Centrum voor Wiskunde en Informatica

REPORT RAPPORT

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

Transient analysis of Brownian queues

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REPORT PNA-R0705 APRIL 2007

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ISSN 1386-3711

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ABSTRACT
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2000 Mathematics Subject Classification: 60G15; 60K25
Keywords and Phrases: Single-node network; Brownian traffic; Large deviations; Schilder's theorem
Note: This research has been funded by the Dutch Bsik/BRICKS (Basic Research in Informatics for Creating the Knowledge Society) project.

Transient Analysis of Brownian Queues

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13th April 2007

Abstract

We analyze a single-node network with Brownian input. We first derive an explicit expression for the joint distribution function of the workloads at two different times, which also allows us to calculate their covariance and exact large-buffer asymptotics. The nature of these asymptotics depends on the model parameters, i.e., there are different regimes. By using sample-path large-deviations (Schilder's theorem) these regimes can be interpreted: we explicitly characterize the most likely way the buffer fills. [1]

[1]This research has been funded by the Dutch BSIK/BRICKS (Basic Research in Informatics for Creating the Knowledge Society) project.

1 Introduction

Consider $\{B(t) - ct, t \geq 0\}$, where $B(t)$ is a standard Brownian motion, and $c > 0$ is a scalar. The reflection of $\{B(t) - ct, t \geq 0\}$ at 0 could be called a *Brownian queue*. It is well-known that the Brownian queue is a natural model for many flow systems, see [10]. The behavior of a queue under heavy-traffic conditions can often be approximated by a Brownian queue.

In this report we analyze the transient behavior of a Brownian queue. We explicitly derive the joint distribution function $\mathbb{P}(Q_0 > b_0, Q_T > b_T)$, where Q_t is the workload of the queue at time t, and $b_0, b_T \geq 0$. This also allows us to explicitly calculate the covariance between Q_0 and Q_T. By setting $b_0 = b$, $b_T = \alpha b$, and $T = \gamma b$, with $\alpha, \gamma \geq 0$, and letting $b \to \infty$, we also obtain exact large-buffer asymptotics of the joint distribution function, i.e., we find a function $f(\cdot)$ such that $\mathbb{P}(Q_0 > b, Q_{\gamma b} > \gamma b)/f(b) \to 1$ as $b \to \infty$. It turns out that the nature of the asymptotics depends on the value of α, γ, and the service rate of the queue, i.e., there are different regimes. These regimes can be further interpreted relying on Schilder's sample-path large-deviations theorem. In particular, we obtain the so-called most probable path, i.e., the most likely way the buffer fills.

The Brownian queue was already studied in [1, 2, 3, 10]. We note that some of the results derived in this report already appeared there, but these results were proved in a completely different manner.

The remainder of the report is organized as follows. In Section 2 we present a description of the model, and we briefly discuss Schilder's sample-path large-deviations theorem. In Section 3 we derive an exact expression for $\mathbb{P}(Q_0 > b_0, Q_T > b_T)$, the covariance between the workloads, large-buffer asymptotics, and the most probable path. We then exploit these results to obtain similar results for $\mathbb{P}(Q_T > b_T | Q_0 = b_0)$ in Section 4. Finally, in Section 5 we further discuss our results, and identify some open research questions.

2 Preliminaries

In this section we first present our queueing model. Subsequently, we discuss a large-deviations theorem that is needed in Sections 3.4 and 4.3.

2.1 Queueing model

We consider a single-node network, with service rate $c > 0$. We assume that the input process is a standard Brownian motion $\{B(t), t \in \mathbb{R}\}$, with $B(0) \equiv 0$. This implies that $B(s,t) = B(t) - B(s) \sim N(0, t-s)$, i.e., the amount of traffic that enters in the interval $(s,t]$ is standard Normally distributed with mean 0 and variance $t - s$. It can be verified that $\Gamma(s,t) := \mathbb{C}\mathrm{ov}(B(s), B(t)) = \min\{|s|, |t|\}$ if $s, t \geq 0$ or $s, t < 0$, and $\Gamma(s,t) = 0$ otherwise. Also, let Q_t denote the workload at time t, $t \in \mathbb{R}$. In this report we focus on the joint distribution of the workloads at time 0 and time $T > 0$. In particular, we derive

$$p(\bar{b}, T) := \mathbb{P}(Q_0 > b_0, Q_T > b_T), \tag{1}$$

with $b_0, b_T \geq 0$, and $\bar{b} = (b_0, b_T)$. In addition, using (1), we also derive

$$q(\bar{b}, T) := \mathbb{P}(Q_T > b_T | Q_0 = b_0).$$

2.2 Large deviations

We continue with a description of the framework of Schilder's sample-path LDP (see [6], and also Thm. 1.3.27 of [8] for a more detailed treatment). Define the path space Ω as

$$\Omega := \left\{ \omega : \mathbb{R} \to \mathbb{R}, \;\text{continuous}, \;\omega(0) = 0, \;\lim_{t\to\infty} \frac{\omega(t)}{1+|t|} = \lim_{t\to-\infty} \frac{\omega(t)}{1+|t|} = 0 \right\}.$$

We note that in [4] it was pointed out that $B(\cdot)$ can be realized on Ω. Then one can construct a reproducing kernel Hilbert space $R \subseteq \Omega$, consisting of elements that are roughly as smooth as the covariance function $\Gamma(s,\cdot)$; for details, see [5]. We start from a 'smaller' space R^*, defined by

$$R^* := \left\{ \omega : \mathbb{R} \to \mathbb{R}, \;\omega(\cdot) = \sum_{i=1}^{n} a_i \Gamma(s_i, \cdot), \;\; a_i, s_i \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

The inner product on this space R^* is, for ω_a, $\omega_b \in R^*$, defined as

$$\langle \omega_a, \omega_b \rangle_R := \left\langle \sum_{i=1}^{n} a_i \Gamma(s_i,\cdot), \sum_{j=1}^{n} b_j \Gamma(s_j,\cdot) \right\rangle_R = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \Gamma(s_i, s_j); \tag{2}$$

notice that this implies $\langle \Gamma(s,\cdot), \Gamma(\cdot,t)\rangle_R = \Gamma(s,t)$. This inner product has the following useful property, which is known as the *reproducing kernel* property,

$$\omega(t) = \sum_{i=1}^{n} a_i \Gamma(s_i, t) = \left\langle \sum_{i=1}^{n} a_i \Gamma(s_i,\cdot), \Gamma(t,\cdot) \right\rangle_R = \langle \omega(\cdot), \Gamma(t,\cdot)\rangle_R.$$

From this we introduce the norm $||\omega||_R := \sqrt{\langle \omega, \omega \rangle_R}$. The closure of R^* under this norm is defined as space R. Now we can define the rate function:

$$I(\omega) := \begin{cases} \frac{1}{2}||\omega||_R^2 & \text{if} \quad \omega \in R; \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

As a side remark we mention that the above framework in fact holds for a general and versatile class of input processes, covering a broad range of correlation structures, viz. the class of centered Gaussian inputs $(A(t), t \in \mathbb{R})$ (which obviously covers standard Brownian input). In that case one should set $\Gamma(s,t) = \mathbb{C}\mathrm{ov}(A(s), A(t))$, $s \le t$. Using (2) and the definition of $\Gamma(s,t)$ in case of standard Brownian inputs (see Section 2.1), we find that, for $\omega(t) = \sum_{i=1}^{n} a_i \Gamma(s_i, t)$, with $s_1 < \ldots < s_n$,

$$\begin{aligned} \frac{1}{2}||\omega||_R^2 &= \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_i a_j \min\{|s_i|, |s_j|\} + \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} a_i a_j \min\{s_i, s_j\} \\ &= \frac{1}{2}\int_{-\infty}^{0} (\omega'(t))^2 \mathrm{d}t + \frac{1}{2}\int_{0}^{\infty} (\omega'(t))^2 \mathrm{d}t, \end{aligned}$$

where $k := \min\{i \in \{1, ..., n\} : s_i \ge 0\}$ if defined, and $k := n+1$ otherwise. It turns out that (3) is equivalent to

$$I(\omega) = \begin{cases} \frac{1}{2}\int_{-\infty}^{\infty} (\omega'(t))^2 \mathrm{d}t & \text{if} \quad \omega \in R; \\ \infty & \text{otherwise,} \end{cases} \tag{4}$$

in case of standard Brownian inputs (see Thm. 5.2.3 of [7]).

Theorem 2.1 **[Schilder]** *For standard Brownian inputs the following sample-path large deviations principle (LDP) holds:*
(a) For any closed set $F \subset \Omega$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in F\right) \leq - \inf_{\omega\in F} I(\omega);$$

(b) For any open set $G \subset \Omega$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in G\right) \geq - \inf_{\omega\in G} I(\omega).$$

Remark: Theorem 2.1 shows that the LDP consists of an upper and lower bound, which apply to closed and open sets, respectively. We will use Theorem 2.1 for the open set S, to be defined in Section 3.4. It can be verified that

$$\inf_{\omega\in S} I(\omega) = \inf_{\omega\in \overline{S}} I(\omega),$$

where $\overline{S}$ is the closure of S. The way to prove this is to show that an arbitrarily chosen path in $\overline{S}$ can be approximated by a path in S. This proof is completely analogously to [14] and Appendix A of [12].

3 Analysis of $p(\overline{b}, T)$

In this section we derive the joint distribution function of the workloads at time 0 and time T, the covariance between these workloads, large-buffer asymptotics, and the most probable path leading to overflow.

3.1 Joint distribution function

In this subsection we derive a closed-form expression for $p(\overline{b}, T)$. It turns out that is easier to first calculate $\overline{p}(\overline{b}, T) := \mathbb{P}(Q_0 \leq b_0, Q_T \leq b_T)$. Let $\Phi(\cdot)$ denote the distribution function of a standard Normal random variable:

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)\mathrm{d}u = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u. \tag{5}$$

According to Reich's formula [15],

$$Q_0 = \sup_{t\geq 0}\{B(-t,0) - ct\} \quad \text{and} \quad Q_T = \sup_{s\geq 0}\{B(T-s,T) - cs\}. \tag{6}$$

Hence, $\overline{p}(\overline{b}, T)$ can be rewritten as

$$\mathbb{P}\left(\sup_{t\geq 0}\{B(-t,0) - ct\} \leq b_0, \sup_{s\geq 0}\{B(T-s,T) - cs\} \leq b_T\right) =$$

$$\mathbb{P}(\forall s,t \geq 0 : B(-t,0) \leq b_0 + ct, B(T-s,T) \leq b_T + cs) =$$

$$\mathbb{P}(\forall s,t \geq 0 : B(T,t+T) \leq b_0 + ct, B(0,s) \leq b_T + cs),$$

where the last line is obtained by using time reversibility arguments. Now, conditioning on the value of $B(0,T)$, we get that $\overline{p}(\overline{b},T)$ is equivalent to

$$\int_{-\infty}^{b_T+cT} \mathbb{P}(N(0,T)=x)\mathbb{P}(\forall s \in [0,T) : B(0,s) \leq b_T + cs | B(0,T) = x)$$

$$\mathbb{P}(\forall t \geq 0 : \forall s \geq T : B(T,t+T) \leq b_0 + ct, B(0,s) \leq b_T + cs | B(0,T) = x)\mathrm{d}x.$$

Let us first focus on the second probability in the integral. Mandjes [11] derived that

$$\mathbb{P}(\forall s \in [0,T) : B(0,s) \leq b_T + cs | B(0,T) = x) = 1 - \exp\left(-2b_T c - 2b_T(b_T - x)/T\right),$$

by showing that this probability can be expressed in terms of the Brownian bridge after some rescaling. Proceeding with the third term in the integral, we find that

$$\mathbb{P}(\forall t \geq 0 : \forall s \geq T : B(T,T+t) \leq b_0 + ct, B(0,s) \leq b_T + cs | B(0,T) = x) =$$

$$\mathbb{P}(\forall t \geq 0 : \forall s \geq T : B(T,T+t) \leq b_0 + ct, B(T,s) \leq b_T + cs - x) =$$

$$\mathbb{P}(\forall s,t \geq 0 : B(T,T+t) \leq b_0 + ct, B(T,T+s) \leq b_T + (s+T)c - x) =$$

$$\mathbb{P}(\forall s,t \geq 0 : B(0,t) \leq b_0 + ct, B(0,s) \leq b_T + (s+T)c - x) =$$

$$\mathbb{P}(\forall t \geq 0 : B(0,t) \leq \min\{b_0, b_T + cT - x\} + ct).$$

Exploiting the well-known result that $\mathbb{P}(\forall t \geq 0 : B(0,t) \leq b + ct) = 1 - \exp(-2bc)$, we finally find that

$$\mathbb{P}(\forall t \geq 0 : B(0,t) \leq \min\{b_0, b_T + cT - x\} + ct) = \begin{cases} 1 - \exp(-2b_0 c) & \text{if } x \leq b_T + cT - b_0; \\ 1 - \exp(-2(b_T + cT - x)c) & \text{if } x > b_T + cT - b_0. \end{cases}$$

Theorem 3.1 *For each $b_0, b_T, T \geq 0$,*

$$p(\overline{b},T) = -\Phi\left(k_1(\overline{b},T)\right) + e^{-2b_T c}\Phi\left(k_2(\overline{b},T)\right) + e^{-2b_0 c}\Phi\left(k_3(\overline{b},T)\right) + e^{-2(b_0+b_T)c}\Phi\left(k_4(\overline{b},T)\right),$$

where

$$k_1(\overline{b},T) = \frac{-b_T - cT - b_0}{\sqrt{T}}; k_2(\overline{b},T) = \frac{b_T - cT - b_0}{\sqrt{T}}; k_3(\overline{b},T) = \frac{-b_T - cT + b_0}{\sqrt{T}}; k_4(\overline{b},T) = \frac{-b_T + cT - b_0}{\sqrt{T}}.$$

Proof: From the above it follows that $\overline{p}(\overline{b},T)$ equals

$$\int_{-\infty}^{b_T+cT-b_0} \mathbb{P}(N(0,T)=x)\left(1 - \exp\left(-2b_T c - 2\frac{b_T(b_T-x)}{T}\right)\right)(1 - \exp(-2b_0 c))\,\mathrm{d}x +$$
$$\int_{b_T+cT-b_0}^{b_T+cT} \mathbb{P}(N(0,T)=x)\left(1 - \exp\left(-2b_T c - 2\frac{b_T(b_T-x)}{T}\right)\right)(1 - \exp(-2(b_T + cT - x)c))\,\mathrm{d}x.$$

It is a straightforward exercise to show that the first integral is equal to

$$(1-\exp(-2b_0c))\left(\Phi\left(\frac{b_T+cT-b_0}{\sqrt{T}}\right)-\exp(-2b_Tc)\Phi\left(\frac{-b_T+cT-b_0}{\sqrt{T}}\right)\right),$$

whereas the second integral equals

$$1-\Phi\left(\frac{-b_T-cT-b_0}{\sqrt{T}}\right)-\Phi\left(\frac{b_T+cT-b_0}{\sqrt{T}}\right)+\exp(-2b_Tc)\left(\Phi\left(\frac{-b_T+cT-b_0}{\sqrt{T}}\right)+\Phi\left(\frac{b_T-cT-b_0}{\sqrt{T}}\right)-1\right).$$

Using the well-known property that $\mathbb{P}(Q_i \leq b_i) = 1-\exp(-2b_ic)$, $i=0,T$, and that $1-\Phi(x) = \Phi(-x)$, the stated follows from

$$p(\bar{b},T) = 1-\mathbb{P}(Q_0\leq b_0)-\mathbb{P}(Q_T\leq b_T)+\bar{p}(\bar{b},T). \qquad \square$$

3.2 Covariance function

In the previous subsection we derived a closed-form expression for $p(\bar{b},T)$, see Theorem 3.1. This result also allows us to calculate the covariance between Q_0 and Q_T, i.e., $\mathbb{C}\mathrm{ov}(Q_0,Q_T)$, which we present in the next theorem.

Theorem 3.2 *For each $T\geq 0$,*

$$\theta(T) := \mathbb{C}\mathrm{ov}(Q_0,Q_T) = \left(-\frac{c^2T^2}{2}-T+\frac{1}{2c^2}\right)\left(1-\Phi(c\sqrt{T})\right)+\phi(c\sqrt{T})\left(\frac{cT\sqrt{T}}{2}+\frac{\sqrt{T}}{2c}\right). \tag{7}$$

Proof: First recall that $\mathbb{C}\mathrm{ov}(Q_0,Q_T) = \mathbb{E}Q_0Q_T - \mathbb{E}Q_0\mathbb{E}Q_T$. Then use the well-known fact that Q_0 and Q_T are both exponentially distributed with mean $1/(2c)$, i.e., $\mathbb{E}Q_0\mathbb{E}Q_T = 1/(4c^2)$. Hence, we are left with $\mathbb{E}Q_0Q_T$. Using Theorem 3.1, we find that

$$\mathbb{E}Q_0Q_T = \int_0^\infty\int_0^\infty p(\bar{b},T)\mathrm{d}b_0\mathrm{d}b_T =$$

$$-\int_0^\infty\int_0^\infty \Phi\left(k_1(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T + \int_0^\infty\int_0^\infty e^{-2b_Tc}\Phi\left(k_2(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T$$
$$+\int_0^\infty\int_0^\infty e^{-2b_0c}\Phi\left(k_3(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T + \int_0^\infty\int_0^\infty e^{-2(b_0+b_T)c}\Phi\left(k_4(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T.$$

By using (5), interchanging the order of integration, and applying integration by parts, straightforward (though tedious) calculus yields that

$$-\int_0^\infty\int_0^\infty \Phi\left(k_1(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T = -\left(\frac{T}{2}+\frac{c^2T^2}{2}\right)\left(1-\Phi(c\sqrt{T})\right)+\frac{cT\sqrt{T}}{2}\phi(c\sqrt{T}); \tag{8}$$

$$\int_0^\infty\int_0^\infty e^{-2b_Tc}\Phi\left(k_2(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T = \left(\frac{1}{2c^2}-\frac{T}{2}\right)\left(1-\Phi(c\sqrt{T})\right)+\frac{\sqrt{T}}{2c}\phi(c\sqrt{T}); \tag{9}$$

$$\int_0^\infty\int_0^\infty e^{-2b_0c}\Phi\left(k_3(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T = \left(\frac{1}{2c^2}-\frac{T}{2}\right)\left(1-\Phi(c\sqrt{T})\right)+\frac{\sqrt{T}}{2c}\phi(c\sqrt{T}); \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{-2(b_0+b_T)c}\Phi\left(k_4(\bar{b},T)\right)\mathrm{d}b_0\mathrm{d}b_T = \left(\frac{T}{2}-\frac{1}{4c^2}\right)\left(1-\Phi(c\sqrt{T})\right)+\frac{1}{4c^2}\Phi(c\sqrt{T})-\frac{\sqrt{T}}{2c}\phi(c\sqrt{T}). \tag{11}$$

Adding up (8), (9), (10) and (11), and subtracting $1/(4c^2)$ yields the stated. □

First note that $\theta(0) = \mathbb{V}\mathrm{ar}(Q_0) = 1/(4c^2)$, i.e., the variance of an exponentially distributed variable with mean $1/(2c)$, as required. Also, note that $\lim_{T\to\infty} \theta(T) \to 0$ as expected, i.e., Q_0 and Q_T become independent as $T \to \infty$. The following proposition summarizes three properties of $\theta(\cdot)$. This proposition implies that $(1 - \theta(\cdot))$ is a distribution function on $[0, \infty)$.

Proposition 3.3 $\theta(\cdot)$ *is non-increasing, convex and non-negative on* $[0, \infty)$.

Proof: $\theta(T)$ is non-increasing on $[0, \infty)$ if $\theta'(T) \le 0$, i.e.,

$$-\left(1 + c^2 T\right)\left(1 - \Phi(c\sqrt{T})\right) + c\sqrt{T}\phi\left(c\sqrt{T}\right) \le 0,$$

which is equivalent to

$$\frac{\phi\left(c\sqrt{T}\right)}{1 - \Phi\left(c\sqrt{T}\right)} \le c\sqrt{T} + \frac{1}{c\sqrt{T}}. \tag{12}$$

Likewise, $\theta(T)$ is convex on $[0, \infty)$ if $\theta''(T) \ge 0$, i.e.,

$$-c^2\left(1 - \Phi(c\sqrt{T})\right) + \frac{c}{\sqrt{T}}\phi\left(c\sqrt{T}\right) \ge 0,$$

or equivalently,

$$\frac{\phi\left(c\sqrt{T}\right)}{1 - \Phi\left(c\sqrt{T}\right)} \ge c\sqrt{T}. \tag{13}$$

Recalling the standard equality (see page 5 of [13])

$$\frac{1}{x + 1/x}\phi(x) \le 1 - \Phi(x) \le \frac{1}{x}\phi(x),$$

it is easily seen that both (12) and (13) hold. The non-negativity of $\theta(T)$ follows from the fact that $\theta(T)$ is non-increasing and $\lim_{T\to\infty} \theta(T) \to 0$. □

The next proposition presents the exact asymptotics of $\theta(T)$. We denote $f(x) \sim g(x)$ when $f(x)/g(x) \to 1$ if $x \to \infty$.

Proposition 3.4 *If* $T \to \infty$,

$$\theta(T) \sim \frac{4}{c^5 T\sqrt{T}}\phi\left(c\sqrt{T}\right). \tag{14}$$

Proof: First use that

$$(1 - \Phi(g(x))) \sim \left(\frac{1}{g(x)} - \frac{1}{(g(x))^3} + \frac{3}{(g(x))^5} - \frac{15}{(g(x))^7}\right)\phi(g(x)) \tag{15}$$

if $g(x)$ is increasing and $x \to \infty$. Using (15) and Theorem 3.2, it can then be verified that

$$\theta(T) \sim \left(\frac{4}{c^5 T \sqrt{T}} + \frac{16\frac{1}{2}}{c^7 T^2 \sqrt{T}} - \frac{7\frac{1}{2}}{c^9 T^3 \sqrt{T}} \right) \phi\left(c\sqrt{T}\right) \sim \frac{4}{c^5 T \sqrt{T}} \phi\left(c\sqrt{T}\right).$$

We note that the correct exact asymptotics of $\theta(T)$ are not obtained, if one uses an approximation of $(1 - \Phi(g(x)))$ that is less accurate than (15). $\square$

Remark: The correlation coefficient between Q_0 and Q_T is given by

$$\rho(T) := \mathbb{C}or(Q_0, Q_T) = \frac{\mathbb{C}ov(Q_0, Q_T)}{\sqrt{\mathbb{V}ar(Q_0)}\sqrt{\mathbb{V}ar(Q_T)}} = 4c^2\theta(T), \tag{16}$$

as both Q_0 and Q_T are exponentially distributed with mean $1/(2c)$. Note that $\rho(0) = 1$ and $\lim_{T\to\infty} \rho(T) \to 0$. Due to (16), we also have that $\rho(T)$ is non-increasing, convex and non-negative on $[0, \infty)$, and that

$$\rho(T) \sim \frac{16}{c^3 T \sqrt{T}} \phi\left(c\sqrt{T}\right).$$

Hence, the exponential decay rate of both $\theta(T)$ and $\rho(T)$ equals $\left(c^2 T\right)/2$.

We note that Theorem 3.2 and Propositions 3.3-3.4 already (partly) appeared (for $\rho(T)$, instead of $\theta(T)$) in [3]. However, we note that our derivations are completely different compared to the ones presented in [3]. We rely on Reich's formula to obtain the results, whereas [3] does not use this formula.

3.3 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of $p(\bar{b}, T)$. Define $\zeta(x) := \left(\sqrt{2\pi}x\right)^{-1} \exp(-x^2/2)$. We first present the following lemma.

Lemma 3.5 *Let $b_0 = b$, $b_T = \alpha b$ and $T = \gamma b$, with $\alpha, \gamma \geq 0$. If $b \to \infty$, then*

$$\Phi(k_1(\bar{b}, T)) \sim -\zeta(k_1(\bar{b}, T));$$

$$\Phi(k_2(\bar{b}, T)) \sim \begin{cases} -\zeta(k_2(\bar{b}, T)) & \text{if } \alpha < 1 + c\gamma; \\ 1/2 & \text{if } \alpha = 1 + c\gamma; \\ 1 & \text{otherwise}; \end{cases}$$

$$\Phi(k_3(\bar{b}, T)) \sim \begin{cases} -\zeta(k_3(\bar{b}, T)) & \text{if } \alpha > 1 - c\gamma; \\ 1/2 & \text{if } \alpha = 1 - c\gamma; \\ 1 & \text{otherwise}; \end{cases}$$

$$\Phi(k_4(\bar{b}, T)) \sim \begin{cases} -\zeta(k_4(\bar{b}, T)) & \text{if } \alpha > c\gamma - 1; \\ 1/2 & \text{if } \alpha = c\gamma - 1; \\ 1 & \text{otherwise}. \end{cases}$$

Proof: First determine for which values of $b_T/b_0 = \alpha$, $k_i(\bar{b},T)$, $i \in \{1,2,3,4\}$, is positive or negative. Note that $k_1(\bar{b})$ is always negative. Hence, we obtain $1+c\gamma$, $1-c\gamma$ and $c\gamma-1$ as critical values from $k_i(\bar{b})$, $i=2,3,4$, respectively. Next use the fact that $\Phi(-u) \sim \zeta(u)$ and $\Phi(u) \sim 1$ as $u \to \infty$. Observe that $\Phi(0) = 1/2$. □

We remark that the $-\zeta(k_i(\bar{b},T)$ terms in Lemma 3.5 are all positive, as $\zeta(k_i(\bar{b},T)$ is negative in the listed cases, $i=1,\ldots,4$. Define

$$\gamma(\bar{b},T) := 2b_0c + \frac{(-b_T - cT + b_0)^2}{2T}.$$

Theorem 3.6 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma > 1$. For $b \to \infty$,*

$$p(\bar{b},T) \sim \begin{cases} e^{-2(b_0+b_T)c} & \text{if } 0 \leq \alpha < \left(\sqrt{c\gamma}-1\right)^2; \\ \left(1 - \frac{1}{\sqrt{2\pi}k_2(\bar{b},T)} - \frac{1}{\sqrt{2\pi}k_3(\bar{b},T)}\right) e^{-2(b_0+b_T)c} & \text{if } \alpha = \left(\sqrt{c\gamma}-1\right)^2; \\ \left(-\frac{1}{\sqrt{2\pi}k_2(\bar{b},T)} - \frac{1}{\sqrt{2\pi}k_3(\bar{b},T)}\right) e^{-\gamma(\bar{b},T)} & \text{if } \left(\sqrt{c\gamma}-1\right)^2 < \alpha < 1 + c\gamma; \\ \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}k_3(\bar{b},T)}\right) e^{-2b_Tc} & \text{if } \alpha = 1 + c\gamma; \\ e^{-2b_Tc} & \text{if } \alpha > 1 + c\gamma. \end{cases}$$

Proof: We only prove the last statement, as the other four statements follow in a similar way. We have to prove that

$$p(\bar{b},T)e^{2b_Tc} \to 1 \text{ as } b \to \infty, \text{ for } \alpha > 1 + c\gamma.$$

From Lemma 3.5 we obtain that for $\alpha > 1 + c\gamma$,

$$\Phi(k_1(\bar{b},T)) \sim -\zeta(k_1(\bar{b},T)); \quad \Phi(k_2(\bar{b},T)) \sim 1;$$

$$\Phi(k_3(\bar{b},T)) \sim -\zeta(k_3(\bar{b},T)); \quad \Phi(k_4(\bar{b},T)) \sim -\zeta(k_4(\bar{b},T)).$$

Now straightforward calculus shows that, as $b \to \infty$,

$$\Phi(k_1(\bar{b},T)) = o\left(e^{-2b_Tc}\right),$$

and the same applies for $\Phi(k_3(\bar{b},T))e^{-2b_0c}$ and $\Phi(k_4(\bar{b},T))e^{-2(b_0+b_T)c}$. With $\Phi(k_2(\bar{b},T)) \sim 1$, Theorem 3.1 implies the stated. □

The following two theorems can be proven in a similar fashion as Theorem 3.6.

Theorem 3.7 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma = 1$. For $b \to \infty$,*

$$p(\bar{b},T) \sim \begin{cases} e^{-2b_0c} & \text{if } \alpha = 0; \\ \left(-\frac{1}{\sqrt{2\pi}k_2(\bar{b},T)} - \frac{1}{\sqrt{2\pi}k_3(\bar{b},T)}\right) e^{-\gamma(\bar{b},T)} & \text{if } 0 < \alpha < 1 + c\gamma; \\ \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}k_3(\bar{b},T)}\right) e^{-2b_Tc} & \text{if } \alpha = 1 + c\gamma; \\ e^{-2b_Tc} & \text{if } \alpha > 1 + c\gamma. \end{cases}$$

Theorem 3.8 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma < 1$. For $b \to \infty$,*

$$p(\bar{b}, T) \sim \begin{cases} e^{-2b_0 c} & \text{if } 0 \leq \alpha < 1 - c\gamma; \\ \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi k_2(\bar{b},T)}}\right) e^{-2b_0 c} & \text{if } \alpha = 1 - c\gamma; \\ \left(-\frac{1}{\sqrt{2\pi k_2(\bar{b},T)}} - \frac{1}{\sqrt{2\pi k_3(\bar{b},T)}}\right) e^{-\gamma(\bar{b},T)} & \text{if } 1 - c\gamma < \alpha < 1 + c\gamma; \\ \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi k_3(\bar{b},T)}}\right) e^{-2b_T c} & \text{if } \alpha = 1 + c\gamma; \\ e^{-2b_T c} & \text{if } \alpha > 1 + c\gamma. \end{cases}$$

3.4 Most probable path

In the previous subsection it was shown that the nature of the large-buffer asymptotics strongly depends on the model parameters α and γ, i.e., there are different regimes. In this subsection we will interpret these regimes by exploiting well-known sample-path large deviations results. Schilder's theorem implies that the exponential decay rate of the joint overflow probability is characterized by the path that minimizes the decay rate. Among all paths such that the queue exceeds b_0 and b_T at time 0 and T respectively, this is the so-called most probable path (MPP): informally speaking, given that this rare event occurs, with overwhelming probability (b_0, b_T) is reached by a path 'close to' the MPP.

In order to apply 'Schilder', we feed the single-node network by n i.i.d. standard Brownian sources. The link rate and buffer thresholds are also scaled by n: nc, nb_0 and nb_T, respectively. Using (6), $p_n(\bar{b}, T)$ can be expressed as

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in S\right),$$

where

$$S := \{f \in \Omega | \exists s, t \geq 0 : -f(-t) > b_0 + ct, f(T) - f(T-s) > b_T + cs\}.$$

From 'Schilder' it follows that

$$J(\bar{b}, T) := -\lim_{n\to\infty} \frac{1}{n} \log p_n(\bar{b}, T) = \inf_{f \in S} I(f).$$

As mentioned in the remark of Section 2.2, we can replace '>' by '$\geq$' in S, without any impact on the decay rate.

Define

$$U := \{f \in \Omega | \exists t \geq 0 : -f(-t) > b_0 + ct\};$$

$$V := \{f \in \Omega | \exists s \geq 0 : -f(T-s) > b_0 + cs\}.$$

Note that $S \subseteq U$ and $S \subseteq V$, which implies that

$$J(\bar{b}, T) \geq \inf_{f \in U} I(f); \tag{17}$$

$$J(\bar{b}, T) \geq \inf_{f \in V} I(f). \tag{18}$$

From the above it follows that if the MPP in U is also contained in set S, then there is equality in (17), and likewise, if the MPP in V is also contained in set S, then there is equality in (18). In [4] it was shown that the MPP in U is given by, for $r \in [-b_0/c, 0]$,

$$f^*(r) = \mathbb{E}(B(r)| - B(-b_0/c) = b_0 + ct).$$

Let (Y_1, Y_2) be bivariate Normally distributed. Now, using that the random variable $(Y_1|Y_2 = y)$, for some $y \in \mathbb{R}$, is Normally distributed with mean

$$\mathbb{E}(Y_1|Y_2 = y) = \mathbb{E}Y_1 + \frac{\mathbb{C}\text{ov}(Y_1, Y_2)}{\mathbb{V}\text{ar}Y_2}(y - \mathbb{E}Y_2),$$

it can be verified that, for $r \in [-b_0/c, 0]$,

$$(f^*)'(r) = 2c.$$

The MPP is only specified in the interval $[-b_0/c, 0]$, because outside this interval the MPP generates traffic with rate 0. Hence, this MPP is such that the queue starts to build up at time $-b_0/c$ with constant rate c, giving $Q_0 = b_0$. Using (4), we find that

$$I(f^*) = \frac{1}{2}\frac{b_0}{c}(2c)^2 = 2b_0c,$$

i.e., the decay rate equals $2b_0c$. The MPP in V has a similar structure as the one above, and it is such that that the queue starts to grow at time $T - b_T/c$ with constant rate c, giving $Q_T = b_T$. The corresponding decay rate equals $2b_Tc$.

We are now ready to provide some explanation for each of the regimes of Theorems 3.6-3.8. Let us start with regime $\alpha \geq 1 + c\gamma$ in Theorems 3.6-3.8. Using that $\alpha = b_T/b_0$ and $\gamma = T/b_0$, it is easily seen that this inequality is equivalent to $b_T - cT \geq b_0$. Consider the MPP in V mentioned above. Recall that this MPP is such that the queue starts to grow at time $T - b_T/c$. Due to $b_T - cT \geq b_0 \geq 0$, it can be verified that $T - b_T/c \leq 0$. It follows that if $\alpha \geq 1 + c\gamma$, then the MPP in V is also contained in S, and therefore it is the MPP in S, i.e., overflow of the queue at time T implies overflow at time 0 without any additional effort. The MPP is depicted in Figure 1 (top, left). Therefore, we find that $J(\bar{b}, T)$ is equal to the decay rate corresponding to the MPP in V, i.e., $2b_Tc$.

Next consider regime $0 \leq \alpha \leq 1 - c\gamma$ in Theorems 3.7-3.8, or equivalently $b_T \leq b_0 - cT$. In this case one can verify that the MPP in set U is also contained in set S, and therefore it is the MPP in S. Thus, overflow at time 0 implies overflow at time T without any extra effort, and $J(\bar{b}, T)$ is therefore equal to $2b_0c$ in case $0 \leq \alpha \leq 1 - c\gamma$. Note that we only see this regime if $c\gamma \leq 1$ (as $\alpha \geq 0$ by definition). The MPP is depicted in Figure 1 (top, right).

We proceed with regime $0 \leq \alpha \leq (\sqrt{c\gamma} - 1)^2$ in Theorem 3.6, or equivalently $T \geq (\sqrt{b_0} + \sqrt{b_T})^2/c$. Consider the path that is such that the queue starts to build up with rate c in the interval $(-b_0/c, 0]$, empties with rate c in the interval $(0, b_0/c]$, is empty in the interval $(b_0/c, T - b_T/c]$, and is growing again with rate c in the interval $(T - b_T/c, T]$, i.e., the MPP of U and V combined. It can be verified that this path is contained in set S if $0 \leq \alpha \leq (\sqrt{c\gamma} - 1)^2$. In Section 3.5 we show that this path is in fact the MPP in S in case $0 \leq \alpha \leq (\sqrt{c\gamma} - 1)^2$, but for the moment assume that this is correct. Then $J(\bar{b}, T)$ can be obtained by using (4), and equals $2b_0c + 2b_Tc$. Clearly, this is no surprise, as the path consists of the MPP of U and V.

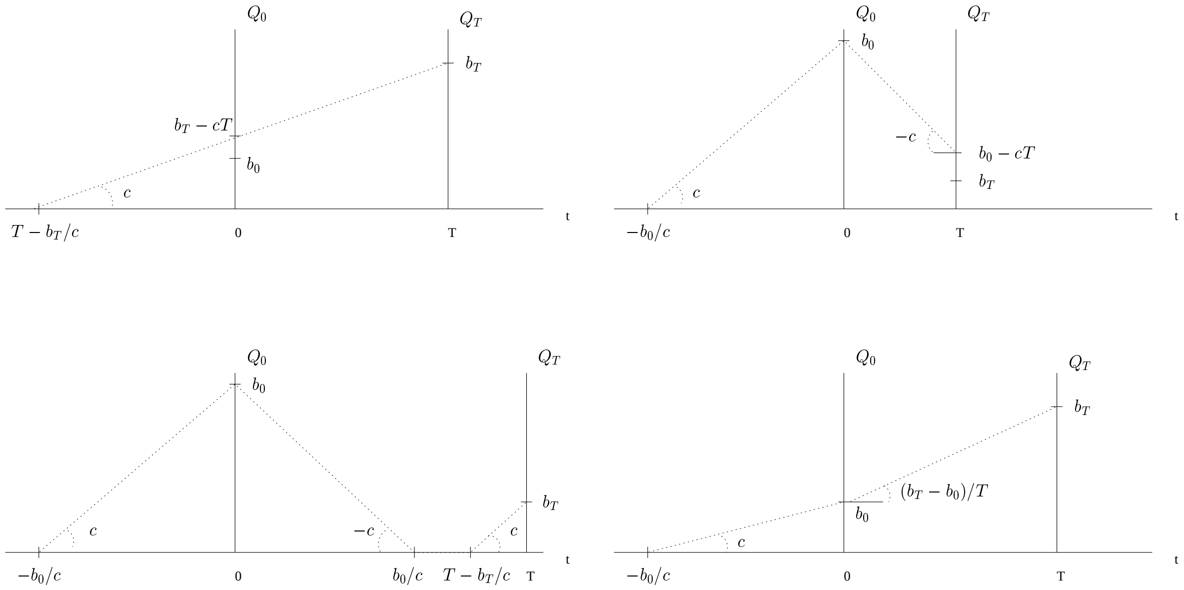

Figure 1: The most probable storage paths in set S.

Note that this suggests that Q_0 and Q_T behave (almost) independently if, compared to b_0 and b_T, T is large enough, as may be expected. The MPP is depicted in Figure 1 (bottom, left).

We now focus on the remaining regimes of Theorems 3.6-3.8. Consider the path that is such that the queue starts to build-up with rate c in the interval $(-b_0/c, 0]$, and in the interval $(0, T]$ builds-up with rate $(b_T - b_0)/T$. Clearly, this path yields $Q_0 = b_0$ and $Q_T = T$, and is thus contained in S. In Section 3.5 we show that this is path is in fact the MPP. Assuming that this is the case, $J(\bar{b}, T)$ is obtained by using (4), and equals $\gamma(\bar{b}, T)$. The MPP is depicted in Figure 1 (bottom, right).

The following two theorems summarize the above mentioned.

Theorem 3.9 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma > 1$. Then it holds that*

$$J(\bar{b}, T) \sim \begin{cases} 2(b_0 + b_T)c & \text{if } 0 \leq \alpha \leq \left(\sqrt{c\gamma} - 1\right)^2; \\ \gamma(\bar{b}, T) & \text{if } \left(\sqrt{c\gamma} - 1\right)^2 < \alpha < 1 + c\gamma; \\ 2b_T c & \text{if } \alpha \geq 1 + c\gamma. \end{cases}$$

Theorem 3.10 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma \leq 1$. Then it holds that*

$$J(\bar{b}, T) \sim \begin{cases} 2b_0 c & \text{if } 0 \leq \alpha \leq 1 - c\gamma; \\ \gamma(\bar{b}, T) & \text{if } 1 - c\gamma < \alpha < 1 + c\gamma; \\ 2b_T c & \text{if } \alpha \geq 1 + c\gamma. \end{cases}$$

3.5 Discussion

Using Theorems 3.6-3.8 also the logarithmic large-buffer asymptotics follow directly. Comparing Theorems 3.6-3.8 with Theorems 3.9-3.10, we find that the logarithmic large-buffer and logarithmic many-sources asymptotics match. Indeed, since we assumed that in the many-sources framework the standard Brownian sources are i.i.d., and because a standard Brownian motion is characterized by independent increments they should match, see for instance Example 7.4 of [9]. This implies that the paths depicted in Figure 1 ((bottom, left) and (bottom, right)) are in fact MPPs in set S.

In the analysis we assumed that the input process was a standard Brownian motion, i.e., no drift and $v(t) = t$. We now show how the results can be extended to general Brownian input, which have drift $\mu > 0$ and variance $v(t) = \lambda t$, $\lambda > 0$. Clearly, we should have that $c > \mu > 0$ to ensure stability. We denote the input process of a general Brownian motion by $\{B^*(t), t \in \mathbb{R}\}$. Note that $B^*(s,t) = N(\mu(t-s), \lambda(t-s)) = \mu(t-s) + \sqrt{\lambda}N(0, t-s) = \mu(t-s) + \sqrt{\lambda}B(s,t)$. This implies that

$$Q_0^* = \sup_{t\geq 0}\{B^*(-t,0) - ct\} = \sup_{t\geq 0}\{\sqrt{\lambda}B(-t,0) - (c-\mu)t\} = \sqrt{\lambda}\sup_{t\geq 0}\left\{B(-t,0) - \frac{c-\mu}{\sqrt{\lambda}}t\right\};$$

$$Q_T^* = \sup_{s\geq 0}\{B^*(T-s,T) - cs\} = \sup_{s\geq 0}\{\sqrt{\lambda}B(T-s,T) - (c-\mu)s\} = \sqrt{\lambda}\sup_{s\geq 0}\left\{B(T-s,T) - \frac{c-\mu}{\sqrt{\lambda}}s\right\}.$$

Hence, in order to generalize the results of this section, it follows that we have to set $c \leftarrow (c-\mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i = 0, T$ there. In addition, in order to generalize the results of Section 3.2 on the covariance, we also need to multiply the right-hand side of (7) and (14) by $\sqrt{\lambda}\sqrt{\lambda} = \lambda$. The results on the correlation coefficient can be generalized in similar way.

4 Analysis of $q(\bar{b}, T)$

In the previous section we derived an closed-form expression for $p(\bar{b}, T)$, large-buffer asymptotics, and the most probable paths to overflow. In this section we focus on $q(\bar{b}, T)$, and we derive similar results by exploiting the results of the previous section.

4.1 Conditional distribution function

In this subsection we derive an exact expression for $q(\bar{b}, T)$. With mild abuse of notation, we also write $q_f(\bar{b}, T) := -\partial p(\bar{b}, T)/\partial b_0 = \mathbb{P}(Q_0 = b_0, Q_T > b_T)$.

Theorem 4.1 *For each $b_0, b_T, T \geq 0$,*

$$q(\bar{b}, T) \quad = \quad \Phi\left(k_3(\bar{b}, T)\right) + \exp(-2b_T c)\Phi\left(k_4(\bar{b}, T)\right).$$

Proof: We have that

$$q(\bar{b}, T) = \frac{\mathbb{P}(Q_0 = b_0, Q_T > b_T)}{\mathbb{P}(Q_0 = b_0)} = \frac{q_f(\bar{b}, T)}{2ce^{-2b_0 c}},$$

as the workload Q_0 is exponentially distributed with mean $1/(2c)$. As mentioned, $q_f(\overline{b},T)$ can be obtained by deriving $-\partial p(\overline{b},T)/\partial b_0$, with $p(\overline{b},T)$ as in Theorem 3.1. This yields

$$\begin{aligned} q_f(\overline{b},T) &= -\frac{1}{\sqrt{T}}\phi(k_1(\overline{b},T)) + \frac{1}{\sqrt{T}}e^{-2b_Tc}\phi(k_2(\overline{b},T)) + \\ & \quad 2ce^{-2b_0c}\Phi(k_3(\overline{b},T)) - \frac{1}{\sqrt{T}}e^{-2b_0c}\phi(k_3(\overline{b},T)) + \\ & \quad 2ce^{-2(b_0+b_T)c}\Phi(k_4(\overline{b},T)) + \frac{1}{\sqrt{T}}e^{-2(b_0+b_T)c}\phi(k_4(\overline{b},T)). \end{aligned}$$

Straightforward calculus also shows that

$$\frac{1}{2ce^{-2b_0c}\sqrt{T}}e^{-2(b_0+b_T)c}\phi(k_4(\overline{b},T)) - \frac{1}{2ce^{-2b_0c}\sqrt{T}}\phi(k_1(\overline{b},T)) = 0,$$

and

$$\frac{1}{2ce^{-2b_0c}\sqrt{T}}e^{-2b_Tc}\phi(k_2(\overline{b},T)) - \frac{1}{2ce^{-2b_0c}}\sqrt{T}e^{-2b_0c}\phi(k_3(\overline{b},T)) = 0,$$

which proves the stated. □

We note that Harrison [10] also obtained Theorem 4.1. However, $q(\overline{b},T)$ was derived in a completely different manner. In [10] the author first calculated the joint distribution of $B_T - cT$ and $\max_{t\in[0,T]}\{B_t - ct\}$ using Martingales, and used this to derive $q(\overline{b},T)$.

4.2 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of $q(\overline{b},T)$. The proof of the following three theorems is similar to the proof of Theorem 3.6. We omit the proofs. Define

$$\delta(\overline{b},T) := \frac{(-b_T - cT + b_0)^2}{2T} = \gamma(\overline{b},T) - 2b_0c.$$

Theorem 4.2 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma > 1$. For $b \to \infty$,*

$$q(\overline{b},T) \sim \begin{cases} e^{-2b_Tc} & \text{if } 0 \leq \alpha < (\sqrt{c\gamma}-1)^2; \\ \left(1 - \frac{1}{\sqrt{2\pi}k_3(\overline{b},T)}\right)e^{-2b_Tc} & \text{if } \alpha = (\sqrt{c\gamma}-1)^2; \\ -\frac{1}{\sqrt{2\pi}k_3(\overline{b},T)}e^{-\delta(\overline{b},T)} & \text{if } \alpha > (\sqrt{c\gamma}-1)^2. \end{cases}$$

Theorem 4.3 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma = 1$. For $b \to \infty$,*

$$q(\overline{b},T) \sim \begin{cases} 1 & \text{if } \alpha = 0; \\ -\frac{1}{\sqrt{2\pi}k_3(\overline{b},T)}e^{-\delta(\overline{b},T)} & \text{if } \alpha > 0. \end{cases}$$

Theorem 4.4 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma < 1$. For $b \to \infty$,*

$$q(\overline{b},T) \sim \begin{cases} 1 & \text{if } 0 \leq \alpha < 1 - c\gamma; \\ 1/2 & \text{if } \alpha = 1 - c\gamma; \\ -\frac{1}{\sqrt{2\pi}k_3(\overline{b},T)}e^{-\delta(\overline{b},T)} & \text{if } \alpha > 1 - c\gamma. \end{cases}$$

4.3 Most probable path

We can interpret the different regimes of the asymptotics of $q(\bar{b},T)$ by using Schilder's theorem. To this end, we feed the single-node network by n i.i.d. standard Brownian sources. The link rate and buffer thresholds are also scaled by n: nc, nb_0 and nb_T, respectively. Let

$$K(\bar{b},T) := -\lim_{n\to\infty} \frac{1}{n} \log q_n(\bar{b},T).$$

The following two theorems present the logarithmic many-sources asymptotics of $q(\bar{b},T)$. The proofs are similar to the ones in Section 3.4, and therefore we omit the proofs.

Theorem 4.5 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma > 1$. Then it holds that*

$$K(\bar{b},T) \sim \begin{cases} 2b_T c & \text{if } 0 \leq \alpha \leq \left(\sqrt{c\gamma}-1\right)^2; \\ \delta(\bar{b},T) & \text{if } \alpha > \left(\sqrt{c\gamma}-1\right)^2. \end{cases}$$

Theorem 4.6 *Let $b_0 = b$, $b_T = \alpha b$, $T = \gamma b$, with $\alpha, \gamma \geq 0$. Suppose $c\gamma \leq 1$. Then it holds that*

$$K(\bar{b},T) \sim \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 1 - c\gamma; \\ \delta(\bar{b},T) & \text{if } \alpha > 1 - c\gamma. \end{cases}$$

The MPPs associated with Theorems 4.5-4.6 are closely related to the ones depicted in Figure 1, however, now only defined on the interval $(0,T]$. The MPP corresponding to regime $\alpha > (\sqrt{c\gamma}-1)^2$ in Theorem 4.5 and regime $\alpha > 1 - c\gamma$ in Theorem 4.6 is depicted in Figure 1 (bottom, right). The MPP associated with regime $0 \leq \alpha \leq 1 - c\gamma$ in Theorem 4.4 is depicted in Figure 1 (top, right). In this regime $K(\bar{b},T) = 0$, because $Q_0 = b_0$ implies that $Q_T \geq b_T$ without additional effort. For regime $0 \leq \alpha \leq (\sqrt{c\gamma - 1})^2$ in Theorem 4.5 the corresponding MPP is illustrated in Figure 1 (bottom, left). Using (4), we find that these MPPs indeed yield Theorems 4.5-4.6.

By comparing Theorems 4.2-4.4 with Theorems 4.5-4.6, it is not hard to see that the logarithmic large-buffer and logarithmic many-sources asymptotics of $q(\bar{b},T)$ match, as was expected. Similar to Section 3.5, the results in this section can be extended to general Brownian input by setting $c \leftarrow (c-\mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i = 0,T$ there.

5 Conclusion

In this report we analyzed a single-node network with Brownian input. We derived the joint distribution function of the workloads at time 0 and T, the covariance between these workloads, the conditional distribution function of the workload at time T given a certain workload at time 0, large buffer asymptotics, and the most probable path leading to overflow.

A natural extension of the present work is to analyze the joint overflow probability in a two-node tandem queue with Brownian input, where we observe queue I at time 0, and queue II at time T, i.e., $\mathbb{P}((Q_{\mathrm{I}})_0 > b_0, (Q_{\mathrm{II}})_T > b_T)$. Another future research direction includes extending the results to other input processes, e.g., light-tailed Lévy processes.

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