Centrum voor Wiskunde en Informatica

REPORTRAPPORT

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

Non-separable 2D wavelets with two-row filters

Yinwei Zhan, Henk J.A.M. Heijmans

REPORT PNA-E0301 OCTOBER 29, 2003

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3711

Non-separable 2D wavelets with two-row filters

ABSTRACT

In the literature 2D (or bivariate) wavelets are usually constructed as a tensor product of 1D wavelets. Such wavelets are called separable. However, there are various applications, e.g. in image processing, for which non-separable 2D wavelets are preferable. In this paper, we investigate the class of compactly supported orthonormal 2D wavelets that was introduced by Belogay and Wang [2]. A characteristic feature of this class of wavelets is that the support of the corresponding filter comprises only two rows. We are concerned with the biorthogonal extension of this kind of wavelets. It turns out that the 2D wavelets in this class are intimately related to some underlying 1D wavelet. We explore this relation in detail, and we explain how the 2D wavelet transforms can be realized by means of a lifting scheme, thus allowing an efficient implementation. We also describe an easy way to construct wavelets with more rows and shorter columns.

2000 Mathematics Subject Classification: 42C40, 94A12

Keywords and Phrases: Perfect reconstruction filter bank, non-separable 2D wavelets, lifting scheme, accuracy, biorthogonality, subband scheme, Laurent polynomial.

Note: Work carried out under project PNA4.2 “Image Representation and Analysis”

Non-separable 2D wavelets with two-row filters

Yinwei Zhan

Henk J.A.M. Heijmans

CWI

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

In the literature 2D (or bivariate) wavelets are usually constructed as a tensor product of 1D wavelets. Such wavelets are called separable. However, there are various applications, e.g. in image processing, for which non-separable 2D wavelets are preferable. In this paper, we investigate the class of compactly supported orthonormal 2D wavelets that was introduced by Belogay and Wang [2]. A characteristic feature of this class of wavelets is that the support of the corresponding filter comprises only two rows. We are concerned with the biorthogonal extension of this kind of wavelets. It turns out that the 2D wavelets in this class are intimately related to some underlying 1D wavelet. We explore this relation in detail, and we explain how the 2D wavelet transforms can be realized by means of a lifting scheme, thus allowing an efficient implementation. We also describe an easy way to construct wavelets with more rows and shorter columns.

2000 Mathematics Subject Classification: 42C40, 94A12

Keywords and Phrases: Perfect reconstruction filter bank, non-separable 2D wavelets, lifting scheme, accuracy, biorthogonality, subband scheme, Laurent polynomial.

Note: Work carried out under project PNA4.2 "Image Representation and Analysis"

1. Introduction

Since their 'invention' in the early 80's, wavelets have been thoroughly investigated by researchers from various fields, in particular applied mathematics and signal and image processing. Depending on the context, wavelets are defined in various ways, e.g., as linear transformations, as perfect reconstruction filter banks, or as bases representations for an underlying Hilbert space. These definitions are all intimately connected, and we refer to [7, 10, 11] for more details. In this paper we regard wavelets as translated and dilated versions of a single mother wavelet function. Thanks to the work of Mallat and Meyer [11], wavelets can be derived from a so-called *multiresolution analysis*, henceforth abbreviated as *MRA*. Within the MRA framework, many useful analytical and approximation properties have been developed.

For practical reasons wavelets with *compact supports* have drawn a great deal of attention. Important families of compactly supported wavelets include Daubechies's orthonormal wavelets [6, 7] and the class *biorthogonal wavelets* introduced by Cohen, Daubechies, and Feauveau [5, 6]. The *lifting scheme* introduced by Sweldens [8, 14] gives an enormous flexibility and freedom in the design of new wavelets from existing ones, and moreover, it allows fast and efficient implementations.

In this paper, we are concerned with 2D (or bivariate) wavelets. Wavelets in two and higher dimensions are often constructed as tensor products of 1D wavelets, resulting in so-called *separable wavelets*. But the tensor product approach has several drawbacks. In fact, this approach is only suited for basic square grids, and cannot cope with arbitrary sampling lattices, such as the quincunx lattice. The emphasis on separable wavelets in the past is understandable if one takes into account that construction of non-separable 2D wavelets is far from trivial [9]. The spectrum factorization method that has been used with great success in the 1D case, is hard to extend to two dimensions.

A typical construction of 2D wavelets follows McClellen's transformation; see [12] and [15]. In general such a transform works on the quincunx grid and results in interpolatory wavelets. In [2], Belogay and Wang construct a family of 2D non-separable orthonormal wavelets related to the dilation matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$, which are not interpolatory.

The support of the associated wavelet filters comprises two rows and the spectrum factorization method can be used thanks to the special structure of these filters. Furthermore, the members of this wavelet family can have any prescribed accuracy. Unfortunately, the wavelets derived by Belogay and Wang [2] lack symmetry, a property which is very useful in various applications, in this paper, we will extend the class of orthonormal wavelets by Belogay and Wang with their biorthogonal counterparts. Furthermore, we investigate the relation between 2D wavelets with the aforementioned dilation matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ and 1D wavelets with scaling factor 2. We also show how to design lifting schemes for our family of 2D wavelets. In [3], Borup and Nielsen utilize the 2-row orthonormal wavelets in wavelet packets. The lifting scheme developed in this paper can provide fast computation in the best basis selection in wavelet packets.

Finally, we present an easy way to construct wavelets with more rows but shorter columns. More precisely, for a given accuracy r, the support of the two-row filters constructed in [2] lie within the range $[0, 4r-1] \times [0, 1]$ whereas the support of the filters with more rows lie within the range $[0, 2r-1] \times [0, r]$.

2. Multiresolution analysis

2.1 Notation

Denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ the sets of real numbers, complex numbers and integers, respectively. For a positive integer d, we use d-tuple column vectors to represent points in $\mathbb{R}^d$ and $\mathbb{C}^d$ or indices in $\mathbb{Z}^d$, e.g., $\mathbf{x} = (x_1, \cdots, x_2)^T$, $\mathbf{n} = (n_1, \cdots, n_2)^T$, and $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_2)^T$, where the superscript T indicates the transpose. The conjugate of $\boldsymbol{\omega}$ is $\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \cdots, \bar{\omega}_2)^T$.

We use $|\cdot|$ to denote the absolute value of a real number, the modulus of a complex number, as well as the (Euclidean) length of a vector in $\mathbb{R}^d$ or $\mathbb{C}^d$. However, if $\mathbf{n}$ is an index, i.e., $\mathbf{n} = (n_1, \cdots, n_2)^T \in (\mathbb{Z}_+)^d$, then we define $|\mathbf{n}| = \sum_{k=1}^d n_k$, called the cardinality of $\mathbf{n}$.

We denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$, i.e., $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$. Given two points $\mathbf{a} = (a_1, \cdots, a_d)^T, \mathbf{b} = (b_1, \cdots, b_d)^T \in \mathbb{C}^d$, define the power $\mathbf{a}^{\mathbf{b}}$ as $\prod_{k=1}^d a_k^{b_k}$.

A $d \times d$ square matrix D with integer entries is said to be a *dilation matrix* if the modules of its eigenvalues are larger than 1. It is easy to show that $N = |\det D|$ is an integer large than 1.

Denote by $L_2(\mathbb{R}^d)$ the space of square integrable functions on $\mathbb{R}^d$ and by $\ell_2(\mathbb{Z}^d)$ the space of square summable sequences indexed by $\mathbb{Z}^d$. Then $\|\cdot\|$ denotes the L_2 norm of a function in $L_2(\mathbb{R}^d)$ or the ℓ_2 norm of a sequence in $\ell_2(\mathbb{Z}^d)$. We use $\langle\cdot,\cdot\rangle$ to denote the inner product of two vectors (or functions) in an inner product space. For a function f in $L_2(\mathbb{R}^d)$, we define its Fourier transform as

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \exp(-i\langle\boldsymbol{\omega}, \mathbf{x}\rangle) d\mathbf{x}, \quad \boldsymbol{\omega} \in \mathbb{R}^d.$$

2.2 A general definition of MRA

Throughout this subsection we assume that D is an arbitrary dilation matrix, and we put $N = |\det D|$.

2.1 Definition. A multiresolution analysis (MRA) with dilation matrix D is a series of closed subspaces V_j in $\mathbb{R}^d$, $j \in \mathbb{Z}$, such that

1. $V_j \subset V_{j+1}$ (monotonicity);
2. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}^d)$ (approximation);
3. $f(\mathbf{x}) \in V_j \Leftrightarrow f(D\mathbf{x}) \in V_{j+1}$ (dilation invariance);
4. $f(\mathbf{x}) \in V_0 \Rightarrow f(\mathbf{x}-\mathbf{n}) \in V_0$, $\mathbf{n} \in \mathbb{Z}^d$ (translation invariance);
5. there exists a scaling function $\phi \in V_0$ such that $\{\phi(\cdot - \mathbf{n}) \mid \mathbf{n} \in \mathbb{Z}^d\}$ is a Riesz basis of V_0, i.e., the span of $\{\phi(\cdot - \mathbf{n}) \mid \mathbf{n} \in \mathbb{Z}^d\}$ is dense in V_0 and there exist constants $0 < c \leq C$ such that
$$c\|a\| \leq \left\| \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} \phi(\cdot - \mathbf{n}) \right\| \leq C\|a\| \tag{2.1}$$
for any sequence $a = \{a_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\} \in \ell_2(\mathbb{Z}^d)$. We say that ϕ *generates* the MRA $\{V_j\}$.

Strictly speaking, property 4 is redundant because of property 5. This last property, in combination with property 3, implies that the scaling function ϕ satisfies the so-called *D-scale dilation equation*

$$\phi(\mathbf{x}) = |\det D|^{1/2} \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} \phi(D\mathbf{x} - \mathbf{n}) . \tag{2.2}$$

In the sequel we use the notation

$$f_{j,\mathbf{n}}(\mathbf{x}) = |\det D|^{j/2} f(D^j \mathbf{x} - \mathbf{n}).$$

for $f \in L_2(\mathbb{R}^d)$. The sequence $\{h_{\mathbf{n}}\} \in \ell_2$ is called the *mask* of ϕ.

In the Fourier domain, the dilation relation becomes

$$\hat{\phi}(D\boldsymbol{\omega}) = |\det D|^{-1/2} H(\boldsymbol{\omega}) \hat{\phi}(\boldsymbol{\omega}), \tag{2.3}$$

where H is called the *symbol* of ϕ :

$$H(\boldsymbol{\omega}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} \exp(-i\langle\mathbf{n}, \boldsymbol{\omega}\rangle). \tag{2.4}$$

Suppose that $\phi \in L_2(\mathbb{R}^d)$ is a scaling function with dilation matrix D which generates an MRA $\{V_j\}$. One can show that there exists a *dual scaling function* $\tilde{\phi} \in L_2(\mathbb{R}^d)$, which satisfies the biorthogonality relations

$$\langle\phi(\cdot - \mathbf{n}), \tilde{\phi}(\cdot - \mathbf{k})\rangle = \delta_{\mathbf{n},\mathbf{k}}, \quad \mathbf{n}, \mathbf{k} \in \mathbb{Z}^d, \tag{2.5}$$

and which generates a dual MRA $\{\tilde{V}_j\}$.

Define the space W_j (resp. $\tilde{W}_j$) as the algebraic complement of V_j (resp. $\tilde{V}_j$) in V_{j+1} (resp. $\tilde{V}_{j+1}$), i.e.,

$$V_j \dot{+} W_j = V_{j+1} \quad \text{and} \quad \tilde{V}_j \dot{+} \tilde{W}_j = \tilde{V}_{j+1}.$$

It is easy to see that both W_j and $\tilde{W}_j$ satisfy the dilation invariance property, i.e.,

$$f(\mathbf{x}) \in W_j \Leftrightarrow f(D\mathbf{x}) \in W_{j+1}$$

(same for $\tilde{W}_j$). Furthermore, the two systems are biorthogonal:

$$W_j \perp \tilde{V}_j \quad \text{and} \quad \tilde{W}_j \perp V_j.$$

Obviously, $\{\phi_{j,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$ is a Riesz basis of V_j, for $j \in \mathbb{Z}$. Recall that $N = |\det D|$. To characterize W_j and $\tilde{W}_j$, we introduce the wavelet functions ψ^s, $s = 1, \cdots, N-1$, and the dual wavelet functions $\tilde{\psi}^s$, $s = 1, \cdots, N-1$. Unfortunately, there is not an unified approach to construct wavelets from the scaling function. In the 1D case, spectral factorization method can be used to construct wavelets with 2-scale relations but in the multivariate case, this method fails in general.

Define $W_j^s = \overline{\mathrm{span}}\{\psi_{j,\mathbf{n}}^s; \mathbf{n} \in \mathbb{Z}^d\}$ and $\tilde{W}_j^s = \overline{\mathrm{span}}\{\tilde{\psi}_{j,\mathbf{n}}^s \mid \mathbf{n} \in \mathbb{Z}^d\}$, $s = 1, \cdots, N-1$, $j \in \mathbb{Z}$, where $\overline{\mathrm{span}} U$ denotes the closed span of the vectors in U. Now we have the following properties:

- dilation invariance: $f(\mathbf{x}) \in W_j^s \Leftrightarrow f(D\mathbf{x}) \in W_{j+1}^s$ and $f(\mathbf{x}) \in \tilde{W}_j^s \Leftrightarrow f(D\mathbf{x}) \in \tilde{W}_{j+1}^s$;
- Riesz basis: $\{\psi_{j,\mathbf{n}}^s; \mathbf{n} \in \mathbb{Z}^d\}$ is a Riesz basis of W_j^s and $\{\tilde{\psi}_{j,\mathbf{n}}^s; \mathbf{n} \in \mathbb{Z}^d\}$ is a Riesz basis of $\tilde{W}_j^s$;
- decomposition: $W_j = \dot{+}_{s=1}^{N-1} W_j^s$ and $\tilde{W}_j = \dot{+}_{s=1}^{N-1} \tilde{W}_j^s$.

Now take $\psi^0 = \phi$ and $\tilde{\psi}^0 = \tilde{\phi}$. Then we have perfect reconstruction formula

$$f = \sum_{s=0}^{N-1} \sum_{j \in \mathbb{Z}, \mathbf{n} \in \mathbb{Z}^d} \langle f, \psi_{j,\mathbf{n}}^s \rangle \tilde{\psi}_{j,\mathbf{n}}^s = \sum_{s=0}^{N-1} \sum_{j \in \mathbb{Z}, \mathbf{n} \in \mathbb{Z}^d} \langle f, \tilde{\psi}_{j,\mathbf{n}}^s \rangle \psi_{j,\mathbf{n}}^s \,.$$

If, in property 5 of Definition 1, 'Riesz basis' is replaced with 'orthonormal basis', then the biorthogonality property above turns into an orthogonality property: the direct sums are orthogonal sums and the dual MRA coincides with the primary MRA.

Finally, we point out that $\{V_{2j}; j \in \mathbb{Z}\}$ is an MRA with dilation matrix D^2 generated by the same scaling function ϕ but with symbol $H^2(\boldsymbol{\omega})$.

2.3 The univariate case

In this subsection we consider the one-dimensional case, i.e., $d = 1$. The dilation matrix, or rather factor, D is taken to be 2. This is the simplest but most useful case and has been fully investigated by various authors; cf. [6, 7]. In this subsection all the functions involved are univariate.

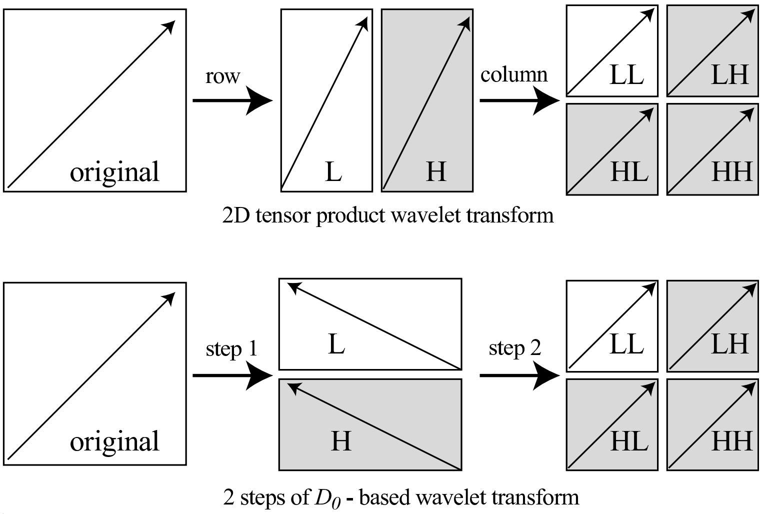

Figure 1: *2-D tensor product wavelet transform vs* D_0*-generated wavelet transform. Here* L *represents the low-pass band and* H *the high-pass band.*

Figure 2: D_0*-generated wavelet transform applied to 'Lenna' image.*

An important research question in wavelet theory is to look for wavelets with prescribed properties like orthogonality, finite support, vanishing moments, accuracies and symmetries. The MRA framework can be very useful to address such questions.

Suppose ϕ and $\tilde{\phi}$ are the scaling functions associated with an MRA in and its dual MRA in $L_2(\mathbb{R})$, and assume that H and $\tilde{H}$, which are both 2π-periodic functions, are the symbols of ϕ and $\tilde{\phi}$ respectively. Then the biorthogonality of ϕ and $\tilde{\phi}$, as given in (2.5), can be written as

$$H(\omega)\overline{\tilde{H}(\omega)} + H(\omega+\pi)\overline{\tilde{H}(\omega+\pi)} = 2\,. \tag{2.6}$$

If ϕ and $\tilde{\phi}$ have compact supports, H and $\tilde{H}$ are trigonometric polynomials and a solution of (2.6) is given in [7]. We omit the details here and refer the reader to [7].

2.4 Bivariate extension by tensor products

In image processing, or more generally two-dimensional signal processing, we are interested in 2D wavelets, also called bivariate wavelets. Such wavelets are often based on the dilation matrix $D = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

Given two[1] univariate MRA's V_j^r, V_j^c with scaling function ϕ^r, ϕ^c and wavelet function ψ^r, ψ^c. By taking tensor products, we can construct an MRA of $L_2(\mathbb{R}^2)$ as $V_j = V_j{}^r \times V_j{}^c$, $j \in \mathbb{Z}$ with scaling function $\phi(\mathbf{x}) = \phi^r(x_1) \cdot \phi^c(x_2)$. The wavelets are $\psi^1(\mathbf{x}) = \phi^r(x_1) \cdot \psi^c(x_2)$, $\psi^2(\mathbf{x}) = \psi^r(x_1) \cdot \phi^c(x_2)$, and $\psi^3(\mathbf{x}) = \psi^r(x_1) \cdot \psi^c(x_2)$. Define $W_j^1 = V_j^r \times W_j^c$, $W_j^2 = W_j^r \times V_j^c$ and $W_j^3 = W_j^r \times W_j^c$ as well as $W_j = W_j^1 \dot{+} W_j^2 \dot{+} W_j^3$. Then $V_{j+1} = V_j \dot{+} W_j$ and ψ^s generates W_j^s, for $s = 1, 2, 3$.

Computation of the corresponding wavelet transform amounts to two successive steps, i.e., first filtering the image row by row and then column by column; see Figure 1.

Both steps are related to matrices $D_r = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $D_c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ respectively. Note that these matrices are not dilation matrices in the sense of the definition in §2.1, as each of them dilates only in one direction.

If we interchange the two columns of D_r or the two rows of D_c we get a new matrix

$$D_0 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that D_0 has eigenvalues $\pm\sqrt{2}$ and therefore D_0 can serve as a dilation matrix.

Since $D = (D_0)^2$, we can apply D_0 twice to obtain tensor-product-like wavelets. However, there is a slight difference: the tensor product wavelet transform results in 4 subbands whereas the D_0-generated wavelet, when applied twice on the low-pass subband, results in 3 subbands. In order to get the same subband decomposition as for the tensor product, we have to apply the D_0-generated wavelet on both the low-pass and the high-pass subband. See Figures 1 and 2 for an illustration.

3. Non-separable cases

3.1 Analysis of dilation matrices

Let $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$ and let $D = (m_{ij})_{2\times 2}$ be a dilation matrix with $|\det D| = 2$. Suppose $\operatorname{adj} D$ is the adjugate of D, i.e.,

$$D^{-1} = \operatorname{adj} D / (\det D).$$

3.1 Lemma. *There exists (at least) one odd entry in D and in $\operatorname{adj} D$.*

Proof. If this were not true, the determinant of D would have an integer factor 4, which contradicts the condition $|\det D| = 2$. *QED*

3.2 Theorem. *Suppose $\operatorname{adj} D$ has an odd entry indexed (l, k). Take $\mathbf{u} = \mathbf{e}_k$ and $\mathbf{v}^T = \mathbf{e}_l^T \operatorname{adj} D$, the l-th row of $\operatorname{adj} D$. Then*

1. *The set $\mathbb{Z}^2$ can be divided into two disjoint sets $D\mathbb{Z}^2$ and $D\mathbb{Z}^2 + \mathbf{u}$:*

$$\mathbb{Z}^2 = D\mathbb{Z}^2 \cup (D\mathbb{Z}^2 + \mathbf{u});$$

[1] ‘r’ stands for ‘row’ and ‘c’ for ‘column’.

2. $\mathbf{v}^T\mathbf{u}$ *is odd and* $\mathbf{v}^T D\mathbf{n}$ *is even for any* $\mathbf{n} \in \mathbb{Z}^2$.

Proof. Notice that any suitable candidate for $\mathbf{u}$ in the partition $\mathbb{Z}^2 = D\mathbb{Z}^2 \cup (D\mathbb{Z}^2 + \mathbf{u})$ is such that $D^{-1}\mathbf{u} \neq \mathbb{Z}^2$. Since $D^{-1} = \pm \mathrm{adj}\, D/2$, this means that $\mathbf{u}$ can be chosen so that $\mathrm{adj}\, D\mathbf{u}$ has an odd entry. Therefore we can take $\mathbf{u} = \mathbf{e}_k$, which proves the first statement.

For the second statement, it is obvious that $\mathbf{v}^T\mathbf{u} = \mathbf{e}_l^T \,\mathrm{adj}\, D\mathbf{e}_k$ is odd. Showing that $\mathbf{v}^T D\mathbf{n}$ is even for every $\mathbf{n} \in \mathbb{Z}^2$ is equivalent to showing that both $\mathbf{v}^T D\mathbf{e}_1$ and $\mathbf{v}^T D\mathbf{e}_2$ are even. This induces that $\mathbf{v}^T D = 2\mathbf{n}^T$ for some $\mathbf{n} \in \mathbb{Z}^2$, that is $\mathbf{v}^T = \mathbf{n}^T \mathrm{adj}\, D$. So it suffices to take $\mathbf{n} = \mathbf{e}_l$.

Example Take $D = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$. Obviously, $\det D = 2$ and $\mathrm{adj}\, D = \left(\begin{smallmatrix} 0 & -2 \\ -1 & 0 \end{smallmatrix}\right)$ has an odd entry -1 indexed $(2,1)^T$. By Theorem 3.2, $\mathbf{u} = \mathbf{e}_1 = (1,0)^T$ and $\mathbf{v} = (-1,0)^T$.

3.2 Subband schemes with perfect reconstruction

In practice, we are interested in compactly supported wavelets. From the dilation equation in an MRA, we see that if a scale function or a wavelet function is compactly supported, its mask is necessarily of finite length and hence its symbol is a trigonometric polynomial. The following discussion is focused only on such cases.

Suppose $T(\omega)$ is a trigonometric polynomial:

$$T(\boldsymbol{\omega}) = \sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} \exp(-i\langle \boldsymbol{\omega}, \mathbf{n}\rangle), \quad \boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2,$$

with a finite index set $\Lambda \subset \mathbb{Z}^2$. Putting $z = \exp(-in_1\omega_1)$ and $w = \exp(-in_2\omega_2)$ we can rewrite T as a function of $\mathbf{z} = (z, w)$:

$$T(\mathbf{z}) = \sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}, \quad \mathbf{z} \in \mathbb{T}^2.$$

In other words, T is a Laurent polynomial in $\mathbf{z} = (z, w) \in \mathbb{C}^2$. The only difference between Laurent polynomials and other polynomials is that the former permits terms with negative powers.

Because of the tight relationship between trigonometric polynomials and Laurent polynomials, we will use the same (capital) letter, i.e., $C(\boldsymbol{\omega}) = \sum c_{\mathbf{n}} \exp(-i\langle \boldsymbol{\omega}, \mathbf{n}\rangle)$, $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$, respectively $C(\mathbf{z}) = \sum c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}, \mathbf{z} \in \mathbb{T}^2$.

3.3 Definition. Define $\underline{C}$ by $\underline{C}(\omega) = C(\omega + \pi\mathbf{v})$ for a trigonometric polynomial, or equivalently $\underline{C}(\mathbf{z}) = C((-1)^{v_1}z, (-1)^{v_2}w)$ for a Laurent polynomial. Here vector $\mathbf{v}$ is taken as in Theorem 1 related to the dilation matrix D.

Suppose two trigonometric polynomials H and G are the symbols of a pair of biorthogonal scaling and wavelet functions and assume that $\tilde{H}$ and $\tilde{G}$ are their duals. The relations between $H, G, \tilde{H}$ and $\tilde{G}$ can be expressed by means of the subband scheme in Figure 3. This scheme, which allows perfect reconstruction, is a two-band system since $|\det D| = 2$.

Let $x \in \ell_2(\mathbb{Z}^d)$ be a signal and let $X(\mathbf{z})$ be its z-form. The output y of downsampling x using dilation matrix D is given by $y_{\mathbf{n}} = x_{D\mathbf{n}}$, and we write $y = D_{\downarrow}(x)$. Similarly, the output y of upsampling x with dilation matrix D is $y_{\mathbf{n}} = x_{\mathbf{k}}$, if $\mathbf{n} = D\mathbf{k}$, and $y_{\mathbf{n}} = 0$ otherwise; we write $y = D_{\uparrow}(x)$. The composition $D^{\uparrow}D_{\downarrow}$ has a compact formulation in the z-form:

$$D^{\uparrow}D_{\downarrow}(X) = (X + \underline{X})/2.$$

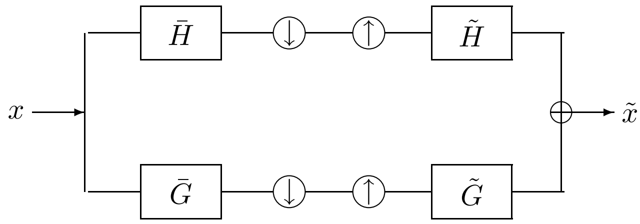

Figure 3: *Subband transformation: ↓ means down-sampling and ↑ means up-sampling.*

Now suppose that X is an input signal of the subband system in Figure 3 and that $\tilde{X}$ the output. According to the above discussion we have

$$\tilde{X} = \frac{1}{2}(\bar{H}X + \underline{\bar{H}X})\tilde{H} + \frac{1}{2}(\bar{G}X + \underline{\bar{G}X})\tilde{G}\,.$$

The perfect reconstruction condition $\tilde{X} = X$ is equivalent to

$$\begin{aligned} \bar{H}\tilde{H} + \bar{G}\tilde{G} &= 2, \\ \underline{\bar{H}}\tilde{H} + \underline{\bar{G}}\tilde{G} &= 0. \end{aligned} \tag{3.1}$$

Now take the modulation matrices

$$M = \begin{pmatrix} H & \underline{H} \\ G & \underline{G} \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} \tilde{H} & \underline{\tilde{H}} \\ \tilde{G} & \underline{\tilde{G}} \end{pmatrix}.$$

Then the perfect reconstruction conditions in (3.1) are equivalent to

$$\tilde{M}^T\bar{M} = 2I, \tag{3.2}$$

where I is the identity matrix. From (3.2) it follows that $\det M$ and $\det\tilde{M}$ are monomials. We get:

$$\tilde{M} = 2(\bar{M}^{-1})^T = \alpha \mathrm{adj}\,\bar{A}^T,$$

where $\alpha = (\det\tilde{M})/2$ is a monomial. From this relation, we get

$$\tilde{H} = \alpha\underline{\bar{G}}, \quad \tilde{G} = \underline{\alpha\bar{H}}, \tag{3.3}$$

and

$$\alpha = -\underline{\alpha}. \tag{3.4}$$

Substituting (3.3) into the first formula in (3.1) we get

$$\bar{H}\tilde{H} + \underline{\bar{H}\tilde{H}} = 2. \tag{3.5}$$

We will use this equation in the following subsection.

3.3 Construction of biorthogonal-like wavelets
In [2], Belogay and Wang developed a kind of non-separable bivariate orthonormal wavelets. Both the scaling functions and the wavelet functions correspond with two-row filters. Because orthogonal wavelets lack symmetry in general, we will present below the biorthogonal versions of their constructions.

From now on, we are focusing wavelets corresponding with the dilation matrix

$$D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

Like in Example in § 3.1, we take $\mathbf{u} = \mathbf{e}_1 = (1,0)^T$ and $\mathbf{v} = (-1,0)^T$.

We can show that for the specific dilation matrix in (3.6), we have

$$\underline{H}(z,w) = H(-z,w).$$

Hence, if $H(\mathbf{z}) = a(z)$, then $\underline{H}(\mathbf{z}) = a(-z)$, and if $H(\mathbf{z}) = a(w)$, then $\underline{H}(\mathbf{z}) = a(w)$. Suppose that $H, \tilde{H}$ are of the form

$$H(\mathbf{z}) = a(z) + (w-1)b(z), \qquad \tilde{H}(\mathbf{z}) = \tilde{a}(z) + (w-1)\tilde{b}(z). \tag{3.7}$$

From (3.5), we get that

$$\bar{a}\tilde{a} + \underline{\bar{a}\tilde{a}} = 2 \tag{3.8}$$

and

$$(\bar{a} - \bar{b})\tilde{b} + (\underline{\bar{a}} - \underline{\bar{b}})\underline{\tilde{b}} = 0, \qquad (\tilde{a} - \tilde{b})\bar{b} + (\underline{\tilde{a}} - \underline{\tilde{b}})\underline{\bar{b}} = 0. \tag{3.9}$$

Let $P(z) = \sum_{n=N_1}^{N_2} p_n z^n$, $N_1 \le N_2$, $p_{N_1}p_{N_2} \ne 0$, be a univariate Laurent polynomial with degree $\deg P = N_2 - N_1$. Suppose P is nontrivial, i.e., $\deg P > 0$. Note that P can be written as

$$P(z) = p_{N_2} z^{N_1} \tilde{P}(z), \quad \tilde{P}(z) = \sum_{n=0}^{N_2-N_1} p_{n+N_1} z^n.$$

so that $\tilde{P}$ is a polynomial in z. The roots of $\tilde{P}(z)$ can be categorized into two sets. A root γ lies in the first set if also $-\gamma$ is a root (which then also lies in the first set). The second set comprises all remaining roots. Thus we can factorize $\tilde{P}$ as

$$\tilde{P}(z) = \prod_{n=1}^{r_1} (z^2 - \gamma_n^2) \prod_{m=1}^{r_2} (z - v_m).$$

Thus we arrive at the following result.

3.4 Lemma. *Let $P(z)$ be a univariate Laurent polynomial. If P is nontrivial, then P can be written as $P(z) = s(z)q(z)$, where the factor $s(z)$ is even in z and has maximal degree, and where $q(z)$ and $q(-z)$ have no common nontrivial factors.*

Using this lemma, we get that condition (3.9) is equivalent to

$$\bar{a} - \bar{b} = s(z^2)l, \quad \tilde{b} = z^{\nu} q(z^2)\underline{l} \tag{3.10}$$

$$\bar{\tilde{a}} - \bar{\tilde{b}} = \tilde{s}(z^2)\tilde{l}, \quad b = z^{\tilde{\nu}} \tilde{q}(z^2)\underline{\tilde{l}}, \tag{3.11}$$

where ν and $\tilde{\nu}$ are odd numbers. Then

$$\bar{a} = \bar{b} + s(z^2)l = z^{-\tilde{\nu}}\tilde{q}(z^{-2})\underline{\bar{\tilde{l}}} + s(z^2)l, \tag{3.12}$$
$$\tilde{a} = \tilde{b} + \tilde{s}(z^{-2})\bar{\tilde{l}} = z^{\nu}q(z^2)\underline{l} + \tilde{s}(z^{-2})\bar{\tilde{l}}. \tag{3.13}$$

Substituting a and $\tilde{a}$ into (3.8), we derive

$$(l\bar{\tilde{l}} + \underline{l}\underline{\bar{\tilde{l}}})\left(z^{\nu-\tilde{\nu}}q(z^2)\bar{\tilde{q}}(z^2) + s(z^2)\bar{\tilde{s}}(z^2)\right) = 2\,.$$

Now both factors are necessarily monomials, and without loss of generality, we may assume that

$$l\bar{\tilde{l}} + \underline{l}\underline{\bar{\tilde{l}}} = 2 \tag{3.14}$$
$$z^{\nu-\tilde{\nu}}q(z^2)\bar{\tilde{q}}(z^2) + s(z^2)\bar{\tilde{s}}(z^2) = 1. \tag{3.15}$$

For (3.14), we just follow the one-dimensional case to find l and $\tilde{l}$. Furthermore, by replacing z^2 by z (3.15) can be rewritten as

$$z^d q\bar{\tilde{q}} + s\bar{\tilde{s}} = 1 \tag{3.16}$$

where $d = (\nu - \tilde{\nu})/2$ is an integer since both ν and $\tilde{\nu}$ are odd numbers.

3.4 Accuracy and symmetry
To find solutions of H and $\tilde{H}$ for which (3.14) and (3.16) hold, we have to impose additional restrictions. One typical restriction concerns the *accuracy.* Depending on the context, it can be formulated in terms of vanishing moments or "sum rules". In the univariate case, if a scaling function ϕ has r vanishing moments i.e. $\int x^t\phi(x)dx = 0$, for $t = 0, \cdots, r-1$, then its mask $H(z)$ has a factor $(1+z)^r$; this property plays an essential role in the formulation and analysis of univariate wavelets. The vanishing moment property can be generalized to higher dimensions. A scaling function ϕ is said to have r vanishing moments if $\int \mathbf{x}^\alpha\phi(\mathbf{x})d\mathbf{x} = 0$, for $|\alpha| < r$. But unfortunately, in this case the mask $H(\mathbf{z})$ cannot be factorized like the univariate case. Instead we can invoke the "sum rules" of its coefficients(cf. [2, 4]) to characterize the accuracy: we assume that H satisfies

$$\frac{\partial^{p+q}}{\partial z^p \partial w^q}H(-1,1) = 0, \quad \forall p, q \geq 0, p+q < r. \tag{3.17}$$

Then, using (3.17), it is easy to obtain the following result.

3.5 Lemma. *Suppose that H and $\tilde{H}$ in (3.7) have accuracies r and $\tilde{r}$ respectively. Then we have*

$$a = \left(\frac{1+z}{2}\right)^{r+1} a_0, \quad b = \left(\frac{1+z}{2}\right)^{r} b_0, \tag{3.18}$$
$$\tilde{a} = \left(\frac{1+z}{2}\right)^{\tilde{r}+1} \tilde{a}_0, \quad \tilde{b} = \left(\frac{1+z}{2}\right)^{\tilde{r}} b_0. \tag{3.19}$$

Using (3.10) and (3.11), we can assume that q and $\tilde{q}$ are of the form

$$\tilde{q}(z^2) = \left(\frac{1+z}{2}\right)^r \left(\frac{1-z}{2}\right)^r q_0(z^2), \qquad q(z^2) = \left(\frac{1+z}{2}\right)^{\tilde{r}} \left(\frac{1-z}{2}\right)^{\tilde{r}} \tilde{q}_0(z^2), \tag{3.20}$$

in order for b and $\tilde{b}$ to have accuracy r and $\tilde{r}$, while l and $\tilde{l}$ are chosen to be

$$l(z) = \left(\frac{1+z}{2}\right)^r l_0(z), \qquad \tilde{l}(z) = \left(\frac{1+z}{2}\right)^{\tilde{r}} \tilde{l}_0(z) \tag{3.21}$$

in order for a and $\tilde{a}$ to have accuracy r and $\tilde{r}$. Furthermore, in order that a and $\tilde{a}$ have accuracy $r+1$ and $\tilde{r}+1$, it is necessary that

$$(-1)^r q_0(1)\tilde{l}_0(1) = s(1)l_0(-1), \qquad (-1)^{\tilde{r}}\tilde{q}_0(1)l_0(1) = \tilde{s}(1)\tilde{l}_0(-1). \tag{3.22}$$

If we put $a(1) = \tilde{a}(1) = \sqrt{2}$, $b(1) = \tilde{b}(1) = 0$, and $l(1) = \tilde{l}(1) = \sqrt{2}$, then $s(1) = \tilde{s}(1) = 1$, and

$$q_0(1) = (-1)^r \frac{\sqrt{2}}{2} l_0(-1), \qquad \tilde{q}_0(1) = (-1)^{\tilde{r}} \frac{\sqrt{2}}{2} \tilde{l}_0(-1). \tag{3.23}$$

By the construction of univariate biorthogonal wavelets in [5], r and $\tilde{r}$ have the same parity. Take $N = (r+\tilde{r})/2$, then

$$l_0(-1)\tilde{l}_0(-1) = 2\sum_{k=0}^{N-1} \binom{N+k-1}{k}$$

or

$$q_0(1)\tilde{q}_0(1) = \sum_{k=0}^{N-1} \binom{N+k-1}{k}.$$

The symmetry of l and $\tilde{l}$ is guaranteed by the symmetry of l_0 and $\tilde{l}_0$ (see [7]). If we can find symmetric solutions of (3.16), then we get a pair of symmetric solutions H and $\tilde{H}$.

3.6 Theorem. *Let l given by (3.21), where l_0 is a Laurent polynomial, be a 1D low-pass filter of accuracy r satisfying (3.14) and $l_0(1) = \sqrt{2}$. Let q be given by (3.20) where q_0 is a Laurent polynomial satisfying (3.23) and let s be a Laurent polynomial such that (3.15) holds and $s(1) = 1$. Then H given by (3.7), where a, b are given by (3.10)–(3.13), defines a 2D low-pass filter with accuracy $r+1$.*

4. Lifting scheme

4.1 Implementation of the wavelet transform

A well-known implementation of a wavelet transformation is provided by Mallat's algorithm [11] which, for the two band case, looks as follows.

Suppose that $\{V_j; j \in \mathbb{Z}\}$ is an MRA of $L_2(\mathbb{R}^2)$ with dilation matrix D, and assume that ϕ and ψ are the respective scaling and wavelet functions that satisfy dilation relations

$$\phi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} g_{\mathbf{n}} \phi(D\mathbf{x} - \mathbf{n}), \quad \psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} h_{\mathbf{n}} \phi(D\mathbf{x} - \mathbf{n}).$$

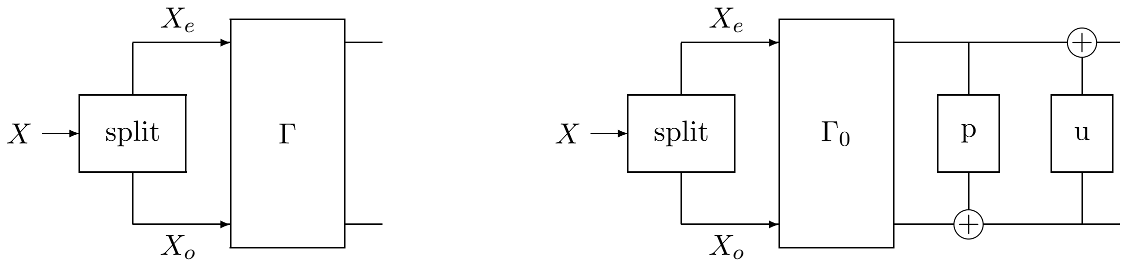

Figure 4: *Lifting scheme. Left: original transformation with polyphase matrix* Γ. *Right: alternative with polyphase matrix* Γ_0 *and lifting steps* p *(prediction) and* u *(update).*

Given $f \in L_2(\mathbb{R}^2)$, denote by $c^j_{\mathbf{n}}$ and $d^j_{\mathbf{n}}$ the coefficients

$$c^j_{\mathbf{n}} = \langle f, \phi_{j,\mathbf{n}} \rangle, \quad d^j_{\mathbf{n}} = \langle f, \psi_{j,\mathbf{n}} \rangle.$$

Then

$$c^{j-1}_{\mathbf{n}} = \sum_{\mathbf{k}} h_{\mathbf{k}-D\mathbf{n}} c^j_{\mathbf{k}}, \quad d^{j-1}_{\mathbf{n}} = \sum_{\mathbf{n}} g_{\mathbf{k}-D\mathbf{n}} c^j_{\mathbf{k}},$$

and

$$c^j_{\mathbf{k}} = \sum h_{\mathbf{k}-D\mathbf{n}} c^{j-1}_{\mathbf{n}} + \sum g_{\mathbf{k}-D\mathbf{n}} d^{j-1}_{\mathbf{n}}.$$

An alternative way to implement the wavelet transform is by using the so-called lifting schemes first developed by Sweldens [14]. The advantages of the lifting scheme over Mallat's algorithm are its flexibility and efficiency. In the following subsections we describe the lifting scheme implementation of our wavelet family.

4.2 The lifting scheme

Let $X(z,w)$ represent an input signal, which is split into its even part X_e and its odd part X_o according to the dilation matrix D. Denote by $\Gamma(z,w)$ the associated polyphase matrix; see also Figure 4. Suppose that we can write

$$\Gamma(z,w) = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \Gamma_0(z,w).$$

Then the transformation Γ is equivalent to the transformation Γ_0 followed by two successive lifting steps:

$X_o = X_o + pX_e$: prediction lifting
$X_e = X_e + uX_o$: update lifting.

See Figure 4 for an illustration.

Any 1D wavelet transform using finite impulse response (FIR) filters can be factorized into lifting steps by means of the Euclidean algorithm [8]. Unfortunately, the factorization results for the 1D case do not have a straightforward generalization to the general non-separable 2D case (see [13]).

For the two-row filters, we will show in the following that the related transformations can be realized in lifting schemes.

4.3 Factorizing the modulation matrix
From the above deduction, we have

$$\begin{aligned} M = \begin{pmatrix} H & \underline{H} \\ G & \underline{G} \end{pmatrix} &= \begin{pmatrix} sl + \beta q \bar{l} w & s\underline{l} - \beta q \bar{l} w \\ \alpha[\bar{s}\underline{\bar{l}} - \bar{\beta}\bar{q} l \bar{w}] & -\alpha[\bar{s}\bar{l} + \bar{\beta}\bar{q}\underline{l}\bar{w}] \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} S(z^2) \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} L(z)\,, \end{aligned} \tag{4.1}$$

where $\alpha(z) = z^\mu$, $\beta(z) = z^\nu$, and

$$Q(z) = z^{(\nu-\mu)/2} q(z), \tag{4.2}$$

$$S(z) = \begin{pmatrix} s(z) & Q(z) \\ -\bar{Q}(z) & \bar{s}(z) \end{pmatrix}, \tag{4.3}$$

$$L(z) = \begin{pmatrix} l(z) & l(-z) \\ \alpha(z) l(-z^{-1}) & -\alpha(z) l(z^{-1}) \end{pmatrix}. \tag{4.4}$$

Recall that both μ and ν are odd, and therefore $(\nu-\mu)/2$ is an integer. It is interesting to note that the matrix in (4.4) is nothing but the modulation matrix of a 1D subband scheme. Therefore, we can follow Daubechies' construction in [7] to obtain l. We will not give any further details here. Note also that the matrix S in (4.3) depends only on the variable z. Thus (4.1) means that the filters with 2-row support can be factorized in terms of one-dimensional filters.

4.4 Lifting implementation
The key point for the lifting realization of a wavelet transform is that its polyphase matrix can be factorized into fundamental matrices, each of which corresponds to a lifting step. For the two-row case, we use the modulation matrix to construct the corresponding polyphase matrix. Recall that the monomial $\alpha(z) = z^\mu$ in (3.3) is odd in z. We may assume without loss of generality (and for the sake of simplicity) that $\alpha(z) = z^{-1}$, i.e., $\mu = -1$.

We split the univariate z-form $l(z)$ into two parts:

$$l(z) = l_e(z^2) + z^{-1} l_o(z^2),$$

where l_e contains the even coefficients and l_o the odd. If $p(z) = z^{-1} l(-z^{-1})$, then

$$p_e(z) = -p_o(z^{-1}), \quad p_o(z) = l_e(z^{-1}).$$

Therefore the modulation matrix L in (4.4) can be written as

$$L(z) = P_1(z^2) \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \text{ with } P_1 = \begin{pmatrix} l_e & l_o \\ -\bar{l}_o & \bar{l}_e \end{pmatrix}.$$

Note that P_1 is the 1D polyphase matrix.

Analogously to the 1D case, the bivariate Laurent polynomial $H(z,w) = a(z) + wb(z)$ can be split into the odd part $H_o(z,w) = a_o(z) + wb_o(z)$ and the even part $H_e(z,w) = a_e(z) + wb_e(z)$. Now the following relation holds:

$$H(z,w) = H_e(z^2,w) + z^{-1} H_o(z^2,w).$$

Suppose that H is the two-row filter defined in §3, which has polyphase matrix

$$P(\mathbf{z}) = \begin{pmatrix} a_e + wb_e & a_o + wb_o \\ -\bar{a}_o - \bar{w}\bar{b}_o & \bar{a}_e + \bar{w}\bar{b}_e \end{pmatrix} . \tag{4.5}$$

It is easy to show that

$$M(z,w) = P(z^2, w) \begin{pmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{pmatrix} .$$

Using that $M^T\bar{M} = 2I$ (see (3.2)) we find that P is unitary. We obtain from (4.1) that

$$P(z,w) = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} S(z) \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} P_1(z). \tag{4.6}$$

In this formulation, matrices S and P_1 are univariate unitary matrices and therefore they can be factorized into fundamental matrices. Thanks to this factorization, we can design an algorithm for the 2D transform based on the underlying 1D wavelet transforms. The algorithm below is given only for the forward transform, in which case we must use $\bar{P}$ rather than P. In the following, '$\star_{\mathsf{R}}$' denotes row-wise convolution. Furthermore, x_0 is the input signal and (x, y) is the transformed signal comprising the approximation band x and the detail band y.

1. Let (x_1, y_1) be the row-wise wavelet transform of x_0 with a 1D wavelet of given accuracy r; this corresponds with the matrix $P_1(z)$ in (4.6).
2. Apply forward vertical shift to y_1; this corresponds with the diagonal matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \bar{w} \end{smallmatrix}\right)$.

3a. Compute $x_2 = x_1 \star_{\mathsf{R}} \bar{s} + y_1 \star_{\mathsf{R}} \bar{Q}$.

3b. Compute $y_2 = -x_1 \star_{\mathsf{R}} Q + y_1 \star_{\mathsf{R}} s$.
Note that these two expressions correspond with the multiplication with matrix S in (4.3).

4. Apply backward vertical shift to y_2.
5. Define $x = x_2^T$ and $y = y_2^T$. This step is necessary because of the transpose in dilation matrix D.

5. Shorter filters with more rows

The factorization in (4.6) shows that the supports of the two-row filters are horizontally stretched versions of the supports of the underlying 1D filter P_1.

Here we will give an alternative factorization with filters that are less stretched. Toward this goal we replace $S(z)$ in factorization (4.6) by $S(w)$. Thus we get a polyphase matrix

$$P(z,w) = \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} S(w) \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} P_1(z). \tag{5.1}$$

This matrix corresponds to the 2D low-pass filter

$$H_s(z,w) = A(z,w) + wB(z,w) \tag{5.2}$$

where A and B are 2D Laurent polynomials defined as

$$A(z,w) = s(w)l(z), \quad B(z,w) = z^{\tilde{\nu}}\tilde{q}(w)\underline{\tilde{l}}(z)\,. \tag{5.3}$$

Here l and $\tilde{l}$ satisfy (3.14), $\tilde{q}$ is the same as in (3.20), but with z replaced by w, and s and $\tilde{s}$ satisfy (3.15). The subscript s in H_s indicates that H_s is a 'short' version of H defined in (3.7), which will be explained below.

The factors $\left(\frac{1+z}{2}\right)^r$ of $l(z)$ and $\left(\frac{1-w}{2}\right)^r$ of $q(w)$ ensure that the filter H_s as defined in (5.2) is of accuracy r. But $\frac{\partial^r}{\partial z^r}H_s(-1,1) = 0$ results in $s(1) = 0$, which is not possible. This means that, unlike the two row case, the filter (5.2) cannot have accuracy $r+1$.

We can prove the following analogue of Theorem 3.6.

5.1 Theorem. *Let l given by (3.21), where l_0 is a Laurent polynomial, be a 1D low-pass filter of accuracy r satisfying (3.14) and $l_0(1) = \sqrt{2}$. Let q be given by (3.20) where q_0 is a Laurent polynomial satisfying (3.23) and let s be a Laurent polynomial such that (3.15) holds and $s(1) = 1$. Then H_s given by (5.2), where A, B are given by (5.3), defines a 2D low-pass filter with accuracy r.*

For a 2D Laurent polynomial

$$A(z,w) = \sum_{I_1 \le i \le I_2} \sum_{J_1 \le j \le J_2} a_{ij} z^i w^j,$$

we can define the degree of A as

$$\deg A = (I_2 - I_1, J_2 - J_1).$$

If A serves as a filter, its degree specifies its filter length. Note that just like in the univariate case, the degree of filter A equals the size of A. The degree of a separable bivariate Laurent polynomial follows immediately from the degrees of its univariate parts. For example, if $A(z,w) = B(z)C(w)$, then $\deg A = (\deg B, \deg C)$. Using this observation, we give an analysis of the degrees of the filters defined in (3.7) and (5.2).

Suppose $B(z) = \sum_{I_1 \le i \le I_2} b_i z^i$ and $C(z) = \sum_{J_1 \le i \le J_2} c_i z^i$. Let $A(z,w) = B(z) + wC(z)$. Then $\deg A = (\max\{I_2, J_2\} - \min\{I_1, J_1\}, 1)$. If, for instance, $I_1 \le J_1 \le J_2 \le I_2$, $\deg A = (\deg B, 1)$.

In order to simplify our discussion, we can assume in (3.7) that with suitably chosen ν and $\tilde{\nu}$ in (3.10) and (3.11)

$$\deg H = (\max\{\deg(a-b), \deg b\}, 1) = (\max\{2\deg s + \deg l, 2\deg \tilde{q} + \deg \tilde{l}\}, 1).$$

Similarly, we can assume

$$\deg H_s = (\max\{\deg s, \deg \tilde{q}\}, \max\{\deg l, \deg \tilde{l}\}).$$

Therefore H has height 2 but its width is large compared to that of H_s which has a support that is more square-shaped.

We can regard the orthonormal case as a special example of the biorthogonal case with primary and dual filters that are the same, i.e., $H = \tilde{H}$, $G = \tilde{G}$, and so on. In this special case, given the assumptions in Theorem 5.1, it is easy to see that

$$\deg A = \deg B = (2r-1, r).$$

i.e., $\deg H_s = (2r-1, r)$, whereas the two-row filter H is supported on $[0, 4r-1] \times [0, 1]$.

If we compare the expressions for A, B in (5.3) with those for a, b in (3.10)–(3.13), we see that $s(z^2)$ has been replaced by $s(w)$. A similar substitution should be used when we compute the modified modulation matrix. However, in the polyphase matrix in (5.1) we encounter the matrix $S(w)$ whereas in (4.6), the matrix $S(z)$ occurs. The algorithm at the end of §4.1 remains unchanged except for steps 3a and 3b which have to be changed into

3a'. Compute $x_2 = x_1 \star_{\mathsf{C}} \bar{s} + y_1 \star_{\mathsf{C}} \bar{Q}$.

3b'. Compute $y_2 = -x_1 \star_{\mathsf{C}} Q + y_1 \star_{\mathsf{C}} s$.

here '$\star_{\mathsf{C}}$' denotes column-wise convolution.

6. Conclusion

We have investigated the class of orthonormal 2D filters that was introduced by Belogay and Wang in [2], and we have given an extension to the biorthogonal case. Furthermore, we have described a factorization of the corresponding modulation matrices, and shown how such a factorization can be used to obtain a realization in terms of 1D filters and allows an efficient implementation based on the lifting scheme.

We have also given a modification of the Belogay-Wang approach so that the resulting filters have a support that is less stretched along the horizontal direction. It uses, in the orthonormal case, $r+1$ rows and $2r$ columns for a decomposition with accuracy r.

References

1. A. N. Akansu and R. A. Haddad, *Multiresolution Signal Decomposition: Transforms, Subbands, and Wavelets,* Academic Press, 2001.
2. E. Belogay and Y. Wang, Arbitrary smooth orthogonal nonseparable wavelets in $\mathbb{R}^2$, *SIAM J. Math. Anal.,* **30**, pp 678–697, 1999.
3. L. Borup and M. Nielsen, Nonseparable wavelet packets, in: *Approximation Theory X: Wavelets, Splines, and Applications*, C.K. Chui, L.L. Schumaker, and J. Stöckler (eds.), Vanderbild University Press, 2002, pp. 51–61.
4. C. Cabrelli, C. Heil, and U. Molter, Accuracy of lattice translates of several multidimensional refinable functions, *J. of Approx. Theory,* **95**, pp 5–52, 1998.
5. A. Cohen, I. Daubechies, and J.C. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure and Applied Mathematics,* **45**, pp 485–500, 1992.
6. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure and Applied Mathematics*, **41**, 909–996, 1988.
7. I. Daubechies, *Ten Lectures on Wavelets,* CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM 61, Philadelphia, PA, 1992.
8. I. Daubechies and W. Sweldens, Factoring wavelet transforms into lifting steps, *J. Fourier Anal. Appl.* **4**, pp 245–269, 1998.
9. J. Kovačević and M. Vetterli, Nonseparable Multidimensional Perfect Reconstruction Filter Banks and Wavelet Bases for $\mathbb{R}^n$, *IEEE Trans. Inform. Theory,* **38**, pp 533-555, 1992.
10. M. Vetterli and J. Kovačević, Wavelets and Subband Coding, Prentice Hall, 1995.
11. S. Mallat *A Wavelet Tour of Signal Processing,* Academic Press, 1999.
12. J.H. McClellen, T.W. Parks, and L.R. Rabiner, A computer program for designing optimum FIR linear phase digital filters, *IEEE Transactions on Audio Electroacoustics* **21**, pp 506–526, 1973 .
13. H. Park and C. Woodburn, An algorithmic proof of Suslin's stability theorem for polynomial rings, *Journal of Algebra,* **178**, pp 277–298, 1995.
14. W. Sweldens, The Lifting Scheme: A New Philosophy in Biorthogonal Wavelet Constructions, in: *Wavelet Applications in Signal and Image Processing III*, editor: A. F. Laine and M. Unser, Proc. SPIE 2569, pp. 68–79, 1995.
15. D.B.H. Tay, Design of Filter Banks / Wavelets using TROV: A Survey, *Digital Signal Processing: A Review Journal* **4**, 1997.
16. D.B. Tay and N.G. Kingsnury, Flexible design of multidimensional perfect reconstruc-

tion FIR 2-band filters using transformations of variables, *IEEE Transactions on Image Processing* **2**, 466–480, 1993.

17. P. Wojtaszczyk, *A Mathematical Introduction to Wavelets,* Cambridge University Press, UK, 1997.