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Applications of factorization embeddings for Lévy processes

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Applications of factorization embeddings for Lévy processes

ABSTRACT
We give three applications of the first factorization identity for Lévy processes: - Phase-type upward jumps: we find the joint distribution of the supremum and the epoch at which it is 'attained' if a Lévy process has phase-type upward jumps. We also find the characteristics of the ladder process. - Perturbed risk models: we establish general properties, and obtain explicit fluctuation identities in case the Lévy process is spectrally positive. - Asymptotics for Lévy processes: we study the tail distribution of the supremum under different assumptions on the tail of the Lévy measure.

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Applications of factorization embeddings for Lévy processes

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Abstract

We give three applications of the first factorization identity for Lévy processes:

- Phase-type upward jumps: we find the joint distribution of the supremum and the epoch at which it is 'attained' if a Lévy process has phase-type upward jumps. We also find the characteristics of the ladder process.
- Perturbed risk models: we establish general properties, and obtain explicit fluctuation identities in case the Lévy process is spectrally positive.
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Key words: first factorization identity, Lévy processes, perturbed risk model, phase-type jumps, ruin probability.

1 Introduction

Fluctuation theory analyzes quantities related to the extrema of a stochastic process. Examples include the distribution of the supremum or infimum, the last (or first) time that the process attains its extremum, first passage times, overshoots, and undershoots. The study of these distributions is often motivated by applications in queueing theory, mathematical finance, or insurance mathematics.

Of particular interest are the fluctuations of a *Lévy process* Z. Such a process has stationary and independent increments, and is defined on the probability space of càdlàg functions with the Borel σ-field generated by the usual Skorokhod topology. The characteristic function of Z_t has necessarily the form $\mathbb{E}e^{i\beta Z_t} = e^{-t\Psi_Z(\beta)}$, $\beta \in \mathbb{R}$, where

$$\Psi_Z(\beta) = \frac{1}{2}\sigma_Z^2\beta^2 + ic_Z\beta + \int_{\mathbb{R}} \left(1 - e^{i\beta z} + i\beta z\mathbb{1}(|z| \leq 1)\right) \Pi_Z(dz),$$

for some $\sigma_Z \geq 0$, $c_Z \in \mathbb{R}$ and a so-called *Lévy measure* Π_Z on $\mathbb{R}\backslash\{0\}$ satisfying $\int(1 \wedge |z|^2)\Pi_Z(dz) < \infty$. In particular, $Z_0 = 0$. Z is called a compound Poisson process if $c_Z = \sigma_Z = 0$ and $\Pi_Z(\mathbb{R}) < \infty$.

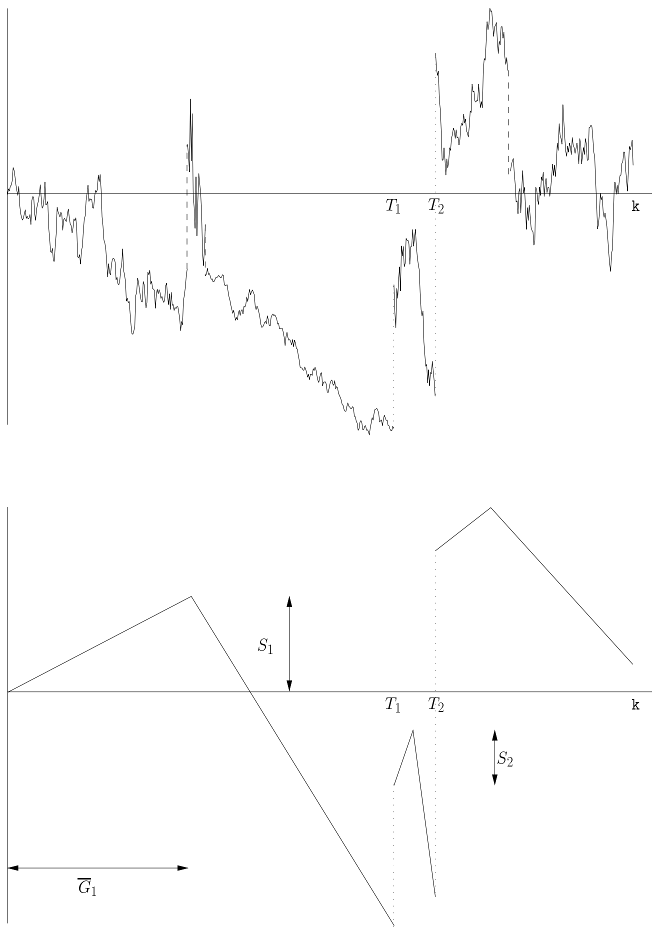

Figure 1: A realization of the killed Lévy process $Z = X + Y$ and the corresponding embedded (piecewise-linear) process. Jumps of Y are dotted and jumps of X are dashed.

Factorization embeddings

It is the aim of this paper to show how embeddings can be used to study the fluctuations of a Lévy process. For this, we consider the sum Z of an arbitrary one-dimensional Lévy process X and a compound Poisson process Y with intensity λ, independent of X. Note that any discontinuous Lévy process Z can be written in this form; in this paper, we are not interested in continuous Lévy processes (i.e., Brownian motions with drift), since their fluctuation theory is well-established. The representation $Z = X + Y$ need not be unique; for instance, there is a continuum of such representations if the Lévy measure has a nonvanishing absolutely continuous part.

Before explaining the idea behind the embedding that we study, we first introduce some notation. Write $T_1, T_2, \ldots$ for the jump epochs of Y, and set $T_0 = 0$. Define the quantities $\overline{G}_i$ and S_i for $i \geq 1$ as follows. $Z_{T_{i-1}} + S_i$ stands for the value of the supremum within $[T_{i-1}, T_i)$, and $T_{i-1} + \overline{G}_i$ is the last epoch in this interval such that the value of Z at $T_{i-1} + \overline{G}_i$ or $(T_{i-1} + \overline{G}_i)-$ is $Z_{T_{i-1}} + S_i$. Although formally incorrect, we say in the remainder that the supremum of Z over $[T_{i-1}, T_i)$ is attained for $T_{i-1} + \overline{G}_i$, with value $Z_{T_{i-1}} + S_i$.

In the first plot of Figure 1, a realization of Z is given. The jumps of Y are dotted and those of X are dashed. The process Z is killed at an exponentially distributed random time $\mathtt{k}$ independent of Z, say with parameter $q \geq 0$ ($q = 0$ corresponds to no killing). The second plot in Figure 1 is obtained from the first by replacing the trajectory of Z between T_{i-1} and T_i by a piecewise straight line consisting of two pieces: one from $(T_{i-1}, Z_{T_{i-1}})$ to $(T_{i-1}+\overline{G}_i, Z_{T_{i-1}}+S_i)$, and one from the latter point to (T_i, Z_{T_i-}). Obviously, by considering the embedded piecewise-linear process, no information is lost on key fluctuation quantities like the global supremum of Z and the epoch at which it is attained for the last time.

The piecewise-linear process, however, has several useful properties. Firstly, by the Markov property, the 'hats' are mutually independent given their starting point. Moreover, obviously, the jumps of Y are independent of the 'hats'. More strikingly, the increasing and decreasing pieces of each 'hat' are also independent; indeed, $(T_i - T_{i-1}, Z_{T_i-} - Z_{T_{i-1}}) = (\overline{G}_i, S_i) + (T_i - T_{i-1} - \overline{G}_i, Z_{T_i-} - Z_{T_{i-1}} - S_i)$, where the two latter vectors are independent, cf. the Pecherskii-Rogozin-Spitzer factorization for Lévy processes (e.g., Thm. VI.5 of Bertoin [6]). This explains the name factorization embedding.

The second plot in Figure 1 can be generated without knowledge of the trajectory of Z. Indeed, since $\{T_i : i \geq 1\}$ is a Poisson point process with intensity λ and killing at rate q, it is equivalent (in law) to the first N points of a Poisson point process with intensity $\lambda + q$, where N is geometrically distributed on $\mathbb{Z}_+$ with parameter $\lambda/(\lambda+q)$ (independent of the point process).

The idea to consider an embedded process for studying fluctuations for Lévy processes is not new. For instance, a classical example with $q = 0$ is the case that X is a negative drift $c < 0$ and Y has only positive jumps, so that $\overline{G}_i \equiv 0$ for every i and $(\overline{G}_i, S_i) + (T_i - T_{i-1} - \overline{G}_i, Z_{T_i-} - Z_{T_{i-1}} - S_i)$ is distributed as (e_λ, ce_λ), where e_λ denotes an exponentially distributed random variable with parameter λ. In that case, a random walk can be studied in order to analyze the fluctuations of Z. To the author's knowledge, nontrivial factorization embeddings have only been used to obtain results in the space domain. We mention the work of Kennedy [21], who studies certain Markov additive processes, and the work of Mordecki [26], who studies supremum of a Lévy process with phase-type upward jumps and general downward jumps. Recently, a slightly different form of this embedding has been used by Doney [14] to derive stochastic bounds on the Lévy processes Z. He defines X and Y such that the supports of Π_X and Π_Y are disjoint, and notices that $\{Z_{T_{i-1}} + S_i\}$ is a random walk with a random starting point, so that it suffices to establish stochastic bounds on the starting point. Doney then uses these to analyze the asymptotic behavior of Lévy processes that converge to $+\infty$ in probability. As an aside, we remark that the factorization embedding is different from the embedding that has been used in [4, 28], where jumps are absorbed by some random environment.

Outline and contribution of the paper; three applications

We now describe how this paper is organized, thereby introducing three problems that are studied using factorization embeddings. All the results in this paper are new, with the only exception of Proposition 1, Theorem 5 and the first claim in Theorem 7.

Section 2 is a preliminary section, in which background is given and the above idea is used to express fluctuation quantities of Z in those of X.

Section 3 uses these results to study the case that Z only has phase-type upward jumps. Then, the Laplace exponent of the bivariate ladder process κ_Z can be given; a quantity that lies at the heart of fluctuation theory for Lévy processes, see Ch. VI of Bertoin [6]. In particular, we give the joint law of the supremum and the epoch at which it is 'attained', generalizing Mordecki's [26] results.

Section 4 studies perturbed risk models, a generalization of classical risk models that has drawn much attention in the literature. We prove a general Pollaczek-Khinchine formula in this framework, but explicit results can only be obtained under further assumptions. Therefore, we impose spectral positivity of the Lévy process underlying the risk model, and extend the recent results of Huzak *et al.* [18] in the following sense. While [18] focuses on quantities related to so-called modified ladder heights, we obtain *joint* distributions related to both the modified ladder epoch and the ladder height. In particular, we obtain the (transform of the) distribution of the first modified ladder epoch.

Section 5 studies the tail of the supremum of Z under three different assumptions on the Lévy measure. We reproduce known results in the Cramér case and the subexponential case, but also give a local variant in the latter case, which is new. Our results for the intermediate case are also new, and complement recent work of Klüppelberg *et al.* [22].

After finishing this paper, the work of Pistorius [28] became available, and there is some overlap between his work and Section 3 in the special case $K = 1$. In [28], the Laplace exponent κ_Z of the ladder process is characterized in terms of the solutions of the equation $\Psi_Z(\beta) = q$. Using a matrix Wiener-Hopf factorization, $\kappa_Z(0, \beta)$ is explicitly found in terms of a vector $\boldsymbol{\alpha}_+^0$, and an algorithm is given to calculate the latter. Section 3 of the present paper, however, is more general in the sense that $\kappa_Z(q, \beta)$ is explicitly found in terms of a vector $\boldsymbol{\alpha}_+^q$, for which an efficient algorithm is given.

2 On factorization identities

In this section, we consider the process $Z = X + Y$, where Y is a compound Poisson process and X is a general Lévy process, independent of Y. After giving some background in Section 2.1, we study the supremum and infimum of Z and the epoch at which they are attained for the first (last) time in Section 2.2. We express their joint distribution in terms of the corresponding distribution of X. Moreover, the characteristics of the bivariate ladder process of Z are expressed in those of X.

2.1 Background

We start with some notation. Given a Lévy process X, we define

$$\begin{aligned} \overline{X}_t &= \sup\{X_s : 0 \le s \le t\}, & \underline{X}_t &= \inf\{X_s : 0 \le s \le t\} \\ \overline{F}_t^X &= \inf\{s < t : X_s = \overline{X}_t \text{ or } X_{s-} = \overline{X}_t\}, & \overline{G}_t^X &= \sup\{s < t : X_s = \overline{X}_t \text{ or } X_{s-} = \overline{X}_t\} \\ \underline{F}_t^X &= \inf\{s < t : X_s = \underline{X}_t \text{ or } X_{s-} = \underline{X}_t\}, & \underline{G}_t^X &= \sup\{s < t : X_s = \underline{X}_t \text{ or } X_{s-} = \underline{X}_t\}. \end{aligned}$$

The following identity, referred to as the Pecherskii-Rogozin-Spitzer (PRS) identity in the remainder, is key to the results in this paper. Here and throughout, e_q denotes an exponentially distributed random variable with parameter q, independent of X and Y. For an account of the history of this identity, we refer to Bertoin [6].

Proposition 1 (Pecherskii-Rogozin-Spitzer) *We have for $\alpha \ge 0$, $\beta \in \mathbb{R}$, and $q > 0$,*

$$\mathbb{E}e^{-\alpha e_q + i\beta X_{e_q}} = \mathbb{E}e^{-\alpha \overline{G}_{e_q}^X + i\beta \overline{X}_{e_q}} \mathbb{E}e^{-\alpha \underline{F}_{e_q}^X + i\beta \underline{X}_{e_q}} = \mathbb{E}e^{-\alpha \overline{F}_{e_q}^X + i\beta \overline{X}_{e_q}} \mathbb{E}e^{-\alpha \underline{G}_{e_q}^X + i\beta \underline{X}_{e_q}}.$$

The second equality follows from the first by considering the dual process $\hat{X} = -X$. The PRS identity is sometimes referred to as the first factorization identity. It can be viewed as a representation of (e_q, X_{e_q}) as as sum of two independent vectors, since $(\underline{F}_{e_q}^X, \underline{X}_{e_q})$ is distributed as $(e_q - \overline{G}_{e_q}^X, X_{e_q} - \overline{X}_{e_q})$.

In order to relate the PRS factors of Z and X, we need an auxiliary random walk. We write $\lambda \in [0, \infty)$ for the intensity of Y, and ξ for its generic jump. For fixed $q > 0$, let $\{S_n^q\}$ be a random walk with step size distribution $\xi + X_{e_{\lambda+q}}$, where the two summands are independent. For this random walk, we define the first strict ascending (descending) ladder epoch $\tau_{s\pm}^q$ as

$$\tau_{s+}^q = \inf\{n \geq 1 : S_n^q > 0\}, \quad \tau_{s-}^q = \inf\{n \geq 1 : S_n^q < 0\},$$

and $\tau_{w\pm}^q$ is defined similarly with a weak inequality. We write $H_{w\pm}^q$ ($H_{s\pm}^q$) for the ladder height $S_{\tau_{w\pm}^q}^q$ ($S_{\tau_{s\pm}^q}^q$).

When integrating with respect to defective distributions, we only carry out the integration over the set where the random variables are both finite and well-defined. For instance, we write $\mathbb{E}e^{-\beta H_{s+}^q}\rho^{\tau_{s+}^q}$ for $\mathbb{E}\left[e^{-\beta H_{s+}^q}\rho^{\tau_{s+}^q}; \tau_{s+}^q < \infty\right]$ in the remainder of this paper, unless indicated otherwise.

2.2 The PRS factorization and ladder characteristics

The main result of this section, which we now formulate, relates the PRS factors of Z and X. When a specific structure is imposed on X and Y, both factors can be computed; see Section 3. Intuitively, a PRS factor of Z is the product of a PRS factor of X and a random-walk PRS factor. The main complication is that the random walk is made into a continuous-time process by 'stretching' time, but this stretching is not done independently of the step size.

Theorem 1 *For every $\alpha, \beta, q > 0$, we have*

$$\mathbb{E}e^{-\alpha\overline{G}_{e_q}^Z - \beta\overline{Z}_{e_q}} = \mathbb{E}e^{-\alpha\overline{G}_{e_{\lambda+q}}^X - \beta\overline{X}_{e_{\lambda+q}}} \frac{1 - \mathbb{E}\left(\frac{\lambda}{\lambda+q}\right)^{\tau_{w+}^q}}{1 - \mathbb{E}e^{-\beta H_{w+}^{q+\alpha}}\left(\frac{\lambda}{\lambda+q+\alpha}\right)^{\tau_{w+}^{q+\alpha}}},$$

$$\mathbb{E}e^{-\alpha\overline{F}_{e_q}^Z - \beta\overline{Z}_{e_q}} = \mathbb{E}e^{-\alpha\overline{F}_{e_{\lambda+q}}^X - \beta\overline{X}_{e_{\lambda+q}}} \frac{1 - \mathbb{E}\left(\frac{\lambda}{\lambda+q}\right)^{\tau_{s+}^q}}{1 - \mathbb{E}e^{-\beta H_{s+}^{q+\alpha}}\left(\frac{\lambda}{\lambda+q+\alpha}\right)^{\tau_{s+}^{q+\alpha}}},$$

and $\mathbb{E}e^{-\alpha\underline{F}_{e_q}^Z + i\beta\underline{Z}_{e_q}}$, $\mathbb{E}e^{-\alpha\underline{G}_{e_q}^Z + i\beta\underline{Z}_{e_q}}$ follow by duality.

Proof. We only prove the first equality; the argument is easily adapted to obtain the second.

The first factor is a direct consequence of the independence of the first straight line in the second plot of Figure 1 and the other pieces; see the remarks accompanying Figure 1. Writing for $i \geq 1$, $w_i = T_{i-1} + \overline{G}_i - \overline{G}_1$ and $W_i = Z_{T_{i-1}+\overline{G}_i} - S_1$, these arguments also yield that $\{W_i : i \geq 1\}$ is a random walk with the same distribution as $\{S_n^q : n \geq 0\}$, except for the killing in every step with probability $\lambda/(\lambda + q)$. Therefore, if we define the first (weak) ascending ladder epoch of this random walk

$$N = \inf\{i \geq 1 : W_i \geq 0\},$$

we have

$$P(N < \infty) = \mathbb{E}\left(\frac{\lambda}{\lambda + q}\right)^{\tau_{w+}^q}.$$

Observe that $(\overline{G}_{e_q}^Z - \overline{G}_1, \overline{Z}_{e_q} - S_1)$ has the same distribution as $\sum_{j=1}^K (w_N^j, W_N^j)$, where K is geometrically distributed on $\mathbb{Z}_+$ with parameter $P(N < \infty)$, and (w_N^j, W_N^j) are independent copies of (w_N, W_N), also independent of K. Note that we consider the weak ladder epoch in the definition of N, since we are interested in $\overline{G}_{e_q}^Z$ (as opposed to $\overline{F}_{e_q}^Z$). This shows that

$$\mathbb{E}e^{-\alpha\left(\overline{G}_{e_q}^Z - \overline{G}_1\right) + i\beta\left(\overline{Z}_{e_q} - S_1\right)} = \frac{1 - \mathbb{E}\left(\frac{\lambda}{\lambda+q}\right)^{\tau_{w+}^q}}{1 - \mathbb{E}\left(\frac{\lambda}{\lambda+q}\right)^N e^{-\alpha w_N + i\beta W_N}},$$

and it remains to study the denominator in more detail.

For this, we rely on Section I.1.12 of Prabhu [29]. The key observation is that $\{(w_i, W_i)\}$ is a random walk in the half-plane $\mathbb{R}_+ \times \mathbb{R}$, with step size distribution characterized by

$$\mathbb{E}e^{-\alpha w_1 + i\beta W_1} = \mathbb{E}e^{-\alpha e_{\lambda+q} + i\beta X_{e_{\lambda+q}}}\mathbb{E}e^{i\beta\xi}.$$

Theorem 27 of [29], which is a Wiener-Hopf factorization for random walks on the half-plane, shows that we may write for $|z| < 1$ and $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$1 - z\mathbb{E}e^{-\alpha e_{\lambda+q} + i\beta X_{e_{\lambda+q}}}\mathbb{E}e^{i\beta\xi} = \left[1 - \mathbb{E}z^N e^{-\alpha w_N + i\beta W_N}\right]\left[1 - \mathbb{E}z^{\bar{N}} e^{-\alpha w_{\bar{N}} + i\beta W_{\bar{N}}}\right],$$

where the bars refer to (strict) descending ladder variables. The actual definitions of these quantities are of minor importance to us; the crucial point is that this factorization is unique. Indeed, an alternative characterization is obtained by conditioning on the value of $e_{\lambda+q}$:

$$1 - z\mathbb{E}e^{-\alpha e_{\lambda+q} + i\beta X_{e_{\lambda+q}}}\mathbb{E}e^{i\beta\xi} = 1 - \frac{(\lambda+q)z}{\lambda+q+\alpha}\mathbb{E}e^{i\beta X_{e_{\lambda+q+\alpha}}}\mathbb{E}e^{i\beta\xi},$$

and the Wiener-Hopf factorization for random walks shows that this can be written as

$$\left[1 - \mathbb{E}\left(\frac{(\lambda+q)z}{\lambda+q+\alpha}\right)^{\tau_{s+}^{q+\alpha}} e^{i\beta H_{s+}^{q+\alpha}}\right]\left[1 - \mathbb{E}\left(\frac{(\lambda+q)z}{\lambda+q+\alpha}\right)^{\tau_{w-}^{q+\alpha}} e^{i\beta H_{w-}^{q+\alpha}}\right]. \tag{1}$$

This decomposition is again unique, so that the claim follows upon substituting $z = \lambda/(\lambda+q)$. □

If $\alpha = 0$, we must have $\mathbb{E}e^{-\alpha\overline{G}_{e_q}^Z - \beta\overline{Z}_{e_q}} = \mathbb{E}e^{-\alpha\overline{F}_{e_q}^Z - \beta\overline{Z}_{e_q}}$, but the formulas in Theorem 1 differ in the sense of weak and strict ladder variables. This is not a contradiction, as Spitzer's identity shows that the fractions are equal for both τ_{w+}^q and τ_{s+}^q.

Let us now verify that the formulas of Theorem 1 are in accordance with the PRS factorization of Proposition 1. Indeed, with the Wiener-Hopf factorization for random walks (1) and Theorem 1 (the transform $\mathbb{E}e^{-\alpha\underline{F}_{e_q}^Z + i\beta\underline{Z}_{e_q}}$ is obtained by duality), we have

$$\mathbb{E}e^{-\alpha\overline{G}_{e_q}^Z + i\beta\overline{Z}_{e_q}}\mathbb{E}e^{-\alpha\underline{F}_{e_q}^Z + i\beta\underline{Z}_{e_q}} = \mathbb{E}e^{-\alpha e_{\lambda+q} + i\beta X_{e_{\lambda+q}}}\frac{1 - \frac{\lambda}{\lambda+q}}{1 - \frac{\lambda}{\lambda+q+\alpha}\mathbb{E}e^{i\beta X_{e_{\lambda+q+\alpha}}}\mathbb{E}e^{i\beta\xi}}.$$

By conditioning on the value of $e_{\lambda+q}$ in the first factor, it is readily seen that this equals

$$\frac{q}{\frac{\lambda+q+\alpha}{\mathbb{E}e^{i\beta X_{e_{\lambda+q+\alpha}}}} - \lambda\mathbb{E}e^{i\beta\xi}} = \frac{q}{\lambda+q+\alpha+\Psi_X(\beta) - \lambda\mathbb{E}e^{i\beta\xi}} = \mathbb{E}e^{-\alpha e_q + i\beta Z_{e_q}}.$$

Given Theorem 1, one can easily deduce the characteristics of the *ladder height process* of Z in terms of those of X; as the notions are standard, we refer to p. 157 of Bertoin [6] for definitions. Further evidence for the importance of this two-dimensional subordinator has recently been given by Doney and Kyprianou [15].

The dual processes of Z and X are defined by $\hat{Z} = -Z$ and $\hat{X} = -X$ respectively.

Corollary 1 *For $\alpha, \beta \geq 0$, we have*

$$\kappa_Z(\alpha,\beta) = \kappa_X(\lambda+\alpha,\beta)\left(1 - \mathbb{E}e^{-\beta H_{s+}^\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\tau_{s+}^\alpha}\right),$$

and

$$\begin{aligned}\hat{\kappa}_Z(\alpha,\beta) &= k\hat{\kappa}_X(\lambda+\alpha,\beta)\left(1-\mathbb{E}e^{\beta H^\alpha_{w-}}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\tau^\alpha_{w-}}\right)\\ &= k\frac{\alpha+\Psi_Z(-i\beta)}{\kappa_X(\lambda+\alpha,-\beta)\left[1-\mathbb{E}e^{\beta H^\alpha_{s+}}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\tau^\alpha_{s+}}\right]},\end{aligned}$$

where k is some meaningless constant.

Proof. It suffices to note that $\kappa_Z(\alpha,-i\beta)\hat{\kappa}_Z(\alpha,i\beta) = k(\alpha+\Psi_Z(\beta))$ by the Wiener-Hopf factorization for random walks, and to continue $\hat{\kappa}_Z$ analytically. □

3 Fluctuation theory with phase-type upward jumps

In this section, we use the results of the previous section to study Lévy processes with phase-type upward jumps, and general downward jumps. According to these results, $(\overline{G}^Z_{e_q}, \overline{Z}_{e_q})$ can be written as the sum of $(\overline{G}^X_{e_{\lambda+q}}, \overline{X}_{e_{\lambda+q}})$ and an (independent) random walk term. In this section, we choose X and Y appropriately, so that the transforms of both vectors can be computed explicitly.

For this, we let X be an arbitrary spectrally negative Lévy process, and Y is a compound Poisson process (not necessarily a subordinator), independent of X, for which the upward jumps have a phase-type distribution. The exact form of the Lévy measure of Y is specified by (2) below.

Apart from their computational convenience, the most important property of phase-type distributions is that they are dense, in the sense of weak convergence, in the class of probability measures (although many phases may be needed to approximate a stable distribution, for instance). A phase-type distribution is the absorption time of a Markov process on a finite state space E. Its intensity matrix matrix is determined by the $|E|\times|E|$-matrix $\boldsymbol{T}$, and its initial distribution is denoted by $\boldsymbol{\alpha}$. For more details on phase-type distributions, we refer to Asmussen [3]. We write $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$, where $\mathbf{1}$ is the vector with ones.

Fluctuation theory for Lévy processes with phase-type jumps has recently been studied by Asmussen *et al.* [4] and Mordecki [26]; see also Kou and Wang [24]. Just like the phase-type distributions are dense in the class of probability measures, this class of Lévy processes is dense in the Skorokhod topology on $D(\mathbb{R}_+)$ (see, e.g., [20, Ch. VI]) in the class of arbitrary Lévy processes. In both [4] and [26], the authors obtain expressions for the Laplace transform of $\overline{Z}_{e_q}$ if Y is a compound Poisson process with only positive (phase-type) jumps.

While the class of processes that we analyze here is slightly more general, the main difference is that we calculate the Laplace transform of the *joint* distribution $(\overline{G}^Z_{e_q}, \overline{Z}_{e_q})$; see Section 3. Hence, if one assumes phase-type upward jumps, one can compute the epoch at which the supremum is attained; the latter is perhaps more surprising than that one can calculate the distribution of $\overline{Z}_{e_q}$. This illustrates why Theorem 1 is interesting.

To the author's knowledge, the results in this section cover any Lévy process for which this joint distribution is known. The only case for which results are available but not covered here is when Z is a certain stable Lévy process; see Doney [13]. Then, only the distribution of the (marginal) law of $\overline{Z}_{e_q}$ is known in a semi-explicit form.

The PRS factorization

We begin with a detailed description of the process Y. Given $K\in\mathbb{N}$, suppose that we have nonnegative random variables $\{A_j : j = 1,\dots,K\}$ and $\{B_j : j = 1,\dots,K\}$, where the distribution P_{B_j} of B_j is phase-type with representation $(E_j, \boldsymbol{\alpha}_j, \boldsymbol{T}_j)$. The distribution P_{-A_j} of $-A_j$ is general; the only restriction we impose is that $P_{-A_j} * P_{B_j}(\{0\}) = 0$ for all j, i.e., A_j and B_j are not both degenerate at zero. We assume that the process Y is a compound Poisson process with Lévy measure given by

$$\Pi_Y = \lambda \sum_{j=1}^{K} \pi_j P_{B_j} * P_{-A_j}, \tag{2}$$

where $\lambda \in (0, \infty)$, $0 \le \pi_j \le 1$ with $\sum \pi_j = 1$, and $*$ denotes convolution. In queueing theory [3, 12], processes of this form arise naturally since the B can be interpreted as the service times and the A as interarrival times. Notice that Y is a subordinator if and only if Π_Y can be written as (2) with $K = 1$ and $A_1 \equiv 0$.

Without loss of generality, we may assume that E_j and $\boldsymbol{T}_j$ do not depend on j. Indeed, if E_j has m_j elements, one can construct an E with $\sum_{j=1}^{K} m_j$ elements and $\boldsymbol{T}$ can then be chosen as a block diagonal matrix with the matrices $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_K$ on its diagonal. The vectors $\boldsymbol{\alpha}_j$ are then padded with zeros, so that they consist of K parts of lengths $m_1, \ldots, m_K$, and only the j-th part is nonzero.

Fix some $q > 0$; our first aim is to study the random walk $\{S_n^q\}$ introduced in Section 2.1, with generic step size distribution (by the PRS factorization)

$$P_{S_1^q} = P_{\overline{X}_{e_{\lambda+q}}} * P_{\underline{X}_{e_{\lambda+q}}} * P_\xi,$$

where $P_\xi = \Pi_Y/\lambda$. We suppose that either $\Pi_Y(\mathbb{R}_+) > 0$ or X is not a (negative) subordinator, so that $P_{S_1^q}$ assigns strictly positive probability to $\mathbb{R}_+$. Throughout this section, we write τ_+^q for τ_{w+}^q. This notation is motivated by the assumption $\Pi_Y(\mathbb{R}_+) > 0$, since then $\tau_{w+}^q = \tau_{s+}^q$.

Since $\overline{X}_{e_{\lambda+q}}$ is either degenerate or exponentially distributed, the law of S_1^q can be written as $\sum \pi_j P_{B_j'(q)} * P_{A_j'(q)}$, where $B_j'(q)$ has again a phase-type distribution, say with representation $(E_q', \boldsymbol{\alpha}_j'(q), \boldsymbol{T}_q')$. It is not hard to express this triple in terms of the original triple $(E, \boldsymbol{\alpha}_j, \boldsymbol{T})$: $(E_q', \boldsymbol{\alpha}_j'(q), \boldsymbol{T}_q') = (E, \boldsymbol{\alpha}_j, \boldsymbol{T})$ if X is a negative subordinator, and otherwise E_q' can be chosen such that $|E_q'| = |E| + 1$, and the dynamics of the underlying Markov chain are unchanged, except for the fact that an additional state is visited before absorption. We set $\boldsymbol{t}_q' = -\boldsymbol{T}_q'\boldsymbol{1}$.

Motivated by Theorem 1, the following lemma calculates the transform of the ladder variables (H_+^q, τ_+^q); recall that the random variables are only integrated over the subset $\{\tau_+^q < \infty\}$ of the probability space.

Lemma 1 *Let $\rho \in (0,1)$ and $\beta \ge 0$. Then there exists some vector $\boldsymbol{\alpha}_+^{\rho,q}$ such that*

$$\mathbb{E}\left[\rho^{\tau_+^q} e^{-\beta H_+^q}\right] = \boldsymbol{\alpha}_+^{\rho,q}(\beta \boldsymbol{I} - \boldsymbol{T}_q')^{-1}\boldsymbol{t}_q'.$$

Proof. The proof is similar to the proofs of Lemma VIII.5.1 and Proposition VIII.5.11 of Asmussen [3]; the details are left to the reader. □

The above lemma shows that it is of interest to be able to calculate $\boldsymbol{\alpha}_+^{\rho,q}$. Therefore, we generalize Theorem VIII.5.12 in [3] to the present setting. We omit a proof, as similar arguments apply; the only difference is that we allow for $K > 1$ and that the random walk can be killed in every step with probability ρ.

Proposition 2 $\boldsymbol{\alpha}_+^{\rho,q}$ *satisfies* $\boldsymbol{\alpha}_+^{\rho,q} = \xi(\boldsymbol{\alpha}_+^{\rho,q})$, *where*

$$\xi(\boldsymbol{\alpha}_+^{\rho,q}) = \rho \sum_{j=1}^{K} \pi_j \boldsymbol{\alpha}_j'(q) \int_0^\infty e^{(\boldsymbol{T}_q' + \boldsymbol{t}_q' \boldsymbol{\alpha}_+^{\rho,q})y} A_j'(q)(dy).$$

It can be computed as $\lim_{n\to\infty} \boldsymbol{\alpha}_+^{\rho,q}(n)$, *where* $\boldsymbol{\alpha}_+^{\rho,q}(0) = \boldsymbol{0}$ *and* $\boldsymbol{\alpha}_+^{\rho,q}(n) = \xi(\boldsymbol{\alpha}_+^{\rho,q}(n-1))$ *for* $n \ge 1$.

The main result of this section follows by combining Theorem 1 with Lemma 1 and using standard fluctuation identities, see for instance [6, Thm. VII.4]. For notational convenience, we write $\boldsymbol{\alpha}_+^q$ for $\boldsymbol{\alpha}_+^{q/(\lambda+q),q}$.

Theorem 2 *Suppose that Z is not a subordinator. Then we have for $\alpha, \beta \geq 0$,*

$$\mathbb{E}e^{-\alpha \overline{G}^Z_{e_q} - \beta \overline{Z}_{e_q}} = \frac{\Phi_X(\lambda+q)\left[1 - \boldsymbol{\alpha}_+^q \mathbf{1}\right]}{\left[\Phi_X(\lambda+q+\alpha)+\beta\right]\left[1 - \boldsymbol{\alpha}_+^{q+\alpha}(\beta \boldsymbol{I} - \boldsymbol{T}'_{q+\alpha})^{-1}\boldsymbol{t}'_{q+\alpha}\right]}$$

and

$$\mathbb{E}e^{-\alpha \overline{F}^Z_{e_q} + \beta \underline{Z}_{e_q}} = \frac{q\left[\Phi_X(\lambda+q+\alpha)-\beta\right]\left[1 + \boldsymbol{\alpha}_+^{q+\alpha}(\beta \boldsymbol{I} + \boldsymbol{T}'_{q+\alpha})^{-1}\boldsymbol{t}'_{q+\alpha}\right]}{\left[q+\alpha+\Psi_Z(-i\beta)\right]\Phi_X(\lambda+q)\left[1 - \boldsymbol{\alpha}_+^q \mathbf{1}\right]}.$$

While Theorem 2 is an immediate consequence of Theorem 1, we now formulate the corresponding analog of Corollary 1. Note that the expression for $\kappa_Z(0,\beta)$ is already visible in the work of Mordecki [26]; here, we obtain a full description of κ_Z.

Corollary 2 *For $\alpha, \beta \geq 0$, we have*

$$\kappa_Z(\alpha,\beta) = \left[\Phi_X(\lambda+\alpha)+\beta\right]\left[1 - \boldsymbol{\alpha}_+^{\alpha}(\beta \boldsymbol{I} - \boldsymbol{T}'_{\alpha})^{-1}\boldsymbol{t}'_{\alpha}\right],$$

and

$$\hat{\kappa}_Z(\alpha,\beta) = k\frac{\alpha+\Psi_Z(-i\beta)}{\left[\Phi_X(\lambda+\alpha)-\beta\right]\left[1 + \boldsymbol{\alpha}_+^{\alpha}(\beta \boldsymbol{I} + \boldsymbol{T}'_{\alpha})^{-1}\boldsymbol{t}'_{\alpha}\right]},$$

where k is a meaningless constant.

4 Perturbed risk models

Let X be an arbitrary Lévy process and Y be a compound Poisson process with intensity λ and generic positive jump ξ. In this section, we suppose that $Z = X + Y$ drifts to $-\infty$. Classical risk theory studies the supremum of Z in case X is a negative drift, i.e., $X_t = -ct$ for some $c > \lambda\mathbb{E}\xi$. Then, its distribution is given by the Pollaczek-Khinchine formula. In this analysis, a key role is played by ladder epochs and heights, i.e., quantities related to the event that Z reaches a new record.

In this section, we replace the negative drift X by an arbitrary Lévy process; in the literature, this is known as a perturbed risk model; see [17, 18, 30] and references therein. To analyze this model, the classical ladder epochs and heights are replaced by so-called modified ladder epochs and heights; these are related to the event that Z reaches a new record as a result of a jump of Y.

In Huzak *et al.* [18], Y is allowed to be a general subordinator, not necessarily of the compound Poisson type. Therefore, the perturbed risk models studied here are slightly less general. However, since any subordinator can be approximated by compound Poisson subordinators, one is led to believe that our results also hold in the general case. Since the approximation argument required for proving this is not in the spirit of this paper, we do not address this issue here. Instead, we shall content ourselves with writing the main results (Proposition 3, Theorem 3, and Theorem 4) in a form that does not rely on Y being compound Poisson, although this assumption is essential for the proofs.

In Section 4.1, we derive a Pollaczek-Khinchine formula for perturbed risk models. Unfortunately, the formula is not so explicit. Therefore, we impose further assumptions in Section 4.2, where we study spectrally positive Z.

As mentioned already, a central role in perturbed risk models is played by the first time χ a new supremum is reached by a jump of Y, i.e.,

$$\chi = \inf\{t > 0 : \Delta Y_t > \overline{Z}_{t-} - Z_{t-}\}.$$

In Figure 1, we have $\chi = T_2$. On the event $\{\chi = \infty\}$, we define $(\overline{G}^Z_{\chi-}, \overline{Z}_{\chi-})$ as $(\overline{G}^Z_\infty, \overline{Z}_\infty)$.

4.1 Generalities

In this subsection, we study the structure of a general perturbed risk model, i.e., we consider a *general* Lévy perturbation X. The results that we obtain are new in this general framework.

The following proposition is crucial for the analysis in this section.

Proposition 3 *We have*

1. $(\overline{G}^Z_{\chi-}, \overline{Z}_{\chi-})$ *is independent of* $\{\chi < \infty\}$*;*
2. $(\overline{G}^Z_{\chi-}, \overline{Z}_{\chi-})$ *is distributed as* $(\overline{G}^Z_\infty, \overline{Z}_\infty)$ *given* $\{\chi = \infty\}$*;*
3. $(Z_\chi - \overline{Z}_{\chi-}, \overline{Z}_{\chi-} - Z_{\chi-}, \chi - \overline{G}^Z_{\chi-})$ *is conditionally independent of* $(\overline{G}^Z_{\chi-}, \overline{Z}_{\chi-})$ *given* $\{\chi < \infty\}$.

Proof. We need some definitions related to the piecewise linear (jump) process of Figure 1, in particular to its excursions. Let $\tilde{P}$ denote the law of the piecewise linear process that is constructed by discarding the first (increasing) piece, and let $\tilde{\mathbb{E}}$ denote the corresponding expectation. Under $\tilde{P}$, there are two possibilities for the process to (strictly) cross the axis: it either crosses continuously or it jumps over it. The event that the first happens is denoted by $\mathcal{X}$, as it is caused by fluctuations in X. We write $\mathcal{Y}$ for the second event. The probability of no crossing (i.e., no new record) is then given by $1 - \tilde{P}(\mathcal{X}) - \tilde{P}(\mathcal{Y})$. Moreover, by the strong Markov property, we have

$$P(\chi < \infty) = \frac{\tilde{P}(\mathcal{Y})}{1 - \tilde{P}(\mathcal{X})}. \tag{3}$$

On $\mathcal{X}$ and $\mathcal{Y}$, we also define the 'excursion lengths' L_e and 'excursion heights' H_e. Moreover, we also define the undershoot U_e on $\mathcal{Y}$; see Figure 2. The dotted line is the piece that is discarded under $\tilde{P}$.

For $\alpha, \beta \geq 0$, by the strong Markov property,

$$\begin{aligned} &\mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\chi-} - \beta \overline{Z}_{\chi-}}; \chi < \infty\right] \\ = \; &\mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda} - \beta \overline{X}_{e_\lambda}} \tilde{P}(\mathcal{Y}) + \tilde{\mathbb{E}}\left[e^{-\alpha L_e - \beta H_e}; \mathcal{X}\right] \mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\chi-} - \beta \overline{Z}_{\chi-}}; \chi < \infty\right], \end{aligned}$$

from which we obtain

$$\mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\chi-} - \beta \overline{Z}_{\chi-}}; \chi < \infty\right] = \mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda} - \beta \overline{X}_{e_\lambda}} \frac{\tilde{P}(\mathcal{Y})}{1 - \tilde{\mathbb{E}}\left[e^{-\alpha L_e - \beta H_e}; \mathcal{X}\right]}.$$

Along the same lines, one can deduce that

$$\mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\infty} - \beta \overline{Z}_{\infty}}; \chi = \infty\right] = \mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda} - \beta \overline{X}_{e_\lambda}} \frac{1 - \tilde{P}(\mathcal{X}) - \tilde{P}(\mathcal{Y})}{1 - \tilde{\mathbb{E}}\left[e^{-\alpha L_e - \beta H_e}; \mathcal{X}\right]},$$

so that

$$\frac{\mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\chi-} - \beta \overline{Z}_{\chi-}}; \chi < \infty\right]}{P(\chi < \infty)} = \mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\chi-} - \beta \overline{Z}_{\chi-}}; \chi < \infty\right] + \mathbb{E}\left[e^{-\alpha \overline{G}^Z_{\infty} - \beta \overline{Z}_{\infty}}; \chi = \infty\right],$$

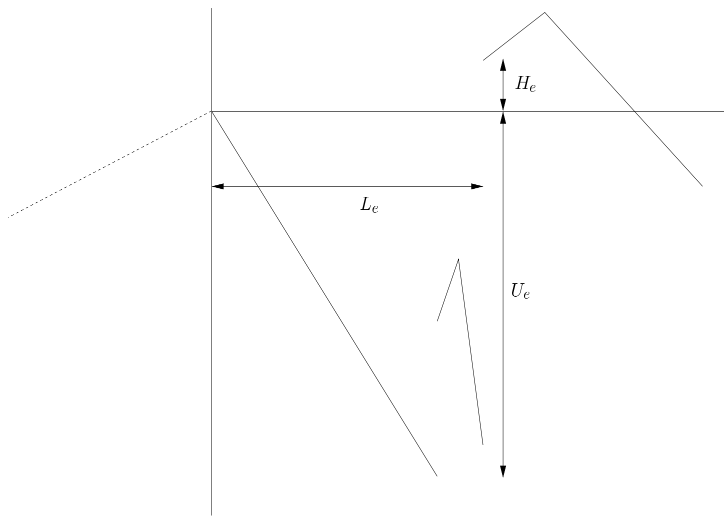

Figure 2: The excursion quantities in the proof of Proposition 3.

which is $\mathbb{E}e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}$; this is the first claim. These calculations also show that

$$\frac{\mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}};\chi=\infty\right]}{P(\chi=\infty)} = \mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}};\chi<\infty\right]+\mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}};\chi=\infty\right],$$

which is the second claim.

For the third claim, it suffices to notice that a variant of the above argument yields for $\alpha,\beta,\gamma,\delta,\varepsilon\geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}e^{-\gamma\left[Z_\chi-\overline{Z}_{\chi-}\right]}e^{-\delta\left[\overline{Z}_{\chi-}-Z_{\chi-}\right]}e^{-\varepsilon\left[\chi-\overline{G}^Z_{\chi-}\right]}\middle|\chi<\infty\right]\\
=\;&\mathbb{E}e^{-\alpha\overline{G}^X_{e_\lambda}-\beta\overline{X}_{e_\lambda}}\frac{1-\tilde{P}(\mathcal{X})}{1-\tilde{\mathbb{E}}\left[e^{-\alpha L_e-\beta H_e};\mathcal{X}\right]}\frac{\tilde{\mathbb{E}}\left[e^{-\gamma H_e}e^{\delta U_e}e^{-\varepsilon L_e};\mathcal{Y}\right]}{\tilde{P}(\mathcal{Y})}.
\end{aligned}$$

□

The formula in the following theorem can be viewed as a generalized Pollaczek-Khinchine formula for perturbed risk models. It is a consequence of the preceding proposition and the observation that by the strong Markov property

$$\mathbb{E}e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}} = \frac{\mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}};\chi=\infty\right]}{1-\mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}};\chi<\infty\right]\mathbb{E}\left[e^{-\alpha\left[\chi-\overline{G}^Z_{\chi-}\right]-\beta\left[Z_\chi-\overline{Z}_{\chi-}\right]}\middle|\chi<\infty\right]}.$$

Theorem 3 *For $\alpha,\beta\geq 0$, we have*

$$\mathbb{E}e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}} = \frac{P(\chi=\infty)\mathbb{E}e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}}{1-P(\chi<\infty)\mathbb{E}e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}\mathbb{E}\left[e^{-\alpha\left[\chi-\overline{G}^Z_{\chi-}\right]-\beta\left[Z_\chi-\overline{Z}_{\chi-}\right]}\middle|\chi<\infty\right]}.$$

4.2 Spectrally positive Z

In this subsection, we analyze the case that Z has only positive jumps. It turns out that the transforms in the previous section can then be computed. As indicated below, this generalizes the results of Huzak *et al.* [18] (modulo the remarks at the beginning of this section). For instance, we obtain the transform of the distribution of (χ, Z_χ). We remark that perturbed risk models with positive jumps are related to $M/G/1$ queueing systems with a second service; see [11].

Throughout, we exclude the case that X is a (negative) subordinator, i.e., that X is a negative drift; the analysis is then classical. By doing so, the standard fluctuation identities can be used for both X and Z, see Theorem VII.4 of Bertoin [6]. In this subsection, we use these identities without further reference.

Our analysis is based on Wiener-Hopf theory for Markov additive processes. Indeed, the Lévy process can be embedded in a Markov additive process in discrete time (e.g., [3, Ch. XI]). To see this, fix some $\alpha \geq 0$, and note that (VI.1) of Bertoin [6] implies that for $q > 0$ and $\beta \in \mathbb{R}$,

$$\mathbb{E}e^{-\alpha \overline{G}^X_{e_q} + i\beta \overline{X}_{e_q}} = \mathbb{E}e^{-\alpha \overline{G}^X_{e_q}} \mathbb{E}e^{i\beta \overline{X}_{e_{q+\alpha}}},$$

and similarly for the joint distribution of $(\underline{F}^X_{e_q}, \underline{X}_{e_q})$. In other words, since α is fixed, the joint distribution can be interpreted as a (defective) marginal distribution. Hence, a 'killing mechanism' has been introduced.

Define a Markov additive process in discrete time $\{(J_n, S_n)\}$ as the Markov process with state space $\{1,2,3\} \times \mathbb{R}$, characterized by the transform matrix

$$\boldsymbol{F}(\alpha, \beta) = \begin{pmatrix} 0 & \mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda}} \mathbb{E}e^{i\beta \overline{X}_{e_{\lambda+\alpha}}} & 0 \\ 0 & 0 & \mathbb{E}e^{-\alpha \underline{F}^X_{e_\lambda}} \mathbb{E}e^{i\beta \underline{X}_{e_{\lambda+\alpha}}} \\ \mathbb{E}e^{i\beta\xi} & 0 & 0 \end{pmatrix}.$$

That is, $S_0 = 0$, and J_n is deterministic given J_0: in every time slot, it jumps from i to $i+1$, unless $i = 3$; then it jumps back to 1. If $J_{n-1} = 1$, the process is killed with probability $1 - \mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda}}$, and otherwise we set $S_n = S_{n-1} + \eta_{n-1}$, where η_{n-1} is independent of S_{n-1} and distributed as $\overline{X}_{e_{\lambda+\alpha}}$. The cases $J_{n-1} = 2$ and $J_{n-1} = 3$ are similar, except for the absence of killing in the latter case. We also write

$$\tau_+ = \inf\{n > 0 : S_n > 0\}, \quad \tau_- = \inf\{n > 0 : S_n \leq 0\}.$$

Expressions of the type $P_2(J_{\tau_+} = 2)$ should be understood as $P(J_{\tau_+} = 2, \tau_+ < \infty | J_0 = 2)$, and similarly for $\mathbb{E}_2$.

In Wiener-Hopf theory for Markov-additive processes, an important role is played by the time-reversed process. To define it, we introduce the Markov chain $\hat{J}$, for which the transitions are deterministic: it jumps from 3 to 2, from 2 to 1, and from 1 to 3. Hence, it jumps into the opposite direction of J. We set $\hat{S}_0 = 0$, and define the transition structure of the time-reversed Markov additive process $(\hat{J}, \hat{S})$ as follows. If $\hat{J}_{n-1} = 2$, the process is killed with probability $1 - \mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda}}$, and otherwise we set $\hat{S}_n = \hat{S}_{n-1} + \hat{\eta}_{n-1}$, where $\hat{\eta}_{n-1}$ is independent of $\hat{S}_{n-1}$ and distributed as $\overline{X}_{e_{\lambda+\alpha}}$. Similarly, if $\hat{J}_{n-1} = 3$, the process is killed with probability $1 - \mathbb{E}e^{-\alpha \underline{F}^X_{e_\lambda}}$, and otherwise the increment is distributed as $\underline{X}_{e_{\lambda+\alpha}}$. If $\hat{J}_{n-1} = 1$, the increment is distributed as $\xi > 0$. The quantities $\hat{\tau}_+$ and $\hat{\tau}_-$ are defined as for $\{(J_n, S_n)\}$. We write $\hat{P}_2$ for the conditional distribution given $\hat{J}_0 = 2$.

Recalling that the dependence on α is 'absorbed' in the killing mechanism, we define

$$G_+^{(k,\ell)}(\alpha, \beta) = \mathbb{E}_k \left[e^{i\beta S_{\tau_+}}; J_{\tau_+} = \ell \right]$$

and

$$\hat{G}_-^{(k,\ell)}(\alpha, \beta) = \hat{\mathbb{E}}_k \left[e^{i\beta \hat{S}_{\hat{\tau}_-}}; \hat{J}_{\hat{\tau}_-} = j \right].$$

Note that $G_+^{(2,2)} = \tilde{\mathbb{E}}\left[e^{-\alpha L_e + i\beta H_e}; \mathcal{X}\right]$ in the notation of the proof of Proposition 3, and similarly for $G_+^{(2,1)}$; then $\mathcal{X}$ is replaced by $\mathcal{Y}$.

The Wiener-Hopf factorization for Markov additive processes (refer to, e.g., Asmussen [3, Thm. XI.2.12] or Prabhu [29, Thm. 5.2]) states that $\boldsymbol{I} - \boldsymbol{F}(\alpha,\beta)$ (where $\boldsymbol{I}$ denotes the identity matrix) equals

$$\begin{pmatrix} 1 & 0 & 0 \\ -\hat{G}_-^{(1,2)} & 1-\hat{G}_-^{(2,2)} & -\hat{G}_-^{(3,2)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbb{E}e^{-\alpha \overline{G}^X_{e_\lambda} + i\beta \overline{X}_{e_\lambda}} & 0 \\ -G_+^{(2,1)} & 1-G_+^{(2,2)} & 0 \\ -\mathbb{E}e^{i\beta\xi} & 0 & 1 \end{pmatrix},$$

where the arguments α and β of G_+ and $\hat{G}_-$ are suppressed for notational convenience.

We start by computing the first matrix. Note that $\hat{G}_-^{(3,2)}(\alpha,\beta) = \mathbb{E}e^{-\alpha \underline{F}_{e_\lambda} + i\beta \underline{X}_{e_\lambda}}$, so that two terms remain. Recall that Φ_{-X} is the inverse of the function $\beta \mapsto \psi_{-X}(\beta) = -\Psi_{-X}(-i\beta)$, and similarly for Φ_{-Z}.

Proposition 4 *For $\beta \in \mathbb{R}$, we have*

$$\hat{G}_-^{(1,2)}(\alpha,\beta) = \mathbb{E}e^{-\Phi_{-Z}(\alpha)\xi} \frac{\Phi_{-X}(\lambda)}{\Phi_{-X}(\lambda+\alpha)+i\beta},$$

and

$$\hat{G}_-^{(2,2)}(\alpha,\beta) = \frac{\Phi_{-X}(\lambda+\alpha) - \Phi_{-Z}(\alpha)}{\Phi_{-X}(\lambda+\alpha)+i\beta}.$$

Proof. We start with $\hat{G}_-^{(2,2)}$. By 'gluing together' the transitions $2 \to 1$ and $1 \to 3$, we see that the killing probability for going from 2 to itself now equals $\lambda/(\lambda+\alpha)$, and the distribution of a jump from 2 to itself can be written as $\xi + \overline{X}_{e_{\lambda+\alpha}} - e_{\Phi_{-X}(\lambda+\alpha)}$, where all three components are independent. Therefore, by standard results on random walks (e.g., Lemma I.4 of Prabhu [29]), we have

$$\hat{G}_-^{(2,2)}(\alpha,\beta) = \hat{\mathbb{E}}_2 \left(\frac{\lambda}{\lambda+\alpha}\right)^{\hat{\tau}_-} e^{i\beta \hat{S}_{\hat{\tau}_-}} = \hat{\mathbb{E}}_2 \left(\frac{\lambda}{\lambda+\alpha}\right)^{\hat{\tau}_-} \frac{\Phi_{-X}(\lambda+\alpha)}{\Phi_{-X}(\lambda+\alpha)+i\beta},$$

and it remains to calculate the expectation, which we write as η_α. For this, we repeat the argument that led to Theorem 1, but now for the minimum and in terms of η_α. We see that $\mathbb{E}e^{i\beta \underline{Z}_\alpha}$ equals

$$\frac{\Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta} = \frac{\Phi_{-X}(\lambda+\alpha)}{\Phi_{-X}(\lambda+\alpha)+i\beta} \frac{1-\eta_\alpha}{1-\eta_\alpha \frac{\Phi_{-X}(\lambda+\alpha)}{\Phi_{-X}(\lambda+\alpha)+i\beta}} = \frac{(1-\eta_\alpha)\Phi_{-X}(\lambda+\alpha)}{(1-\eta_\alpha)\Phi_{-X}(\lambda+\alpha)+i\beta},$$

so that $1-\eta_\alpha = \Phi_{-Z}(\alpha)/\Phi_{-X}(\lambda+\alpha)$.

Now we study $\hat{G}_-^{(1,2)}$. A descending ladder epoch occurs either at the first time that $\hat{J}$ visits 2, or in subsequent visits. The contribution to $\hat{G}_-^{(1,2)}$ of the first term is

$$\begin{aligned}
\hat{\mathbb{E}}_1\left[e^{i\beta \hat{S}_2}; \hat{S}_2 < 0\right] &= \mathbb{E}e^{-\alpha \underline{F}^X_{e_\lambda}} \mathbb{E}\left[e^{i\beta(\xi + \underline{X}_{\lambda+\alpha})}; \xi + \underline{X}_{\lambda+\alpha} < 0\right] \\
&= \int_0^\infty \Phi_{-X}(\lambda) e^{-(\Phi_{-X}(\lambda+\alpha)+i\beta)t} \mathbb{E}\left[e^{i\beta\xi}; \xi < t\right] dt \\
&= \mathbb{E}e^{i\beta\xi} \int_0^\xi \Phi_{-X}(\lambda) e^{-(\Phi_{-X}(\lambda+\alpha)+i\beta)t} dt \\
&= \frac{\Phi_{-X}(\lambda)}{\Phi_{-X}(\lambda+\alpha)+i\beta} \mathbb{E}e^{-\Phi_{-X}(\lambda+\alpha)\xi}.
\end{aligned}$$

To compute the contribution to $\hat{G}_-^{(1,2)}$ of paths for which $\hat{S}_2$ is positive, we apply results of Arjas and Speed [1] on random walks with a random initial point. Although we use their notation and arguments, we do not repeat them; the details are left to the reader.

In the notation of [1], using the previously computed $\hat{G}_-^{(2,2)}$, (again, the transform depends on α through the killing mechanism), we have

$$\bar{w}_{z-}(\beta) = \frac{1}{1 - \frac{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}{\Phi_{-X}(\lambda+\alpha)+i\beta}} = 1 + \frac{\Phi_{-X}(\lambda+\alpha) - \Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta}.$$

Therefore, using the projection operator $\mathcal{P}$ as defined by (2.4) of [1], the second contribution to $\hat{G}_-^{(1,2)}$ equals by Theorem 1(b) of [1],

$$\frac{1}{\bar{w}_{z-}(\beta)}\mathcal{P}\left[\mathbb{E}e^{-\alpha \underline{F}^X_{e_\lambda}}\mathbb{E}\left[e^{i\beta(\xi+\underline{X}_{\lambda+\alpha})}; \xi + \underline{X}_{\lambda+\alpha} > 0\right]\bar{w}_{z-}(\beta)\right]. \tag{4}$$

A similar reasoning as before shows that

$$\begin{aligned}
\hat{\mathbb{E}}_1\left[e^{i\beta\hat{S}_2}; \hat{S}_2 > 0\right] &= \mathbb{E}e^{-\alpha \underline{F}^X_{e_\lambda}}\mathbb{E}\left[e^{i\beta\left(\xi+\underline{X}_{e_{\lambda+\alpha}}\right)}; \xi + \underline{X}_{e_{\lambda+\alpha}} > 0\right] \\
&= \Phi_{-X}(\lambda)\int_0^\infty e^{-(\Phi_{-X}(\lambda+\alpha)+i\beta)t}\mathbb{E}\left[e^{i\beta\xi}; \xi > t\right]dt \\
&= \Phi_{-X}(\lambda)\frac{\mathbb{E}e^{i\beta\xi} - \mathbb{E}e^{-\Phi_{-X}(\lambda+\alpha)\xi}}{\Phi_{-X}(\lambda+\alpha)+i\beta}.
\end{aligned}$$

As this is the transform of a positive random variable, the first observation in the proof of Corollary 1 in [1] shows that

$$\mathcal{P}\left[\frac{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta}\hat{\mathbb{E}}_1\left[e^{i\beta\hat{S}_2}; \hat{S}_2 > 0\right]\right] = \frac{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta}\hat{\mathbb{E}}_1\left[e^{-\Phi_{-Z}(\alpha)\hat{S}_2}; \hat{S}_2 > 0\right].$$

Therefore, (4) equals

$$\begin{aligned}
&\Phi_{-X}(\lambda)\frac{\mathbb{E}e^{-\Phi_{-Z}(\alpha)\xi} - \mathbb{E}e^{-\Phi_{-X}(\lambda+\alpha)\xi}}{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}\frac{\frac{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta}}{1+\frac{\Phi_{-X}(\lambda+\alpha)-\Phi_{-Z}(\alpha)}{\Phi_{-Z}(\alpha)+i\beta}} \\
= &\frac{\Phi_{-X}(\lambda)}{\Phi_{-X}(\lambda+\alpha)+i\beta}\left[\mathbb{E}e^{-\Phi_{-Z}(\alpha)\xi} - \mathbb{E}e^{-\Phi_{-X}(\lambda+\alpha)\xi}\right].
\end{aligned}$$

The claim follows by summing the two contributions. □

With the preceding proposition at our disposal, the Wiener-Hopf factorization yields that

$$G_+^{(2,1)}(\alpha,\beta) = \Phi_{-X}(\lambda)\frac{\mathbb{E}e^{i\beta\xi} - \mathbb{E}e^{-\Phi_{-Z}(\alpha)\xi}}{\Phi_{-Z}(\alpha)+i\beta},$$

and

$$1 - G_+^{(2,2)}(\alpha,\beta) = \frac{\Phi_{-X}(\lambda+\alpha)+i\beta - \Phi_{-X}(\lambda)\mathbb{E}e^{-\Phi_Z(\alpha)\xi}\mathbb{E}e^{-\alpha\overline{G}^X_{e_\lambda}+i\beta\overline{X}_{e_\lambda}}}{\Phi_{-Z}(\alpha)+i\beta},$$

and $\mathbb{E}e^{-\alpha\overline{G}^X_{e_\lambda}+i\beta\overline{X}_{e_\lambda}}$ is explicitly known in terms of Φ_{-X}.

From these expressions, upon choosing $\alpha = \beta = 0$, one obtains that $\tilde{P}(\mathcal{X}) = 1 + \frac{\Phi_{-X}(\lambda)}{\lambda}\mathbb{E}X_1$ and $\tilde{P}(\mathcal{Y}) = \Phi_{-X}(\lambda)\mathbb{E}\xi$. In particular, $1 - \tilde{P}(\mathcal{X}) - \tilde{P}(\mathcal{Y}) = -\frac{\Phi_{-X}(\lambda)}{\lambda}\mathbb{E}Z_1$.

Our next goal is to characterize distributions related to modified ladder epochs and heights, leading to the main result of this subsection.

Theorem 4 *Let X be spectrally positive, but not a negative drift.*

1. *For $\alpha, \beta \geq 0$,*

$$\begin{aligned} \mathbb{E}e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}} &= \mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}\middle|\chi<\infty\right] = \mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\infty}-\beta\overline{Z}_{\infty}}\middle|\chi=\infty\right] \\ &= -\mathbb{E}X_1\frac{\Phi_{-Z}(\alpha)-\beta}{\alpha-\psi_{-X}(\beta)-\psi_{-Y}(\Phi_{-Z}(\alpha))}, \end{aligned}$$

which should be interpreted as $-\mathbb{E}X_1/\psi'_{-Z}(\beta)$ for $\beta = \Phi_{-Z}(\alpha)$.

In particular, $\overline{Z}_{\chi-}$ has the same distribution as $\overline{X}_\infty$.

2. *For $\alpha, \beta \geq 0$,*

$$\mathbb{E}\left[e^{-\alpha\left[\chi-\overline{G}^Z_{\chi-}\right]-\beta\left[Z_\chi-\overline{Z}_{\chi-}\right]}\middle|\chi<\infty\right] = \frac{1}{\mathbb{E}Y_1}\frac{\psi_{-Y}(\beta)-\psi_{-Y}(\Phi_{-Z}(\alpha))}{\Phi_{-Z}(\alpha)-\beta},$$

which should be interpreted as $-\psi'_{-Y}(\beta)/\mathbb{E}Y_1$ for $\beta = \Phi_{-Z}(\alpha)$.

In particular, for $y, z > 0$,

$$P\left(Z_\chi - \overline{Z}_{\chi-} > x, \overline{Z}_{\chi-} - Z_{\chi-} > y \,|\chi < \infty\right) = \frac{1}{\mathbb{E}\xi}\int_{x+y}^\infty P(\xi > u)du.$$

3. *For $\alpha, \beta \geq 0$,*

$$\mathbb{E}\left[e^{-\alpha\chi-\beta Z_\chi};\chi<\infty\right] = \frac{\psi_{-Y}(\beta)-\psi_{-Y}(\Phi_{-Z}(\alpha))}{\alpha-\psi_{-X}(\beta)-\psi_{-Y}(\Phi_{-Z}(\alpha))},$$

which should be interpreted as $-\psi'_{-Y}(\beta)/\psi'_{-Z}(\beta)$ for $\beta = \Phi_{-Z}(\alpha)$.

In particular, $P(\chi < \infty) = 1 - \mathbb{E}X_1/\mathbb{E}Z_1$.

Proof. To compute the transform of the joint distribution of $(\overline{G}^Z_{\chi-}, \overline{Z}_{\chi-})$, we use elements of the proof of Proposition 3:

$$\begin{aligned} \mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}\right] &= \frac{-\mathbb{E}X_1}{\lambda\mathbb{E}\xi}\mathbb{E}e^{-\alpha\overline{G}^X_{e_\lambda}-\beta\overline{X}_{e_\lambda}}\frac{\tilde{P}(\mathcal{Y})}{1-G_+^{(2,2)}(\alpha,\beta)} \\ &= -\mathbb{E}X_1\frac{\frac{\Phi_{-X}(\lambda)}{\lambda}\left[\Phi_{-Z}(\alpha)+i\beta\right]}{\frac{\Phi_{-X}(\lambda+\alpha)+i\beta}{\mathbb{E}e^{-\alpha\overline{G}^X_{e_\lambda}-\beta\overline{X}_{e_\lambda}}}-\Phi_{-X}(\lambda)\mathbb{E}e^{-\Phi_{-Z}(\alpha)\xi}}, \end{aligned}$$

from which the first claim follows.

The second claim is a consequence of the observation that the transform equals $G_+^{(2,1)}(\alpha,\beta)/\tilde{P}(\mathcal{Y})$. The second statement follows upon choosing $\alpha = 0$, and noting that

$$P\left(\overline{Z}_{\chi-} - Z_{\chi-} > x\middle| Z_\chi - \overline{Z}_{\chi-} = y, \chi < \infty\right) = P(\xi > x + y|\xi > y).$$

The third claim is obtained from the identity

$$\mathbb{E}\left[e^{-\alpha\chi-\beta Z_\chi}\middle|\chi<\infty\right] = \mathbb{E}\left[e^{-\alpha\overline{G}^Z_{\chi-}-\beta\overline{Z}_{\chi-}}\middle|\chi<\infty\right]\mathbb{E}\left[e^{-\alpha\left[\chi-\overline{G}^Z_{\chi-}\right]-\beta\left[Z_\chi-\overline{Z}_{\chi-}\right]}\middle|\chi<\infty\right],$$

and from (3). □

Let us now calculate the transform of $(G^Z_\infty, \overline{Z}_\infty)$ with Theorem 3: for $\alpha, \beta \geq 0$:

$$\mathbb{E}e^{-\alpha \overline{G}^Z_\infty - \beta \overline{Z}_\infty} = \frac{-\mathbb{E}Z_1 \frac{\Phi_{-Z}(\alpha)-\beta}{\alpha - \psi_{-X}(\beta) - \psi_{-Y}(\Phi_{-Z}(\alpha))}}{1 - \frac{\psi_{-Y}(\beta) - \psi_{-Y}(\Phi_{-Z}(\alpha))}{\alpha - \psi_{-X}(\beta) - \psi_{-Y}(\Phi_{-Z}(\alpha))}} = \frac{-\mathbb{E}Z_1 \left[\Phi_{-Z}(\alpha) - \beta\right]}{\alpha - \psi_{-X}(\beta) - \psi_{-Y}(\beta)},$$

in accordance with standard fluctuation identities.

Note that Theorem 4.7 of [18] is recovered upon combining the 'in particular'-statements of this theorem with Proposition 3, at least if Y is compound Poisson. There is also another way to see that $P(\overline{Z}_{\chi-} \leq x | \chi < \infty) = P(\overline{X}_\infty \leq x)$. Indeed, one can 'cut away' certain pieces of the path of Z to see that $\overline{X}_\infty$ is distributed as $\overline{Z}_\infty$ given the event $\{\chi < \infty\}$. Schmidli [30] makes this argument precise by time reversal of Z. However, this argument cannot be used to find the distribution of $\overline{G}^Z_{\chi-}$.

We end this subsection by remarking that similar formulas can be derived if ξ is not necessarily positive. However, the system of Wiener-Hopf relations then becomes larger and no explicit results can be obtained, unless some structure is imposed; for instance, that ξ have downward phase-type jumps.

5 Asymptotics of the maximum

In this section, we study the probability $P(\overline{Z}_\infty > x)$ as $x \to \infty$ for a Lévy process Z that drifts to $-\infty$, under several conditions on the tail of the Lévy measure. The motivation for studying this problem stems from risk theory; the probability $P(\overline{Z}_\infty > x)$ is often called the *ruin probability.*

It is our aim to show that the embedding approach is a natural yet powerful method for studying tail asymptotics. Relying on random walk results (see Korshunov [23] for an overview), we study these asymptotics in the Cramér case, the intermediate case, and the subexponential case. To our knowledge, this method has not been applied to this problem before, yet the asymptotics in the Cramér and subexponential case have been obtained elsewhere. Our results for the intermediate case, however, are new; see Section 5.2.

More results (and references) on tail asymptotics for Lévy processes can be found in [15, 22].

In order to apply the embedding approach, we write Z as a sum of two independent processes X and Y; one with small jumps ($\Pi_X((1,\infty)) = 0$), and a compound Poisson subordinator Y with jumps exceeding 1. This decomposition has recently been used by Doney [14] and Pakes [27] in the context of asymptotics. We write $\lambda = \Pi_Z([1,\infty)) \in [0,\infty)$, and ξ denotes a generic jump of Y. If $\lambda = 0$, we set $\xi = 0$. The random walk $\{S^q_n\}$ introduced in Section 2.1 plays an important role for $q = 0$. For notational convenience, we write S_n for S^0_n, i.e., S is a random walk with step size distribution $\xi + X_{e_\lambda}$.

The process X has a useful property: for any $\eta > 0$, we have $\mathbb{E}e^{\eta \overline{X}_{e_\lambda}}, \mathbb{E}e^{\eta X_{e_\lambda}} < \infty$. As a result, both $P(\overline{X}_{e_\lambda} > x)$ and $P(X_{e_\lambda} > x)$ decay faster than any exponential (by Chernoff's inequality). To see that the moment generating functions are finite, first observe that for $\Re\beta = 0$, by the Wiener-Hopf factorization,

$$\mathbb{E}e^{\beta X_{e_\lambda}} = \mathbb{E}e^{\beta \overline{X}_{e_\lambda}} \mathbb{E}e^{\beta \underline{X}_{e_\lambda}}.$$

This identity can be extended to $\Re\beta > 0$ by analytic continuation, since on this domain

$$\mathbb{E}e^{\beta X_{e_\lambda}} = \frac{\lambda}{\lambda + \Psi_X(-i\beta)} < \infty,$$

where the finiteness follows from the fact that Π_X is supported on $(-\infty, 1]$. It is trivial that $\mathbb{E}e^{\beta \underline{X}_{e_\lambda}}$ is analytic for $\Re\beta > 0$, hence the claim is obtained.

5.1 The Cramér case

First we deal with the Cramér case, i.e., when there exists some $\omega \in (0,\infty)$ for which $\mathbb{E}e^{\omega Z_1} = 1$.

Given ω, one can define an associate probability measure P^ω, such that Z is a Lévy process under P^ω with Lévy exponent $\Psi_Z(u - i\omega)$. This measure plays an important role in the following result, which is due to Bertoin and Doney [7]. The case that Z has a discrete ladder structure is excluded, as random walk identities then directly apply.

Theorem 5 *Let Z be a Lévy process for which 0 is regular for $(0,\infty)$. Moreover, suppose that there is some $\omega \in (0,\infty)$ such that $\mathbb{E}e^{\omega Z_1} = 1$, and that $\mathbb{E}Z_1 e^{\omega Z_1} < \infty$.*

Then, as $x \to \infty$, we have

$$P(\overline{Z}_\infty > x) \sim \frac{C_\omega}{\omega \mathbb{E} Z_1 e^{\omega Z_1}} e^{-\omega x},$$

where

$$\log C_\omega = -\int_0^\infty t^{-1}(1-e^{-t})\left[P(Z_t > 0) + P^\omega(Z_t \le 0)\right] dt. \tag{5}$$

Moreover, for any $T > 0$, we have as $x \to \infty$,

$$P(\overline{Z}_\infty \in (x, x+T]) \sim \frac{C_\omega}{\omega \mathbb{E} Z_1 e^{\omega Z_1}} \left(1 - e^{-\omega T}\right) e^{-\omega x}.$$

Proof. As the reader readily verifies, the second claim follows immediately from the first.

Let us study the random walk S_n under the present assumptions. First note that $\mathbb{E}e^{\omega Z_1} = 1$ is equivalent with $\mathbb{E}e^{\omega X_{e_\lambda}} \mathbb{E}e^{\omega \xi} = 1$, so that by Lemma 1 of Iglehart [19] (the step size distribution is nonlattice),

$$P\left(\sup_{n\ge 1} S_n > x\right) \sim e^{-\sum_{n=1}^\infty \frac{1}{n}\left\{P(S_n>0)+\mathbb{E}\left[e^{\omega S_n}; S_n \le 0\right]\right\}} \frac{1}{\omega \mathbb{E} S_1 e^{\omega S_1}} e^{-\omega x}.$$

Since $\overline{X}_{e_\lambda}$ has a finite moment generating function, we have by Lemma 2.1 of Pakes [27] that

$$P(\overline{Z}_\infty > x) = P\left(\overline{X}_{e_\lambda} + \sup_{n\ge 1} S_n > x\right) \sim e^{-\sum_{n=1}^\infty \frac{1}{n}\left\{P(S_n>0)+\mathbb{E}\left[e^{\omega S_n}; S_n \le 0\right]\right\}} \frac{\mathbb{E}e^{\omega \overline{X}_{e_\lambda}}}{\omega \mathbb{E} S_1 e^{\omega S_1}} e^{-\omega x}.$$

The rest of the proof consists of translating 'random walk language' into 'Lévy language'. For this, we suppose that the ladder process of X is normalized such that for $\alpha > 0, \beta \in \mathbb{R}$,

$$\alpha + \Psi_X(\beta) = \kappa_X(\alpha, -i\beta)\hat{\kappa}_X(\alpha, i\beta),$$

and similarly for Z.

The quantity $1 - \mathbb{E}e^{i\beta S_1} = \Psi_Z(\beta)/(\lambda + \Psi_X(\beta))$ has both a 'random walk' Wiener-Hopf decomposition and a 'Lévy' Wiener-Hopf decomposition, and their uniqueness leads to the identity

$$\exp\left(-\sum_{n=1}^\infty \frac{1}{n}\mathbb{E}\left[e^{i\beta S_n}; S_n > 0\right]\right) = \frac{\kappa_Z(0,-i\beta)}{\kappa_X(\lambda,-i\beta)}.$$

Similarly, since $1 - \mathbb{E}e^{(\omega+i\beta)S_1} = \Psi_Z(\beta - i\omega)/[\lambda + \Psi_X(\beta - i\omega)]$, we have

$$\exp\left(-\sum_{n=1}^\infty \frac{1}{n}\mathbb{E}\left[e^{(\omega+i\beta)S_n}; S_n \le 0\right]\right) = \frac{\hat{\kappa}_Z(0, i\beta+\omega)}{\hat{\kappa}_X(\lambda, i\beta+\omega)}.$$

Using the fact that $\mathbb{E}e^{\omega \overline{X}_{e_\lambda}} = \kappa_X(\lambda,0)/\kappa_X(\lambda,-\omega)$ (cf. (VI.1) of Bertoin [6]), and that

$$\mathbb{E}S_1 e^{\omega S_1} = \frac{\mathbb{E}Z_1 e^{\omega Z_1}}{\lambda \mathbb{E}e^{\omega\xi}} = \frac{\mathbb{E}Z_1 e^{\omega Z_1}}{\lambda + \Psi_X(-i\omega)} = \frac{\mathbb{E}Z_1 e^{\omega Z_1}}{\kappa_X(\lambda,-\omega)\hat{\kappa}_X(\lambda,\omega)},$$

the claim is obtained with $C_\omega = \kappa_Z(0,0)\hat{\kappa}_Z(0,\omega)$. Corollary VI.10 of Bertoin [6] shows that $\log C_\omega$ is given by (5). □

5.2 The intermediate case

This subsection studies the tail asymptotics of $\overline{Z}_\infty$ under the condition

$$\delta = \sup\{\theta > 0 : \mathbb{E}e^{\theta Z_1} < \infty\} > 0, \tag{6}$$

but we now suppose that we are in the intermediate case, i.e., that $\delta < \infty$ and $\mathbb{E}e^{\delta Z_1} < 1$. These assumptions imply that $\lambda \in (0, \infty)$.

If $D = 1 - \overline{D}$ is a probability distribution on $\mathbb{R}$, we write $D \in \mathcal{S}(\alpha)$, $\alpha > 0$, if

1. $\lim_{x\to\infty} \overline{D}(x+y)/\overline{D}(x) = e^{-\alpha y}$ for all $y \in \mathbb{R}$,
2. $\int_{-\infty}^{\infty} e^{\alpha y} D(dy) < \infty$,
3. $\lim_{x\to\infty} \left\{\overline{D^{(2)}}(x)/\overline{D}(x)\right\} = 2\int_{-\infty}^{\infty} e^{\alpha y} D(dy)$,

where $D^{(2)} = D * D$ is the convolution of D with itself. Note that the first requirement excludes the case that D is concentrated on a lattice. More generally, if μ is a measure with $\mu([1,\infty)) < \infty$, we write $\mu \in \mathcal{S}(\alpha)$ if $\mu([1,\cdot])/\mu([1,\infty)) \in \mathcal{S}(\alpha)$.

We remark that if (6) holds and $\Pi_Z \in \mathcal{S}(\alpha)$, then α necessarily equals δ, as the reader easily verifies.

The following theorem builds on results of Bertoin and Doney [8]. It is closely related to Theorem 4.1 of Klüppelberg *et al.* [22], where the tail asymptotics are expressed in terms of characteristics of the ladder process. Here, it is found directly in terms of the Lévy measure of Z.

Theorem 6 *Let Z be a Lévy process that drifts to $-\infty$, for which $\delta \in (0,\infty)$ and $\mathbb{E}e^{\delta Z_1} < 1$. If $\Pi_Z \in \mathcal{S}(\delta)$, then $\mathbb{E}e^{\delta \overline{Z}_\infty} < \infty$ and $P(\overline{Z}_\infty \le \cdot) \in \mathcal{S}(\delta)$; in fact, as $x \to \infty$, we have*

$$P(\overline{Z}_\infty > x) \sim -\frac{\mathbb{E}e^{\delta \overline{Z}_\infty}}{\log \mathbb{E}e^{\delta Z_1}} \Pi_Z((x,\infty)) \sim -\frac{\mathbb{E}e^{\delta \overline{Z}_\infty}}{\mathbb{E}e^{\delta Z_1} \log \mathbb{E}e^{\delta Z_1}} P(Z_1 > x).$$

Moreover, under these assumptions, we have as $x \to \infty$, for any $T > 0$,

$$P(\overline{Z}_\infty \in (x, x+T]) \sim -\frac{\mathbb{E}e^{\delta \overline{Z}_\infty}}{\log \mathbb{E}e^{\delta Z_1}} \Pi_Z((x,x+T]) \sim -\frac{\mathbb{E}e^{\delta \overline{Z}_\infty}}{\mathbb{E}e^{\delta Z_1} \log \mathbb{E}e^{\delta Z_1}} P(Z_1 \in (x,x+T]).$$

Proof. It suffices to prove the first asymptotic equivalences; for the relationship between the tail of the Lévy measures and the tail of the marginal distribution, we refer to Theorem 3.1 of Pakes [27].

With the embedding in our mind, we first note that by Lemma 2.1 of [27], we have $P(\xi + X_{e_\lambda} > x) \sim \mathbb{E}e^{\delta X_{e_\lambda}} P(\xi > x)$. Since $\mathbb{E}e^{\delta Z_1} < 1$ implies $\mathbb{E}e^{\delta S_1} < 1$ (and vice versa), we can apply Theorem 1 of Bertoin and Doney [8], which states (using Spitzer's identity) that

$$P\left(\sup_{n\ge1} S_n > x\right) \sim \frac{\mathbb{E}e^{\delta X_{e_\lambda}}}{1 - \mathbb{E}e^{\delta\xi}\mathbb{E}e^{\delta X_{e_\lambda}}} \mathbb{E}\exp\left(\delta \sup_{n\ge1} S_n\right) P(\xi > x).$$

It is easy to see that

$$\frac{\mathbb{E}e^{\delta X_{e_\lambda}}}{1 - \mathbb{E}e^{\delta\xi}\mathbb{E}e^{\delta X_{e_\lambda}}} = \frac{1}{\frac{\lambda + \Psi_X(-i\delta)}{\lambda} - \mathbb{E}e^{\delta\xi}} = \frac{\lambda}{\Psi_Z(-i\delta)} = -\frac{\lambda}{\log \mathbb{E}e^{\delta Z_1}}.$$

Using the fact that the moment generating function of $\overline{X}_{e_\lambda}$ is finite, we can again apply Lemma 2.1 of [27] to see that

$$\begin{aligned} P\left(\overline{X}_{e_\lambda} + \sup_{n\geq 1} S_n > x\right) &\sim -\frac{\lambda}{\log \mathbb{E}e^{\delta Z_1}} \mathbb{E}e^{\delta \overline{X}_{e_\lambda}} \mathbb{E}\exp\left(\delta \sup_{n\geq 1} S_n\right) P(\xi > x) \\ &= -\frac{\lambda \mathbb{E}e^{\delta \overline{Z}_\infty}}{\log \mathbb{E}e^{\delta Z_1}} P(\xi > x), \end{aligned}$$

as claimed.

The second assertion is a consequence of the first claim and the observations $P(\overline{Z}_\infty > x+T) \sim e^{-\gamma T} P(\overline{Z}_\infty > x)$ and $\Pi_Z((x+T,\infty)) \sim e^{-\gamma T}\Pi_Z((x,\infty))$. □

It is readily checked that the statements of this theorem are equivalent to

$$P(\overline{Z}_\infty > x) \sim -\frac{\delta \mathbb{E}e^{\delta \overline{Z}_\infty}}{\log \mathbb{E}e^{\delta Z_1}} \int_x^\infty \Pi_Z((y,\infty))dy.$$

In this expression, one can formally let $\delta \to 0$, so that the pre-integral factor tends to $1/\mathbb{E}Z_1$. This naturally leads to the subexponential case.

5.3 The subexponential case

A distribution function D on $\mathbb{R}_+$ is called *subexponential*, abbreviated as $D \in \mathcal{S}$, if, in the notation of the previous subsection, $\overline{D^{(2)}}(x) \sim 2\overline{D}(x)$. An important subclass of subexponential distributions have finite mean and satisfy $\int_0^x \overline{D}(y)\overline{D}(x-y)dy \sim 2\int_0^\infty \overline{D}(y)dy\overline{D}(x)$; we then write $D \in \mathcal{S}^*$. More generally, for a measure μ, we write $\mu \in \mathcal{S}$ (or $\mathcal{S}^*$) if $\mu([1,\infty)) < \infty$ and $\mu([1,\cdot])/\mu([1,\infty)) \in \mathcal{S}$ ($\mathcal{S}^*$).

In this subsection, we suppose that the integrated tail of the Lévy measure

$$\Pi_I((x,\infty)) = \int_x^\infty \Pi_Z((y,\infty))dy$$

is subexponential, i.e., $\Pi_I \in \mathcal{S}$. This property is known to be implied by $\Pi \in \mathcal{S}^*$. Additionally, we suppose that Z_1 is integrable. The first assertion in the following result is due to Asmussen [2, Cor. 2.5]; see also Maulik and Zwart [25], Chan [10], and Braverman *et al.* [9]. As opposed to the Cramér and intermediate case, a local version of this theorem does not follow from the global version, and that part of the theorem is new.

Theorem 7 *Let Z be a Lévy process that drifts to $-\infty$, for which $\Pi_I \in \mathcal{S}$. Then $P(\overline{Z}_\infty \leq \cdot) \in \mathcal{S}$; in fact, as $x \to \infty$, we have*

$$P(\overline{Z}_\infty > x) \sim -\frac{\int_x^\infty \Pi_Z((y,\infty))dy}{\mathbb{E}Z_1} \sim -\frac{\int_x^\infty P(Z_1 > y)dy}{\mathbb{E}Z_1}.$$

Moreover, if $\Pi_Z \in \mathcal{S}^*$ and Π_Z is (ultimately) nonlattice, then we have as $x \to \infty$, for any $T > 0$,

$$P(\overline{Z}_\infty \in (x, x+T]) \sim -\frac{\int_x^{x+T} \Pi_Z((y,\infty))dy}{\mathbb{E}Z_1} \sim -\frac{\int_x^{x+T} P(Z_1 > y)dy}{\mathbb{E}Z_1}.$$

Proof. We have $\Pi_Z((x,\infty)) \sim P(Z_1 > x)$ (see, e.g., [27]); hence, it suffices to prove only the first equivalences.

Since $\Pi_I \in \mathcal{S}$, it is in particular long-tailed, so that for $z \in \mathbb{R}$, $\int_x^\infty P(\xi > y+z)dy \sim \int_x^\infty P(\xi > y)dy$. Fix some $\eta > 0$. The latter observation implies that the function $x \mapsto$

$x^\eta \int_{1\vee\log x}^\infty P(\xi > y)dy$ is locally bounded and regularly varying at infinity with index η, so that by the Uniform Convergence Theorem, for large x,

$$\int_x^\infty P(\xi > y - z)dy \le (1 + e^{\eta z}) \int_x^\infty P(\xi > y)dy,$$

uniformly for $z \in [0, x]$. Since $X_{e_\lambda} \le \overline{X}_{e_\lambda}$ and $\mathbb{E}e^{\eta \overline{X}_{e_\lambda}} < \infty$, this implies that $\int_x^\infty P(\xi + X_{e_\lambda} > y)dy = O\left(\int_x^\infty P(\xi > y)dy\right)$. This shows that one can apply dominated convergence after Veraverbeke's theorem to see that

$$\begin{aligned} P\left(\sup_{n\ge 1} S_n > x\right) &\sim -\frac{1}{\mathbb{E}[X_{e_\lambda} + \xi]} \int_x^\infty P(X_{e_\lambda} + \xi > y)dy \\ &\sim -\frac{1}{\mathbb{E}[X_{e_\lambda} + \xi]} \int_x^\infty P(\xi > y)dy. \end{aligned}$$

By definition of ξ, the right hand side is easily seen to be equivalent to $\int_x^\infty \Pi_Z((y, \infty))dy/|\mathbb{E}Z_1|$. Since this is the tail of a subexponential random variable, the first claim follows from the fact that $\overline{X}_{e_\lambda}$ has a lighter tail.

The second assertion is proven similarly, but with Veraverbeke's theorem replaced by its local counterpart, see Equation (18) in Asmussen *et al.* [5]. The rest of the argument is simpler than for the 'global' version, since $P(X_{e_\lambda} + \xi > x) \sim P(\xi > x)$ as $\Pi \in \mathcal{S}^* \subset \mathcal{S}$. A lattice version can also be given. □

A different proof for the first claim can be given based on recent results of Foss and Zachary [16]. Indeed, as noted in Section 4, a Markov modulated random walk is embedded in the second plot of Figure 1. In order to verify the assumptions of [16] we suppose that Z is not spectrally positive, so that there exist $M_- \le 0$ and $M_+ \ge 0$ such that $\lambda_\pm = \Pi_Z(\mathbb{R}\backslash(M_-, M_+)) < \infty$ and $\int_{\mathbb{R}\backslash(M_-,M_+)} z\Pi_Z(dz) < 0$. One can write Z as a sum of X and Y, where Y is now a compound Poisson process with Lévy measure Π_Z restricted to $\mathbb{R}\backslash(M_-, M_+)$. Further details are left to the reader.

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References

[1] E. Arjas and T. P. Speed. A note on the second factorization identity of A. A. Borovkov. *Theory Prob. Appl.*, 18:576–578, 1973.

[2] S. Asmussen. Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities. *Ann. Appl. Probab.*, 8:354–374, 1998.

[3] S. Asmussen. *Applied probability and queues.* Springer-Verlag, New York, second edition, 2003.

[4] S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.*, 109:79–111, 2004.

[5] S. Asmussen, S. Foss, and D. Korshunov. Asymptotics for sums of random variables with local subexponential behaviour. *J. Theoret. Probab.*, 16:489–518, 2003.

[6] J. Bertoin. *Lévy processes.* Cambridge University Press, Cambridge, 1996.

[7] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21:363–365, 1994.

[8] J. Bertoin and R. A. Doney. Some asymptotic results for transient random walks. *Adv. in Appl. Probab.*, 28:207–226, 1996.

[9] M. Braverman, T. Mikosch, and G. Samorodnitsky. Tail probabilities of subadditive functionals of Lévy processes. *Ann. Appl. Probab.*, 12:69–100, 2002.

[10] T. Chan. Some applications of Lévy processes in insurance and finance. Preprint, 2005.

[11] G. Choudhury. Some aspects of an $M/G/1$ queueing system with optional second service. *Top*, 11:141–150, 2003.

[12] J. W. Cohen. *The single server queue.* North-Holland Publishing Co., Amsterdam, second edition, 1982.

[13] R. A. Doney. On Wiener-Hopf factorisation and the distribution of extrema for certain stable processes. *Ann. Probab.*, 15, 1987.

[14] R. A. Doney. Stochastic bounds for Lévy processes. *Ann. Probab.*, 32:1545–1552, 2004.

[15] R. A. Doney and A. E. Kyprianou. Overshoots and undershoots of Lévy processes. Preprint, 2005.

[16] S. Foss and S. Zachary. Asymptotics for the maximum of a modulated random walk with heavy-tailed increments. In *Analytic methods in applied probability*, pages 37–52. Amer. Math. Soc., Providence, RI, 2002.

[17] H. J. Furrer. Risk processes perturbed by α-stable Lévy motion. *Scand. Actuar. J.*, pages 59–74, 1998.

[18] M. Huzak, M. Perman, H. Šikić, and Z. Vondraček. Ruin probabilities and decompositions for general perturbed risk processes. *Ann. Appl. Probab.*, 14:1378–1397, 2004.

[19] D. L. Iglehart. Extreme values in the $GI/G/1$ queue. *Ann. Math. Statist.*, 43:627–635, 1972.

[20] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes.* Springer-Verlag, Berlin, second edition, 2003.

[21] J. Kennedy. A probabilistic view of some algebraic results in Wiener-Hopf theory for symmetrizable Markov chains. In *Stochastics and quantum mechanics*, pages 165–177. World Sci. Publishing, River Edge, NJ, 1992.

[22] C. Klüppelberg, A. E. Kyprianou, and R. A. Maller. Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.*, 14:1766–1801, 2004.

[23] D. Korshunov. On distribution tail of the maximum of a random walk. *Stochastic Process. Appl.*, 72:97–103, 1997.

[24] S. Kou and H. Wang. First passage times of a jump diffusion process. *Adv. in Appl. Probab.*, 35:504–531, 2003.

[25] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. Preprint, 2004.

[26] E. Mordecki. The distribution of the maximum of a Lévy processes with positive jumps of phase-type. In *Proceedings of the Conference Dedicated to the 90th Anniversary of Boris Vladimirovich Gnedenko (Kyiv, 2002)*, volume 8, pages 309–316, 2002.

[27] A. G. Pakes. Convolution equivalence and infinite divisibility. *J. Appl. Probab.*, 41:407–424, 2004.

[28] M. Pistorius. On maxima and ladder processes for a dense class of Lévy processes. Preprint, 2005.

[29] N. U. Prabhu. *Stochastic storage processes.* Springer-Verlag, New York, 1998.

[30] H. Schmidli. Distribution of the first ladder height of a stationary risk process perturbed by α-stable Lévy motion. *Insurance Math. Econom.*, 28:13–20, 2001.