Centrum voor Wiskunde en Informatica
REPORT *RAPPORT*

Grey-scale Morphology Based on Fuzzy Logic

T.-Q.n Deng, H.J.A.M. Heijmans

Probability, Networks and Algorithms (PNA)

PNA-R0012 October 31, 2000

Report PNA-R0012
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Grey-scale Morphology Based on Fuzzy Logic

Ting-Quan Deng
Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, People's Republic of China

Henk J.A.M. Heijmans
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

There exist several methods to extend binary morphology to grey-scale images. One of these methods is based on fuzzy logic and fuzzy set theory. Another approach starts from the complete lattice framework for morphology and the theory of adjunctions. In this paper, both approaches are combined. The basic idea is to use (fuzzy) conjunctions and implications which are adjoint in the definition of dilations and erosions, respectively. This gives rise to a large class of morphological operators for grey-scale images. It turns out that this class includes the often used grey-scale Minkowski addition and subtraction.

2000 Mathematics Subject Classification: 03B52, 03E72, 68U10
Keywords and Phrases: Mathematical morphology, image processing, fuzzy logic, complete lattice, adjunction, conjunction, implication, dilation, erosion, negation.
Note: This work was carried out during a one-year visit of the first author to CWI. He gratefully acknowledges financial support of the Netherlands Organization for International Cooperation in Higher Education (NUFFIC). Furthermore, he wants to thank CWI for its hospitality. The work of the second author was carried out under project PNA4.2 "Wavelets and Morphology".

1. Introduction

Mathematical morphology was invented in the early sixties by Matheron [12] and Serra [17] as a novel geometry-based technique for image processing and analysis. Originally, mathematical morphology was developed for binary images and used simple concepts from set theory and geometry such as set inclusion, intersection, union, complementation, and translation. This resulted in a collection of tools, called *morphological operators*, which are eminently suited for the analysis of shape and structure in binary images [21]. The most well-known of these operators are erosion and dilation.

Soon thereafter, mathematical morphology was extended to grey-scale images. Such an extension requires rules for the 'combination' of different grey-values. In the binary case, the set paradigm leads in a natural way to 'combinations' based on concepts from Boolean logic. In the grey-scale case, the set paradigm is no longer valid, and as a consequence it is not a priori clear which 'combination mechanism' should be used. Furthermore, the choice of the 'combination mechanism' may also depend on the physical interpretation of the grey-values.

To extend binary morphology to grey-scale images (i.e., functions) people have chosen different approaches. Here we briefly discuss the most important ones.

- *The umbra approach:* here, a grey-scale image is considered as a 3-dimensional landscape which can then be transformed using tools which are known from the binary (2-dimensional) case [7, 8, 17, 22].

- *The threshold set approach:* a grey-scale image is decomposed in terms of its threshold (or level) sets. To each of these sets one can apply a binary operator, after which the resulting sets can be used to synthesize a transformed grey-scale image [8, 11, 17].
- *The complete lattice approach:* today, complete lattices are considered as the right mathematical framework for morphology. This framework can be used to formulate (minimal) conditions that need to be satisfied by grey-scale morphological operators [8, 16, 18].

It should be clear that these different approaches, rather than leading to different families of operators (which they do only to a limited extent), mainly affect the way we conceive the resulting operators. It cannot be denied, for example, that the introduction of the complete lattice approach has resulted in an algebraization of morphology. Although this has been very much to the benefit of this area, it has some drawbacks too. The major drawback, from our perspective, is the abandonment (to a high degree) of the set paradigm. However, there exists yet a fourth approach, not mentioned so-far, which enables a return to the set paradigm.

- *The fuzzy logic approach:* in this approach, one uses concepts from fuzzy logic and fuzzy set theory for the design of morphological operators.

To our knowledge, the first author using concepts from fuzzy logic in mathematical morphology is Goetcherian [5]. Since then, several authors have advocated this approach, for example Sinha and Dougherty [19, 20], Bloch and Maitre [1, 2], and De Baets, Nachtegael and Kerre [3, 4, 13]. Rather than discussing their work here, and the differences between the various approaches, we refer to the paper of Nachtegael and Kerre [13] which presents an excellent survey of the existing literature on fuzzy morphology. Another good source is the recent volume edited by Kerre and Nachtegael [9].

To our point of view, one key ingredient is missing in all existing approaches of fuzzy morphology: *adjunctions*. Adjunctions, which do arise naturally in a complete lattice framework, can be considered as one of the most (if not the most) important concepts in mathematical morphology. It provides a unique pairing between dilations and erosions, in contrast to complementation (or negation). Although it is true that negations can be used to construct erosions from dilations (and vice versa), this technique has some serious drawbacks. First of all, negations are, in most cases, not unique, meaning that the aforementioned pairing is not unique either. Furthermore, the pairing provided by negations does not necessarily lead to dilations and erosions that, when they are composed, yield openings and closings.

In this paper we adopt the fuzzy logic approach to grey-scale morphology and combine it with the concept of adjunctions. It turns out that this amounts to considering (fuzzy) conjunctions and implications which, themselves, satisfy the adjunction relation. Although grey-scale images and fuzzy sets do have the same mathematical representation in this paper (both are functions from a given set U into $[0, 1]$), they have a different interpretation. Although it is true that fuzzy set theory has become an important tool in image processing, it does not make sense to think of an image as being a fuzzy set unless the grey-values are known to represent a measure of uncertainty.

In this paper we use notions from fuzzy logic to extend binary morphology (which can be described in terms of classical binary logic) to grey-scale images. In doing so, it turns out that existing grey-scale operators such as grey-scale Minkowski addition and subtraction can be considered as a very special case of the fuzzy logic framework.

This paper is organized as follows. In Sections 2 and 3, we recall, respectively, binary morphology and morphology on complete lattices. In Section 4, we discuss various concepts from fuzzy logic that are needed in the sequel of the paper. Particular attention will be given to adjunctions. Then, in Section 5, we show how such notions can be used to define morphological operators for grey-scale images, preserving as much as possible the geometrical interpretation characteristic for binary morphology. We will also explain there how the classical grey-scale operators can be fit within our framework. In Section 6, we explain the role of negations. Some experimental results are presented in Section 7, and we end with some conclusions and final remarks in Section 8.

2. Binary Morphology

Let U be a nonempty set called a universe, and let $\mathcal{P}(U)$ be the family of all subsets of U. Often, we choose $U = \mathbb{R}^d$, the d-dimensional Euclidean space, in which case a subset X of U represents a continuous binary image on U, or we take $U = \mathbb{Z}^d$, in which case $X \subseteq U$ represents a discrete binary image on U. Given a subset A of U and a vector $h \in U$, we denote by A_h the translation of A along the vector h, that is

$$A_h = \{a + h \mid a \in A\}.$$

The *Minkowski addition* of two sets X and A is defined as

$$X \oplus A = \bigcup_{a \in A} X_a = \{y \in U \mid \check{A}_y \cap X \neq \emptyset\}, \tag{2.1}$$

where $\check{A} = \{-a \mid a \in A\}$ is the reflection of set A around the origin. The *Minkowski subtraction* is

$$X \ominus A = \bigcap_{a \in A} X_{-a} = \{y \in U \mid A_y \subseteq X\}. \tag{2.2}$$

Given $A \subseteq U$, the dilation δ_A and erosion ε_A of an image $X \in \mathcal{P}(U)$ are, respectively, defined as

$$\delta_A(X) = X \oplus A \quad \text{and} \quad \varepsilon_A(X) = X \ominus A.$$

The set A is called *structuring element* in the morphological literature. The most important relation between dilation and erosion is

$$Y \oplus A \subseteq X \iff Y \subseteq X \ominus A, \quad X, Y, A \in \mathcal{P}(U), \tag{2.3}$$

which is called the *adjunction* relation. Another duality relation between dilation and erosion is given by means of complementation It is easy to show that

$$X^{\mathrm{c}} \oplus A = (X \ominus \check{A})^{\mathrm{c}} \ \text{ and } \ X^{\mathrm{c}} \ominus A = (X \oplus \check{A})^{\mathrm{c}}, \tag{2.4}$$

where X^{c} denotes the set complement of X and $\check{A}$ is the reflection of A. In fact, some authors have used this duality to construct erosions from dilations or vice versa. However, from our point of view the adjunction relation in (2.3) is the most general as well as the most powerful duality relation between dilations and erosions. We will return to this issue in Section 6.

In general, dilation and erosion are not inverse operators. More specifically, if a set X is eroded by a set A and then dilated by A, the resulting set is not the original set X but a subset of it. In most cases this set is smaller than X. It is called the *opening* of X by A, denoted by $X \circ A$:

$$X \circ A = (X \ominus A) \oplus A.$$

Dually, if X is first dilated by A and then eroded we get the *closing*

$$X \bullet A = (X \oplus A) \ominus A,$$

a set which contains the original set X, and in most cases, is larger than this set.

For the opening one can easily show that

$$X \circ A = \bigcup \{A_y \mid y \in U\,,\ A_y \subseteq X\}\,.$$

The most important property of the opening and closing operator is their idempotence. Recall that an operator ψ is called *idempotent* if $\psi^2 = \psi$.

3. Morphology on Complete Lattices

The formal mathematical framework for morphology is based on complete lattices and operators between them. In this section we briefly recall this framework. For more details, the reader may refer to [8].

A nonempty set $\mathcal{L}$ with a partial ordering $\leq$ is called a *complete lattice* if every subset $M \subseteq \mathcal{L}$ has an infimum $\bigwedge M$ and a supremum $\bigvee M$ in $\mathcal{L}$. The least and greatest element of a complete lattice are denoted by $\bot$ and $\top$, respectively. Suppose that $\mathcal{L}$ and $\mathcal{M}$ are complete lattices; an operator $\psi : \mathcal{L} \to \mathcal{M}$ is said to be *increasing* if $X_1 \leq X_2$ implies that $\psi(X_1) \leq \psi(X_2)$, for $X_1, X_2 \in \mathcal{L}$.

An operator $\delta : \mathcal{M} \to \mathcal{L}$ is called a *dilation* if $\delta(\bigvee_{j\in J} Y_j) = \bigvee_{j \in J} \delta(Y_j)$ for every collection $\{Y_j \mid j \in J\} \subseteq \mathcal{M}$. An operator $\varepsilon : \mathcal{L} \to \mathcal{M}$ is called an *erosion* if $\varepsilon(\bigwedge_{j\in J} X_j) = \bigwedge_{j\in J} \varepsilon(X_j)$ for every collection $\{X_j \mid j \in J\} \subseteq \mathcal{L}$. Note in particular that, by choosing J to be the empty set, we get $\delta(\bot) = \bot$ and $\varepsilon(\top) = \top$. Two operators $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$ are said to form an *adjunction* if

$$\delta(Y) \leq X \iff Y \leq \varepsilon(X)\,,$$

for any $X \in \mathcal{L}$ and $Y \in \mathcal{M}$. In this case, we say that $(\varepsilon\,, \delta)$ is an *adjunction* between $\mathcal{L}$ and $\mathcal{M}$. We list some elementary results.

3.1. Proposition. *Let $\mathcal{L}$ and $\mathcal{M}$ be two complete lattices, and let $(\varepsilon\,, \delta)$ be an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then ε is an erosion and δ is a dilation.*

3.2. Proposition. *If $\delta : \mathcal{M} \to \mathcal{L}$ is a dilation, then there exists a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ such that $(\varepsilon\,, \delta)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$. Dually, if $\varepsilon : \mathcal{L} \to \mathcal{M}$ is an erosion, then there exists a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ such that $(\varepsilon\,, \delta)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$.*

3.3. Proposition. *Assume that $(\varepsilon_1\,, \delta_1)$ is an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$, and that $(\varepsilon_2\,, \delta_2)$ is an adjunction between complete lattices $\mathcal{M}$ and $\mathcal{N}$, then $(\varepsilon_2\varepsilon_1\,, \delta_1\delta_2)$ is an adjunction between $\mathcal{L}$ and $\mathcal{N}$.*

3.4. Proposition. *Let $(\varepsilon_j\,, \delta_j)$ be an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$ for any $j \in J$, then $(\bigwedge_{j\in J} \varepsilon_j\,, \bigvee_{j\in J} \delta_j)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$ as well.*

Given a complete lattice $\mathcal{L}$, an operator $\alpha : \mathcal{L} \to \mathcal{L}$ is called an *opening* if α is increasing, idempotent, and anti-extensive (that is $\alpha(X) \leq X$, for every $X \in \mathcal{L}$). Dually, an operator β is called a *closing* if it is increasing, idempotent and extensive (that is $X \leq \beta(X)$, for every $X \in \mathcal{L}$).

3.5. Proposition. *If $(\varepsilon\,, \delta)$ is an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$, then $\varepsilon\delta$ is a closing on $\mathcal{M}$, whereas $\delta\varepsilon$ is an opening on $\mathcal{L}$.*

A simple example of a complete lattice is $\mathcal{P}(U)$. In this case, the least and greatest element are $\emptyset$ and U, respectively. If $U = \mathbb{R}^d$ or $\mathbb{Z}^d$, then the dilations δ_A and the erosion ε_A introduced in the previous section, form an adjunction. Moreover, it is not difficult to show that every translation invariant adjunction is of this form.

4. Fuzzy Logic

In this section, we discuss some basic concepts from fuzzy logic which are important for the sequel of this paper. In particular, we will discuss the conjunction and implication, and we will explain how these two notions can be paired by means of the adjunction relation discussed in the previous section.

There is a huge literature on fuzzy logic and fuzzy set theory (see e.g. [10, 15, 24]), and it should be clear that the discussion presented here is far from complete, and only touches upon the issues we believe are important here.

4.1 Conjunction and implication

Two important binary operations known from predicate logic are the conjunction and the implication, in this paper denoted by C and I, respectively. For the conjunction $C(s,t) = s \wedge t$ we have

$$C(0,0) = C(1,0) = C(0,1) = 0 \quad \text{and} \quad C(1,1) = 1\,, \tag{4.1}$$

and for the implication $I(s,t) = s \to t$ we have

$$I(0,0) = I(0,1) = I(1,1) = 1 \quad \text{and} \quad I(1,0) = 0\,. \tag{4.2}$$

In fuzzy logic, the operations C and I are extended from the Boolean domain $\{0,1\} \times \{0,1\}$ to the rectangle $[0,1] \times [0,1]$.

4.1. Definition. A mapping $C: [0,1] \times [0,1] \to [0,1]$ is called a *fuzzy conjunction* (briefly, *conjunction*) if it is increasing in both arguments and (4.1) holds. A mapping $I: [0,1] \times [0,1] \to [0,1]$ is called a *fuzzy implication* (briefly, *implication*) if it is decreasing in the first argument, increasing in the second, and satisfies (4.2).

Evidently, every conjunction satisfies

$$C(s,0) = C(0,s) = 0\,, \quad s \in [0,1]\,,$$

and every implication satisfies

$$I(s,1) = I(0,s) = 1\,, \quad s \in [0,1]\,.$$

4.2. Definition. An implication I and a conjunction C are said to be *adjoint* (on $[0,1]$) if

$$C(a,t) \le s \iff t \le I(a,s) \tag{4.3}$$

for all $a, s, t \in [0,1]$.

Thus an implication I and a conjunction C are adjoint if for every $a \in [0,1]$, the pair $(I(a,\cdot), C(a,\cdot))$ forms an adjunction on $[0,1]$. Alternatively, we can say that the pair (I,C) is an adjunction. This means in particular that $I(a,\cdot)$ is an erosion on $[0,1]$, or alternatively, *continuous from the right*, and that $C(a,\cdot)$ is a dilation, or alternatively, *continuous from the left*. The reader should take notice of the fact that the supposition that (I,C) forms an adjunction does not necessarily mean that I is an implication and that C is a conjunction.

Substituting $a = 1$ in (4.3) we get that

$$C(1,t) = t \iff I(1,t) = t\,, \text{ for } t \in [0,1]\,. \tag{4.4}$$

Let us give some examples of conjunctions and their adjoint implications.

Gödel-Brouwer :

$$\begin{aligned} C(a\,,t) &= a\wedge t \\ I(a\,,s) &= \begin{cases} s\,, & s<a \\ 1\,, & s\geq a \end{cases} \end{aligned}$$

Lukasiewicz :

$$\begin{aligned} C(a\,,t) &= 0\vee(a+t-1) \\ I(a\,,s) &= 1\wedge(s-a+1) \end{aligned}$$

Kleene-Dienes :

$$\begin{aligned} C(a\,,t) &= \begin{cases} 0\,, & t\leq 1-a \\ t\,, & t>1-a \end{cases} \\ I(a\,,s) &= (1-a)\vee s \end{aligned}$$

Reichenbach :

$$\begin{aligned} C(a\,,t) &= \begin{cases} 0\,, & t\leq 1-a \\ (t+a-1)/a\,, & t>1-a \end{cases} \\ I(a\,,s) &= 1-a+as \end{aligned}$$

Hamacher family : If $r>1$, define

$$\begin{aligned} C(a\,,t) &= \begin{cases} \frac{t}{r+(1-r)(a+t-at)} & a>0 \\ 0\,, & a=0 \end{cases} \\ I(a\,,s) &= \begin{cases} \frac{(1-r)as+rs}{r+(1-r)(1-s+as)} & a>0 \\ 1\,, & a=0 \end{cases} \end{aligned}$$

If $0\leq r\leq 1$, define

$$\begin{aligned} C(a\,,t) &= \begin{cases} \frac{(1-r)at+rt}{r+(1-r)(1-t+at)} & a>0 \\ 0\,, & a=0 \end{cases} \\ I(a\,,s) &= \begin{cases} \frac{s}{r+(1-r)(a+s-as)} & a>0 \\ 1\,, & a=0 \end{cases} \end{aligned}$$

From Proposition 3.4, the following result follows.

4.3. Proposition. *Let implication I_j and conjunction C_j form an adjunction for $j\in J$, then $\bigvee_{j\in J}C_j$ is a conjunction, $\bigwedge_{j\in J}I_j$ is an implication, and the pair $(\bigwedge_{j\in J}I_j\,,\bigvee_{j\in J}C_j)$ is an adjunction.*

4.4. Proposition. *Let implication I_j and conjunction C_j be adjoint, respectively, for $j=1\,,2$. Define*

$$I(a\,,t)=I_1(a\,,I_2(a\,,t)) \quad \text{and} \quad C(a\,,t)=C_2(a\,,C_1(a\,,t))\,,$$

for $a\,,t\in[0,1]$, then I is an implication and C is a conjunction, and moreover, the pair $(I(a\,,\cdot)\,,C(a\,,\cdot))$ forms an adjunction.

Proof. It is easy to demonstrate that I is an implication and that C is a conjunction. From the fact that $(I_j\,,C_j)$ is an adjunction for $j=1\,,2$, we find

$$\begin{aligned} C(a\,,t)\le s &\iff C_2(a\,,C_1(a\,,t))\le s \\ &\iff C_1(a\,,t)\le I_2(a\,,s) \\ &\iff t\le I_1(a\,,I_2(a\,,s))=I(a\,,s)\,, \end{aligned}$$

for any $a\,,s\,,t\in[0,1]$. This means that the pair $(I(a\,,\cdot)\,,C(a\,,\cdot))$ forms an adjunction for every $a\in[0,1]$. □

4.5. Proposition. *Let $\sigma:\ [0\,,1]\to[0\,,1]$ be a continuous, increasing mapping with $\sigma(0)=0$ and $\sigma(1)=1$, and with inverse σ^{-1}. Assume that the implication I and the conjunction C are adjoint, and define*

$$I_\sigma(a\,,t)=\sigma^{-1}(I(\sigma(a)\,,t)) \quad \textit{and} \quad C_\sigma(a\,,t)=C(\sigma(a)\,,\sigma(t))\,,$$

for all $a\,,t\in[0\,,1]$, then I_σ is an implication, whereas C_σ is a conjunction. Furthermore, the pair $(I_\sigma\,,C_\sigma)$ is adjoint.

Proof. The proof is straightforward. □

4.6. Proposition. *Let $(I\,,C)$ be adjoint, then*

$$C(C(a\,,b)\,,s)=C(b\,,C(a\,,s)) \iff I(a\,,I(b\,,s))=I(C(a\,,b)\,,s)\,.$$

for all $a\,,b\,,s\in[0\,,1]$.

Proof. $\Rightarrow$: Since $C(C(a\,,b)\,,t)=C(b\,,C(a\,,t))$, for any $a\,,b\,,t\in[0\,,1]$, we find that

$$C(C(a\,,b)\,,t)\le s \iff C(b\,,C(a\,,t))\le s\,,$$

for every $s\in[0\,,1]$. By the adjunction relation

$$C(C(a\,,b)\,,t)\le s \iff t\le I(C(a\,,b)\,,s)\,.$$

On the other hand

$$C(b\,,C(a\,,t))\le s \iff C(a\,,t)\le I(b\,,s) \iff t\le I(a\,,I(b\,,s))\,.$$

Combination of both relations yields $I(a\,,I(b\,,s))=I(C(a\,,b)\,,s)$.

$\Leftarrow$: Analogous. □

4.7. Corollary. *Let $(I\,,C)$ be adjoint, and assume that C is commutative, then C is associative if and only if*

$$I(a\,,I(b\,,s))=I(C(a\,,b)\,,s)\,,$$

for any $a\,,b\,,s\in[0\,,1]$.

Proof. $\Leftarrow$: By Proposition 4.6 and the condition that C is commutative, it is obvious that C is associative.

$\Rightarrow$: Since C is commutative and associative, we have $C(C(a\,,b)\,,s)=C(C(b\,,a)\,,s))=C(b\,,C(a\,,s))$, for all $a\,,b\,,s\in[0\,,1]$. By Proposition 4.6, the equality $I(a\,,I(b\,,s))=I(C(a\,,b)\,,s)$ holds for all $a\,,b\,,s\in[0\,,1]$. □

4.8. Proposition. *Let $(I\,,C)$ be adjoint. If C is continuous from the left with respect to the first argument, then,*

$$C(\bigvee_{j\in J} s_j\,,t)=\bigvee_{j\in J} C(s_j\,,t) \quad \textit{and} \quad I(\bigvee_{j\in J} s_j\,,t)=\bigwedge_{i\in J} I(s_j\,,t)\,,$$

for every family $\{s_j\}_{j\in J}\subseteq[0\,,1]$ and $t\in[0\,,1]$.

Proof. Let $\bigvee_{j\in J} s_j = s$, then $C(s_j\,,t) \le C(s\,,t)$ for any $j \in J$. So, $\bigvee_{j\in J} C(s_j\,,t) \le C(s\,,t) = C(\bigvee_{j\in J} s_j\,,t)$.

On the other hand, given $\epsilon > 0$, there exists $j_0 \in J$ such that $s - \epsilon = \bigvee_{j\in J} s_j - \epsilon < s_{j_0}$. Then $C(s-\epsilon\,,t) \le C(s_{j_0}\,,t) \le \bigvee_{i\in J} C(s_j\,,t)$. By the continuity of C, we have that $C(s\,,t) \le \bigvee_{i\in J} C(s_j\,,t)$. Therefore, $C(\bigvee_{j\in J} s_j\,,t) = \bigvee_{i\in J} C(s_j\,,t)$.

To prove the second relation, take $r \in [0\,,1]$. We have

$$\begin{aligned} r \le I(\bigvee_{j\in J} s_j\,,t) &\iff C(\bigvee_{j\in J} s_j\,,r) \le t \\ &\iff \bigvee_{j\in J} C(s_j\,,r) \le t \\ &\iff \forall\, j \in J\,,\; C(s_j\,,r) \le t \\ &\iff \forall\, j \in J\,,\; r \le I(s_j\,,t) \\ &\iff r \le \bigwedge_{j\in J} I(s_j\,,t)\,. \end{aligned}$$

Hence, $I(\bigvee_{j\in J} s_j\,,t) = \bigwedge_{j\in J} I(s_j\,,t)\,.$ □

If $(I\,,C)$ is an adjunction and C is commutative, then the condition that C is continuous from the left with respect to the first argument, is trivially satisfied.

4.2 Conjunctions and implications on arbitrary complete lattices

The mapping $\sigma : [0\,,1] \to [0\,,1]$ in Proposition 4.5 may be extended to a more general mapping $\sigma : [0\,,1] \to \mathcal{T}$, where $\mathcal{T}$ is a complete lattice. This leads us to the definition of conjunctions and implications on $\mathcal{T}$. This extension plays a role later in this paper when we consider grey-scale images.

Let us present here the definition of conjunctions and implications on arbitrary complete lattices.

4.9. Definition. Let $\mathcal{T}$ be a complete lattice with least and greatest element denoted by $\bot$ and $\top$, respectively. An operator $C : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is called a *conjunction on* $\mathcal{T}$ if C is increasing in both arguments, and

$$C(\bot\,,\bot) = C(\bot\,,\top) = C(\top\,,\bot) = \bot \quad\text{and}\quad C(\top\,,\top) = \top\,.$$

An operator $I : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is called an *implication on* $\mathcal{T}$ if I is decreasing in the first argument, increasing in the second and satisfies

$$I(\bot\,,\bot) = I(\bot\,,\top) = I(\top\,,\top) = \top \quad\text{and}\quad I(\top\,,\bot) = \bot\,.$$

An implication I and a conjunction C on $\mathcal{T}$ are said to be *adjoint* if

$$C(s\,,t) \le r \iff t \le I(s\,,r)\,, \qquad s\,,t\,,r \in \mathcal{T}\,.$$

4.10. Proposition. *Let $\mathcal{T}$ be a complete lattice, and let $\sigma :\; [0\,,1] \to \mathcal{T}$ be a continuous increasing mapping such that $\sigma(0) = \bot$ and $\sigma(1) = \top$. Let I and C be two functions from $[0\,,1] \times [0\,,1] \to [0\,,1]$, respectively. For any $s\,,t \in \mathcal{T}$, define $C_\sigma(s\,,t) = \sigma(C(\sigma^{-1}(s)\,,\sigma^{-1}(t)))$ and $I_\sigma(s\,,t) = \sigma(I(\sigma^{-1}(s)\,,\sigma^{-1}(t)))$, then the following assertions hold.*

(1) *I is an implication on $[0,1]$ if and only if I_σ is an implication on $\mathcal{T}$.*

(2) *C is a conjunction on $[0,1]$ if and only if C_σ is a conjunction on $\mathcal{T}$.*

(3) *$(I\,,C)$ is an adjunction on $[0\,,1]$ if and only if $(I_\sigma\,,C_\sigma)$ is an adjunction on $\mathcal{T}$.*

In the context of grey-scale morphology, Definition 4.9 is useful when we are dealing with other grey-value sets than $[0,1]$, in particular when there does not exist an isomorphism between $\mathcal{T}$ and $[0,1]$. This is the case, for example, if $\mathcal{T}$ is a discrete grey-value set, such as $\mathcal{T} = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ or $\mathcal{T} = \{a_0, a_1, a_2, \ldots, a_N\}$, where a_k are real numbers. The latter case will be discussed in more detail in § 5.5.

5. Grey-scale Morphology

In this section we define a general class of grey-scale morphological operators based on concepts from fuzzy logic.

5.1 Introduction

Let U be a given set called the universe, and let $\mathcal{T}$ be a set of grey-values. Then $\mathcal{T}^U$ is the set of functions mapping U into $\mathcal{T}$. If $\mathcal{T}$ is a complete lattice with a partial ordering $\leq$, then $\mathcal{T}^U$ is also a complete lattice with a partial ordering, also denoted by $\leq$, defined as

$$F_1 \leq F_2 \iff F_1(x) \leq F_2(x), \quad \forall\, x \in U, \tag{5.1}$$

for any $F_1, F_2 \in \mathcal{T}^U$.

In morphology, $\mathcal{T}^U$ is used as the basic model for grey-scale images. Within this model, morphological operators are considered as operators on $\mathcal{T}^U$ with some additional properties such as increasingness and translation invariance.

In this paper we represent grey-scale images as fuzzy sets (but keeping in mind that they have a different interpretation), and we try to extend classical binary morphological operators (see Section 2) to the fuzzy set framework by taking into account the underlying set-theoretic interpretation. To be specific, we want to define an erosion on $\mathcal{F}(U)$ by fuzzifying the set inclusion and extending definition (2.2). For other morphological operators we want to find similar extensions. As far as dilation and erosion are concerned, we also want to preserve the adjunction relation. In view of what we have said before, this may sound obvious, but this happens to be the issue where our approach deviates substantially from other existing approaches on fuzzy morphology [1–4,13,19,20].

5.2 Fuzzy sets

Fuzzy sets were introduced by Zadeh [23] in 1965. The definition of a fuzzy set is based on the principle that meaning in natural languages is a matter of degree. The "set of young people" does not have a sharp boundary (i.e., it is not the same as the set of people which are not older than 18) but a transition region where "the degree of being young" varies.

5.1. Definition. A *fuzzy set* on U is a function $F : U \to [0,1]$. The function F is called *membership function*. The family of all fuzzy sets on U is denoted by $\mathcal{F}(U)$. A fuzzy set F is called *crisp* if it takes only the values 0 and 1.

It is obvious that the family $\mathcal{P}(U)$ of subsets A of U can be embedded into $\mathcal{F}(U)$ by means of the characteristic function F_A given by $F_A(x) = 0$ if $x \notin A$ and $F_A(x) = 1$ if $x \in A$. Note that F_A is crisp. Since $\mathcal{F}(U) = [0,1]^U$ we conclude from the previous that $\mathcal{F}(U)$ provided with the pointwise partial ordering defined in (5.1) is a complete lattice.

Fuzzy set theory is largely concerned with the extension of notions from classical set theory such as union, intersection, inclusion, complementation, et cetera, to the context of fuzzy sets. Since this theory is treated comprehensively in various textbooks such as [10,14,15], we will restrict ourselves here to those concepts that are important for our purposes. This holds in particular for the so-called Extension Principle.

Extension Principle. Let V be a set and $p \geq 1$ an integer. Every mapping $\Omega : U^p \to V$ can be extended to a mapping $\overline{\Omega} : \mathcal{F}(U)^p \to \mathcal{F}(V)$ in the following way:

$$\overline{\Omega}(F_1, \ldots, F_p)(y) = \bigvee \{F_1(x_1) \wedge \cdots \wedge F_p(x_p) \mid x_j \in U \text{ and } \Omega(x_1, \ldots, x_p) = y\} .$$

Note that $\overline{\Omega}(F_1, \ldots, F_p)(y) = 0$ if no solution of $\Omega(x_1, \ldots, x_p) = y$ exists, since the supremum of the empty set equals the least element.

Assume, for example, that $*$ is one of the arithmetic operations addition, subtraction, or multiplication. We get that

$$(F_1 * F_2)(x) = \bigvee \{F_1(x_1) \wedge F_2(x_2) \mid x_1 * x_2 = x\} .$$

If both F_1 and F_2 are crisp, then $F_1 * F_2$ is crisp, too, and

$$x \in F_1 * F_2 \;\text{ iff }\; x_1 * x_2 = x \text{ for some } x_1 \in F_1 \text{ and } x_2 \in F_2 .$$

In this paper, we will go a step further and generalise the Extension Principle by replacing the inf-operation by the more general concept of *conjunction.* As we shall see later, this leads us to a new definition of the *dilation.*

5.3 Morphological operators on fuzzy sets

As we said before, we model grey-scale images with domain U as fuzzy sets on U. For the definition of morphological operators on $\mathcal{F}(U)$ we need to fuzzify some well-known concepts from set theory, starting with set inclusion. Let $A, B \in \mathcal{P}(U)$, then

$$\begin{aligned} A \subseteq B \quad &\Longleftrightarrow \quad \forall y \in U\,,\; y \in A \Rightarrow y \in B \\ &\Longleftrightarrow \quad \forall y \in U\,,\; A(y) \to B(y) \\ &\Longleftrightarrow \quad \forall y \in U\,,\; I(A(y)\,, B(y)) = 1\,, \end{aligned}$$

where I is the classical binary implication $\to$. We extend the above set inclusion relation from the family $\mathcal{P}(U)$ of all subsets of U to the family $\mathcal{F}(U)$ of all fuzzy sets. Given $F, G \in \mathcal{F}(U)$, we define $|G \subseteq F|$ as the degree (ranging from 0 to 1) of fuzzy set G being included in fuzzy set F:

$$|G \subseteq F| := \bigwedge_{y \in U} I(G(y)\,, F(y))\,, \tag{5.2}$$

where $I : [0,1] \times [0,1] \to [0,1]$ is a given fuzzy implication.

Analogously, let $A, B \in \mathcal{P}(U)$, then

$$\begin{aligned} A \cap B \neq \emptyset \quad &\Longleftrightarrow \quad \exists y \in U\,,\; y \in A \text{ and } y \in B \\ &\Longleftrightarrow \quad \exists y \in U\,,\; C(A(y)\,, B(y)) = 1\,, \end{aligned}$$

where C is the classical binary conjunction $C(a,b) = a \wedge b$ for $a, b \in \{0,1\}$.

Sometimes we shall denote $A \cap B \neq \emptyset$ by $A \Uparrow B$, and we say that A 'hits' B. We extend the latter set relation to fuzzy sets. Let $F, G \in \mathcal{F}(U)$, we denote by $|G \Uparrow F|$ the degree of fuzzy set G hitting fuzzy set F:

$$|G \Uparrow F| := \bigvee_{y \in U} C(G(y)\,, F(y))\,, \tag{5.3}$$

where C is a given fuzzy conjunction.

The fuzzy relations (5.2)–(5.3) can be used to extend the binary dilation and erosion in (2.1)-(2.2) to $\mathcal{F}(U)$. To do so we let G take the role of a 'fuzzy structuring element' or 'structuring function'.

The erosion of a grey-scale image F by the structuring function G, at point $x \in U$ equals $|G_x \subseteq F|$, where G_x is the translation of G along x, i.e., $G_x(y) = G(y-x)\,,\ y \in U$. Thus

$$\mathcal{E}_G(F)(x) = |G_x \subseteq F|\,. \tag{5.4}$$

Similarly, we find for the dilation:

$$\Delta_G(F)(x) = |\check{G}_x \Uparrow F|\,. \tag{5.5}$$

Here $\check{G}(y) = G(-y)$ and $\check{G}_x(y) = G(x-y)\,,\ y \in U$.

Assuming that the implication I and conjunction C are adjoint, we may conclude that both the erosion $\mathcal{E}_G$ and the dilation Δ_G are determined by C (and, of course, by G). Introducing the notation

$$\Delta_G(F) = F \oplus_C G \text{ and } \mathcal{E}_G(F) = F \ominus_C G\,,$$

we get

$$\Delta_G(F)(x) = (F \oplus_C G)(x) = \bigvee_{y \in U} C(G(x-y)\,, F(y))\,, \tag{5.6}$$

$$\mathcal{E}_G(F)(x) = (F \ominus_C G)(x) = \bigwedge_{y \in U} I(G(y-x)\,, F(y))\,. \tag{5.7}$$

We can now state the main conclusion of this subsection.

5.2. Proposition. *Let I be an implication and C be a conjunction on $[0\,,1]$. The pair $(I\,,C)$ is an adjunction on $[0\,,1]$ if and only if $(\mathcal{E}_G\,,\Delta_G)$ given by (5.6)–(5.7) is an adjunction on $\mathcal{F}(U)$ for every $G \in \mathcal{F}(U)$.*

Proof. $\Rightarrow$: For any $a\,,s\,,t \in [0\,,1]$, we have $C(a\,,t) \le s \iff t \le I(a\,,s)$. So for any $F\,,G\,,H \in \mathcal{F}(U)$,

$$\begin{aligned}
\Delta_G(F) \le H \quad &\iff \quad \forall\, x \in U\,,\ \Delta_G(F)(x) \le H(x) \\
&\iff \quad \forall\, x \in U\,, \bigvee_{y \in U} C(G(x-y)\,, F(y)) \le H(x) \\
&\iff \quad \forall\, x \in U\,,\ \forall\, y \in U\,,\ C(G(x-y)\,, F(y)) \le H(x) \\
&\iff \quad \forall\, x \in U\,,\ \forall\, y \in U\,,\ F(y) \le I(G(x-y)\,, H(x)) \\
&\iff \quad \forall\, y \in U\,,\ F(y) \le \bigwedge_{x \in U} I(G(x-y)\,, H(x)) = \mathcal{E}_G(H)(y) \\
&\iff \quad F \le \mathcal{E}_G(H)\,.
\end{aligned}$$

$\Leftarrow$: Given $a\,,s\,,t \in [0\,,1]$, define the constant functions $G \equiv a\,, F \equiv s\,,$ and $H \equiv t$. Then $\Delta_G(F)(x) = \bigvee_{y \in U} C(G(x-y)\,, F(y)) = C(a\,,s)$ and $\mathcal{E}_G(H)(x) = \bigwedge_{y \in U} I(G(y-x)\,, H(y)) = I(a\,,t)$, for every $x \in U$. Therefore, $C(a\,,s) \le t \iff \Delta_G(F) \le H \iff F \le \mathcal{E}_G(H) \iff s \le I(a\,,t)$. □

It is obvious that the operators $\mathcal{E}_G$ and Δ_G are translation invariant. Furthermore, if the definitions of $\oplus_C$ and $\ominus_C$ are restricted to binary (structuring) functions, they correspond with the classical binary definitions of Minkowski addition and subtraction given in Section 2.

It follows immediately that

$$\bigvee_{j \in J} F_j \oplus_C G = \bigvee_{j \in J} (F_j \oplus_C G) \quad \text{and} \quad \bigwedge_{j \in J} F_j \ominus_C G = \bigwedge_{j \in J} (F_j \ominus_C G)\,,$$

for an arbitrary family $\{F_j \mid j \in J\} \subseteq \mathcal{F}(U)$. Furthermore, Proposition 4.8 gives us that

$$F \oplus_C (\bigvee_{j\in J} G_j) = \bigvee_{j\in J} (F \oplus_C G_j) \quad \text{and} \quad F \ominus_C (\bigvee_{j\in J} G_j) = \bigwedge_{j\in J} (F \ominus_C G_j)\,,$$

for an arbitrary family $\{G_j \mid j \in J\} \subseteq \mathcal{F}(U)$ if we assume in addition that the conjunction C is continuous from the left with respect to the first argument.

The following result is a straightforward consequence of Proposition 4.6 and Corollary 4.7.

5.3. Proposition. *Let $(I\,,C)$ be an adjunction, and assume moreover that the conjunction C is commutative and associative. Then*

$$(F \oplus_C G_1) \oplus_C G_2 = F \oplus_C (G_1 \oplus_C G_2)$$

and

$$(F \ominus_C G_1) \ominus_C G_2 = F \ominus_C (G_1 \oplus_C G_2)\,,$$

for all $F\,,\ G_1\,,\ G_2 \in \mathcal{F}(U)$.

This proposition shows that structuring elements are decomposable. The previous derivation of grey-scale dilation and erosion mimics the corresponding definitions in the binary case. Our next result shows that, indeed, the case where both the structuring function and the input function are crisp, coincides with the original binary case, regardless of the particular choice of the underlying conjunction and implication.

Recall that for a set $A \subseteq U$, we denote by F_A its characteristic function.

5.4. Proposition. *Let $(I\,,C)$ be an adjunction, and let $A, X \subseteq U$, then*

$$F_X \oplus_C F_A = F_{X \oplus A} \quad \textit{and} \quad F_X \ominus_C F_A = F_{X \ominus A}\,.$$

Proof. Using relation (4.4), we find that

$$\begin{aligned} (F_X \oplus_C F_A)(x) = 1 \quad &\Longleftrightarrow \quad \exists y_0 \in U\,,\ C(F_A(x - y_0)\,, F_X(y_0)) = 1 \\ &\Longleftrightarrow \quad F_A(x - y_0) = 1 \quad \text{and} \quad F_X(y_0) = 1 \\ &\Longleftrightarrow \quad x - y_0 \in A \quad \text{and} \quad y_0 \in X \\ &\Longleftrightarrow \quad x \in X \oplus A\,, \end{aligned}$$

for every $x \in U$.

The second equality can be proved analogously. □

From Proposition 3.5 we know that the composition of the dilation Δ_G and the erosion $\mathcal{E}_G$ yields an opening and a closing on $\mathcal{F}(U)$.

5.4 Additive case

The representation of a grey-scale image as a fuzzy set requires that the set of grey-values is $[0\,,1]$. In this and the next subsection we will show how the previous results can be extended to other grey-value sets $\mathcal{T}$. In this subsection we assume that $\mathcal{T} = \overline{\mathbb{R}} = [-\infty\,,+\infty]$, and we show that classical grey-scale morphological operators using additive structuring functions can be fit within the framework based on concepts from fuzzy logic.

Assume that $\theta : \mathbb{R} \to [0\,,1]$ is a continuous, strictly decreasing function with

$$\lim_{t\to-\infty} \theta(t) = 1 \quad \text{and} \quad \lim_{t\to+\infty} \theta(t) = 0\,.$$

For example, $\theta(t) = \frac{1}{2} + \frac{1}{\pi} \arctan t$. Defining $\theta(-\infty) = 1$ and $\theta(+\infty) = 0$, we get that θ is an automorphism between the complete lattices $\overline{\mathbb{R}}$ and $[0,1]$, and we denote its inverse by θ^{-1}. Now, we define two functions $C, I : [0,1] \times [0,1] \to [0,1]$ as follows:

$$C(s,t) = \theta(\theta^{-1}(s) + \theta^{-1}(t)) \tag{5.8}$$

$$I(s,t) = \theta(\theta^{-1}(t) - \theta^{-1}(s)) \tag{5.9}$$

Note that C is unambiguously characterized by (5.8) except at points $(0,1)$ and $(1,0)$. We define

$$C(1,0) = C(0,1) = 0 \tag{5.10}$$

Similarly, I has to be defined explicitly at the points $(0,0)$ and $(1,1)$ as:

$$I(0,0) = I(1,1) = 1 \tag{5.11}$$

In fact, definitions (5.10)-(5.11) are based on the following conventions:

$$(-\infty) + (+\infty) = (+\infty) + (-\infty) = +\infty,$$

$$(-\infty) - (-\infty) = (+\infty) - (+\infty) = -\infty.$$

Observe that the conjunction C defined by (5.8) and (5.10) is commutative and associative.

5.5. Proposition. *Let C, I be defined as in (5.8)-(5.11), then C is a conjunction and I is an implication, and moreover, (I, C) is an adjunction.*

Proof. It is easy to check that C is a conjunction and that I is an implication by (5.8)-(5.11).

For any $s, t, r \in [0,1]$

$$\begin{aligned} C(s,t) \le r &\iff \theta(\theta^{-1}(s) + \theta^{-1}(t)) \le r \\ &\iff \theta^{-1}(s) + \theta^{-1}(t) \ge \theta^{-1}(r) \\ &\iff \theta^{-1}(t) \ge \theta^{-1}(r) - \theta^{-1}(s) \\ &\iff t \le \theta(\theta^{-1}(r) - \theta^{-1}(s)) = I(s,r). \end{aligned}$$

Thus, (I, C) is an adjunction. □

With every function $F \in \overline{\mathbb{R}}^U$, we can associate a fuzzy set $\theta(F) \in \mathcal{F}(U)$ defined by

$$\theta(F)(x) = \theta(F(x)), \qquad x \in U.$$

Thus, if $G \in \overline{\mathbb{R}}^U$ is a structuring function, we can define operators $\overline{\Delta}_G$ and $\overline{\mathcal{E}}_G$ on $\overline{\mathbb{R}}^U$ in the following way

$$\overline{\Delta}_G(F) = \theta^{-1}(\theta(F) \oplus_C \theta(G)),$$
$$\overline{\mathcal{E}}_G(F) = \theta^{-1}(\theta(F) \ominus_C \theta(G)).$$

A straightforward computation shows that $\overline{\Delta}_G(F)$ and $\overline{\mathcal{E}}_G(F)$ are the classical grey-scale dilation and erosion given by

$$\overline{\Delta}_G(F)(x) = \bigvee_{y \in U} (F(x-y) + G(y)),$$

and

$$\overline{\mathcal{E}}_G(F)(x) = \bigwedge_{y \in U} (F(x+y) - G(y)).$$

Therefore the conclusion of the results obtained in this subsection is that classical additive grey-scale morphology can be regarded merely as a special case of grey-scale morphology based on fuzzy logic.

5.5 Finite grey-value sets

In practical applications, the grey-value set is usually a finite point set, for instance, $\mathcal{T} = \{a_0, a_1, a_2, \ldots, a_N\}$, where N is a positive integer. We assume that $a_0 \le a_1 \le a_2 \le \cdots \le a_N$. In this case, it is no longer appropriate to model grey-scale functions as fuzzy sets, and we have to take recourse to the extension to arbitrary complete lattices as explained in § 4.2.

Define C and I on $\mathcal{T}$ as follows:

$$C(a_i, a_j) = a_{\max(0, i+j-N)} \quad \text{and} \quad I(a_i, a_j) = a_{\min(N, j-i+N)},$$

for any $i, j \in \{0, 1, 2, \ldots, N\}$. It is not difficult to verify that C is a conjunction on $\mathcal{T}$ and that I is an implication on $\mathcal{T}$. Moreover, it can easily be shown that (I, C) defines an adjunction on $\mathcal{T}$, that is,

$$C(a_i, a_j) \le a_k \iff a_j \le I(a_i, a_k),$$

for $1 \le i, j, k \le N$. Assume, for simplicity, that $a_i = i$. Then we get that

$$C(i, j) = \max(0, i+j-N) \quad \text{and} \quad I(i, j) = \min(N, j-i+N). \tag{5.12}$$

Introducing the notation

$$\begin{aligned} j \dot{+} i &= C(i,j) = \max(0, i+j-N) \\ j \dot{-} i &= I(i,j) = \min(N, j-i+N), \end{aligned}$$

we arrive at the following expressions for the dilation and erosion for functions in $\mathcal{T}^U$:

$$\Delta_G(F)(x) = \bigvee_{y \in U} (F(x-y) \dot{+} G(y)),$$

and

$$\mathcal{E}_G(F)(x) = \bigwedge_{y \in U} (F(x+y) \dot{-} G(y)).$$

Here $G : U \to \mathcal{T}$ is the structuring function.

In [6] (see also [8, Section 11.9]) one of us has derived similar expressions for the so-called *truncated* grey-scale dilations and erosions. However, there we defined the truncated addition $\dot{+}$ and subtraction $\dot{-}$ in a different way:

$$j \dot{+} i = \begin{cases} 0, & \text{if } j = 0, \\ j+i, & \text{if } 0 < j+i \le N, \\ N, & \text{if } j+i > N, \end{cases} \quad \text{and} \quad j \dot{-} i = \begin{cases} 0, & \text{if } j-i < 0, \\ j-i, & \text{if } 0 \le j-i < N, \\ N, & \text{if } j = N. \end{cases}$$

Defining $C(i, j) = j \dot{+} i$ and $I(i, j) = j \dot{-} i$ using these expressions, we find that (I, C) is an adjunction in the sense that

$$C(i,j) \le k \iff j \le I(i,k),$$

for $0 \le i, j, k \le N$. However, in this case C is *not* a conjunction (for example, $C(N, 0) = N$) and I is *not* an implication (for example, $I(0, 0) = 0$).

The major conclusion of this subsection is that the approach based on fuzzy logic suggests a new class of grey-scale dilations and erosions in the case of a finite set of grey-values.

6. Negations

In this section we will briefly discuss the role of negations in mathematical morphology. The simplest example of a negation operator is the complementation $X \mapsto X^c$ on $\mathcal{P}(U)$. As we have observed in Section 2, a number of authors use negations to construct erosions from dilations and vice versa. But the theory developed in this and other papers illustrates clearly that adjunctions provide the right framework for 'pairing' dilations and erosions. Despite this fact, the role of negations in mathematical morphology is quite important: negations have the effect of transforming the image foreground into the image background and vice versa.

It is a well-known fact in mathematical morphology that erosions and dilations (as well as closings and openings) are dual operators in the sense of negation (or complementation in the binary case); see for example (2.4). Let us first briefly discuss a formalisation of this property in the general complete lattice case.

An operator ν on a complete lattice $\mathcal{L}$ is called a *negation* if ν is bijective, decreasing (we say that ν is a *dual automorphism*), and $\nu^2 = id$, that is ν is an *involution*. When no confusion is possible, we will henceforth denote a negation with the symbol $*$, and instead of $\nu(X)$, we write X^*.

Given an operator Ψ on $\mathcal{L}$ and a negation $*$, we define the negation Ψ^* of operator Ψ, also called the *negative operator*, as

$$\Psi^*(X) = (\Psi(X^*))^*, \quad X \in \mathcal{L}. \tag{6.1}$$

It is easy to show that $(\mathcal{E}, \Delta)$ is an adjunction on $\mathcal{L}$ iff $(\Delta^*, \mathcal{E}^*)$ is an adjunction on $\mathcal{L}$. Let us, henceforth, restrict ourselves to morphological operators constructed from fuzzy logic concepts.

6.1. Definition. A mapping $\nu : [0,1] \to [0,1]$ is a *negation* if it is decreasing and satisfies $\nu^2(t) = t$ for $t \in [0,1]$.

6.2. Proposition. *Every negation ν on $[0,1]$ is a dual automorphism. This means in particular that ν is bijective, that $\nu(0) = 1$ and $\nu(1) = 0$, and that*

$$\nu(\bigvee_{j \in J} t_i) = \bigwedge_{j \in J} \nu(t_j) \quad \text{and} \quad \nu(\bigwedge_{j \in J} t_j) = \bigvee_{j \in J} \nu(t_j)$$

for every family $\{t_j\}_{j \in J} \subseteq [0,1]$.

As before, we write $*$ rather than ν when no confusion is possible. Given a fuzzy set $F \in \mathcal{F}(U)$, we define its *negation* (or *complement*) F^* by

$$F^*(x) = \nu(F(x)), \quad x \in U. \tag{6.2}$$

The negation Ψ^* of an operator Ψ on $\mathcal{F}(U)$ is defined as in (6.1). It is evident that Ψ is increasing iff Ψ^* is increasing, and that

$$(\Psi^*)^* = \Psi.$$

Furthermore, Ψ^* is a dilation (resp. closing) if and only if Ψ is an erosion (resp. opening).

6.3. Definition. Given a function $H : [0,1] \times [0,1] \to [0,1]$, we define

$$H_*(s,t) = H(s,t^*)^*, \quad s,t \in [0,1].$$

It is evident that $(H_*)_* = H$ for every given function H. We can prove the following result.

6.4. Proposition. *(a) C is a conjunction if and only if C_* is an implication.*

(*b*) *I is an implication if and only if I_* is a conjunction.*

(*c*) *Let C be a conjunction and I an implication, then $(I\,,C)$ is an adjunction iff $(C_*\,,I_*)$ is an adjunction.*

Proof. (*a*): $\Rightarrow$: Since ν is decreasing and C is increasing in both arguments, we derive that C_* is decreasing in the first argument and increasing in the second. Furthermore, we get

$$C_*(0\,,0) = \nu(C(0\,,\nu(0))) = \nu(C(0\,,1)) = \nu(0) = 1\,.$$

Similarly, we get

$$C_*(0\,,1) = C_*(1\,,1) = 1 \quad \text{and} \quad C_*(1\,,0) = 0\,.$$

Thus C_* is an implication.

$\Leftarrow$: Analogous.

(*b*): The proof is similar to that for (*a*).

(*c*): $\Rightarrow$: From the fact that $(I(a,\cdot), C(a,\cdot))$ forms an adjunction, we get

$$\begin{aligned} I_*(a\,,t) \le s \quad &\Longleftrightarrow \quad I(a\,,t^*)^* \le s \\ &\Longleftrightarrow \quad s^* \le I(a\,,t^*) \\ &\Longleftrightarrow \quad C(a\,,s^*) \le t^* \\ &\Longleftrightarrow \quad t \le C(a\,,s^*)^* \\ &\Longleftrightarrow \quad t \le C_*(a\,,s)\,, \end{aligned}$$

for $a, s, t \in [0,1]$. This means that $(C_*\,,I_*)$ is an adjunction.

$\Leftarrow$: Analogous proof. □

A straightforward computation show that for $\nu(t) = 1 - t$, the negation C_* of the *Gödel-Brouwer* conjunction C is the *Kleene-Dienes* implication, and the negations of the *Lukasiewicz* conjunction and the *Hamacher* conjunction are their adjoint implications, respectively. For the discrete conjunction and implication in (5.12) we also get $C_*(i,j) = I(i,j)$ and $I_*(i,j) = C(i,j)$ for the negation $i^* = N - i$.

The next results show the effect of taking the negation of the dilation $F \mapsto F \oplus_C G$ and the erosion $F \mapsto F \ominus_C G$.

6.5. Proposition. *Let C be a conjunction and I an implication such that $(I\,,C)$ is an adjunction, and let ν be a negation. For every structuring element $G \in \mathcal{F}(U)$ we have*

$$\Delta_G^*(F) = (F^* \oplus_C G)^* = F \ominus_{I_*} \check{G}\,,$$

$$\mathcal{E}_G^*(F) = (F^* \ominus_C G)^* = F \oplus_{I_*} \check{G}\,,$$

for $F \in \mathcal{F}(U)$; here $\check{G}(x) = G(-x)$ for $x \in U$.

Proof. For any $F \in \mathcal{F}(U)$ and any $x \in U$,

$$\begin{aligned} (F^* \oplus_C G)^*(x) \quad &= \quad \nu(\bigvee_{y\in U} C(G(x-y)\,,\nu(F(y)))) \\ &= \quad \bigwedge_{y\in U} \nu(C(G(x-y)\,,\nu(F(y)))) \\ &= \quad \bigwedge_{y\in U} C_*(G(x-y)\,,F(y)) \\ &= \quad (F \ominus_{I_*} \check{G})(x)\,. \end{aligned}$$

The second statement can be proved similarly. □

7. Experimental results

In this section we present some experiments showing the differences between basic morphological operators using different conjunctions. We compare the outcomes of the operators based on fuzzy logic with the corresponding flat operators using a crisp structuring element. Our input image is depicted in Fig. 1. The structuring function G used for the 'fuzzy' operators

Figure 1: Original 256×256 grey-scale image.

is represented by the matrix

$$G = \frac{1}{20} * \begin{pmatrix} 0 & 2 & 4 & 5 & 4 & 2 & 0 \\ 2 & 6 & 9 & 10 & 9 & 6 & 2 \\ 4 & 9 & 13 & 15 & 13 & 9 & 4 \\ 5 & 10 & 15 & 20 & 15 & 10 & 5 \\ 4 & 9 & 13 & 15 & 13 & 9 & 4 \\ 2 & 6 & 9 & 10 & 9 & 6 & 2 \\ 0 & 2 & 4 & 5 & 4 & 2 & 0 \end{pmatrix}$$

Observe that this matrix approximates a cone in the sense that an entry is approximately given by $1 - \frac{1}{4}(i^2 + j^2)^{\frac{1}{2}}$, where (i, j) are the coordinates of the entry relative to the centre of the matrix. This approximation is based on the 5-7-11 chamfer distance [8]. The binary structuring element is obtained by thresholding the (fuzzy) structuring function at level 0.5, and is given by

$$A = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

In Fig. 2 we show the dilation and the erosion and three gradient operators derived from these two operators. The first column represents the operators with the flat structuring element A, the second column represents the operators using the structuring function G in combination with the Gödel-Brouwer conjunction and implication, and the third column represents the operators using the structuring function G in combination with the Kleene-Dienes conjunction and implication.

The rows represent, respectively, dilation, erosion, dilation minus erosion, dilation minus original, and original minus erosion.

The fuzzy nature of the operators in the second and third column is clearly reflected by the images, especially close to the edges. Furthermore, there are various differences (again near the edges) between these two columns, indicating that the particular choice of the conjunction does have a serious impact on the results. Similar observations can be made for the images shown in Fig. 3. Here the columns have the same interpretation as in Fig. 2, but the rows represent respectively the closing, the opening, the closing minus the opening, the closing minus the original, and the original minus the opening.

8. Concluding remarks

We have shown that basic concepts from fuzzy logic can be used to build a large class of grey-scale morphological operators. Such operators use two ingredients: the structuring element, which does also play a role in classical grey-scale morphology, and the conjunction and implication which are typical for the fuzzy logic framework. The choice of the conjunction and implication, which has to be made in such a way that the two form an adjunction, gives us additional freedom in the design of our operators. We have shown that the classical grey-scale operators using additive structuring functions are nothing but a special member of our enlarged class based on fuzzy logic.

The results of this paper have to be considered as a very first step towards a general theory of grey-scale morphology using concepts from fuzzy logic. In future work we want to address various other important issues such as

- generalisation of Matheron's representation theorem;
- systematic study of grey-scale granulometries and the role of convex fuzzy sets therein;
- geodesic and connected operators;
- design of morphological filters.

The validity and usefulness of the fuzzy logic based approach to grey-scale morphology depends largely on the outcomes and conclusions of such future investigations. Nevertheless, we hope that we have succeeded in giving the reader an impression of the potential of this approach.

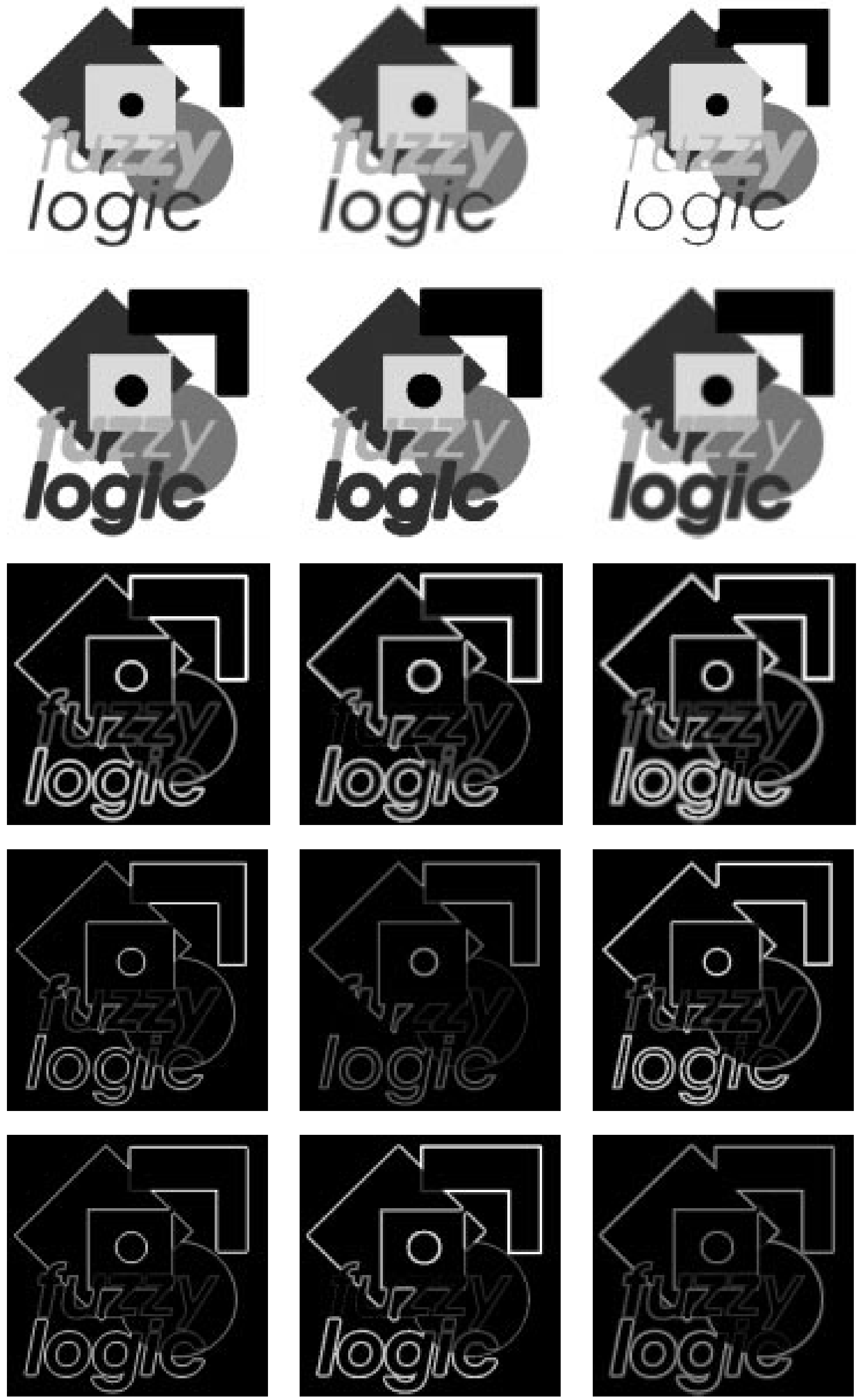

Figure 2: Top to bottom: dilation, erosion, dilation minus erosion, dilation minus original, and original minus erosion. Left to right: flat operator, fuzzy operator using Gödel-Brouwer conjunction and implication, and fuzzy operator using Kleene-Dienes conjunction and implication.

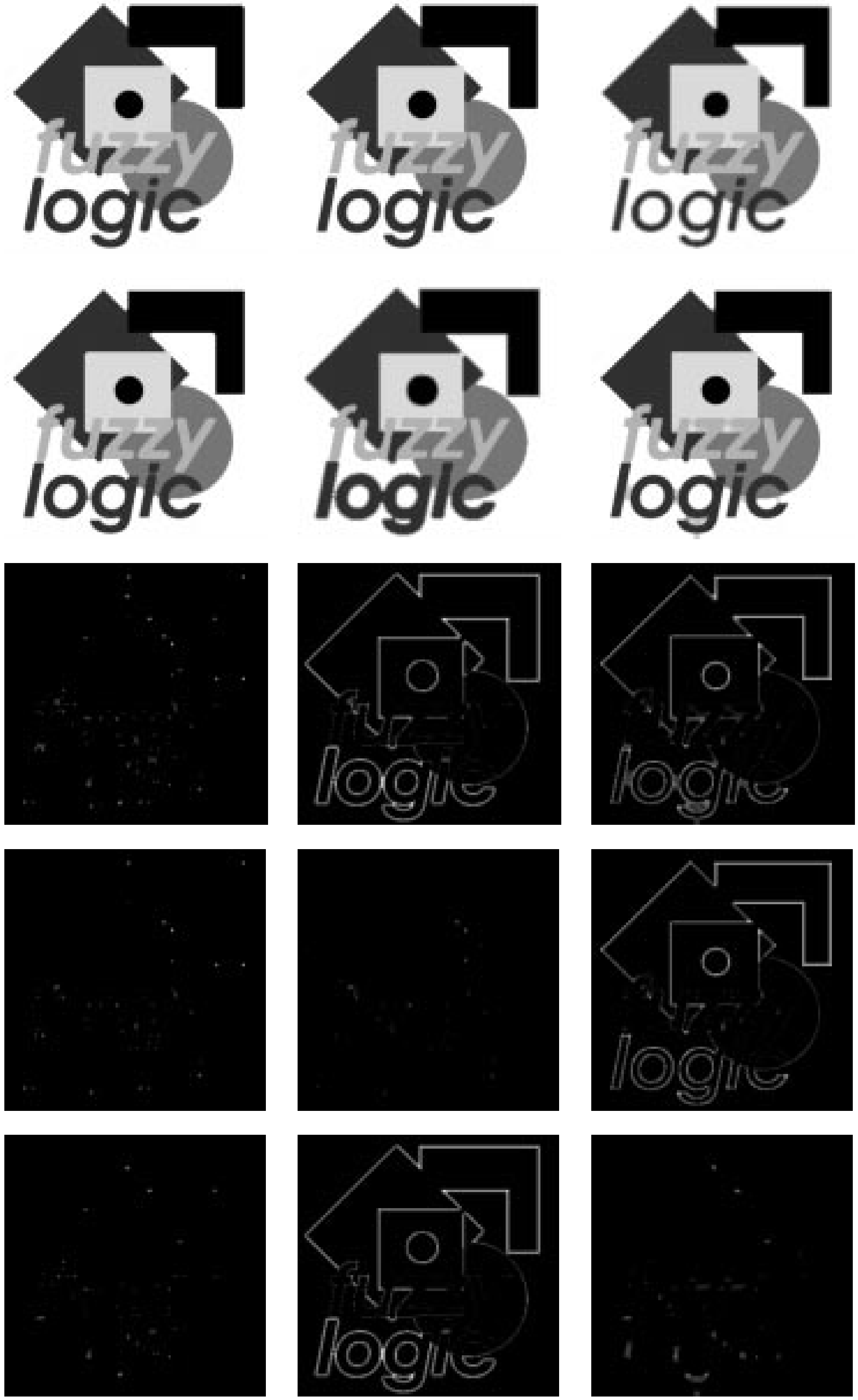

Figure 3: Top to bottom: closing, opening, closing minus opening, closing minus original, and original minus opening. Left to right: flat operator, fuzzy operator using Gödel-Brouwer conjunction and implication, and fuzzy operator using Kleene-Dienes conjunction and implication.

References

1. Bloch, I., and Maitre, H. Fuzzy mathematical morphology. *Annals of Mathematics and Artificial Intelligence*, 10 (1994), 55–84.
2. Bloch, I., and Maitre, H. Fuzzy mathematical morphologies: a comparative study. *Pattern Recognition 28*, 9 (1995), 1341–1387.
3. De Baets, B. Fuzzy morphology: A logical approach. In *Uncertainty Analysis in Engineering and Science: Fuzzy Logic, Statistics, and Neural Network Approach*, B. M. Ayyub and M. M. Gupta, Eds. Kluwer Academic Publishers, Norwell, 1997, pp. 53–67.
4. De Baets, B., and Kerre, E. The fundamentals of fuzzy mathematical morphology part 1: basic concepts. *International Journal of General Systems 23* (1995), 155–171.
5. Goetcherian, V. From binary to grey tone image processing using fuzzy logic concepts. *Pattern Recognition 12* (1980), 7–15.
6. Heijmans, H. J. A. M. Theoretical aspects of gray-level morphology. *IEEE Transactions on Pattern Analysis and Machine Intelligence 13* (1991), 568–582.
7. Heijmans, H. J. A. M. A note on the umbra transform in gray-scale morphology. *Pattern Recognition Letters 14* (1993), 877–881.
8. Heijmans, H. J. A. M. *Morphological Image Operators.* Academic Press, Boston, 1994.
9. Kerre, E., and Nachtegael, M. *Fuzzy Techniques in Image Processing : Techniques and Applications.* Studies in Fuzziness and Soft Computing, Vol. 52. Physica Verlag, 2000.
10. Klir, G. J., and Yuan, B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice Hall, Upper Saddle River, 1995.
11. Maragos, P., and Ziff, R. D. Threshold superposition in morphological image analysis systems. *IEEE Transactions on Pattern Analysis and Machine Intelligence 12* (1990), 498–504.
12. Matheron, G. *Random Sets and Integral Geometry.* John Wiley & Sons, New York, 1975.
13. Nachtegael, M., and Kerre, E. Connections between binary, grey-scale and fuzzy mathematical morphology. *Fuzzy Sets and Systems* (submitted).
14. Nguyen, H. T., and Walker, E. A. *A First Course in Fuzzy Logic (2nd ed).* Chapman & Hall/CRC, Boca Raton, Florida, 1994.
15. R. Kruse, J. G., and Klawonn, F. *Foundations of Fuzzy Systems.* John Wiley & Sons, England, 1994.
16. Ronse, C. Why mathematical morphology needs complete lattices. *Signal Processing 21* (1990), 129–154.
17. Serra, J. *Image Analysis and Mathematical Morphology.* Academic Press, London, 1982.
18. Serra, J., Ed. *Image Analysis and Mathematical Morphology. II: Theoretical Advances.* Academic Press, London, 1988.

19. Sinha, D., and Dougherty, E. R. Fuzzy mathematical morphology. *Journal of Visual Communication and Image Representation 3*, 3 (1992), 286–302.

20. Sinha, D., and Dougherty, E. R. Fuzzification of set inclusion: theory and applications. *Fuzzy Sets and Systems*, 55 (1993), 15–42.

21. Soille, P. *Morphological Image Analysis.* Springer-Verlag, Berlin, 1999.

22. Sternberg, S. R. Grayscale morphology. *Computer Vision, Graphics and Image Processing 35* (1986), 333–355.

23. Zadeh, L. A. Fuzzy sets. *Information and Control 8* (1965), 338–353.

24. Zimmerman, H. J. *Fuzzy set theory and its applications.* Academic Press, Boston, 1991.