Centrum voor Wiskunde en Informatica

REPORT RAPPORT

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

A fluid model for a relay node in an ad-hoc network: the case of heavy-tailed input

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REPORT PNA-E0703 SEPTEMBER 2007

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ISSN 1386-3711

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2000 Mathematics Subject Classification: 60K25
Keywords and Phrases: fluid queue - ad hoc network - regular variation
Note: Part of the research was conducted while the first author was affiliated to CWI. Hans van den Berg (TNO Telecom) is acknowledged for bringing this model under our attention.

A fluid model for a relay node in an ad-hoc network: the case of heavy-tailed input

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Abstract

Relay nodes in an ad-hoc network can be modelled as fluid queues, in which the available service capacity is shared by the input and output. In this paper such a relay node is considered; jobs arrive according to a Poisson process and bring along a random amount of work. The total transmission capacity is fairly shared, meaning that, when n jobs are present, each job transmits traffic into the queue at rate $1/(n+1)$ while the queue is drained at the same rate of $1/(n+1)$. Where previous studies mainly concentrated on the case of exponentially distributed job sizes, the present paper addresses regularly varying jobs. The focus lies on the tail asymptotics of the sojourn time S. Using sample-path arguments, it is proven that $\mathbb{P}\{S > x\}$ behaves roughly as the *residual* job size, i.e., if the job sizes are regularly varying of index $-\nu$, the tail of S is regularly varying of index $1-\nu$. In addition, we address the tail asymptotics of other performance metrics, such as the workload in the queue, the flow transfer time and the queueing delay.

1 Introduction

Ad-hoc networks are self-configuring networks of mobile routers, connected by wireless links. They enable infrastructure-free communication: no fixed equipment is needed, but instead each client acts as a hub. When information needs to be transmitted across

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the network, it is sent from the sender to the receiver by relaying the packets along intermediate nodes. An excellent survey on ad-hoc networks, with special emphasis on Quality-of-Service aspects, is [9].

On an abstract level one could model nodes in an ad-hoc network as *queues* [2]. Indeed, data packets arrive and are served to be relayed, and during periods in which the arrival rate (temporarily) exceeds the departure rate, the buffer content of the queue grows. There is one property, however, that distinguishes these queues from ordinary queues: they have the interesting feature that, at any node, the available transmission capacity is used both to (i) 'pull' information packets from the 'predecessor nodes' into the queue, and (ii) 'push' information packets from the queue towards 'successor nodes' (and eventually the destination client).

Now consider the situation that at some point in time n stations send traffic through the same relay node. Then each 'sending node' is assigned an equal share $1/(n+1)$ of the available medium capacity (which we may normalize to 1), which is the same fraction as is allocated to serve the queue. In other words, the total input rate is $n/(n+1)$, so that net rate of growth of the queue is $(n-1)/(n+1)$. We conclude that as soon as $n > 1$, the node's input rate exceeds its output rate, and hence the excess traffic accumulates in the node's buffer; only when $n = 0$ the queue drains. Interestingly, this entails that relay nodes are prone to becoming bottlenecks. We remark that the queue is served on a First In First Out (FIFO) basis.

To study the relay node described above, we consider a flow-level model. Jobs arrive at the relay node, for instance, according to a Poisson process, and bring along a random amount of work, say i.i.d. (independent and identically distributed) samples from a distribution $B(\cdot)$. At the flow level, traffic then arrives as a *fluid* to the queue.

There are several interesting performance measures to consider. In the first place, one may be interested in the time F before a job is completely 'pulled out of the predecessor node', in that all traffic has reached the queue of the relay node. It then still takes some time, however, before the job has gone through the relay node: the sojourn time S equals F, increased by the delay D of the last particle of the flow. Previous work [7] focused on the case that $B(\cdot)$ corresponds to an exponential distribution, and it is a natural question whether other distributions are amenable for analysis as well.

In this paper we consider the relevant, and technically interesting, case of heavy-tailed jobs. More precisely, we assume that the jobs are i.i.d. samples from a regularly-varying distribution of index $-\nu$, i.e., $\mathbb{P}\{B > x\}$ behaves roughly like $x^{-\nu}$, for some $\nu > 0$; we write $B(\cdot) \in \mathcal{R}_{-\nu}$. For standard queueing models (i.e., models of M/G/1 or GI/G/1 type) with regularly-varying input, a wealth of interesting contributions have appeared; early papers are for instance [5, 8]. Generally speaking, under FIFO scheduling the sojourn time (just like the workload) is in $\mathcal{R}_{1-\nu}$ (that is, the tail is as heavy as a *residual* job size), whereas under processor-sharing [12] it is in $\mathcal{R}_{-\nu}$ (and hence the tail is essentially as heavy as that of the jobs themselves). As is clear from the model description we gave

above, our relay-node has both PS and FIFO elements, and therefore it is an interesting fundamental question whether the sojourn time is in $\mathcal{R}_{1-\nu}$ or $\mathcal{R}_{-\nu}$ (or perhaps regularly-varying of another index).
In our analysis, we rely on sample-path methods, comparable to those developed in [1]. A lower bound is derived by identifying a most likely scenario, and computing its tail asymptotics. Then, in the upper bound we split the event of our interest into a number of sub-events, and show that among these, asymptotically, only the most likely scenario is relevant.

This paper is organized as follows. Section 2 details the model, and present some preliminaries. Section 3 describes the main results, which are proven in Section 4. Section 5 concludes.

2 Model and preliminaries

In this section, we first give a description of the fluid-flow queueing system that is used to model a relay node in an ad-hoc network. Second, we give some preliminary results that are mainly used in the proofs of our results.

Model description
Consider a queueing system at which flows arrive according to a Poisson process with rate λ. Each active flow brings along an amount of work; we assume that the service requirements are i.i.d. with distribution $B(\cdot)$ and mean $\beta < \infty$. The flow transmits traffic into the queue (according to a procedure detailed below) until it has sent out its full service requirement; then we say that the flow becomes inactive. Evidently, the mean amount of work generated per unit time is $\varrho := \lambda\beta$.
The total transmission capacity is, without loss of generality, normalized to 1. This transmission capacity is fairly shared between all nodes present. This means that when there are $N(u)$ active flows at time u, each active flow transmits traffic into the queue at rate $1/(N(u)+1)$. The service rate of the queue then also equals $1/(N(u)+1)$, implying that the queue is only drained when there are no active flows (and remains constant when there is one flow present).
In this paper, particularly in the proofs, we frequently use terminology of fluid-tandem queues; quantities associated with traffic of active flows that is not yet in the queue are labelled class 1, while quantities associated with traffic present in the queue are labelled class 2. The total available transmission capacity for the active flows (that is, the input rate of the queue) during the time interval $[s,t]$ thus equals

$$C_1(s,t) := \int_s^t \frac{N(u)}{N(u)+1}\mathrm{d}u$$

and the total service rate for the queue equals

$$C_2(s,t) := \int_s^t \frac{1}{N(u)+1}\mathrm{d}u.$$

Since the input into the buffer during the interval $[s,t]$ equals the total transmissions of active flows, we obtain the following useful representation of the class-2 workload:

$$V_2(t) = \sup_{s\le t}\{C_1(s,t) - C_2(s,t)\} = \sup_{s\le t}\left\{\int_s^t \frac{N(u)-1}{N(u)+1}\mathrm{d}u\right\}. \tag{1}$$

In this paper, we assume that the service requirement distribution is heavy-tailed. Let B denote a generic service requirement, and let B^r be a random variable distributed as the residual lifetime of B, i.e.,

$$B^r(x) := \mathbb{P}\{B^r < x\} = \frac{1}{\beta}\int_0^x (1-B(y))\mathrm{d}y. \tag{2}$$

More specifically, we assume that the service requirement distribution is regularly varying of index $-\nu$ (denoted as $B(\cdot)\in\mathcal{R}_{-\nu}$), i.e., $1-B(x)\sim L(x)x^{-\nu}, \nu>1$ (so that $\beta<\infty$), with $L(x)$ some slowly varying function. Here, and throughout the paper, we use the notation $f(x)\sim g(x)$ to indicate that $f(x)/g(x)\to 1$ as $x\to\infty$. (A function $L(\cdot)$ is called slowly varying if $L(\eta x)\sim L(x)$ for all $\eta>1$.) It follows from Karamata's Theorem [3, Thm. 5.1.11] that $x\mathbb{P}\{B>x\}\sim(\nu-1)\beta\,\mathbb{P}\{B^r>x\}$, and thus $B^r(\cdot)\in\mathcal{R}_{1-\nu}$.

Preliminaries

Denote by $B_i(s,t)$, $i=1,2$, the amount of service received by class i during the interval $[s,t]$. The amounts of service satisfy the following evident inequality

$$B_1(s,t) + B_2(s,t) \le t-s, \tag{3}$$

with equality iff $V_1(u)+V_2(u)>0$ for all $u\in[s,t]$.

Similarly, define $A_i(s,t)$, $i=1,2$, as the total input for class i during the interval $[s,t]$. For the workloads the following obvious identity relation holds, for $i=1,2$ and $s<t$,

$$V_i(t) = V_i(s) + A_i(s,t) - B_i(s,t). \tag{4}$$

Furthermore, using the fact that $A_2(s,t)=B_1(s,t)$, we have

$$V_2(t) = V_2(s) + B_1(s,t) - B_2(s,t). \tag{5}$$

$A_1(s,t)$ is distributed as a Poisson number (with mean $\lambda(t-s)$) of i.i.d. service requirements, each with distribution $B(\cdot)$, and the class-1 workload obeys

$$V_1(t) = \sup_{s\le t}\left\{A_1(s,t) - \int_s^t \frac{N(u)}{N(u)+1}\mathrm{d}u\right\}.$$

We note that the above entails that the workload in the *overall* system can be directly related to the stationary workload in an M/G/1 queueing model. This can be seen as follows. In our tandem-queueing model, arriving flows are essentially served twice [2]; flows that belong to the class-1 workload require two stages of service, while for work in the class-2 workload (that is, the queue) there is only a single stage of service left. This entails that the stability constraint of the model is $\varrho < \frac{1}{2}$. Furthermore, it holds that the total service capacity (at a constant rate of 1 per unit time) is used as long as there is any work present, which entails that the system is work-conserving. According to Reich's formula [10], the steady-state overall-workload representation therefore reads

$$2V_1 + V_2 =_{\mathrm{d}} \sup_{t\geq 0}\{2A_1(-t,0) - t\}.$$

The distribution of $2V_1 + V_2$ thus equals the steady-state workload distribution of an M/G/1 queue with generic service requirement $2B$. Applying the well-known asymptotic result for the standard M/G/1 queue [5, 8], we directly obtain the asymptotic tail distribution of the overall workload:

Theorem 2.1. *Assume that $\varrho < 1/2$. Then, $B(\cdot) \in \mathcal{R}_{-\nu}$ iff $\mathbb{P}\{2V_1 + V_2 < \cdot\} \in \mathcal{R}_{1-\nu}$, and then*

$$\mathbb{P}\{2V_1 + V_2 > x\} \sim \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{x}{2}\right\}.$$

Finally, we focus on the time required to serve an amount of work in the queue. Let time 0 be an arbitrary instant at which a flow becomes active and let

$$W_0 := \arg\inf_{t\geq 0}\{B_2(0,t) = V_2(0)\}$$

be the time required to serve the amount of work present in the queue at flow initiation. Since active flows initiate their transmission immediately upon arrival, W_0 also corresponds to the epoch at which the first packet (to be interpreted as infinitesimally small fluid particle) of the flow leaves the buffer. In queueing terminology, this quantity is frequently referred to as the *waiting time*. Because the total transmission capacity is used during $[0, W_0]$, we have the following identity:

$$W_0 = V_2(0) + B_1(0, W_0). \tag{6}$$

We note that we interchangeably use W and W_0 to denote such a generic waiting time.

3 Results

In this section, we present the main results of the paper. In particular, we give exact asymptotics for the steady-state workload, flow-transfer delay, queueing delay, and sojourn time. For the former two quantities this section also provides the proofs; the proofs for the latter two quantities (which are considerably more involved) are given in the next section. For each quantity, we also provide the underlying heuristics; these turn out to be extremely useful in understanding the model's properties (and play an important role in the proofs of Section 4).

3.1 Steady-state workload

We first consider the steady-state class-2 workload distribution V_2. For this quantity we can rely on the main result of [1] giving the workload asymptotics of (in the terminology of [1]) the streaming traffic, sharing bandwidth with a second class of elastic flows according to the PS discipline. The result indicates that the tail of the steady-state workload is as heavy as that of B^r, i.e., regularly varying of index $1-\nu$.

Theorem 3.1. *If $B(\cdot) \in \mathcal{R}_{-\nu}$ and $0 < \varrho < \frac{1}{2}$, then*

$$\mathbb{P}\{V_2 > x\} \sim \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{1-\varrho}{2\varrho}x\right\}.$$

Proof. Using (1), we may rewrite the workload representation as

$$\begin{aligned} V_2(t) &= \sup_{s\le t}\left\{t-s-\int_s^t \frac{2}{N(u)+1}\mathrm{d}u\right\} \\ &= 2\sup_{s\le t}\left\{\frac{1}{2}(t-s)-\int_s^t \frac{1}{N(u)+1}\mathrm{d}u\right\} =: 2V^\star(t). \end{aligned} \tag{7}$$

We note that $V^\star(t)$ equals the workload of the streaming class in case $K=1$ and $r=1/2$, using the terminology of the workload representation in [1]. In that case, the condition $Kr < 1-\varrho < (K+1)r$ translates into $0 < \varrho < 1/2$. Now, using (7) and applying [1, Thm. 4.1] with $K=1$ and $r=1/2$ gives the result. □

Heuristic arguments

The heuristics behind the workload asymptotics are as follows. Consider the workload at an arbitrary instant, say, at time 0. The most likely way for $V_2(0)$ to become large is the arrival of one exceptionally large job (also referred to as *tagged job*) of size B_{tag} before time $-\tilde{t}_1$ (which is defined below). Suppose that this job arrives at time $-y$. For any value of $y \ge \tilde{t}_1$ one can determine the minimal size of the tagged job to make sure that $V_2(0) > x$. We first observe that y cannot be smaller than

$$\tilde{t}_1 := \frac{x}{\varrho}.$$

This can be seen as follows. The amount of work stored in the queue by all jobs except for the tagged one is close to its average amount of work generated, i.e., roughly ϱy arrives to the queue due to the other job arrivals during the interval $[-y, 0]$. Assuming that the tagged job is still transmitting into the buffer at time 0 (otherwise $V_2(0)$ would even be smaller), it follows from the PS discipline that the tagged job brings along as much work as is served by the queue; the buffer content at time 0 is then about ϱy. For $V_2(0) > x$ it is thus required that $y > \tilde{t}_1$.

Suppose that the tagged job has size B_{tag}. Over the duration of the transmission of the tagged job, it equally shares the remaining capacity with the queue at a rate $(1-\varrho)/2$. It then takes $2B_{\text{tag}}/(1-\varrho)$ time to fully put the tagged job into the buffer, i.e., at time

$-y + 2B_{\text{tag}}/(1-\varrho)$ the tagged job has transmitted its full service request. Note that for the minimal size of B_{tag} it holds that this time should be before time 0. When the tagged job has fully arrived at the queue, the buffer drains at a rate $1-2\varrho$. Thus, the buffer content at time 0 is

$$V_2(0) = \frac{2B_{\text{tag}}}{1-\varrho}\varrho - \left(y - \frac{2B_{\text{tag}}}{1-\varrho}\right)(1-2\varrho) = 2B_{\text{tag}} - (1-2\varrho)y.$$

To make sure that $V_2(0) > x$, we have

$$B_{\text{tag}} > \frac{x}{2} + \frac{1-2\varrho}{2}y.$$

Integrating with respect to y (and neglecting the asymptotically small probability of having two or more large job arrivals), it follows that the probability of a large workload roughly equals

$$\int_{y=\tilde{t}_1}^{\infty} \lambda \mathbb{P}\left\{B_{\text{tag}} > \frac{x}{2} + \frac{1-2\varrho}{2}y\right\} \mathrm{d}y.$$

After the change of variable $z = y - \tilde{t}_1$, dividing and multiplying with β, and using (2), we obtain the desired expression.

3.2 Flow transfer delay

Here we consider the time F it takes for an arbitrary arriving flow to transmit its traffic into the buffer. Since the available transmission capacity at time u for each individual active flow equals $1/(N(u)+1)$ it trivially follows that the flow transfer delay equals the sojourn time of a non-permanent customer in a M/G/1 PS queue with one permanent customer. Applying [6, Thm. 3], it follows directly that the flow transfer delay is regularly varying of index $-\nu$:

Proposition 3.2. *If $B(\cdot) \in \mathcal{R}_{-\nu}$ and $0 < \varrho < 1$, then*

$$\mathbb{P}\{F > x\} \sim \mathbb{P}\left\{B > \frac{1-\varrho}{2}x\right\}.$$

Proof. The result follows directly from [6, Thm. 3] with the identification $\gamma^f := (1-\varrho)/2$, see also [1, Prop. 3.1]. □

Heuristic arguments

Clearly, the heuristics behind the flow-transfer delay asymptotics are the same as the heuristics for the asymptotic sojourn time in a M/G/1 PS queue with one permanent customer. That is, a large flow-transfer delay is due to a large service requirement of the flow itself. The ratio $(1-\varrho)/2$ is simply the average service rate received by the large flow; over the duration of the large flow, the other flows transmission rate roughly equals their average input rate ϱ. The remaining capacity of $1-\varrho$ is equally shared between the large flow and the relay node (i.e., the buffer).

3.3 Sojourn time and queueing delay

In this part, we consider the queueing delay D and sojourn time $S = F+D$ of an arbitrary arriving flow. The queueing delay is here defined as the time it takes the last packet (recall that a packet is to be interpreted as an infinitesimally small fluid particle) of the flow to go through the queue. The sojourn time is the time between the arrival of a flow until the last packet leaves the buffer.
In fact, the queueing delay and sojourn time are asymptotically equivalent, as presented in the following theorem:

Theorem 3.3. *Define* $\phi_{\max} := \max\{1-\varrho, (1+\varrho)/2\}$ *and* $\phi_{\min} := \min\{1-\varrho, (1+\varrho)/2\}$. *If* $B(\cdot) \in \mathcal{R}_{-\nu}$ *and* $0 < \varrho < \frac{1}{2}$ ($\varrho \neq \frac{1}{3}$), *then*

$$\begin{aligned}\mathbb{P}\{S > x\} \sim \mathbb{P}\{D > x\} \sim \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{(1-\varrho)^2}{2\varrho}x\right\} \\ + \frac{2\varrho}{|1-3\varrho|}\mathbb{P}\left\{\phi_{\min}\frac{(1-\varrho)}{2\varrho}x < B^r < \phi_{\max}\frac{(1-\varrho)}{2\varrho}x\right\}. \end{aligned} \tag{8}$$

Proof. For any flow, the waiting time is evidently less than its sojourn time. Lower and upper bounds for the sojourn time S that asymptotically coincide are then given by Propositions 4.2 and 4.3, respectively, providing the asymptotic tail of S.
Since the delay of a flow is bounded by its sojourn time, Proposition 4.3 also gives an asymptotic upper bound for the tail of D. For the lower bound, write

$$\begin{aligned}\mathbb{P}\{D > x\} &\geq \mathbb{P}\{S > (1+\epsilon)x; F < \epsilon x\} \\ &\geq \mathbb{P}\{S > (1+\epsilon)x\} - \mathbb{P}\{F > \epsilon x\}.\end{aligned}$$

Using Proposition 3.2 and the fact that $B(\cdot) \in \mathcal{R}_{-\nu}$ it follows that

$$\mathbb{P}\{F > \epsilon x\} = o(\mathbb{P}\{B^r > x\}).$$

The lower bound of D now follows directly by letting $\epsilon \downarrow 0$ and using the fact that $S \in \mathcal{R}_{1-\nu}$, completing the proof. □

Corollary 3.4. *If* $\mathbb{P}\{B > x\} = L(x)x^{-\nu}$ *for some slowly varying* $L(\cdot)$ *and* $0 < \varrho < \frac{1}{2}$, *then* S *and* $D \in \mathcal{R}_{1-\nu}$, *and, in particular,*

$$\mathbb{P}\{S > x\} \sim \mathbb{P}\{D > x\} \sim \frac{1}{(\nu-1)\beta}L(x)x^{1-\nu} \cdot (\xi_1(\varrho) + \xi_2(\varrho)),$$

where

$$\xi_1(\varrho) := \frac{2\varrho}{1-2\varrho}\left(\frac{2\varrho}{(1-\varrho)^2}\right)^{\nu-1};$$

$$\xi_2(\varrho) := \frac{2\varrho}{|1-3\varrho|}\left(\left(\frac{2\varrho}{\phi_{\min}(1-\varrho)}\right)^{\nu-1} - \left(\frac{2\varrho}{\phi_{\max}(1-\varrho)}\right)^{\nu-1}\right).$$

Remark 3.5. Notice that the value $\varrho = \frac{1}{3}$ plays a special role in Corollary 3.4 (and Theorem 3.3). First observe that $\varrho < \frac{1}{3}$ holds iff $1 - \varrho > (1 + \varrho)/2$; for $\varrho = \frac{1}{3}$ we have that $\phi_{\min}$ and $\phi_{\max}$ are equal. Elementary calculus shows that this entails that $\xi_2(\varrho)$ can be alternatively written as

$$\xi_2(\varrho) = \frac{2\varrho}{1-3\varrho}\left(\left(\frac{4\varrho}{1-\varrho^2}\right)^{\nu-1} - \left(\frac{2\varrho}{(1-\varrho)^2}\right)^{\nu-1}\right).$$

L'Hôptital's rule yields that

$$\xi_2\left(\frac{1}{3}\right) = \frac{3}{4}(\nu-1)\left(\frac{3}{2}\right)^{\nu-2},$$

so that for $\varrho = \frac{1}{3}$

$$\mathbb{P}\{S > x\} \sim \mathbb{P}\{D > x\} \sim \frac{1}{(\nu-1)\beta}L(x)x^{1-\nu} \cdot \left(2\left(\frac{3}{2}\right)^{\nu-1} + \frac{3}{4}(\nu-1)\left(\frac{3}{2}\right)^{\nu-2}\right).$$

Remark 3.6. In [7], the (virtual) queueing delay is defined as the delay experienced by a fluid particle arriving at the buffer at a random point in time. Using PASTA, it follows that the buffer content and number of flows present at the Poisson instants of flow arrivals are equal to these quantatities at arbitrary instants (time averages). Hence, the above definition of the queueing delay distribution is identical to the 'waiting time' distribution of the present paper and thus has the same asymptotic behavior.

Heuristic arguments

The heuristics of the sojourn time and queueing delay are as follows. Consider the job that arrives at, say, time 0. This job has an exceptionally long sojourn time if it sees upon arrival an exceptionally large workload, while the job itself is relatively small. This large workload is in turn due to a single exceptionally large job (to which we refer to as the *tagged job*) that arrived in the past at time, say, $-y$. Because the job itself is small, the flow transfer delay can be neglected compared to the queueing delay, yielding the same asymptotic behavior for the queueing delay, sojourn time, and also the waiting time. In the heuristics we henceforth focus on the waiting time W_0, i.e., the time required to serve $V_2(0)$.

For any value of $y > 0$ one can determine the minimal size of the tagged job to make sure that work arriving at the queue at time 0 does not leave the system before time x.

We first observe that y cannot be smaller than

$$t_0 := \frac{(1-\varrho)x}{2\varrho}.$$

This can be seen as follows. As long as the tagged job is in the system, the queue grows at a rate of roughly ϱ (because the tagged job brings along as much work as is served by the queue, and hence all 'usual input', arriving at an average rate of ϱ, is stored in

the queue). In other words, the buffer content at time 0 is about ϱy. Now consider a fluid packet arriving at time 0. To maximize the time before this packet leaves the queue, assume that the tagged job stays in the system; then the buffer drains at a rate $(1-\varrho)/2$. This means that the queue is empty at time $2\varrho y/(1-\varrho)$, which cannot be larger than x if y is smaller than t_0. Therefore we assume from now on $y \geq t_0$.
First we consider the situation that the tagged job has transmitted its full service requirement into the buffer at time 0. The capacity used by all other customers to store traffic into the queue roughly equals the amount of work generated, which is close to average. Hence, after time 0, the buffer is drained at a rate of $1-\varrho$, implying that

$$W_0 \approx \frac{V_2(0)}{1-\varrho}.$$

For $W_0 > x$ it is thus sufficient that $V_2(0) > x(1-\varrho)$. Using the asymptotic results for $V_2(0)$ this directly provides the first term in the rhs of (8). We note that it follows directly from the heuristics of the workload asymptotics (see Subsection 3.1) that the most likely scenario for $V_2(0) > x(1-\varrho)$ to occur is the arrival of a large job at time $-y$, with

$$y \geq t_1 := \frac{1-\varrho}{\varrho}x,$$

and with a service requirement

$$B_{\text{tag}} > \frac{1-\varrho}{2}x + \frac{1-2\varrho}{2}y.$$

The heuristic arguments for this scenario are also depicted in the first figure of Figure 1.

Now consider the situation that the tagged job is still transmitting into the queue at time 0. The tagged job can send traffic into the queue at a rate of about $(1-\varrho)/2$. Supposing the tagged job has size B_{tag}, then it has been put into the buffer at time $-y+2B_{\text{tag}}/(1-\varrho) > 0$; since there is roughly ϱy in the buffer at time 0, the amount of work in front of the job arriving at time 0 left at that particular instant then equals

$$\varrho y - \left(\frac{2B_{\text{tag}}}{1-\varrho} - y\right)\left(\frac{1-\varrho}{2}\right) = \varrho y - B_{\text{tag}} + \frac{1}{2}y(1-\varrho).$$

In other words, the waiting time W_0 (or time to serve the amount of work at time 0) is

$$-y + \frac{2B_{\text{tag}}}{1-\varrho} + \left(\varrho y - B_{\text{tag}} + \frac{1}{2}y(1-\varrho)\right)\Big/(1-\varrho) = \frac{B_{\text{tag}}}{1-\varrho} + \frac{3\varrho-1}{2(1-\varrho)}y;$$

this time is larger than x if

$$B_{\text{tag}} > (1-\varrho)x + \frac{1}{2}(1-3\varrho)y.$$

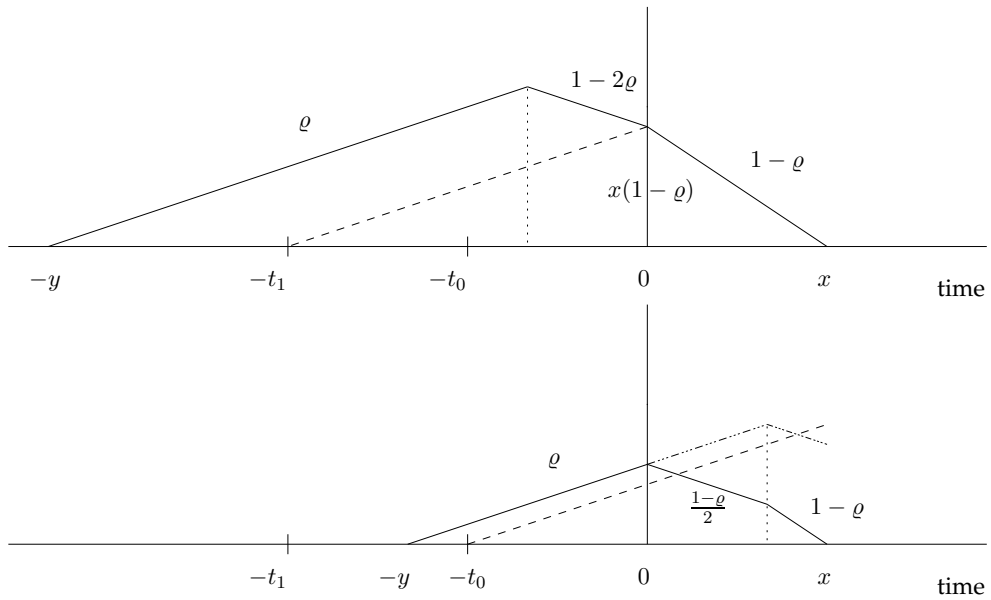

Figure 1: Two realizations of a large waiting time x; before time 0 (i.e., *left* of the vertical axis) the total amount of work in the buffer is depicted, while after time 0 (i.e., *right* of the vertical axis) the amount of work in the buffer in front of the fluid particle arriving at time 0 is drawn.

See also the second figure of Figure 1 for a graphical illustration of this scenario. It is readily verified that these y should lie in the interval $[t_0, t_1]$.
Now, realizing that the probability that a job arriving in interval $\mathrm{d}y$ around y is $\lambda \mathrm{d}y$, the probability of a long delay (and the probability of a long sojourn time) roughly equals

$$\int_{t_0}^{t_1} \lambda \mathbb{P}\left\{B > (1-\varrho)x + \frac{1}{2}(1-3\varrho)y\right\} \mathrm{d}y + \int_{t_1}^{\infty} \lambda \mathbb{P}\left\{B > \frac{1}{2}(1-\varrho)x + \frac{1-2\varrho}{2}y\right\} \mathrm{d}y.$$

After a change of variable, dividing and multiplying with β, and using the definition of the residual lifetime of B, we obtain the desired expression.

4 Proofs

In this section we derive the asymptotics of the sojourn time and queueing delay, i.e., we prove the asymptotics of $\mathbb{P}\{S > x\}$ and $\mathbb{P}\{W > x\}$, as stated in Theorem 3.3. We consider the system at an arbitrary instant at which a flow becomes active, say at time 0. As indicated earlier, we denote by $W \equiv W_0$ its "waiting time" (defined as the time until the first packet of the flow leaves the buffer). In Section 4.1, we obtain an asymptotic lower bound for $\mathbb{P}\{W_0 > x\}$ and thus for $\mathbb{P}\{S_0 > x\}$ (with S_0 denoting the sojourn time of our flow), while in Section 4.2 an asymptotic upper bound for $\mathbb{P}\{S_0 > x\}$ that asymptotically coincides is derived.

4.1 Lower bound

In this subsection, we derive an asymptotic lower bound for $\mathbb{P}\{S > x\}$ and $\mathbb{P}\{W > x\}$. First, we sketch two scenarios which enable us to show that these two scenarios provide sufficient sample-path conditions for the event $W_0 > x$ to occur, see Lemma 4.1; it is instructive to compare these scenarios with those heuristically derived in the previous section. Next, we convert these sample-path inclusions into a probabilistic lower bound for $\mathbb{P}\{W_0 > x\}$.

Define two time instants $t_0 < t_1$ as

$$t_0 := \frac{1-\varrho+\delta+5\epsilon}{2(\varrho-\delta)}x \qquad \text{and} \qquad t_1 := \frac{1-\varrho+\delta+3\epsilon}{\varrho-\delta}x,$$

that is, t_0 is close to $x \cdot (1-\varrho)/2\varrho$, and t_1 is close to $x \cdot (1-\varrho)/\varrho$. Now, consider the following events:

1. Either $\exists y \in [t_0, t_1]$ such that at time $-y$ a tagged flow arrives with service requirement

$$\begin{aligned} B_{\text{tag}} &\geq \frac{1-\varrho+\delta+5\epsilon}{2(\varrho-\delta)}\frac{1+\varrho-\delta}{2}x + \frac{1-3(\varrho-\delta)}{2}(y-t_0) - \epsilon x \\ &= (1-\varrho+\delta+5\epsilon)x + \frac{1-3(\varrho-\delta)}{2}y - \epsilon x, \end{aligned} \tag{9}$$

   or $\exists y \geq t_1$ such that at time $-y$ a tagged flow arrives with service requirement

$$\begin{aligned} B_{\text{tag}} &\geq \frac{1-\varrho+\delta}{2(\varrho-\delta)}(1-\varrho+\delta+3\epsilon)x + \frac{1-2(\varrho-\delta)}{2}(y-t_1) + \epsilon x \\ &= \frac{1}{2}(1-\varrho+\delta+3\epsilon)x + \frac{1-2(\varrho-\delta)}{2}y + \epsilon x. \end{aligned} \tag{10}$$

2. For the amount of arriving traffic it holds that

$$A_1(-y,0) \geq (\varrho-\delta)y - \epsilon x \qquad \text{and} \qquad A_1(0,W_0) \geq (\varrho-\delta)W_0 - \epsilon x. \tag{11}$$

3. The workload of class 1, except from the tagged flow, satisfies

$$V_1^-(0) \leq \epsilon x \qquad \text{and} \qquad V_1^-(W_0) \leq \epsilon x. \tag{12}$$

The next lemma gives a sample-path relation between the scenarios given above and the event $W_0 > x$.

Lemma 4.1. *If either the events* $\{(9),(11),(12)\}$ *or* $\{(10),(11),(12)\}$ *occur simultaneously, then* $W_0 > x$.

Proof. Let us first consider the case that the events (9), (11), and (12) occur simultaneously. We now distinguish between two cases: (i) The large tagged flow — as defined through (9) — is still present at time W_0; and (ii) the tagged flow already left before time W_0. Denote by $B_1^{\text{tag}}(s,t)$ and $B_1^-(s,t)$ the amount of service received by the tagged flow and by class 1 except for the tagged flow, respectively, during the interval $[s,t]$.
First consider case (i). Because the tagged flow is still present at time 0, it follows from the PS discipline that $B_1^{\text{tag}}(-y,0) = B_2(-y,0)$. Combining this with (5) and (4), yields

$$\begin{aligned} V_2(0) &= V_2(-y) + B_1^-(-y,0) + B_1^{\text{tag}}(-y,0) - B_2(-y,0) \\ &\geq A_1(-y,0) - V_1^-(0). \end{aligned} \tag{13}$$

From the PS discipline, it follows that $B_1^{\text{tag}}(0,W_0) = V_2(0)$ since the tagged flow is still present at time W_0. Combining this with (13) and using (6) and (4), we obtain

$$\begin{aligned} W_0 &= V_2(0) + B_1^{\text{tag}}(0,W_0) + B_1^-(0,W_0) \\ &\geq 2A_1(-y,0) + A_1(0,W_0) - V_1^-(0) - V_1^-(W_0). \end{aligned}$$

Rewriting gives

$$\begin{aligned} (1-\varrho+\delta)W_0 &\geq 2A_1(-y,0) + A_1(0,W_0) - (\varrho-\delta)W_0 - V_1^-(0) - V_1^-(W_0) \\ &\overset{\text{(a)}}{\geq} 2(\varrho-\delta)y - 5\epsilon x \\ &\geq 2(\varrho-\delta)t_0 - 5\epsilon x = (1-\varrho+\delta)x, \end{aligned}$$

where the equality follows from the definition of t_0. Notice that in Inequality (a) we have used (11) and (12), whereas (9) does not need to be invoked in this case.
Next, consider case (ii). Applying (3) and (4), we have

$$\begin{aligned} B_1^{\text{tag}}(-y,0) + B_2(-y,0) &\leq y - B_1^-(-y,0) \\ &\leq y - A_1(-y,0) + V_1^-(0). \end{aligned}$$

Because of the PS discipline, it holds that $B_1^{\text{tag}}(-y,0) \leq B_2(-y,0)$ and, hence,

$$B_1^{\text{tag}}(-y,0) \leq \frac{1}{2}(y - A_1(-y,0) + V_1^-(0)).$$

Observe that

$$\begin{aligned} B_1^{\text{tag}}(-y,0) &\leq (\varrho-\delta)y + \epsilon x + \frac{1-3(\varrho-\delta)}{2}y \\ &\leq (\varrho-\delta)t_1 + \epsilon x + \frac{1-3(\varrho-\delta)}{2}y \\ &\leq (1-\varrho+\delta)x + 4\epsilon x + \frac{1-3(\varrho-\delta)}{2}y \leq B_{\text{tag}}, \end{aligned}$$

implying that the tagged flow is still active at time 0. The lower bound (13) for $V_2(0)$ thus applies.

Also, because the tagged flow already left at time W_0, we have

$$\begin{aligned} B_1^{\text{tag}}(0, W_0) &= B_{\text{tag}} - B_1^{\text{tag}}(-y, 0) \\ &\geq B_{\text{tag}} - \frac{1}{2}(y - A_1(-y, 0) + V_1^-(0)). \end{aligned}$$

Thus, using (4),

$$\begin{aligned} B_1(0, W_0) &= B_1^{\text{tag}}(0, W_0) + B_1^-(0, W_0) \\ &\geq B_{\text{tag}} - \frac{1}{2}(y - A_1(-y, 0)) + \frac{1}{2}V_1^-(0) + A_1(0, W_0) - V_1^-(W_0). \end{aligned} \tag{14}$$

Hence, upon combining (6), (13), and (14), in addition to some rewriting, we obtain

$$\begin{aligned} (1 - \varrho + \delta)W_0 &\geq B_{\text{tag}} + \frac{3}{2}A_1(-y, 0) - \frac{1}{2}y + A_1(0, W_0) \\ &\qquad - (\varrho - \delta)W_0 - \frac{1}{2}V_1^-(0) - V_1^-(W_0) \\ &\overset{\text{(b)}}{\geq} B_{\text{tag}} - \frac{1}{2}(1 - 3(\varrho - \delta))y - 4\epsilon x \overset{\text{(c)}}{\geq} (1 - \varrho + \delta)x, \end{aligned}$$

where step (b) follows from (11) and (12), and step (c) from (9). This completes the analysis of the first scenario.

We now turn to the case that the events (10), (11), and (12) occur simultaneously. Again, we distinguish between two cases: (i) The large tagged flow is still present at time 0; and (ii) the tagged flow already left before time 0.
In case (i) the tagged flow is present at time 0 and the lower bound (13) for $V_2(0)$ thus applies again. Using (4), it follows that

$$B_1(0, W_0) \geq B_1^-(0, W_0) = V_1^-(0) + A_1(0, W_0) - V_1^-(W_0). \tag{15}$$

Hence, combining (6), (13), and (15) gives

$$\begin{aligned} (1 - \varrho + \delta)W_0 &\geq A_1(-y, 0) + A_1(0, W_0) - (\varrho - \delta)W_0 - V_1^-(W_0) \\ &\overset{\text{(d)}}{\geq} (\varrho - \delta)y - 3\epsilon x \\ &\geq (\varrho - \delta)t_1 - 3\epsilon x = (1 - \varrho + \delta)x, \end{aligned}$$

where (d) follows from (11) and (12).
Next, consider case (ii). From (4) we obtain

$$\begin{aligned} B_1(-y, 0) &= B_1^-(-y, 0) + B_1^{\text{tag}}(-y, 0) \\ &\geq A_1(-y, 0) - V_1^-(0) + B_{\text{tag}}. \end{aligned}$$

Applying the above together with (5) and (3), yields

$$\begin{aligned} V_2(0) &\geq B_1(-y, 0) - B_2(-y, 0) \\ &\geq 2B_1(-y, 0) - y \\ &\geq 2B_{\text{tag}} + 2A_1(-y, 0) - y - 2V_1^-(0). \end{aligned} \tag{16}$$

Now, it follows from (6), (15), and (16) that

$$\begin{aligned}(1-\varrho+\delta)W_0 &\geq 2B_{\text{tag}} + 2A_1(-y,0) - y + A_1(0,W_0) \\ &\qquad - (\varrho-\delta)W_0 - V_1^-(0) - V_1^-(W_0) \\ &\overset{\text{(e)}}{\geq} 2B_{\text{tag}} - (1-2(\varrho-\delta))y - 5\epsilon x \overset{\text{(f)}}{=} (1-\varrho+\delta)x;\end{aligned}$$

here (e) follows from (11) and (12), and (f) from (10). This completes the sample-path analysis of the second scenario and the proof of the lemma. □

In the next proposition, we convert the sample-path relation of Lemma 4.1 into a probabilistic lower bound for the tail distribution of W_0.

Proposition 4.2. *(lower bound) If $B(\cdot) \in \mathcal{R}_{-\nu}$ and $0 < \varrho < \frac{1}{2}$ ($\varrho \neq \frac{1}{3}$), then*

$$\begin{aligned}\mathbb{P}\{W > x\} &\geq \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{(1-\varrho)^2}{2\varrho}x\right\}(1+o(1)) \\ &\quad + \frac{2\varrho}{|1-3\varrho|}\mathbb{P}\left\{\phi_{\min}\frac{1-\varrho}{2\varrho}x < B^r < \phi_{\max}\frac{1-\varrho}{2\varrho}x\right\}(1+o(1)), \qquad x \to \infty.\end{aligned}$$

Proof. For notational convenience, we define

$$g_1(\delta,\epsilon) := \frac{1-\varrho+\delta+5\epsilon}{2(\varrho-\delta)}\frac{1+\varrho-\delta}{2} - \epsilon; \qquad g_2(\delta,\epsilon) := \frac{1-\varrho+\delta}{2(\varrho-\delta)}(1-\varrho+\delta+3\epsilon) + \epsilon.$$

To bound the two probabilities of (12) from below, we apply the M/G/1 PS model with 2 permanent customers and denote the workload at time t in the latter model by $V_{\text{perm}}(t)$. We thus have that $V_1^-(t) \leq V_{\text{perm}}(t)$.
Using Lemma 4.1 and the observations above, we have

$$\begin{aligned}\mathbb{P}\{W_0 > x\} &\geq \mathbb{P}\left\{\exists y \in [t_0,t_1] : B_{\text{tag}} > g_1(\delta,\epsilon)x + \frac{1-3(\varrho-\delta)}{2}(y-t_0); (11); (12)\right\} \\ &\quad + \mathbb{P}\left\{\exists y \geq t_1 : B_{\text{tag}} > g_2(\delta,\epsilon)x + \frac{1-2(\varrho-\delta)}{2}(y-t_1); (11); (12)\right\} \\ &\geq \left(\mathbb{P}\left\{\exists y \in [t_0,t_1] : B_{\text{tag}} > g_1(\delta,\epsilon)x + \frac{1-3(\varrho-\delta)}{2}(y-t_0)\right\}\right. \\ &\qquad \left. + \mathbb{P}\left\{\exists y \geq t_1 : B_{\text{tag}} > g_2(\delta,\epsilon)x + \frac{1-2(\varrho-\delta)}{2}(y-t_1)\right\}\right) \\ &\quad \times \mathbb{P}\{(11); V_{\text{perm}}(0) \leq \epsilon x; V_{\text{perm}}(W_0) \leq \epsilon x\}.\end{aligned} \tag{17}$$

We now treat the three probabilities on the right-hand side of (17) separately. Starting with the second probability, we obtain, by integrating with respect to y, using the short-

hand notation $\bar{g}(\varrho,\delta) := \frac{1}{2} - (\varrho - \delta)$,

$$\begin{aligned}
&\mathbb{P}\left\{\exists y \geq t_1 : B_{\text{tag}} > g_2(\delta,\epsilon)x + \frac{1-2(\varrho-\delta)}{2}(y-t_1)\right\} \\
&\geq \int_0^\infty \lambda \mathbb{P}\left\{B_{\text{tag}} > g_2(\delta,\epsilon)x + \bar{g}(\varrho,\delta)y\right\} \mathrm{d}y \\
&\quad - \int_0^\infty \int_y^\infty \lambda^2 \mathbb{P}\left\{B_{\text{tag}} > g_2(\delta,\epsilon)x + \bar{g}(\varrho,\delta)y; B_{-z} > g_2(\delta,\epsilon)x + \bar{g}(\varrho,\delta)z\right\} \mathrm{d}z\mathrm{d}y \\
&\sim \frac{2\varrho}{1-2(\varrho-\delta)} \mathbb{P}\left\{B^r > g_2(\delta,\epsilon)x\right\}(1+o(1)),
\end{aligned}$$

where B_{-z} denotes the service requirement of a second large customer arriving at time $-z$ (see e.g. [4] for details on the asymptotically small probability of having two or more large flow arrivals).

The first probability on the right-hand side of (17) can be treated similarly. Neglecting the asymptotically small probability of two or more large customer arrivals again, we have

$$\begin{aligned}
&\mathbb{P}\left\{\exists y \in [t_0,t_1] : B_{\text{tag}} > g_1(\delta,\epsilon)x + \frac{1-3(\varrho-\delta)}{2}(y-t_0)\right\} \\
&= \int_0^\infty \lambda \mathbb{P}\left\{B_{\text{tag}} > g_1(\delta,\epsilon)x + \frac{1-3(\varrho-\delta)}{2}y\right\} \mathrm{d}y \\
&\qquad - \int_{t_1-t_0}^\infty \lambda \mathbb{P}\left\{B_{\text{tag}} > g_1(\delta,\epsilon)x + \frac{1-3(\varrho-\delta)}{2}y\right\} \mathrm{d}y + o(\mathbb{P}\left\{B^r > x\right\}) \\
&\sim \frac{2\varrho}{1-3(\varrho-\delta)} \left(\mathbb{P}\left\{B^r > g_1(\delta,\epsilon)x\right\} - \mathbb{P}\left\{B^r > g_2(\delta,\epsilon)x\right\}\right)(1+o(1)),
\end{aligned}$$

where we used that $g_2(\delta,\epsilon)x = g_1(\delta,\epsilon)x + (t_1-t_0)(1-3(\varrho-\delta))/2$ in the final step.

For the third probability on the rhs of (17), we note that $A_1(-y,0)$, $A_1(0,W_0)$, $V_{\text{perm}}(0)$, and $V_{\text{perm}}(W_0)$ are not independent. We therefore write

$$\begin{aligned}
&\mathbb{P}\left\{(11); V_{\text{perm}}(0) \leq \epsilon x; V_{\text{perm}}(W_0) \leq \epsilon x\right\} \\
&\geq \mathbb{P}\left\{A_1(-y,0) \geq (\varrho-\delta)y - \epsilon x\right\} \times \mathbb{P}\left\{A_1(0,W_0) \geq (\varrho-\delta)W_0 - \epsilon x\right\} \\
&\qquad - \mathbb{P}\left\{V_{\text{perm}}(0) > \epsilon x\right\} - \mathbb{P}\left\{V_{\text{perm}}(W_0) > \epsilon x\right\}.
\end{aligned}$$

Due to the (weak) law of large numbers we have that $\mathbb{P}\left\{A_1(-y,0) \geq (\varrho-\delta)y - \epsilon x\right\}$ converges to 1 as $x \to \infty$; similarly $\mathbb{P}\left\{A_1(0,W_0) \geq (\varrho-\delta)W_0 - \epsilon x\right\} \to 1$ as $x \to \infty$. Since $V_{\text{perm}}(0)$ and $V_{\text{perm}}(W_0)$ have proper (that is, non-defective) distribution functions, it also holds that

$$\lim_{x\to\infty} \mathbb{P}\left\{V_{\text{perm}}(0) > \epsilon x\right\} = 0, \quad \text{and} \quad \lim_{x\to\infty} \mathbb{P}\left\{V_{\text{perm}}(W_0) > \epsilon x\right\} = 0.$$

Finally, combining the above and using the fact that $B^r(\cdot) \in \mathcal{R}_{1-\nu}$, we have, as $x \to \infty$,

$$\begin{aligned}
\mathbb{P}\{W > x\} &\gtrsim \frac{2\varrho}{1-2(\varrho-\delta)}\mathbb{P}\{B^r > g_2(\delta,\epsilon)x\} \\
&\quad + \frac{2\varrho}{1-3(\varrho-\delta)}\left(\mathbb{P}\{B^r > g_1(\delta,\epsilon)x\} - \mathbb{P}\{B^r > g_2(\delta,\epsilon)x\}\right) \\
&\to \frac{2\varrho}{1-2\varrho}\mathbb{P}\{B^r > g_2(0,0)x\} \\
&\quad + \frac{2\varrho}{1-3\varrho}\left(\mathbb{P}\{B^r > g_1(0,0)x\} - \mathbb{P}\{B^r > g_2(0,0)x\}\right), \quad \delta,\epsilon \downarrow 0;
\end{aligned}$$

here $f(x) \gtrsim g(x)$, $x \to \infty$, indicates that $\liminf_{x\to\infty} f(x)/g(x) \geq 1$. The proof is now completed by distinguishing between the cases $0 < \varrho < \frac{1}{3}$ and $\frac{1}{3} < \varrho < \frac{1}{2}$. □

4.2 Upper bound

In this subsection, we let time 0 correspond to an arbitrary flow arrival, and derive an asymptotic upper bound for $\mathbb{P}\{S_0 > x\}$. In the proofs, we use a representation of the sojourn time S_0 that is similar to the waiting time representation (6). Let F_0 correspond to the flow transfer delay of the flow that arrives at time 0 and note that the events $S_0 \geq x$ and $B_2(F_0, x) \leq V_2(F_0)$ are equivalent. Since the total service capacity is then used during (F_0, x), the latter event can be rewritten as

$$x - F_0 \leq V_2(F_0) + B_1(F_0, x).$$

Moreover, we show in Proposition 4.3 that the most likely way for the sojourn time to become large is due to the arrival of a large tagged flow while the actual flow itself is small. It may be seen from Proposition 3.2 that the flow transfer delay F_0 of the 'small' flow is 'small' as well. Thus, assuming that $S_0 > x$ and $F_0 \leq \epsilon x$, we obtain the following relation:

$$x(1-2\epsilon) \leq V_2(0) + B_1(0, x). \tag{18}$$

This inequality relation will be the starting point for most of the sample-path relations in the proofs below.

Before turning to the proof of the asymptotic upper bound for $\mathbb{P}\{S_0 > x\}$ (i.e. Proposition 4.3), we first introduce some notation that will be used throughout the section. Let $N_b(s,t)$ be the number of flows arriving in the interval $[s,t]$ with a service requirement satisfying b. In particular, we are interested in so-called 'large' flows that have a service requirement larger than κx, for some $\kappa > 0$ independent of x (in which case we say that the service requirement is "$> \kappa x$"). Also, for $t > 0$, define

$$W^c(0,t) := \sup_{0\leq s\leq t}\{A_1(0,s) - cs\}, \tag{19}$$

and, for $u \le v < 0$,

$$W^c(u,v) := \sup_{u \le s \le v} \{A_1(s,v) - c(v-s)\}. \tag{20}$$

In case the input process is modified such that only the flows with service requirements of at most κx are admitted, we add a subscript "$\le \kappa x$", i.e., we write $W^c_{\le \kappa x}(0,t)$ and $W^c_{\le \kappa x}(u,v)$, respectively.

Finally, similar to Subsection 4.1, define two time instants $0 < s_0 < s_1$ as

$$s_0 := \frac{1-\varrho-\delta-2\eta-10\epsilon}{2(\varrho+\delta)}x \quad \text{and} \quad s_1 := \frac{1-\varrho-\delta-\eta-\frac{13}{2}\epsilon}{\varrho+\delta}x;$$

observe that s_0 is close to t_0, and s_1 to t_1. Let $s^\star := \inf\{0 \le t < s_1 : V_2(-t) = 0\}$ be the last epoch in $(-s_1, 0]$ that the system was empty, and let $s^\star = s_1$ in case $V_2(-t) > 0$ for all $t \in (-s_1, 0]$. We have thus enforced that the total service capacity is used during the interval $[-s^\star, 0]$.

We are now settled for the proofs.

Proposition 4.3. *(upper bound) If $B(\cdot) \in \mathcal{R}_{-\nu}$ and $0 < \varrho < \frac{1}{2}$ ($\varrho \ne \frac{1}{3}$), then*

$$\begin{aligned} \mathbb{P}\{S > x\} &\le \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{(1-\varrho)^2}{2\varrho}x\right\}(1+o(1)) \\ &+ \frac{2\varrho}{|1-3\varrho|}\mathbb{P}\left\{\phi_{\min}\frac{1-\varrho}{2\varrho}x < B^r < \phi_{\max}\frac{1-\varrho}{2\varrho}x\right\}(1+o(1)), \qquad x \to \infty. \end{aligned} \tag{21}$$

Proof. First, we note that the premise for a large sojourn time to occur is that the arriving flow finds a large workload in the buffer, while the actual flow itself is small. Using Proposition 3.2, this implies that the flow transfer delay is small as well:

$$\begin{aligned} \mathbb{P}\{S_0 > x\} &\le \mathbb{P}\{S_0 > x; F_0 \le \epsilon x\} + \mathbb{P}\{F_0 > \epsilon x\} \\ &= \mathbb{P}\{S_0 > x; F_0 \le \epsilon x\} + o(\mathbb{P}\{B^r > x\}). \end{aligned}$$

We henceforth assume that $F_0 \le \epsilon x$.

There are in fact two 'most likely scenarios' for the event $S_0 > x$ to occur; all the other scenarios are asymptotically negligible. To identify these most likely scenarios, decompose

$$\begin{aligned} &\mathbb{P}\{S_0 > x; F_0 \le \epsilon x\} \\ &\le \mathbb{P}\{2V_1(-s_1) + V_2(-s_1) > (1-\varrho-\delta-5\epsilon)x + (1-2(\varrho+\delta))s_1\} \\ &\quad + \mathbb{P}\{2V_1(-s_1) + V_2(-s_1) \le (1-\varrho-\delta-5\epsilon)x + (1-2(\varrho+\delta))s_1; S_0 > x; F_0 \le \epsilon x\}. \end{aligned} \tag{22}$$

The first probability on the right-hand side of (22) contains the first most likely scenario. Note that the system is in steady state at time $-s_1$. Application of Theorem 2.1, in conjunction with the definition of s_1, then provides

$$\begin{aligned} &\mathbb{P}\{2V_1(-s_1) + V_2(-s_1) > (1-\varrho-\delta-5\epsilon)x + (1-2(\varrho+\delta))s_1\} \\ &\sim \frac{2\varrho}{1-2\varrho}\mathbb{P}\left\{B^r > \frac{1-\varrho-\delta-\eta-\frac{13}{2}\epsilon}{2(\varrho+\delta)}(1-\varrho-\delta)x + \left(\eta + \frac{3}{2}\epsilon\right)x\right\}. \end{aligned} \tag{23}$$

We now turn to the second probability on the right-hand side of (22). Distinguishing between 0, 1, and 2 or more large-flow arrivals during $(-s_1, x]$ and the value of $2V_1(-s_1)+V_2(-s_1)$ in case of 1 large-flow arrival during $(-s_1, x]$, we obtain

$$\begin{aligned}&\mathbb{P}\left\{2V_1(-s_1)+V_2(-s_1)\le(1-\varrho-\delta-5\epsilon)x+(1-2(\varrho+\delta))s_1; S_0>x; F_0\le\epsilon x\right\}\\ &\quad= P_1(x)+P_2(x)+P_3(x)+P_4(x),\end{aligned}$$

where

$$\begin{aligned}P_1(x) &:= \mathbb{P}\left\{\begin{array}{l}2V_1(-s_1)+V_2(-s_1)\le(1-\varrho-\delta-5\epsilon)x+(1-2(\varrho+\delta))s_1;\\ \quad N_{>\kappa x}(-s_1,x)=0; S_0>x; F_0\le\epsilon x\end{array}\right\},\\ P_2(x) &:= \mathbb{P}\left\{\begin{array}{l}2V_1(-s_1)+V_2(-s_1)\le\eta x;\\ \quad N_{>\kappa x}(-s_1,x)=1; S_0>x; F_0\le\epsilon x\end{array}\right\},\\ P_3(x) &:= \mathbb{P}\left\{\begin{array}{l}\eta x<2V_1(-s_1)+V_2(-s_1)\le(1-\varrho-\delta-5\epsilon)x+(1-2(\varrho+\delta))s_1;\\ \quad N_{>\kappa x}(-s_1,x)=1; S_0>x; F_0\le\epsilon x\end{array}\right\},\\ P_4(x) &:= \mathbb{P}\left\{\begin{array}{l}2V_1(-s_1)+V_2(-s_1)\le(1-\varrho-\delta-5\epsilon)x+(1-2(\varrho+\delta))s_1;\\ \quad N_{>\kappa x}(-s_1,x)\ge 2; S_0>x; F_0\le\epsilon x\end{array}\right\}.\end{aligned}$$

The probability $P_2(x)$ contains the second most likely scenario, in that the probabilities $P_1(x)$, $P_3(x)$, and $P_4(x)$ of the other scenarios are negligible relative to $P_2(x)$. The asymptotic behavior of $P_2(x)$ can be found in Lemma 4.4. Lemmas 4.7, 4.8, and 4.9 show that the terms $P_1(x)$, $P_3(x)$, and $P_4(x)$, respectively, are negligible compared to $P_2(x)$ (and to the tail of the other most likely scenario).

To complete the proof, we let $\delta, \epsilon, \eta \downarrow 0$ (in (23) and (24)) and use the fact that $B^r(\cdot) \in \mathcal{R}_{1-\nu}$. The equivalence with (21) may be seen by distinguishing between $\varrho < \frac{1}{3}$ and $\varrho > \frac{1}{3}$ and some straightforward rewriting. □

Lemma 4.4. *For $\delta, \epsilon, \eta, \kappa > 0$ sufficiently small ($\varrho+\delta \neq \frac{1}{3}$) and $\varrho < \frac{1}{2}$, we have, as $x \to \infty$,*

$$\begin{aligned}P_2(x) \lesssim \frac{2\varrho}{1-3(\varrho+\delta)}\Bigg(&\mathbb{P}\left\{B^r > \frac{1-\varrho-\delta-2\eta-10\epsilon}{2(\varrho+\delta)}\frac{1+\varrho+\delta}{2}x+\left(\eta+\frac{7}{2}\epsilon\right)x\right\}\\ &-\mathbb{P}\left\{B^r > \frac{1-\varrho-\delta-\eta-\frac{13}{2}\epsilon}{2(\varrho+\delta)}(1-\varrho-\delta)x\right\}\Bigg).\end{aligned} \tag{24}$$

Proof. As in Subsection 4.1, denote the service requirement of the large tagged flow by B_{tag} and let $-y$, $y \in [-x, s_1]$, be its arrival instant. We bound $P_2(x)$ by distinguishing between $y \ge s_0$ and $y < s_0$, and by the size of B_{tag} in case $y \ge s_0$:

$$P_2(x) \le P_{21}(x) + P_{22}(x) + P_{23}(x),$$

where

$$
\begin{aligned}
P_{21}(x) &:= \mathbb{P}\left\{ \begin{array}{l} 2V_1(-s_1)+V_2(-s_1) \le \eta x; \\ \quad N_{>\kappa x}(-s_1,-s_0)=0; N_{>\kappa x}(-s_0,x)=1; S_0>x; F_0 \le \epsilon x \end{array} \right\} \\
P_{22}(x) &:= \mathbb{P}\left\{ \begin{array}{l} 2V_1(-s_1)+V_2(-s_1) \le \eta x; \\ \quad B_{\text{tag}} \le (1-\varrho-\delta-\eta-\frac{13}{2}\epsilon)x+\frac{1}{2}(1-3(\varrho+\delta))y, \quad y\in[s_0,s_1]; \\ \quad N^-_{>\kappa x}(-s_1,x)=0; S_0>x; F_0\le \epsilon x \end{array} \right\} \\
P_{23}(x) &:= \mathbb{P}\left\{ \exists y\in[s_0,s_1] : B_{\text{tag}} > (1-\varrho-\delta-\eta-\tfrac{13}{2}\epsilon)x+\tfrac{1}{2}(1-3(\varrho+\delta))y \right\}.
\end{aligned}
$$

The probabilities $P_{21}(x)$ and $P_{22}(x)$ can be bounded by Lemmas 4.5 and 4.6, respectively, and are thus negligible compared to the dominant scenarios. For $P_{23}(x)$ we obtain, by integrating with respect to y,

$$
\begin{aligned}
P_{23}(x) &= \int_{s_0}^{\infty} \lambda \mathbb{P}\left\{ B_{\text{tag}} > \left(1-\varrho-\delta-\eta-\frac{13}{2}\epsilon\right)x+\frac{1}{2}(1-3(\varrho+\delta))y \right\} \mathrm{d}y \\
&\quad - \int_{s_1}^{\infty} \lambda \mathbb{P}\left\{ B_{\text{tag}} > \left(1-\varrho-\delta-\eta-\frac{13}{2}\epsilon\right)x+\frac{1}{2}(1-3(\varrho+\delta))y \right\} \mathrm{d}y \\
&= \frac{2\varrho}{1-3(\varrho+\delta)} \Bigg(\mathbb{P}\left\{ B^r > \frac{1-\varrho-\delta-2\eta-10\epsilon}{2(\varrho+\delta)}\frac{1}{2}(1+\varrho+\delta)x + \left(\eta+\frac{7}{2}\epsilon\right)x \right\} \\
&\qquad - \mathbb{P}\left\{ B^r > \frac{1-\varrho-\delta-\eta-\frac{13}{2}\epsilon}{2(\varrho+\delta)}(1-\varrho-\delta)x \right\} \Bigg),
\end{aligned}
$$

where the second equality follows from the definitions of s_0 and s_1. This completes the proof. □

Lemma 4.5. *For $\delta,\epsilon,\eta,\kappa>0$ sufficiently small and $\varrho<\frac{1}{2}$, we have*

$$
P_{21}(x) = o(\mathbb{P}\{B^r > x\}) \qquad \text{as } x\to\infty.
$$

Proof. Recall that $-s^\star$ represents the last epoch before time 0 that the system was empty. In case $s^\star \ge s_0$, we obtain from (5), (3), and (4) that

$$
\begin{aligned}
V_2(-s_0) &= V_2(-s^\star)+B_1(-s^\star,-s_0)-B_2(-s^\star,-s_0) \\
&= V_2(-s^\star)+2B_1(-s^\star,-s_0)-(s^\star-s_0) \\
&= 2V_1(-s^\star)+V_2(-s^\star)+2\left(A_1(-s^\star,-s_0)-\frac{1}{2}(s^\star-s_0)\right)-2V_1(-s_0).
\end{aligned}
$$

Hence,

$$
2V_1(-s_0)+V_2(-s_0) \le 2V_1(-s_1)+V_2(-s_1)+2W^{\frac{1}{2}}(-s_1,-s_0). \tag{25}
$$

Define $s' := \min\{s^\star, s_0\}$, such that the full service capacity is used during $[-s',0]$. Due to the PS discipline, it holds that $B_2(-s',0) \ge B_1^{\text{tag}}(-s',0)$. Combining the above with (5)

and (4) gives

$$\begin{aligned} V_2(0) &= V_2(-s') + B_1^-(-s',0) + B_1^{\text{tag}}(-s',0) - B_2(-s',0) \\ &\leq V_2(-s') + V_1^-(-s') + A_1^-(-s',0) - (\varrho+\delta)s' + (\varrho+\delta)s' - V_1^-(0) \\ &\leq 2V_1(-s_1) + V_2(-s_1) + 2W^{\frac{1}{2}}(-s_1,-s_0) \\ &\qquad + W_{-\text{tag}}^{\varrho+\delta}(-s_0,0) + (\varrho+\delta)s_0 - V_1^-(0), \end{aligned} \tag{26}$$

where $W_{-\text{tag}}^c(u,v)$ corresponds to $W^c(u,v)$ with the modification that the large tagged flow is excluded from the arrival process ($W_{-\text{tag}}^c(0,t)$ is defined similarly).
By again applying the properties of the PS discipline we find that $B_1^{\text{tag}}(F_0,x) \leq V_2(F_0)$ in case $S_0 > x$. Under the condition that $F_0 \leq \epsilon x$, we have $B_1^{\text{tag}}(0,x) \leq V_2(0) + \epsilon x$. Together with (4), this yields

$$\begin{aligned} B_1(0,x) &= B_1^-(0,x) + B_1^{\text{tag}}(0,x) \\ &\leq V_1^-(0) + A_1^-(0,x) + V_2(0) + \epsilon x. \end{aligned} \tag{27}$$

Now, by combining (26) and (27), we have

$$\begin{aligned} V_2(0) + B_1(0,x) &\leq 2V_2(0) + V_1^-(0) + A_1^-(0,x) - (\varrho+\delta)x + (\varrho+\delta)x + \epsilon x \\ &\leq 2\left(2V_1(-s_1) + V_2(-s_1) + 2W^{\frac{1}{2}}(-s_1,-s_0) + W_{-\text{tag}}^{\varrho+\delta}(-s_0,0)\right) \\ &\qquad + 2(\varrho+\delta)s_0 + W_{-\text{tag}}^{\varrho+\delta}(0,x) + (\varrho+\delta+\epsilon)x. \end{aligned}$$

Using (18) to convert this sample-path relation into a probabilistic bound gives

$$\begin{aligned} P_{21}(x) &\leq \mathbb{P}\left\{ \begin{array}{c} 2\left(2V_1(-s_1) + V_2(-s_1) + 2W^{\frac{1}{2}}(-s_1,-s_0) + W_{-\text{tag}}^{\varrho+\delta}(-s_0,0)\right) \\ + 2(\varrho+\delta)s_0 + W_{-\text{tag}}^{\varrho+\delta}(0,x) + (\varrho+\delta+\epsilon)x \geq x(1-2\epsilon); \\ 2V_1(-s_1) + V_2(-s_1) \leq \eta x; N_{>\kappa x}(-s_1,-s_0) = 0; N_{>\kappa x}(-s_0,x) = 1 \end{array} \right\} \\ &\leq \mathbb{P}\left\{ \begin{array}{c|l} 4W^{\frac{1}{2}}(-s_1,-s_0) + 2W_{-\text{tag}}^{\varrho+\delta}(-s_0,0) & N_{>\kappa x}(-s_1,-s_0) = 0; \\ + W_{-\text{tag}}^{\varrho+\delta}(0,x) \geq 7\epsilon x & N_{>\kappa x}(-s_0,x) = 1 \end{array} \right\} \\ &\leq \mathbb{P}\left\{W_{\leq\kappa x}^{\frac{1}{2}}(-s_1,-s_0) \geq \epsilon x\right\} + \mathbb{P}\left\{W_{\leq\kappa x}^{\varrho+\delta}(-s_0,0) \geq \epsilon x\right\} + \mathbb{P}\left\{W_{\leq\kappa x}^{\varrho+\delta}(0,x) \geq \epsilon x\right\}, \end{aligned}$$

where the second step follows from the definition of s_0. Each of the three terms can now be controlled by applying Lemma A.2, thus completing the proof. □

Lemma 4.6. *For $\delta, \epsilon, \eta > 0$ sufficiently small and $\varrho < \frac{1}{2}$, we have*

$$P_{22}(x) = o(\mathbb{P}\{B^r > x\}) \qquad \text{as } x \to \infty.$$

Proof. Again, let $-y$, $y \in [s_0, s_1]$, be the arrival epoch of the tagged flow. The upper bound for $V_2(0)$ is similar to the bound for $V_2(0)$ given by (26), see Lemma 4.5. Specifically, it is first readily verified that (25) can be modified into

$$2V_1(-y) + V_2(-y) \leq 2V_1(-s_1) + V_2(-s_1) + 2W_{-\text{tag}}^{\frac{1}{2}}(-s_1,-s_0). \tag{28}$$

Because of the PS discipline, we have $B_2(-y,0) \geq B_1^{\text{tag}}(-y,0)$. Combining the above with (5) and (4) yields

$$\begin{aligned} V_2(0) &= V_2(-y) + B_1^-(-y,0) + B_1^{\text{tag}}(-y,0) - B_2(-y,0) \\ &\leq V_2(-y) + V_1(-y) + A_1^-(-y,0) - (\varrho+\delta)y + (\varrho+\delta)y - V_1^-(0) \\ &\leq V_2(-y) + V_1(-y) + W_{-\text{tag}}^{\varrho+\delta}(-s_1,0) + (\varrho+\delta)y - V_1^-(0). \end{aligned} \tag{29}$$

We now distinguish between two different cases: (i) The large tagged flow already left before time 0, and (ii) the large tagged flow is still present at time 0.
First consider case (i). Using (4), it holds that

$$\begin{aligned} B_1(0,x) &\leq V_1^-(0) + A_1(0,x) \\ &\leq V_1^-(0) + W^{\varrho+\delta}(0,x) + (\varrho+\delta)x. \end{aligned} \tag{30}$$

Thus, using (28)–(30),

$$\begin{aligned} &V_2(0) + B_1(0,x) \\ &\leq 2V_1(-s_1) + V_2(-s_1) + 2W_{-\text{tag}}^{\frac{1}{2}}(-s_1,-s_0) \\ &\qquad + W_{-\text{tag}}^{\varrho+\delta}(-s_1,0) + (\varrho+\delta)s_1 + W^{\varrho+\delta}(0,x) + (\varrho+\delta)x \\ &\leq (\varrho+\delta+\eta)x + (\varrho+\delta)s_1 + 2W_{-\text{tag}}^{\frac{1}{2}}(-s_1,-s_0) + \frac{3}{2}W_{-\text{tag}}^{\varrho+\delta}(-s_1,0) + W^{\varrho+\delta}(0,x) \\ &\leq \left(1 - \frac{13}{2}\epsilon\right)x + 2W_{-\text{tag}}^{\frac{1}{2}}(-s_1,-s_0) + \frac{3}{2}W_{-\text{tag}}^{\varrho+\delta}(-s_1,0) + W^{\varrho+\delta}(0,x), \end{aligned}$$

where, in the second step, we added the term $\frac{1}{2}W_{-\text{tag}}^{\varrho+\delta}(-s_1,0)$ for consistency with case (ii) below and we used that $2V_1(-s_1) + V_2(-s_1) \leq \eta x$, while the final step follows from the definition of s_1.
Next consider case (ii). Due to the PS discipline, we have $B_1^{\text{tag}}(-y,0) = B_2(-y,0)$. Moreover, because the total service capacity is used during $[-y,0]$, we obtain, using (3) and (4), that

$$\begin{aligned} B_1^{\text{tag}}(-y,0) &= \frac{1}{2}\left(y - B_1^-(-y,0)\right) \\ &= \frac{1}{2}\left(y - V_1(-y) - A_1^-(-y,0) + V_1^-(0)\right). \end{aligned}$$

Note that $B_1^{\text{tag}}(0,x) \leq B_{\text{tag}} - B_1^{\text{tag}}(-y,0)$. Combining the above and using (4) yields

$$\begin{aligned} B_1(0,x) &= B_1^-(0,x) + B_1^{\text{tag}}(0,x) \\ &\leq V_1^-(0) + A_1(0,x) + B_{\text{tag}} - \frac{1}{2}\left(y - V_1(-y) - A_1^-(-y,0) + V_1^-(0)\right) \\ &\leq V_1^-(0) + W^{\varrho+\delta}(0,x) + (\varrho+\delta)x + B_{\text{tag}} \\ &\qquad + \frac{1}{2}V_1(-y) + \frac{1}{2}\left(W_{-\text{tag}}^{\varrho+\delta}(-s_1,0) + (\varrho+\delta-1)y\right). \end{aligned} \tag{31}$$

Thus, applying (28), (29), and (31),

$$\begin{aligned} V_2(0)+B_1(0,x) &\leq 2V_1(-s_1)+V_2(-s_1)+2W^{\frac{1}{2}}_{-\text{tag}}(-s_1,-s_0)+\frac{3}{2}W^{\varrho+\delta}_{-\text{tag}}(-s_1,0) \\ &\quad + W^{\varrho+\delta}(0,x)+(\varrho+\delta)x+B_{\text{tag}}+\frac{1}{2}\left(3(\varrho+\delta)y-1\right) \\ &\leq \left(1-\frac{13}{2}\epsilon\right)x+2W^{\frac{1}{2}}_{-\text{tag}}(-s_1,-s_0)+\frac{3}{2}W^{\varrho+\delta}_{-\text{tag}}(-s_1,0)+W^{\varrho+\delta}(0,x), \end{aligned}$$

where the final step follows from $2V_1(-s_1)+V_2(-s_1)\leq \eta x$ and the upper bound of the service requirement of the tagged flow B_{tag}.
Combining the sample-path relations for the cases (i) and (ii) with (18) provides the following bound:

$$\begin{aligned} P_{22}(x) &\leq \mathbb{P}\left\{ \begin{array}{l} \left(1-\frac{13}{2}\epsilon\right)x+2W^{\frac{1}{2}}_{-\text{tag}}(-s_1,-s_0) \\ +\frac{3}{2}W^{\varrho+\delta}_{-\text{tag}}(-s_1,0)+W^{\varrho+\delta}(0,x)\leq(1-2\epsilon)x \end{array} \middle| \begin{array}{l} N_{>\kappa x}(-s_1,-s_0)=1; \\ N_{>\kappa x}(-s_0,x)=0 \end{array} \right\} \\ &\leq \mathbb{P}\left\{W^{\frac{1}{2}}_{\leq\kappa x}(-s_1,-s_0)\geq\epsilon x\right\}+\mathbb{P}\left\{W^{\varrho+\delta}_{\leq\kappa x}(-s_1,0)\geq\epsilon x\right\}+\mathbb{P}\left\{W^{\varrho+\delta}_{\leq\kappa x}(0,x)\geq\epsilon x\right\}. \end{aligned}$$

Again, each of the three above terms can be controlled by an application of Lemma A.2, which completes the proof. □

Lemma 4.7. *For $0<\delta<\frac{1}{2}-\varrho$, $0<\epsilon<(1-\varrho-\delta)/5$, and $\varrho<\frac{1}{2}$, we have*

$$P_1(x)=o(\mathbb{P}\{B^r>x\}) \qquad \text{as } x\to\infty.$$

Proof. Using (4) and (5) in addition to the fact that the total service capacity is used during $[-s^\star,0]$, we have

$$\begin{aligned} V_1(0)+V_2(0) &= V_1(-s^\star)+V_2(-s^\star)+A_1(-s^\star,0)+B_1(-s^\star,0)-s^\star \\ &= 2V_1(-s^\star)+V_2(-s^\star)+2A_1(-s^\star,0)-s^\star-V_1(0) \\ &= 2V_1(-s^\star)+V_2(-s^\star)+2(A_1(-s^\star,0)-(\varrho+\delta)s^\star) \\ &\quad -(1-2(\varrho+\delta))s^\star-V_1(0) \\ &\leq \max\{2V_1(-s_1)+V_2(-s_1)-(1-2(\varrho+\delta))s_1,0\}+2W^{\varrho+\delta}(-s_1,0)-V_1(0), \end{aligned}$$

where the last step follows by distinguishing between $s^\star=s_1$ and $s^\star<s_1$ (in addition to $\delta\leq\frac{1}{2}-\varrho$). Using (4) once more yields

$$B_1(0,x)\leq V_1(0)+A_1(0,x).$$

Combining the above and using the definitions (19) and (20), we obtain

$$\begin{aligned} V_2(0)+B_1(0,x) &\leq \max\{2V_1(-s_1)+V_2(-s_1)-(1-2(\varrho+\delta))s_1,0\} \\ &\quad +2W^{\varrho+\delta}(-s_1,0)+W^{\varrho+\delta}(0,x)+(\varrho+\delta)x. \end{aligned}$$

Using (18), this sample-path relation can now be converted into a probabilistic bound:

$$\begin{aligned} P_1(x) &\leq \mathbb{P}\left\{\begin{array}{c} \max\{2V_1(-s_1)+V_2(-s_1)-(1-2(\varrho+\delta))s_1,0\}+2W^{\varrho+\delta}(-s_1,0) \\ +W^{\varrho+\delta}(0,x)+(\varrho+\delta)x \geq x(1-2\epsilon); \\ 2V_1(-s_1)+V_2(-s_1) \leq (1-\varrho-\delta-5\epsilon)x+(1-2(\varrho+\delta))s_1; \\ N_{>\kappa x}(-s_1,x)=0 \end{array}\right\} \\ &\leq \mathbb{P}\left\{2W^{\varrho+\delta}(-s_1,0)+W^{\varrho+\delta}(0,x) \geq 3\epsilon x \mid N_{>\kappa x}(-s_1,x)=0\right\} \\ &\leq \mathbb{P}\left\{W^{\varrho+\delta}_{\leq \kappa x}(-s_1,0) \geq \epsilon x\right\}+\mathbb{P}\left\{W^{\varrho+\delta}_{\leq \kappa x}(0,x) \geq \epsilon x\right\}, \end{aligned}$$

where we used that $\epsilon \leq (1-\varrho-\delta)/5$ in the second step. Both terms can be controlled by Lemma A.2. This completes the proof. □

Lemma 4.8. *For all $\eta, \kappa > 0$ sufficiently small, $\kappa > 0$ and $\varrho < \frac{1}{2}$, we have*

$$P_3(x) = o(\mathbb{P}\{B^r > x\}) \qquad \text{as } x \to \infty.$$

Proof. This probability corresponds to the combination of two unlikely events. Specifically, since $2V(-s_1)+V_2(-s_1)$ and $N_{>\kappa x}(-s_1,x)$ are independent, we have

$$P_3(x) \leq \mathbb{P}\{2V(-s_1)+V_2(-s_1) > \eta x\}\,\mathbb{P}\{N_{>\kappa x}(-s_1,x)=1\}.$$

It follows from Theorem 2.1 and Lemma A.1 that $P_3(x)$ is bounded by $o(\mathbb{P}\{B^r > x\})$ as $x \to \infty$. □

Lemma 4.9. *For any $\kappa > 0$, we have*

$$P_4(x) = o(\mathbb{P}\{B^r > x\}) \qquad \text{as } x \to \infty.$$

Proof. This follows directly from Lemma A.1. □

5 Conclusion and discussion

The main conclusion of our paper is that if B is regularly varying of index ν, then so is the flow transfer delay; the steady-state workload, sojourn time, and queueing delay, however, are regularly varying of index $1-\nu$. The results for the flow transfer delay and workload followed in a rather straightforward fashion from earlier results; the derivation of the asymptotics of the sojourn time and queueing delay turned out to be substantially more involved. The proof relies on the following principles: (1) First a most likely scenario is identified; (2) a lower bound follows from computing the asymptotics corresponding to the most likely scenario; (3) then it is shown that all other scenarios provide negligible contributions compared to the most likely scenario.

It is interesting to compare the sojourn-time asymptotics of this system with those of corresponding FIFO and PS systems. Under FIFO a sojourn time is extremely long essentially because the job under consideration finds an extremely long queue, and this long

queue is the result of one of the previous jobs being long. This explains why the tail of the sojourn time resembles the tail of the workload, which [8] is known to behave as the tail of B^r, i.e., regularly varying of index $1-\nu$. Under PS, on the contrary, the sojourn time is long because the job itself is large, and therefore the tail behaves as the tail of B, i.e., regularly varying of index $-\nu$. The heuristics behind the sojourn-time asymptotics of our model reveal that the sojourn time is large mainly due to finding a long queue, thus explaining why the corresponding tail probability vanishes as a regularly varying function of index $1-\nu$. However, in addition to a long queue, the long flow also affects the sojourn time in this system by reducing the service capacity of the buffer. These effects lead to interesting most likely scenarios for a large sojourn time to occur.
In the model we considered in this paper the queue is allocated the same share of the service capacity as each of the transmitting flows. It could be expected that such a policy may lead to relatively large buffer content of the queue. Alternatively, one may decide to assign a higher weight to the queue than to the flows; one could for instance serve the queue at rate $2/(n+2)$ when there are n flows present. Under such a policy multiple extremely large jobs are needed to cause a long sojourn time. This will be reflected in the corresponding asymptotics, cf. [11].

A Technical lemmas

Here we present two technical lemma's that can be frequently encountered in studies of queues with regularly varying service times, and that are used in Subsection 4.2. For proofs, we refer to e.g. [1, 11].

Lemma A.1. *For any $k \in \mathbb{N}$, $\kappa > 0$, and $\gamma > 0$,*

$$\mathbb{P}\left\{N_{>\kappa x}(-\gamma x, 0) \geq k\right\} = O(\mathbb{P}\left\{B^r > x\right\}^k), \qquad \text{as} \quad x \to \infty.$$

Lemma A.2. *There exists a $\kappa^* > 0$ such that for all $\kappa \in (0, \kappa^*]$, as $x \to \infty$,*

$$\mathbb{P}\left\{\sup_{0 \leq s \leq \gamma x} \{A_2(-s, 0) - (\varrho + \delta)s\} > \epsilon x \mid N_{>\kappa x}(-\gamma x, 0) = 0\right\} = o(\mathbb{P}\left\{B^r > x\right\}).$$

The same holds for the time-reversed case, i.e., there exists a $\kappa^* > 0$ such that for all $\kappa \in (0, \kappa^*]$, as $x \to \infty$,

$$\mathbb{P}\left\{\sup_{0 \leq s \leq \gamma x} \{A_2(0, s) - (\varrho + \delta)s\} > \epsilon x \mid N_{>\kappa x}(0, \gamma x) = 0\right\} = o(\mathbb{P}\left\{B^r > x\right\}),$$

References

[1] Bekker, R., Borst, S.C., Núñez-Queija, R. (2005). Performance of TCP-friendly streaming sessions in the presence of heavy-tailed elastic flows. *Perf. Eval.* **61**, 143–162.

[2] Van den Berg, J.L., Mandjes, M.R.H., Roijers F. (2006). Performance modeling of a bottleneck node in an IEEE 802.11 ad-hoc network. In: T. Kunz and S.S. Ravi (eds.): *Ad Hoc Now 2006. 5th International Conference on Ad hoc Networks & Wireless "Ad Hoc Now"*, Ottawa, Canada. Lecture Notes in Computer Science (LNCS) Series, 4104, 321–336.

[3] Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987). *Regular Variation.* Cambridge University Press, Cambridge, UK.

[4] Boxma, O.J., Foss, S.G., Lasgouttes, J.-M., Núñez-Queija, R. (2003). Waiting time asymptotics in the single server queue with service in random order. *Queueing Systems* **46**, 35–73.

[5] Cohen, J.W. (1973). Some results on regular variation for distributions in queueing and fluctuation theory. *J. Appl. Probab.* **10**, 343–353.

[6] Guillemin, F., Robert, Ph., Zwart, A.P. (2003). Tail asymptotics for processor sharing queues. *Adv. Appl. Probab.* **36**, 525–543.

[7] Mandjes, M.R.H., Roijers, F. (2007). A fluid system with coupled input and output, and its application to bottlenecks in ad hoc networks. To appear in *Queueing Systems*.

[8] Pakes, A.G. (1975). On the tails of waiting-time distributions. *J. Appl. Probab.* **12**, 555–564.

[9] Reddy, T.B., Karthigeyan, I., Manoj, B.S., Murthy, C.S.R. (2006). Quality-of-Service provisioning in ad hoc wireless networks: a survey of issues and solutions. *Ad Hoc Networks* **4**, 83–124.

[10] Reich, E. (1958). On the integrodifferential equation of Takács. I. *Ann. Math. Statist.* **29**, 563–570.

[11] Zwart, A.P., Borst, S.C., Mandjes, M.R.H. (2004). Exact asymptotics for fluid queues fed by multiple heavy-tailed on-off flows. *Ann. Appl. Probab.* **14**, 903–957

[12] Zwart, A.P., Boxma, O.J. (2000). Sojourn time asymptotics in the M/G/1 processor sharing queue. *Queueing Systems* **35**, 141-166.