_a,\{0\})$, and so $[-1,0] \subset \mathrm{ChainReach}(F_a,X_0)$, since $F_a([a-1,a+1]) \subset [-1,0]$. Thus $\mathrm{ChainReach}(F_a,\{0\}) = [-1,\infty)$ for any a.

We therefore have a situation in which $F_a \to F$ in $\mu^{\mathcal{K}}$ as $a \to \infty$, but $\mathrm{ChainReach}(F_a,\{0\})$ does not converge to $\mathrm{ChainReach}(F,\{0\})$ as $a \to \infty$ in $\tau_{>}^{\mathcal{A}}$.

4.2 Computability of reachable sets

We now consider the computability of the closed reachability operator and the chain reachability operator. We find that the closed reachability operator is lower-computable in all cases, and the chain reachability operator is upper-computable if the chain-reachable set is compact. Using these results, we can obtain semi-decision algorithms for verification of system properties.

Theorem 4.8 (Computability of closed reachability).

1. *The closed reachability operator for lower-semicontinuous discrete-time systems is* $(\mu_{<}^{\mathcal{A}}, \psi_{<}; \psi_{<})$*-computable.*
2. *The closed reachability operator for bounded discrete-time systems is not* $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau_{>}^{\mathcal{K}})$*-continuous.*

Proof. 1. Since $(F, A) \mapsto \mathrm{cl}(F(A))$ is $(\mu_{<}^{\mathcal{A}}, \psi_{<}, \psi_{<})$-computable, and the function $(F, A) \mapsto A_i := \mathrm{cl}(F^i(A))$ is $(\mu_{<}^{\mathcal{A}}, \psi_{<}, \psi_{<})$-computable for all $i \in \mathbb{N}$. Since $\overline{\mathrm{Reach}}(F, A) := \mathrm{cl}(\bigcup_{i=0}^{\infty} F^i(A)) = \mathrm{cl}(\bigcup_{i=0}^{\infty} A_i)$, and countable closed union is $(\psi_{<}, \psi_{<}, \ldots; \psi_{<})$-computable, the result follows.

2. Consider the system f_ϵ defined in Section 1.2. Then $f_\epsilon \to f_0$ in $\tau^{\mathcal{MK}}$, and $\{q_-(\epsilon)\} \to \{q_-(0)\}$ in $\tau^{\mathcal{K}}$, but $\overline{\mathrm{Reach}}(f_\epsilon, [q_-(\epsilon), 0]\}) = [q_-(\epsilon), q_+(\epsilon)]$ for $\epsilon > 0$, which does not converge to $[q_-(0), 0] = \overline{\mathrm{Reach}}(F_0, [q_-(0), 0])$ in $\tau_{>}^{\mathcal{K}}$.

□

We can Theorem 4.8(1) to verify system controllability. Suppose we wish to check whether it is possible to reach an open set U starting from some initial point x. We compute a $\psi_{<}$-name of Reach$(F, \{x\})$, and verify controllability if the $\psi_{<}$-name contains some set J with $\overline{J} \subset U$. If the set is not reachable, then the procedure does not terminate.

Theorem 4.9 (Computability of chain reachability).

1. *If* ChainReach(F, C) *is compact, then* $(F, C) \mapsto$ ChainReach(F, C) *is* $(\mu_{>}^{\mathcal{K}}, \kappa_{>}; \kappa_{>})$*-computable.*
2. *The map* $(F, C) \mapsto$ ChainReach(F, C) *is not* $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}, \tau_{>}^{\mathcal{A}})$*-continuous.*
3. *The map* $(F, C) \mapsto$ ChainReach(F, C) *is not* $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau_{<}^{\mathcal{A}})$*-continuous.*

Proof. 1. A $\kappa_{>}$ name of C encodes a list of all basic open covers of C. A $\mu_{>}^{\mathcal{K}}$-name of F encodes a list of all tuples $(\overline{I}, J_1, \ldots, J_k)$ such that $F(\overline{I}) \subset \bigcup_{i=1}^{k} J_i$ For each basic open cover $\{I_1, \ldots, I_k\}$ of C, we let $U = \bigcup_{j=1}^{k} I_j$. Then $F(\mathrm{cl}(U)) \subset U$ if, and only if, $F(\overline{I}_i) \subset \bigcup_{j=1}^{k} I_j$ for all i. Hence we can compute a list of all open U with $A \subset U$ such that $U = \bigcup_{j=1}^{k} I_j$ and $F(\mathrm{cl}(U)) \subset U$. By Theorem 4.5, the intersection of all such U equals ChainReach(F, A), hence we have computed a $\kappa_{>}$-name of ChainReach(F, A).

2. Consider the systems F_a of Example 4.7. Then $F_a \to F_\infty$ in $\tau^{\mathcal{MK}}$ as $a \to \infty$. However, ChainReach$(F_a, \{0\}) = [1, \infty)$ whereas ChainReach$(F_\infty, \{0\}) = [0, \infty)$, so ChainReach$(F_a, \{0\})$ does not converge to ChainReach$(F_\infty, \{0\})$ in $\tau_{>}^{\mathcal{A}}$. Hence $(F, C) \mapsto$ ChainReach(F, C) is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}, \tau_{>}^{\mathcal{A}})$-continuous. (A similar example can be made in two-dimensions with a single-valued continuous map.)

3. Consider the map f_ϵ defined in Example 1.2. Then $\{p_-(\epsilon)\} \to \{0\}$ in τ^{κ} as $\epsilon \to 0$. We have ChainReach$(f_\epsilon, \{p_-(\epsilon)\}) = \{p_-(\epsilon)\}$ for $\epsilon < 0$, and ChainReach$(f_0, \{0\}) = [0, q_+(\epsilon)]$. Hence ChainReach$(f_\epsilon, \{p_-(\epsilon)\})$ does not converge to ChainReach$(f_0, 0)$ in $\tau_{<}^{\mathcal{A}}$. Therefore $(F, C) \mapsto$ ChainReach(F, C) is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau_{<}^{\mathcal{A}})$-continuous.

□

We can use the chain reachable set to check safety properties of a system, that is, whether it is possible to leave an open set S of safe states starting from some initial set X_0. We compute a $\kappa_{>}$-representation of ChainReach(F, X_0), and verify safety if there exists some open cover $\{J_1, \ldots, J_k\}$ of ChainReach(F, X_0) such that the set $U := \bigcup_{i=1}^{k} J_i$ with ChainReach$(F, X_0) \subset U$ is a subset of S.

We say that reachable set is *robust* if $\overline{\mathrm{Reach}}(F, A) = \mathrm{ChainReach}(F, A)$. We have seen that we can compute inner and outer approximations to $\overline{\mathrm{Reach}}(F, A)$ if the reachable set is robust. The following result shows that this condition is sharp.

Theorem 4.10 (Uncomputability of reachability). *The closed reachable set is $(\mu^{\mathcal{K}}, \kappa; \kappa)$-computable if and only if it is robust.*

Proof. We have already shown that $\overline{\mathrm{Reach}}(F, A)$ is computable if $\overline{\mathrm{Reach}}(F, A) = \mathrm{ChainReach}(F, A)$.

Conversely, suppose $\overline{\mathrm{Reach}}(F, A) \neq \mathrm{ChainReach}(F, A)$. Let F_n be a sequence of continuous multivalued maps converging to F such that $\mathrm{Graph}(F) \subset \mathrm{int}(\mathrm{Graph}(F_n))$ for all n. Then $\mathrm{ChainReach}(F, A) \subset \mathrm{Reach}(F_n, A)$ for all n, and $\mathrm{Reach}(F_n, A) \to \mathrm{ChainReach}(F, A)$ as $n \to \infty$.

For any name p of F, there is a sequence p_n of names of F_n such that $p_n \to p$, since any elements in a $\mu^{\mathcal{A}}_{<}$-name of F are present in a $\mu^{\mathcal{A}}_{<}$-name of F_n, and for n sufficiently large, the first m elements of p give rise to sets disjoint from $\mathrm{Graph}(F)$. Any computation of the first m elements of $\psi_{>}$-name of $\overline{\mathrm{Reach}}(F, A)$ depends only on the first k elements of a ψ-name of F, and hence is equal to the first m elements of a $\psi_{>}$-name of $\overline{\mathrm{Reach}}(F_n, A)$ for n sufficiently large. In particular, the first m elements of a $\psi_{>}$-name of $\overline{\mathrm{Reach}}(F, A)$ are disjoint from $\mathrm{Reach}(F_n, A)$, and hence from $\mathrm{ChainReach}(F, A)$. Since this is true for any $m \in \mathbb{N}$, we see that it is impossible to compute a lower bound for $\overline{\mathrm{Reach}}(F_n, A)$ smaller than $\mathrm{ChainReach}(F, A)$. □

4.3 Closure-interior systems

A set which is the closure of its interior may be both inner- and outer-approximated.

Definition 4.11 (Closure-interior sets). A set A is a *closure-interior* or *clint* set if $A = \mathrm{cl}(\mathrm{int}(A))$. We denote the set of all closure-interior subsets of X by $\mathcal{CI}(X)$.

If $A \in \mathcal{CI}(X)$, then the set $U := \mathrm{int}(A)$ satisfies $U = \mathrm{int}(\mathrm{cl}(U))$. Conversely, if $U = \mathrm{int}(\mathrm{cl}(U))$, then $A := \mathrm{cl}(U) \in \mathcal{CI}(X)$.

Unlike general closed sets which admit outer approximations but not inner approximations (we use lower approximations instead), clint sets admit natural outer and inner approximations. We use a representation combining a $\theta_{<}$-name for $\mathrm{int}(A)$ (as defined in Section 2.6) and either a $\psi_{>}$-name or a $\kappa_{>}$-name for A, as appropriate.

A continuous function F such that $\mathrm{Graph}(F)$ is a clint set may be specified by a $\psi_{>}$-name or $\kappa_{>}$-name for $\mathrm{Graph}(F)$, and by a $\theta_{<}$-name for $\mathrm{Graph}(G)$ where G is defined by $\mathrm{Graph}(G) = \mathrm{int}(\mathrm{Graph}(F))$. Note that a function G is lower-semicontinuous with open values if, and only if, $\mathrm{Graph}(G)$ is open, and if G_1 and G_2 have open graphs, then so does $G_1 \circ G_2$.The following theorem shows that continuous clint systems behave nicely when operating on sets.

Theorem 4.12. *If G is a continuous, open-valued function, and U is open, then* $\mathrm{cl}(G(U)) = F(\mathrm{cl}(U))$, *where* $\mathrm{Graph}(F) = \mathrm{cl}(\mathrm{Graph}(G))$.

Proof. Clearly $\mathrm{cl}(G(U)) \subset F(\mathrm{cl}(U))$ since $F(\mathrm{cl}(U))$ is closed. Suppose $y \not\in \mathrm{cl}(G(U))$. Then there exists a neighbourhood Z of y such that $Z \cap G(U) = \emptyset$. Then $G^{-1}(Z) \cap U = \emptyset$, and since Z is open and G is lower-semicontinuous, $G^{-1}(Z) \cap \mathrm{cl}(U) = \emptyset$. Choose a neighbourhood W of y with $\mathrm{cl}(W) \subset Z$. Then $G^{-1}(\mathrm{cl}(W)) \cap \mathrm{cl}(U) = \emptyset$, and since G is lower-semicontinuous, $G^{-1}(\mathrm{cl}(W))$ is closed. Hence there exists open V with $\mathrm{cl}(U) \subset V$ such that $G^{-1}(\mathrm{cl}(W)) \cap V = \emptyset$. Then $W \cap G(V) = \emptyset$, so $\mathrm{Graph}(G) \cap V \times W = \emptyset$, and so $\mathrm{Graph}(F) \cap V \times W = \emptyset$, and hence $y \not\in F(\mathrm{cl}(U))$. □

If G is not continuous, then the result may not be true, as the following example shows.

Example 4.13. Let $G(x) = (0, 1)$ if $x \in (0, 1]$, $F(x) = (0, 2)$ if $x \in (1, 2)$. Then $F(x) = [0, 1]$ if $x \in [0, 1)$, $F(x) = [0, 2]$ if $x \in [1, 2]$. Let $U = (0, 1)$ and $A = \mathrm{cl}(U) = [0, 1]$. Then $G(U) = (0, 1)$ but $F(A) = [0, 2] \neq \mathrm{cl}(G(U))$.

Corollary 4.14. *If F, G are continuous, closure-interior systems, then so is $F \circ G$.*

The following result shows that the reachable set is inner-computable for closure-interior systems.

Theorem 4.15. *Let G be a continuous, open-valued multivalued function, and U an open set. Then the operator $(G, U) \mapsto \mathrm{Reach}(G, U)$ is $(\mu_{<}^{\mathcal{O}}, \theta_{<}; \theta_{<})$-computable.*

Proof. We first show that the map $(G, U) \mapsto G(U)$ is $(\mu_{<}^{\mathcal{O}}, \theta_{<}; \theta_{<})$-computable. Output I with $\overline{I} \subset G(U)$ if there exist $J_1, \ldots J_k$ and $K_1, \ldots, K_k$ such that $\overline{J}_i \subset U$ and $\overline{J}_i \times \overline{K}_i \subset \mathrm{Graph}(G)$ for $i = 1, \ldots, k$, and $\overline{I} \subset \bigcup_{i=1}^{k} K_i$. It is straightforward to check that these I encode a $\theta_{<}$-name of $G(U)$.

The function $(G, U) \mapsto G^n(U)$ is then $(\mu_{<}^{\mathcal{O}}, \theta_{<}; \theta_{<})$-computable for all n. It is straightforward to check that countable union $\mathcal{O}^{\mathbb{N}} \to \mathcal{O}$ is $(\theta_{<}, \theta_{<}, \ldots; \theta)$-computable. □

The chain reachable set is $(\mu_{>}^{\mathcal{K}}, \kappa_{>}; \kappa_{>})$-computable as before. However, by modifying the Example 1.2, it is straightforward to show that it is still impossible to compute a better upper approximation for the reachable set than the chain reachable set.

Thus closure-interior systems admit inner approximations to the reachable set, which may be useful in verifying certain reachability properties and in the construction of algorithms, but the reachable set may still be uncomputable.

4.4 Continuous-time systems

Up to now, we have considered reachability for discrete-time systems. We can also consider continuous-time systems described by a differential inclusions

$$\dot{x}(t) \in F(x(t)) \tag{24}$$

where F is a multivalued section of the tangent bundle $\mathrm{T}X$. Then $\mathrm{Graph}(F)$ is a subset of $\mathrm{T}X$, and define the differential inclusion by $(x, \dot{x}) \in \mathrm{Graph}(F)$.

The following result of Puri, Varaiya and Borkar [21] shows that computable Lipschitz differential inclusions may be integrated to give computable continuous multivalued maps.

Theorem 4.16 (Puri, Varaiya, Borkar). *Suppose $\dot{x} \in F(x)$ is a Lipschitz differential inclusion. Then for any $\gamma > 0$ and any $t \geqslant 0$, we can compute a set R as a union of polyhedrons such that $\mathrm{Reach}(F, X_0, t) \subset R$ and $d_H(\mathrm{Reach}(F, X_0, t), R) < \gamma$.*

We define the *flow* Φ of F by $\Phi_t(x) := \mathrm{Reach}(F, \{x\}, t)$, and $\Phi_{\leqslant t}(x) := \bigcup_{\tau \in [0,t]} \Phi_\tau(x)$. It is immediate that

$$\mathrm{Reach}(F, X_0) = \Phi_{\leqslant t}(\mathrm{Reach}(\Phi_t, X_0)) = \mathrm{Reach}(\Phi_{\leqslant t}, X_0). \tag{25}$$

The following result follows from Theorem 4.16

Corollary 4.17. *For any rational t, and for F a Lipschitz differential inclusion, the functions $F \mapsto \Phi_t$ and $F \mapsto \Phi_{\leqslant t}$ are computable.*

Proof. That Φ_t is computable is immediate. To show that $\Phi_{\leqslant t}$ is computable, consider the system $\widetilde{F}$ with $\widetilde{F}(x) = \mathrm{conv}\,(F(x) \cup \{0\})$ for all x. Then $\Phi_{\leqslant t}(x) = \mathrm{Reach}(F, \{x\}, [0, t])$. □

We define the chain reachable set for an upper-semicontinuous Lipschitz differential inclusion by

$$\mathrm{ChainReach}(F, X_0) := \bigcap \{\mathrm{Reach}(G, X_0) : G \subset \mathcal{O}(\mathrm{T}X) \text{ and } F \subset G\} \tag{26}$$

Since $\mathrm{ChainReach}(F, X_0) = \Phi_{\leqslant t}(\mathrm{ChainReach}(\Phi_t, X_0))$, we obtain the following result.

Theorem 4.18 (Reachability of Lipschitz differential inclusions).

1. *The map* $(F, X_0) \mapsto \overline{\text{Reach}}(F, X_0)$ *is* $(\mu^{\mathcal{K}}, \psi_<; \psi_<)$*-computable for Lipschitz* F*.*
2. *The map* $(F, X_0) \mapsto \text{ChainReach}(F, X_0)$ *is* $(\mu^{\mathcal{K}}, \kappa_>; \kappa_>)$*-computable for Lipschitz* F*.*

Notice that the results presented here have only been proved for Lipschitz differential inclusions and for the $\mu^{\mathcal{K}}$ representation. We would expect that the map $(F, X_0) \mapsto \overline{\text{Reach}}(F, X_0)$ to be $(\mu^{\mathcal{A}}_<, \psi_<; \psi_<)$-computable and $(\mu^{\mathcal{K}}_<, \kappa_>; \kappa_>)$-computable for appropriate classes of differential inclusion. It may be possible to weaken the Lipschitz restriction slightly, but the following example shows that Hölder continuity is insufficient for lower-computability.

Example 4.19. Consider the Hölder-continuous differential equation

$$\dot{x} = f_\epsilon(x) := \sqrt{|x|} + \epsilon. \tag{27}$$

For $\epsilon < 0$, we have $\text{Reach}(f_\epsilon, \{0\}) = (-\epsilon^2, 0]$ and $\text{ChainReach}(f_\epsilon, \{0\}) = [-\epsilon^2, 0]$. For $\epsilon > 0$, we have $\text{Reach}(f_\epsilon, \{0\}) = \text{ChainReach}(f_\epsilon, \{0\}) = [0, \infty)$.

The interesting case is $\epsilon = 0$, where $\dot{x} = \sqrt{|x|}$. Here, the solutions are not unique; indeed, for any $a \geqslant 0$, we have a solution

$$x(t) = 0 \text{ for } t \leqslant a; \qquad x(t) = \tfrac{1}{4}(t-a)^2 \text{ for } t \geqslant a \tag{28}$$

Then the time-t reachable set $\text{Reach}(f_0, \{0\}, t)$ is therefore $[0, t^2/4]$, and so the time-t reachable set does not vary continuously with ϵ. The reachable set $\text{Reach}(f_0, \{0\})$ is therefore $[0, \infty)$, and therefore $\overline{\text{Reach}}$ is not $(\tau^{\mathcal{MA}}_<, \tau^{\mathcal{A}}_<; \tau^{\mathcal{A}}_<)$-continuous.

Lipschitz continuity of the right-hand side is therefore a necessary condition for the time-t reachable set to be $(\mu^{\mathcal{A}}_<, \psi_<; \psi_<)$-computable. However, we expect that the time-t chain-reachable set to be $(\mu^{\mathcal{K}}_>, \kappa_>; \kappa_>)$-computable with only a continuous F.

5 Approximation methods

Although the representations of sets given in Section 2 are convenient for a general analysis of computability properties, they require an infinite amount of data. We often want to describe a set by giving an approximation using a finite amount of data. To do this, we first choose a denumerable collection of *denotable sets*, which can be described exactly, and describe other sets by giving an approximating denotable set and an error bound. Such approximations are used in existing software for performing set-based analysis, including GAIO [11] and the ellipsoidal calculus of Kurzhanski and Valyi [15].

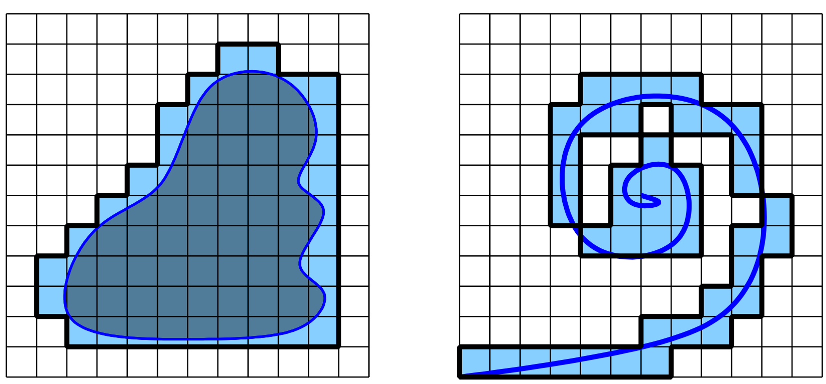


Figure 4: Approximations of compact sets on a grid.

A particularly important class of denotable sets in applications is that based on cuboidal grids, as shown in Figure 4. We can take a decreasing sequence of grids $\mathcal{G}_q$ based on the *dyadic rationals* $\mathbb{Q}_2 := \{p/2^q : p \in \mathbb{Z},\ q \in \mathbb{N}\}$ as unions of cuboids of the form

$$\overline{I} = \left[\tfrac{p_1}{2^q}, \tfrac{p_1+1}{2^q}\right] \times \left[\tfrac{p_2}{2^q}, \tfrac{p_2+1}{2^q}\right] \times \cdots \times \left[\tfrac{p_n}{2^q}, \tfrac{p_n+1}{2^q}\right] \tag{29}$$

By taking finer and finer grids, better approximations can be computed.

From a computability viewpoint, we are interested in whether it is possible to compute approximations to a set to arbitrary precision. We therefore consider *approximation representations*, in which we represent a set by a convergent sequence of denotable sets. The advantage of approximation representations over the standard representations is that the approximating denotable elements have the same type as the element being represented. For real numbers, points in Euclidean space, and open and closed sets, we can find approximation representations equivalent to the standard representations. Hence the computability results for reachable and chain reachable sets in the standard representations are also valid for the approximation representations.

We first give an outline of approximation methods in a more general setting, with the example being that of the real numbers and points in Euclidean space. We then consider different types of denotable closed sets, with particular emphasis on those defined on cuboidal grids. Finally, we consider those approximation representations which correspond to the standard representations $\psi_<$ of $\mathcal{A}$ and $\kappa_>$ and κ of $\mathcal{K}$. The material in this section is only an introduction to the use of approximation methods; a complete treatment is beyond the scope of this paper.

5.1 Approximation representations

We first define a general framework for considering approximations.

Definition 5.1 (Denotable element). Let (X,τ) be a second-countable Hausdorff space, and $\xi :\subset \Sigma^* \to X$ be a function whose range is a dense subset of X. We say an element $x \in X$ is *denotable* if $x = \xi(w)$ for some $w \in \mathrm{dom}(\xi)$. The triple (X,τ,ξ) is a *denotable topological space.*

Appropriate choices for denotable real numbers are the rationals $\mathbb{Q}$ or the dyadic rationals $\mathbb{Q}_2$. Appropriate choices for denotable points in Euclidean space $\mathbb{R}^n$ are $\mathbb{Q}^n$ and $\mathbb{Q}_2^n$.

We can now define representations of elements of X by convergent sequences.

Definition 5.2 (Approximation representation). Let (X,τ,ξ) be a denotable topological space. An *approximation representation* of (X,τ,ξ) is a function $\delta :\subset \Sigma^\omega \to X$ such that

$$\delta\langle w_1, w_2, \ldots\rangle = x \quad :\Longleftrightarrow \quad \langle w_1, w_2, \ldots\rangle \in \mathrm{dom}(\delta) \text{ and } \lim_{i\to\infty} \xi(w_i) = x. \tag{30}$$

In other words, an approximation representation encodes a convergent sequence of denotable elements (x_i), where $x_i := \xi(w_i)$.

The main difficulty when considering approximation representations is that no finite portion of a general convergent sequence gives any information about its limit. The main challenge is therefore to restrict the domain of the representation δ to sequences with appropriate properties, so that meaningful approximations can be extracted. We henceforth often restrict approximation representations to (strictly) increasing or decreasing sequences, or *effective Cauchy sequences* with $d(x_i,x_j) \leqslant \epsilon_i$ whenever $j > i$, where (ϵ_i) is a strictly decreasing sequence of rationals with $\lim_{i\to\infty} \epsilon_i = 0$, a typical choice being $\epsilon_i = 2^{-i}$

Many of the representations of real numbers $\mathbb{R}$ given in [23, Section 4.1] are approximation representations, most notably the *Cauchy representation* ρ_C given by

$$\rho_C\langle w_1, w_2, \ldots\rangle = x \quad :\Longleftrightarrow \quad |\xi(w_i) - \xi(w_j)| \leqslant 2^{-i} \text{ for } i < j \text{ and } x = \lim_{i\to\infty} \xi(w_i).$$

The Cauchy representation is an approximation representation with domain given by

$$\langle w_1, w_2, \ldots\rangle \in \mathrm{dom}(\rho_C) \quad :\Longleftrightarrow \quad x_i := \xi(w_i) \text{ satisfy } |x_i - x_j| < 2^{-i} \text{ for } i < j.$$

The Cauchy representation is equivalent to the standard representation ρ. An alternative approximation representation of $\mathbb{R}$ which is equivalent to ρ is that by alternating sequences (x_i) satisfying $x_{2i} < x_{2i+2} < x_{2i+3} < x_{2i+1}$ for all i.

An approximation representation equivalent to the standard representation ρ^n of $\mathbb{R}^n$ is given by taking strongly convergent sequences, where $||x - x_i|| < 2^{-i}i$ for all i. Here, the most natural norm to take is the sup-norm $||\cdot||_\infty$.

From an approximation representation, we often wish to derive a single approximation to the represented element. We can specify an approximation by giving an approximating denotable element, and specifying the type of approximation.

Definition 5.3 (Approximation). An *approximation type* is a function $e : \mathrm{range}(\xi) \to \tau$. An *approximation* to an element $x \in X$ is a pair $(\tilde{x}, e)$, where $\tilde{x}$ is a denotable element, and e is an approximation type, such that $x \in e(\tilde{x})$. We say that $\tilde{x}$ is an *e-approximation* to x.

A *lower approximation* of a real number is specified by the approximation type $e_<(\tilde{x}) = (\tilde{x}, \infty)$, since $\tilde{x}$ is a lower approximation to x if $\tilde{x} < x$, which is equivalent to $x \in (\tilde{x}, \infty)$. An *upper approximation* is specified by $e_>(\tilde{x}) = (-\infty, \tilde{x})$. An *$\epsilon$-approximation* is specified by $e_\epsilon(\tilde{x}) = (\tilde{x} - \epsilon, \tilde{x} + \epsilon)$. ϵ-approximations can be extracted from effective Cauchy sequences, since if (x_i) is an effective Cauchy sequence converging to x, then $|x - x_i| \leqslant \epsilon_i$ for all i.

The concept of lower approximation generalises to any partially ordered topological space, and that of ϵ-approximation to any metric space. For general topological spaces, we can define approximations in terms of an open cover. A *$\mathcal{U}$-approximation* is specified by $e_\mathcal{U}(\tilde{x}) = \bigcup\{U \in \mathcal{U} : \tilde{x} \in U\}$, so $\tilde{x}$ is a $\mathcal{U}$-approximation to x if there exists $U \in \mathcal{U}$ such that $x, \tilde{x} \in U$.

5.2 Approximations of sets

We now consider approximation representations of closed and compact sets. Let (X, τ, β, ν) be a computable Hausdorff space. Then the topological spaces $(\mathcal{A}(X), \tau^\mathcal{A})$, $(\mathcal{O}(X), \tau^\mathcal{O})$ and $(\mathcal{K}(X), \tau^\mathcal{K})$ are second-countable Hausdorff spaces. An appropriate notion of a denotable set is one which can be written as a finite union of basic (open or closed) sets of X.

Definition 5.4 (Denotable set).

1. A closed set A is denotable if there are finitely many basic closed sets $\overline{I}_1, \ldots, \overline{I}_k$ such that $A = \bigcup_{i=1}^k \overline{I}_i$. The function $\widehat{\nu} :\subset \Sigma^* \to \mathcal{A}(X)$ defined by
$$\widehat{\nu}\langle w_1, \ldots, w_k\rangle := \bigcup_{i=1}^k \overline{\nu}(w_i) = \mathrm{cl}(\widetilde{\nu}\langle w_1, \ldots, w_k\rangle). \tag{31}$$
is a notation for the denotable closed sets.
2. An open set U is denotable if there are finitely many basic open sets $J_1, \ldots, J_k$ such that $U = \bigcup_{i=1}^k J_i$. The function $\widetilde{\nu} :\subset \Sigma^* \to \mathcal{O}(X)$ defined by
$$\widetilde{\nu}\langle w_1, \ldots, w_k\rangle := \bigcup_{i=1}^k \nu(w_i) \tag{32}$$
is a notation for the denotable open sets.
3. Since the denotable closed sets are compact, compact set C is denotable if it is a denotable closed set, $C = \bigcup_{i=1}^k \overline{I}_i$.

There are a number of useful approximation representations of open, closed and compact sets, each based on a type of sequence.

Definition 5.5 (Monotonic sequences). A sequence of open sets (U_i) is *increasing* if $U_i \subset U_j$ whenever $i < j$, and *strictly increasing* if $\mathrm{cl}(U_i) \subset U_j$. A sequence of compact sets (C_i) is *decreasing* if $C_j \subset C_i$ whenever $i < j$, and *strictly decreasing* if $C_j \subset \mathrm{int}(C_i)$.

Since a closed set A may have nonempty interior, we cannot in general find an increasing sequences of denotable closed sets converging to A. Instead, we consider approximations by Cauchy sequences. For the rest of this section, we suppose X is a metric space with metric d. Recall that if A is a closed set, the *ϵ-neighbourhood* of A is $N_\epsilon(A) := \{x : d(x, A) < \epsilon\}$. We also fix a strictly decreasing sequence of rationals (ϵ_i) converging to 0.

Definition 5.6 (Cauchy sequences).

1. A sequence of closed sets (A_i) is a *lower Cauchy sequence* if $A_i \subset N_{\epsilon_i}(A_j)$ whenever $i < j$.
2. A sequence of compact sets (C_i) is an *upper Cauchy sequence* if $C_j \subset N_{\epsilon_i}(C_i)$ whenever $i < j$.
3. A sequence of compact sets (C_i) is a *decreasing Cauchy sequence* if $C_j \subset C_i \subset N_{\epsilon_i}(C_j)$ whenever $i < j$, and a *strictly decreasing Cauchy sequence* if $C_j \subset \mathrm{int}(C_i)$ and $C_i \subset N_{\epsilon_i}(C_j)$ whenever $i < j$.

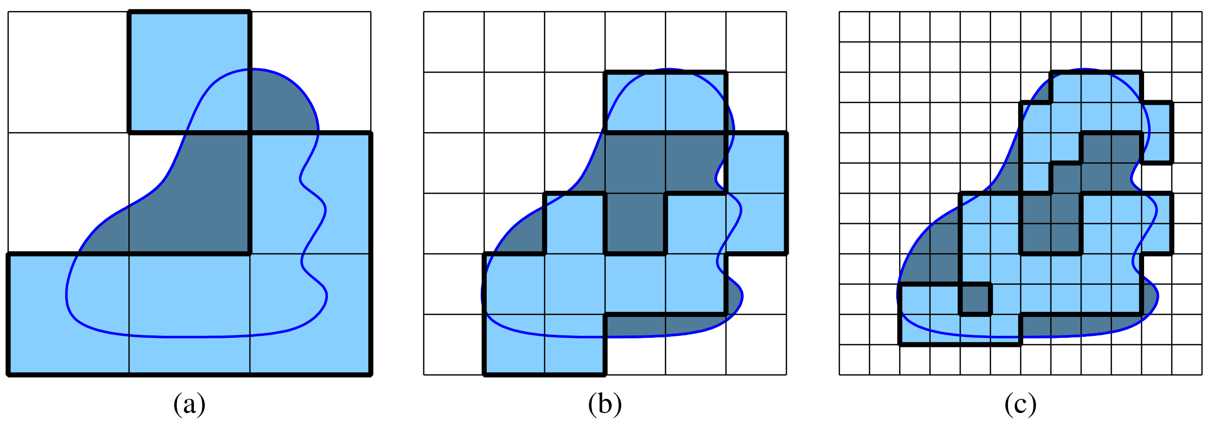


Figure 5: Convergence of lower Cauchy sequences. The sets illustrated in (a), (b) and (c) form three terms of a lower Cauchy approximation representation.

We can use monotone and Cauchy sequences to define approximation representations

Definition 5.7 (Approximation representations).

1. The approximation representation of open sets by increasing sequences sequences is the *inner approximation representation*, $\theta_<^{\mathrm{approx}}$.
2. The approximation representation of closed sets by lower Cauchy sequences is the *lower Cauchy approximation representation*, $\psi_<^{\mathrm{approx}}$.
3. The approximation representation of compact sets by decreasing sequences is the *outer approximation representation* $\kappa_>^{\mathrm{approx}}$.
4. The approximation representation of compact sets by decreasing Cauchy sequences is the *outer Cauchy approximation representation* κ^{approx}.

These approximation representations are equivalent to the standard representations.

Theorem 5.8 (Approximation representations).

1. *The inner approximation representation $\theta_<^{\mathrm{approx}}$ and the standard representation $\theta_<$ are equivalent.*
2. *The lower approximation representation $\psi_<^{\mathrm{approx}}$ and the standard representation $\psi_<$ are equivalent.*
3. *The outer approximation representation $\kappa_>^{\mathrm{approx}}$ and the standard representation $\kappa_>$ are equivalent.*
4. *The outer Cauchy approximation representation κ^{approx} and the standard representation κ are equivalent.*

(sketch).

1. Let $(\overline{I}_1, \overline{I}_2, \ldots)$ encode a $\theta_<$-name for an open set U. Then the sequence (U_i) defined by $U_i := \bigcup_{i=1}^{k}$ is an increasing sequence converging to U.

   Conversely, given an increasing sequence of denotable sets (U_i) converging to U, we output $\overline{I}$ if $\overline{I} \subset U$ (covering).

2. Let $(J_1, J_2, \ldots)$ encode a $\psi_<$-name for a closed set A. For simplicity, we only consider the case where A is compact. We let $A_{j,k} = \mathrm{cl}(\bigcup\{J_m : m \leqslant k \text{ and } \mathrm{diam}(\overline{J}_m) < \epsilon_j\})$. We take the jth approximation $A_j = A_{j,k}$ whenever $A_{i,k} \subset N_{\epsilon_i}(A_{j,k})$ for all $i < j$. The resulting sequence is a lower Cauchy sequence converging to A.

   Conversely, given a lower Cauchy sequence (A_i) converging to A, we have that $A_i \subset \mathrm{cl}(N_{\epsilon_i}(A))$ for all i. Hence if J contains a basic closed set $\overline{I}$ such that $I \cap A_i \neq \emptyset$ and $\mathrm{cl}(N_{\epsilon_i}(\overline{I})) \subset J$ for some i, then $J \cap A \neq \emptyset$. If $J \cap A \neq \emptyset$, then there exists I with $\overline{I} \subset J$, and $I \cap A \neq \emptyset$. There exists i such that $I \cap A_i \neq \emptyset$ (by convergence) and $\mathrm{cl}(N_{\epsilon_i}(I)) \subset J$. Hence J is output.

3. We use the representation $\kappa_>^{\mathrm{cv}}$. Given a sequence of all open covers $\mathcal{U}_i$ of containing C, we can take (C_i) to be a subsequence of $\bigcup\{\overline{I} : I \in \mathcal{U}_i\}$. Given a decreasing sequence (C_i) converging to C, we can take all open covers of the C_i to obtain all open covers of C.

4. For any i, we compute C_i such that $C \subset \mathrm{int}(C_i)$ and $C_i \subset N_{\epsilon_i}(C)$. The construction is that of [23, page 127]. Then if $i < j$, we have $C_j \subset N_{\epsilon_i}(C) \subset N_{\epsilon_i}(C_j)$. By taking a subsequence if necessary, we obtain $C_j \subset C_i$ for $i < j$.

□

As usual, the approximating sequences give rise to approximation concepts. A *ϵ-lower approximation* to a closed set A is a denotable closed set $\widetilde{A}$ such that $\widetilde{A} \subset N_\epsilon(A)$. An *outer approximation* to a compact set C is a denotable compact set $\widetilde{C}$ such that $C \subset \widetilde{C}$, and a *strict outer approximation* is a denotable compact set $\widetilde{C}$ such that $C \subset \mathrm{int}(\widetilde{C})$. An *outer ϵ-approximation* to a compact set C is a denotable compact set $\widetilde{C}$ such that $C \subset \widetilde{C} \subset N_\epsilon(C)$.

5.3 Approximations on grids

A natural way of approximating a compact set in Euclidean space is to construct a *grid* of closed cuboids with disjoint interiors, and to take denotable sets which are a union of finitely many cuboids, as shown previously in Figure 4. This notion generalises to arbitrary computable Hausdorff spaces.

Definition 5.9 (Grid). A *grid* is a collection $\mathcal{G}$ of basic closed sets $\overline{I}$ such that $X = \bigcup \mathcal{G}$ and $I \cap J = \emptyset$ whenever $\overline{I}, \overline{J} \in \mathcal{G}$.

Similarly to refinements of open covers, we can consider refinements of grids. However, refinements of grids are more restricted, since we require that each element of the larger grid be a union of elements of the refinement.

Definition 5.10 (Proper refinement). A *proper refinement* of a grid $\mathcal{G}_1$ is a grid $\mathcal{G}_2$ such that for every $\overline{I}_2 \in \mathcal{G}_2$, there exists $\overline{I}_1 \in \mathcal{G}_1$ such that $\overline{I}_2 \subset \overline{I}_1$, and for every $\overline{I}_1 \in \mathcal{G}_1$, we have $\overline{I}_1 = \bigcup\{\overline{I}_2 \in \mathcal{G}_2 : \overline{I}_2 \subset \overline{I}_1\}$.

We would like to construct approximations to a compact set C as finite unions of grid elements. Clearly the best such outer approximation on a given grid is $\widetilde{C}_{\mathcal{G}} := \bigcup\{\overline{I} \in \mathcal{G} : \overline{I} \cap C \neq \emptyset\}$. Unfortunately this concept cannot be directly effectivised, since although we can use $\psi_<$ to show $I \cap C \neq \emptyset$, and $\kappa_>$ to show $\overline{I} \cap C = \emptyset$, we cannot effectively decide whether C intersects the boundary of I.

To overcome this difficulty, we consider a neighbourhood $N(\overline{I})$ for each grid element $\overline{I}$. Assuming that $N(\overline{I})$ is a basic open set for every grid element $\overline{I}$, we define a function

$$\eta :\subset \Sigma^* \to \Sigma^*, \quad \nu(\eta(w)) := N(\overline{\nu}(w)), \tag{33}$$

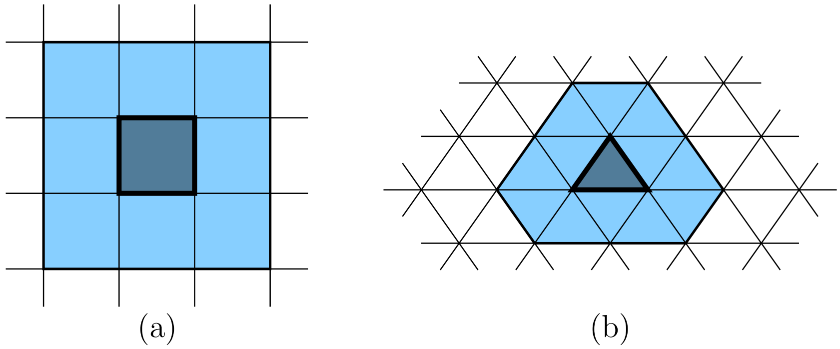


Figure 6: One box neighbourhoods. (a) A cuboidal grid, and (b) a simplicial grid.

so that $\eta(w)$ is a name for $N(\overline{I})$ if w is a name for $\overline{I}$. The natural neighbourhoods to consider are the *one-box neighbourhoods*, defined as follows:

Definition 5.11 (One-box neighbourhood). The *one-box neighbourhood* of $\overline{I}$ is the set

$$N(\overline{I}) := X \setminus \bigcup\{\overline{J} \in \mathcal{G} : \overline{J} \cap \overline{I} = \emptyset\}. \tag{34}$$

Examples of one-box neighbourhoods for cubical and simplicial grids are shown in Figure 6. Note that the one-box neighbourhoods for a simplicial grid are not simplexes.

Given a $\psi_<$-name of C, we can eventually find all grid elements $\overline{I} \in \mathcal{G}$ such that $N(\overline{I}) \cap C \neq \emptyset$. Hence we can build up a sequence of sets $\widetilde{C}_i$ such that $\widetilde{C}_i \subset \widetilde{C}_j \subset N_\epsilon(C)$, for all $i < j$, where $\epsilon = \sup\{\mathrm{diam}(\overline{I}) : \overline{I} \in \mathcal{G}\}$. Given a $\kappa_>$-name of C, we can eventually find all grid elements $\overline{I} \in \mathcal{G}$ such that $\overline{I} \cap C = \emptyset$. Hence we can construct a sequence of sets $\widetilde{C}_i$ such that $C \subset \widetilde{C}_j \subset \widetilde{C}_i$ for all $j < i$.

Now for each grid element $\overline{I} \in \mathcal{G}$, we either have $N(\overline{I}) \cap C \neq \emptyset$ or $\overline{I} \cap C = \emptyset$, or both. Hence, given a κ-name of C, we can compute, in finite time, a finite set of grid elements $\{I_1, \ldots, I_k\}$ such that $\widetilde{C} := \bigcup_{i=1}^{k} \overline{I}_i$ satisfies $C \subset \mathrm{int}(\widetilde{C})$ and $\widetilde{C} \subset N_\epsilon(C)$, as described in Weihrauch [23, Figure 5.2].

The above discussion has focused on a single grid. We now consider the construction of lower approximating sequences and outer approximating sequences on a sequence of grids $(\mathcal{G}_i)$, where $\mathcal{G}_j$ is a proper refinement of $\mathcal{G}_i$ for $i < j$.

Definition 5.12. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be grids, where $\mathcal{G}_2$ is a proper refinement of $\mathcal{G}_1$, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be finite sets of basic closed sets of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. We say $\mathcal{L}_2 < \mathcal{L}_1$ if for all $\overline{L}_2 \in \mathcal{L}_2$, there exists $\overline{L}_1 \in \mathcal{L}_1$ such that $\overline{L}_2 \subset \overline{L}_1$. We say $\mathcal{L}_1 \prec \mathcal{L}_2$ if for all $\overline{L}_1 \in \mathcal{L}_1$, there exists $\overline{L}_2 \in \mathcal{L}_2$ such that $N(\overline{L}_2) \subset N(\overline{L}_1)$, and $\mathcal{L}_1 \precsim \mathcal{L}_2$ if for all $\overline{L}_1 \in \mathcal{L}_1$, there exists $\overline{L}_2 \in \mathcal{L}_2$ such that $\overline{L}_2 \cap N(\overline{L}_1) \neq \emptyset$.

If $\mathcal{L}_2 < \mathcal{L}_1$, then $\bigcup \mathcal{L}_2 \subset \bigcup \mathcal{L}_1$, and if $\mathcal{L}_1 \precsim \mathcal{L}_2$, then $\bigcup \mathcal{L}_1 \subset N_{\epsilon_1}(\bigcup \mathcal{L}_2)$. The relation $\mathcal{L}_1 \prec \mathcal{L}_2$ is stronger than $\mathcal{L}_1 \precsim \mathcal{L}_2$. The relations $<$ and $\prec$ are partial orders.

The relations $\prec$ and $\precsim$ can be used to compute lower approximating sequences to C. Given a $\psi_<$-name of C as a list $(J_1, J_2, \ldots)$ of basic open sets intersecting C, we let $\mathcal{L}_{j,k} := \{\overline{I} \in \mathcal{G}_j : N(\overline{I}) \in J_1, \ldots, J_k\}$. We take $\mathcal{L}_j = \mathcal{L}_{j,k}$ if $k \geqslant j$ and $\mathcal{L}_{i,k} \precsim \mathcal{L}_{j,k}$ for all $i < j$. This is guaranteed to terminate, since if $x \in N(\overline{I}_i) \cap C$, then $x \in \overline{I}_j$ for some $\overline{I}_j \in \mathcal{G}_j$, so $N(\overline{I}_j \cap C \neq \emptyset$ and $\overline{I}_j \cap N(\overline{I}_i) \neq \emptyset$. The sets $C_j := \bigcup \overline{L}_j$ are then a lower approximating sequence to C.

If the open covers $\mathcal{U}_i := \{N(\overline{I}) : \overline{I} \in \mathcal{G}_i\}$ are proper refinements of each other, then whenever $i < j$ and $N(\overline{I}_i) \cap C \neq \emptyset$ for some $\overline{I} \in \mathcal{G}_i$, then there exists $\overline{I}_j \in \mathcal{G}_j$ such that $N(\overline{I}_j) \subset N(\overline{I}_i)$ and $N(\overline{I}_j) \cap C \neq \emptyset$. We can then find a sequence $(\mathcal{L}_i)$ such that $\mathcal{L}_i \subset \mathcal{G}_i$ and $\mathcal{L}_i \prec \mathcal{L}_j$ whenever $i < j$.

The relation $<$ can be used to define an outer approximating sequence to C. Given a $\kappa_>$-name of C as an open cover $J_1, \ldots, J_k$ of C and a list of basic closed sets $(\overline{K}_1, \overline{K}_2, \ldots)$ disjoint from C, we construct finite subsets $\mathcal{L}_{j,k} \subset \mathcal{G}_j$ as follows. We start by taking $\mathcal{L}_{j,0}$ such that $\bigcup_{i=1}^{j} J_i \subset \bigcup \mathcal{L}_{j_0}$, and take $\mathcal{L}_{j,k} = \mathcal{L}_{j,0} \setminus \{\overline{I}_1, \ldots, \overline{I}_k\}$ for $k > 0$. We take $\mathcal{L}_j = \mathcal{L}_{j,k}$ if $j > k$ and $\mathcal{L}_{j,k} < \mathcal{L}_{i,k}$ for all $i < j$. The sets $C_j := \bigcup \overline{L}_j$ are then a decreasing approximating sequence to C.

5.4 Approximation of functions

Semicontinuous multivalued functions can be described in terms of their graphs. In particular, a lower-semicontinuous, open-valued function has an open graph, and an upper-semicontinuous, closed-valued functions has a closed graph. We can define classes of denotable function as follows.

Definition 5.13 (Denotable function). A lower-semicontinuous, compact-valued function $F : X \rightrightarrows Y$ is *denotable* if $\text{Graph}(F) = \bigcup_{i=1}^{k} I_i \times \overline{J}_i$, where $I_i \in \beta_X$ and $J_i \in \beta_Y$ for $i = 1, \ldots, k$. Denotable functions in $USC_{\mathcal{K}}$, $LSC_{\mathcal{O}}$ and $USC_{\mathcal{O}}$ are defined similarly.

A notation for $LSC_{\mathcal{K}}$ is given by the function $(\nu \times \overline{\nu})$ given by

$$\text{Graph}(F) = (\nu \times \overline{\nu}) \big\langle \langle v_1, w_1 \rangle, \ldots, \langle v_k, w_k \rangle \big\rangle := \textstyle\bigcup_{i=1}^{k} \nu(v_i) \times \overline{\nu}(w_i). \tag{35}$$

Similarly, we can give notations $\overline{\nu} \times \overline{\nu}$, $\nu \times \nu$ and $\overline{\nu} \times \nu$ denote elements of $USC_{\mathcal{K}}$, $LSC_{\mathcal{O}}$ and $USC_{\mathcal{O}}$, respectively. It is clear that if S is a denotable (open or closed) set and F is a denotable function, then $F(S)$, $F^{-1}(S)$ and $F^{\Leftarrow}(S)$ are all denotable sets.

We have seen that a word $p = \langle u_1, \ldots, u_k \rangle$ is a name for both a denotable open set $\widetilde{\nu}\langle u_1, \ldots, u_k \rangle$ and a denotable compact set $\widehat{\nu}\langle u_1, \ldots, u_k \rangle$. Similarly, we think of a word $q = \big\langle \langle v_1, w_1 \rangle, \ldots, \langle v_l, w_l \rangle \big\rangle$ may denote lower-semicontinuous open-valued or closed-valued functions. If the elements $\overline{\nu}(u_i)$, $\overline{\nu}(v_j)$ and $\overline{\nu}(w_j)$ lie in some common grid $\mathcal{G}$, we can describe q by a finite graph on the elements of $\mathcal{G}$. Thinking of p as abstractly denoting a set, and q as an abstract function, we can define an abstract image $r = q(p)$ by

$$w \lhd r \iff \exists v \in \Sigma^*,\ v \lhd p \text{ and } \langle v, w \rangle \lhd q. \tag{36}$$

The abstract image is particularly useful when working with outer approximations. Since an upper-semicontinuous compact-valued map with compact domain has a compact graph, we can use the approximation representation of Graph(F) as an outer approximation of F. Given a strict outer approximation $p = \langle u_1, \ldots u_l \rangle$ for a compact set set C, and a strict outer approximation $q = \big\langle \langle v_1, w_1 \rangle, \ldots \langle v_k, w_k \rangle \big\rangle$ for a function F, then $F(A) \subset \text{int}(\widehat{\nu}(r))$, where $r = q(p)$ is defined by (36). Since if $\overline{\nu}(u), \overline{\nu}(v) \in \mathcal{G}$ for some grid $\mathcal{G}$ we have $\nu(u) \cap \nu(v) \neq \emptyset \iff u = v$, this is equivalent to

$$w \lhd r \iff \exists u, v \in \Sigma^*,\ u \lhd p,\ \langle v, w \rangle \lhd q \text{ and } \nu(u) \cap \nu(v) \neq \emptyset. \tag{37}$$

It is then immediate that r is an outer approximation of $F(C)$.

Hence the abstract image reduces a problem of computing the image of a set to a simple combinatorial exercise.

6 Conclusions and further research

In this paper, we have considered the computation of reachable sets in the setting of computable analysis and topology. We have shown that the reachable set is in generally uncomputable in this approximative setting, but that lower approximations to the reachable set and upper approximations to the chain reachable set can be computed. Further, in the case that the closure of the reachable set and the chain reachable set coincide, then the reachable set can be approximated to any specified accuracy. These computations can be used for the verification of controllability and of safety properties.

The difference between the reachable and the chain reachable sets can be viewed as a measure of the "robustness" of the system, or its sensitivity to noise. Thus, even when the reachable set is not computable, we obtain useful information about the system.

The type-two effectivity theory used has a number of features which we believe make it the most appropriate theory for the analysis of system properties. It provides a formal model of computation which can be realised on digital computers, and hence algorithms expressed in this theory can be practically realised. There is already considerable material on the representation of open, closed and compact sets and continuous functions in this theory. The theory deals with quite general topological spaces, allowing computations on manifolds as well as Euclidean spaces, and also allows for the study of semicontinuous multivalued maps and differential inclusions. As well as providing a framework for representing the standard objects of topology and analysis, and for computing approximations, it also allows us to deduce that certain computations are not possible, simply by showing that they attempt to compute a discontinuous function.

Given the power of the type-two effectivity theory, the results in this paper barely scratch the surface of what we believe can be achieved. We now give some possible directions for future work.

The results presented here have mostly been developed for *discrete-time* systems, though we have also presented results for *continuous-time systems*. We would like to extend the results further to deal with *hybrid-time* systems, in which evolution occurs in both continuous time (differential equations or inclusions) and discrete time (reset maps). We expect much greater problems when considering hybrid systems, since here the evolution may be discontinuous even over finite time intervals.

We have only presented an analysis of reachability problems. Another area of study is that of *viability theory* and *invariant sets* [4]. For a discrete-time multivalued system, a set A is viable if $\forall x \in A$, $F(x) \cap A \neq \emptyset$. The *viability kernel* of a set A is the maximal viable subset of A. A set is *invariant* if $F(A) \subset A$. The *invariance kernel* of a set A is the maximal invariant subset of A.

One promising tool for the study of viability problems is the *Conley index* [19], which computes *isolated invariant sets*. The Conley index requires the computation of homology groups related to the system dynamics. Hence, it is important to study the formal computability properties of homology groups in the setting of type-two effectivity.

It would also be interesting to develop these ideas further from a computational viewpoint into a "timed logic of approximation". In this thesis, fundamental notions of timed logic (e.g. UNTIL quantifiers) and topological notions of approximation, closure and interior, should be combined to give a consistent framework for the approximative study of system properties [18]. Of particular interest is the complementation operator, which takes closed sets to open sets and vice-versa, and timed unions and intersections (a union over infinite times takes an open set to an open set, but need not respect closedness).

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