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ABSTRACT
This paper proposes a general algebraic construction technique for image scale-spaces. The basic idea is to first downscale the image by some factor using an invertible scaling, then apply an image operator (linear or morphological) at a unit scale, and finally resize the image to its original scale. It is then required that the resulting one-parameter family of image operators satisfies the semigroup property. Such an approach encompasses linear as well as nonlinear (morphological) operators. Furthermore, there exists some freedom as to which semigroup operation on the scale- (or time-) axis is being chosen. Particular attention is given to additive and supremal semigroups. A large part of the paper is devoted to morphological scale-spaces, in particular to scale-spaces associated with an erosion or an opening. In these cases, classical tools from convex analysis, such as the (Young-Fenchel) conjugate, play an important role.

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Keywords and Phrases: atlas principle, Cauchy kernel, convex function, erosion, Gaussian kernel, homogeneous function, infimal convolution morphological operator, (naturally) linearly ordered semigroup, parabolic structuring function, scale-space, scaling, semigroup, sinc-kernel, sublinear function, Young-Fenchel conjugate.

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1. Introduction

Scale-space is an accepted and often used formalism in image processing and computer vision. Today, this formalism plays an important role because of the necessity to specify explicitly at what scale visual observations (i.e. measurements in the context of *computer* vision) are to be made. What is an edge at small scale might be a corner at larger scale, or vice versa. Before the popularization of the scale-space concept, the choice for an observation scale was often hidden somewhere in the definition of the operators.

The notion of multi-scale operators has a very long history in image processing. The standard 'first reference' is the paper by Witkin [32] in which the author shows that the Gaussian convolution is the unique operator that satisfies general principles of spatial symmetry and scale invariance. Koenderink [18] was the first to show that these symmetry and invariance principles are compatible with a causality principle requiring that new details cannot be formed when moving from fine to coarser scales. Weickert *et al.* [30, 31] only recently 'discovered' that the concept of linear Gaussian scale-space dates back to the sixties and was first invented by Iijima, who published his results not only in Japanese but also in English. For some reason, at that time the idea did not catch on. Lindeberg [19] was the first to consider the discrete equivalent of the Gaussian linear scale-space. Instead of specifying a scale-space operator in the continuous domain and then discretizing the continuous operator, Lindeberg 'discretized' the scale-space requirements. Fortunately, only for very small scales the two approaches differ significantly. In this paper we take the classical continuous (and, admittedly, mathematical easier) route. Weickert's overview [30] of the many possible ways to derive the Gaussian scale-space from (somewhat different) basic principles shows that the (linear) scale-space concept in a sense is overdetermined. Many sets of reasonable requirements lead to the same answer. In the work of Pauwels *et al.* [23] a nice account of linear multi-scale operators can be found. Their work will be our starting point when we discuss linear scale-spaces.

In mathematical morphology the notion of scale (or size) dependent observations was pioneered by Matheron [22] in his study of *granulometries*, designed to capture the notion of size and size distributions of spatial observations (focused on subsets of the plane at that time). The class of non-linear morphological scale-dependent operators that follow from his study (the openings and closings) are later often suggested in the literature as the morphological scale-space operators; see Chen and Yan [4]. In this paper we will show that the use of openings and closings to construct multi-scale operators leads to the special class of the so-called *supremal* scale-spaces.

Brockett and Maragos [3] were the first to show that (flat) morphological operators like dilations and erosions can be described in terms of (nonlinear) PDE's. Jackway [14] and van den Boomgaard [27] independently showed that a morphological analogue of the Gaussian linear scale-space does exist: the parabolic erosions and dilations. The same set of basic principles is shown to lead to both the linear Gaussian scale-space in case a linear operator is sought for, whereas the the parabolic erosions (dilations) are 'found' in case a morphological operator is sought for. Van den Boomgaard and Smeulders [28] also showed that the morphological parabolic scale-space can be viewed upon as the solution of a (nonlinear) partial differential equation, just like the linear scale-space is the solution of the diffusion equation.

In this paper we look at the definition of scale-spaces from an algebraic point of view. Our guiding principle nevertheless is a physical one. In our view a scale-space is the mathematical construct that describes the scale-dependent observation (probing) of images. Therefore we only look at the scale-space operators that are able of making observations at a finite scale *without* the necessity to make all observations at smaller scale as well. We thus take a quite different approach to scale-space which is quite different from the approach that has so elegantly been put forward by Alvarez *et al.* [1], who take the evolution of the zero scale image modeled by a partial differential equation as their starting point.

Let f be the image at scale zero[†] and let $T(s)$ be the operator such that $T(s)f$ is the observation at scale s. The family of operators $\{T(s)\}_{s>0}$ is collectively known as a scale-space. We will also call the collection of images $\{T(s)f\}_{s>0}$ a scale-space, although we will always refer to scale-space properties that are independent of the zero-scale image.

In our definition of scale-space we will also follow the idea of Koenderink [18] (later his ideas were given a very firm mathematical and physical basis in the work of Florack [7, 8]; see also [26]) that it should be possible to build a scale-space incrementally. Koenderink called this the *atlas principle*. In words, it says that if we observe an image at scale s and take that observation (which is an image itself) as the input for another observation at scale t, an observation at scale $r \geq \max\{s,t\}$ results. The atlas principle thus can be mathematically formulated with a *semigroup* property of the one-parameter family of scale-space operators $T(t)$.

The atlas principle allows a very precise mathematical interpretation where one demands that the operator to observe the image at scale $s+t$ given image observed at scale s is independent of the 'starting scale' s . Under rather mild (mathematical) restrictions on the operators it has been shown that this interpretation of the atlas principle leads to the notion of an *infinitesimal generator* that takes an image at scale s and subsequently derives the image at scale $s+ds$ using only the local differential structure of the image at scale s. This is the approach advocated by Alvarez *et al.* [1].

Although some of the scale-space operators presented in this paper do have an infinitesimal generator it is not our starting point. By concentrating on the PDE description of an evolutionary process one runs into the risk of defining a sequence of images indexed with a continuous 'time' parameter which cannot be linked intuitively with the notion of scale. Furthermore these evolutionary processes most often do not have the property that an observation at finite scale t can be done without 'running' the evolution from time 0 to t and thus generating all 'observations' at smaller scales as well. We want our scale-space operator to be a macroscopic one, in the sense that $T(t)$ only needs the zero scale image without the need to calculate all intermediate images $T(s)f$ for $s<t$.

We take the scale to be a positive scalar corresponding with our intuitive notion of size. The connection between ordered semigroups and the physical notion of measurements is well-known. In the beginning of this century O. Hölder studied the axiomatic foundations for the theory of magnitude and measure within the context of orderable semigroups; see the paper by Hofmann and Lawson [13]. In § 2.2 we state some results from semigroup theory which are relevant for this paper.

The existence of an infinitesimal generator does *not* follow from the atlas principle. The atlas principle (in its most general form) states that the family of scale-space operators should be an orderable semigroup. The existence of an infinitesimal generator needs the semigroup to be *naturally ordered*. This excludes scale-spaces based on morphological openings (or closings) as the corresponding semigroup operation on $(0,\infty)$ is the supremum, and this is not naturally ordered.

In this paper we propose a construction technique for scale-space operators where we first

[†] The zero scale image is of course a mathematical construct. It cannot be observed as all observations are inherently done at some finite scale.

downscale the image by a factor t using an invertible scaling or magnification operator $S(t)^{-1}$, then apply an image operator ψ at unit scale and finally resize the image to its original scale using $S(t)$. That is we will propose $T(t) = S(t)\psi S(t)^{-1}$ as the scale-space operator. We will show that scale invariance is guaranteed through this construction. The orderable semigroup property needs to be looked into for each proposed image operator ψ.

In this paper we will concentrate on linear and morphological operators as the choice for ψ. Because the literature on linear scale-space operators is vast and comprehensive we only sketch a brief outline closely following the work of Pauwels *et al.* [23]. Doing so, one observes that morphological scale-space operators have the same algebraic structure as their linear counterparts. This algebraic consistency becomes very apparent by introducing the slope transform (also called Fenchel conjugate or Legendre transform) in the morphological domain. For our purposes it suffices to look at convex functions. Although the theory of convex functions is well-established in the mathematical literature, it is less known to researchers in mathematical morphology, and for that reason we have included a short but concise introduction in § 4.2.

2. Scale-space: a formal definition

The scale-space operators considered in this paper are constructed by first scaling down the image by a factor t using an invertible scaling or magnification operator $S(t)^{-1}$, then applying an image operator ψ at unit scale and finally resizing the image to its original scale using $S(t)$. That is we propose $T_\psi(t) = S(t)\psi S(t)^{-1}$ as a scale-space operator subject to the condition that $T_\psi(t)$ obeys the semigroup property.

In the first subsection we define the image scalings as used in the construction of scale-spaces. In § 2.2 we summarize some known results on semigroups on $\mathcal{T} = (0, \infty)$ which are linearly or naturally ordered. Finally, in § 2.3 we present the formal scale-space construction linking the family of operators $\{T_\psi(t)\}_{t>0}$ with semigroup theory.

§ 2.1. Scalings

Let $\mathcal{L}$ be the family of images under consideration. In this paper an image is defined as a mapping from the spatial domain $\mathbb{R}^d$ to the range $\overline{\mathbb{R}}$ of (grey level) pixel values: i.e. $\mathcal{L} = \mathrm{Fun}(\mathbb{R}^d))$. Depending on the context, $\mathcal{L}$ will be assumed to have some additional structure, e.g. a vector space when we consider linear image operators or a complete lattice when we deal with morphological operators.

2.1. Definition. A one-parameter family $S = \{S(t) \mid t > 0\}$ of operators on $\mathcal{L}$ is called a *scaling* (or *multiplication*) if

(*i*) $S(1) = \mathsf{id}$

(*ii*) $S(t)S(s) = S(ts), \quad s, t > 0.$

This means in particular that S is a commutative group and that the inverse of $S(t)$ is given by

$$S(t)^{-1} = S(1/t),\ t > 0.$$

If S_1, S_2 are two scalings whose members commute mutually, i.e. $S_1(t)S_2(s) = S_2(s)S_1(t)$ for $s, t > 0$, then the composition $\{S_2(t)S_1(t) \mid t > 0\}$ is a scaling too. In particular, if $S(t)$ defines a scaling, then for $p \in \mathbb{R}$,

$$S^p(t) := S(t^p), \quad t > 0 \tag{2.1}$$

does so as well. We denote this scaling by S^p.

2.2. Definition. Two scalings S and S' are said to be *anamorphic* if there exists an increasing bijection γ on $(0,\infty)$ such that $S'(t) = S(\gamma(t))$ for $t > 0$.

In particular, S^p and S^q, where $pq > 0$ and S is a given scaling, are anamorphic.

The following result, the proof of which is straightforward, provides another method for constructing new scalings from existing ones.

2.3. Proposition. *If S is a scaling on $\mathcal{L}$ and $\lambda : \mathcal{L} \to \mathcal{L}$ is an invertible operator, then S_λ given by*

$$S_\lambda(t) = \lambda^{-1} S(t) \lambda \,, \ t > 0 \,,$$

is also a scaling.

If $\{A(t) \mid t \in \mathbb{R}\}$ is an additive group of operators on $\mathcal{L}$, that is, $A(t)A(s) = A(t+s)$ for $s, t \in \mathbb{R}$ and $A(0) = \mathsf{id}$, then S given by

$$S(t) = A(\log t) \,, \tag{2.2}$$

defines a scaling on $\mathcal{L}$. The next two examples, which are based on the construction in (2.2), show that the word 'scaling' should not be taken too literally.

2.4. Example.

(*i*) Let $a_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around the origin over the angle $t\alpha$, where $\alpha > 0$ is given. Then the operator $A(t)$ on $\mathrm{Fun}(\mathbb{R}^2)$ given by

$$(A(t)f)(x) = f(a_t(x)), \ x \in \mathbb{R}^2, \ t > 0 \,,$$

defines an additive group. Alternatively, one can choose for a_t the translation over a vector tv, where $v \in \mathbb{R}^2$ is fixed, that is $a_t(x) = x + tv$.

(*ii*) The family $A(t)$ given by $A(t)f = f + ct$, where $c \in \mathbb{R}$ is fixed, defines an additive group. This gives rise to a scaling $S(t)f = f + c \log t$, which corresponds with a grey-level shift.

In the sequel of this paper the two parameter family of scalings $S^{p,q}$ (see below) will play an important role. To verify that a particular family defines also a scaling on a subcollection of $\mathrm{Fun}(\mathbb{R}^d)$, e.g. $L^2(\mathbb{R}^d)$, the square integrable functions, it needs only to be verified that this subcollection is invariant under each scaling.

Given $p, q \in \mathbb{R}$, define

$$S^{p,q}(t)f = t^q f(\cdot/t^p) \text{ for } f \in \mathrm{Fun}(\mathbb{R}^d), \ t > 0 \,. \tag{2.3}$$

It is obvious that every family $\{S^{p,q}(t) \mid t > 0\}$ defines a scaling. The p-parameter is related to the spatial scaling whereas the q-parameter is related to the grey value scaling. The following special cases play a prominent role in the sequel:

- $p = 1, \ q = 1$: umbral scaling $S^{1,1}(t)f = tf(\cdot/t)$
- $p = 1, \ q = 0$: spatial scaling $S^{1,0}(t)f = f(\cdot/t)$
- $p = 1/2, \ q = 0$: quadratic scaling $S^{1/2,0}(t)f = f(\cdot/\sqrt{t})$
- $p = 0, \ q = 1$: grey-level scaling $S^{0,1}(t)f = tf$

Refer to Figure 2.1 for an illustration.

We can also use Proposition 2.3 to find scalings. For example, the invertible operator $\lambda : \mathrm{Fun}(\mathbb{R}^d) \to \mathrm{Fun}(\mathbb{R}^d)$ given by $\lambda(f) = f(\cdot) + 1$ in combination with the umbral scaling yields the scaling

$$S(t)f(x) = tf(x/t) + t - 1 \,.$$

Obviously, there exist many variations on this theme.

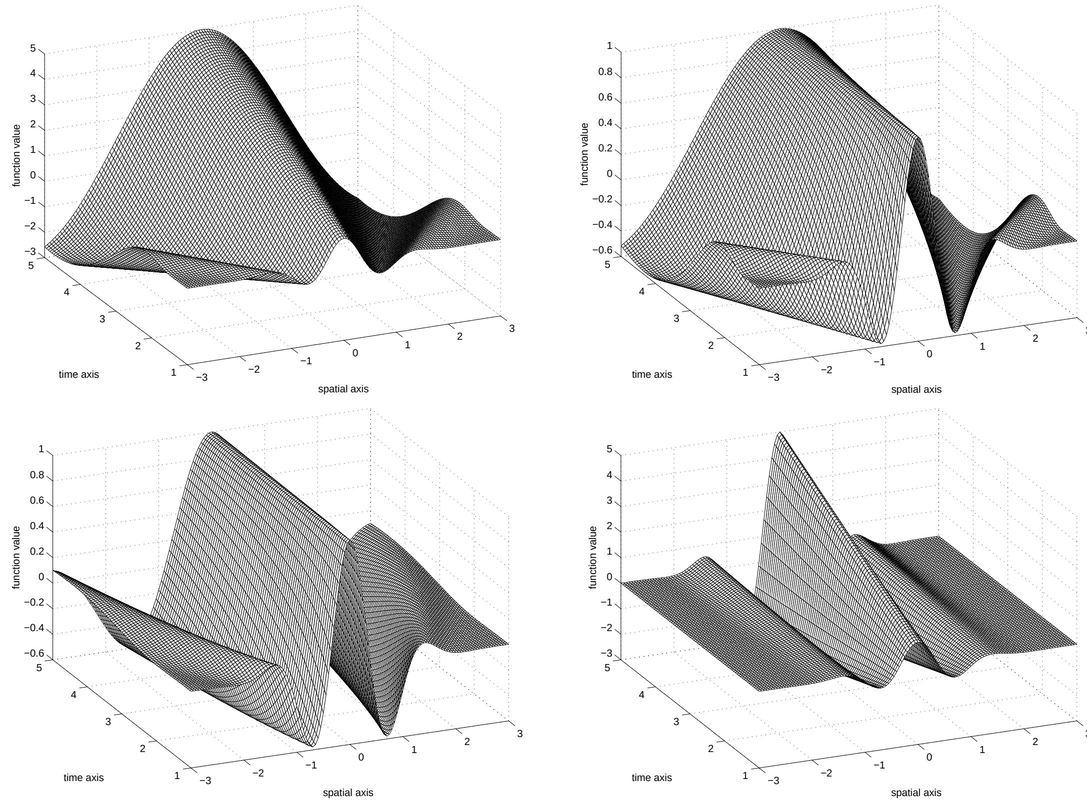

Fig. 2.1. From left to right and top to bottom: a function f and its umbral, spatial, quadratic, and grey-level scaling.

§ 2.2. The semigroup property

The atlas principle introduced by Koenderink [18] states that it should be possible to build a scale-space incrementally. That is, if we consider the image observed at scale s and take that image as the input for an observation at scale t an image at some higher scale r should result:

$$T(t)T(s) = T(r)\,. \tag{2.4}$$

Mathematically this is equivalent with the requirement that the family of scale-space operators $T(t)$ is a semigroup. The semigroup property of the scale-space operators thus corresponds with a semigroup property of the set of scale values, i.e. the positive real numbers. We can define a semigroup operation on $(0, \infty)$ by putting $s \dotplus t = r$ where, for given $s, t \in (0, \infty)$, the value $r \in (0, \infty)$ is determined by (2.4), presumed that r is uniquely determined.

In this section we present a more formal discussion of semigroups with an underlying ordering. We refer to Fuchs [9] for some further background information. Let $\mathcal{T} = (0, \infty)$ and assume that the binary operation $\dotplus$ on $\mathcal{T}$ defines a commutative semigroup, that is,

- the operation $\dotplus$ is associative: $(r \dotplus s) \dotplus t = r \dotplus (s \dotplus t)$, for $r, s, t \in \mathcal{T}$;
- the operation $\dotplus$ is commutative: $t \dotplus s = s \dotplus t$, for $s, t \in \mathcal{T}$.

The simplest example is the standard addition

$$t \dotplus s = t + s\,.$$

We refer to this example as the *additive* case. A second example is given by

$$t \dotplus s = t \vee s\,,$$

the *supremal* case. Of course, the dual operation $t \dot{+} s = t \wedge s$ defines a commutative semigroup too, but for our purposes this is of no interest. Note that, when $\dot{+}$ is a commutative semigroup and $u : \mathcal{T} \to \mathcal{T}$ is a bijection, then

$$t \hat{+} s = u^{-1}(u(t) \dot{+} u(s)) \tag{2.5}$$

defines a commutative semigroup as well. Choosing $u(t) = t^\nu$ in (2.5), where $\nu > 0$, we arrive at the semigroup operation

$$s \dot{+}_\nu t = (s^\nu \dot{+} t^\nu)^{1/\nu}, \quad s, t > 0\,. \tag{2.6}$$

Note that $s \dot{+}_1 t = s \dot{+} t$. Obviously, in the context of image scale-spaces, where $t \in \mathcal{T}$ has the interpretation of scale, the ordering in magnitude on $\mathcal{T}$ is extremely important: the content of the image is likely to decrease if the scale parameter t increases. The following algebraic definition restricts the class of semigroups by imposing a relation between the semigroup operation and the ordering.

2.5. Definition. If the semigroup operation $\dot{+}$ is such that the monotonicity condition

$$r, s, t \in (0, \infty) \text{ and } s \le t \;\Rightarrow\; s \dot{+} r \le t \dot{+} r$$

holds, then the triple $(\mathcal{T}, \dot{+}, \le)$ is called a linearly ordered semigroup.

In this definition, as well as in the remainder of this paper, it has been assumed that the semigroup operation $\dot{+}$ is commutative. If $\dot{+}$ is a semigroup on $\mathcal{T}$ and $u : \mathcal{T} \to \mathcal{T}$ is a bijection which is order-preserving, that is, $s \le t \iff u(s) \le u(t)$ for $s, t \in \mathcal{T}$, then $\hat{+}$ given by (2.5) defines a semigroup as well, and we say that $(\mathcal{T}, \dot{+}, \le)$ and $(\mathcal{T}, \hat{+}, \le)$ are *isomorphic*. If the first triple defines a linearly ordered semigroup, then the second defines a linearly ordered semigroup too, and vice versa.

2.6. Definition. Assume that $(\mathcal{T}, \dot{+}, \le)$ is linearly ordered. We say that it is *naturally linearly ordered* if for $s, t \in \mathcal{T}$:

(i) $s, t < s \dot{+} t$;
(ii) $s < t \Rightarrow s \dot{+} r = t$, for some $r \in \mathcal{T}$.

In fact, r in condition (ii) is uniquely determined by s and t. For, assume that $s \dot{+} r_1 = s \dot{+} r_2 = t$ and $r_1 \ne r_2$. Without loss of generality, assume that $r_1 < r_2$. There exists an $r \in \mathcal{T}$ such that $r_1 \dot{+} r = r_2$. Then $t = s \dot{+} r_2 = s \dot{+} (r_1 \dot{+} r) = (s \dot{+} r_1) \dot{+} r = t \dot{+} r$. But $t < t \dot{+} r$ by condition (i), which yields a contradiction. We denote the unique element r in (ii) by $r = t \dot{-} s$. Please note that only for a *naturally* linearly ordered semigroup, this notation makes sense.

Both $(\mathcal{T}, +, \le)$ and $(\mathcal{T}, \vee, \le)$ are linearly ordered semigroups, but only the first triple is naturally linearly ordered. Furthermore, if $(\mathcal{T}, \dot{+}, \le)$ is a naturally linearly ordered semigroup which is isomorphic to $(\mathcal{T}, \hat{+}, \le)$, then $(\mathcal{T}, \hat{+}, \le)$ is naturally linearly ordered as well. In particular, we have that $(\mathcal{T}, +_\nu, \le)$, where $+_\nu$ is defined by (2.6) (with $\dot{+}$ replaced by $+$), is naturally linearly ordered.

The following result is a special case of a result which is originally due to O. Hölder; see [13] as well as Theorem XI.2 in Fuchs [9].

2.7. Proposition. *Every naturally linearly ordered semigroup $(\mathcal{T}, \dot{+}, \le)$ is isomorphic to $(\mathcal{T}, +, \le)$.*

We conclude this section with some additional examples of semigroup operations.

2.8. Examples.

(a) The multiplication $s \dot{+} t = st$ defines a semigroup on $\mathcal{T}$. It is linearly ordered but not naturally linearly ordered.

(b) The operation $s \dot{+} t = s + t + st$ defines a naturally linearly ordered semigroup with subtraction $t \dot{-} s = (t-s)/(s+1)$. Thus, on behalf of Proposition 2.7, this semigroup is isomorphic with $(\mathcal{T}, +, \leq)$. Indeed, choosing $u(t) = \log(1+t)$ we find that $s \dot{+} t = u^{-1}(u(s) + u(t))$.

(c) A semigroup on $\mathcal{T}$ which is a combination of the additive and the supremal semigroup is given by

$$s \dot{+} t = s \vee t \vee ((s+t) \wedge 1)\,.$$

It is linearly ordered but not naturally linearly ordered.

(d) Define the operation $\dot{+}$ on $\mathcal{T}$ by taking a bitwise supremum with respect to the decimal decomposition. For example, $16.432 \dot{+} 8.1723 = 18.4723$. This operation is associative and commutative, hence $(\mathcal{T}, \dot{+})$ is a semigroup. However, this semigroup is not linearly ordered. We can replace the bitwise supremum of the previous example by a truncated addition, for example, $16.432 \dot{+} 8.1723 = 19.594$. Again the resulting semigroup is not linearly ordered.

§ 2.3. Scale-space

Now we are ready to give a formal definition of a scale-space. In our approach the starting assumption is that the scale-space is a semigroup of operators on the image space $\mathcal{L}$ under composition, compatible with a given semigroup $\dot{+}$ on $\mathcal{T}$ and invariant with respect to a given scaling S.

2.9. Definition. Let $(\mathcal{T}, \dot{+}, \leq)$ be a linearly ordered semigroup and let S be a scaling on $\mathcal{L}$. The family $\{T(t)\}_{t>0}$ of operators on $\mathcal{L}$ is called an *$(S, \dot{+})$-scale-space* if

$$T(t)T(s) = T(t \dot{+} s), \;\; s, t > 0\,, \tag{2.7}$$

$$T(t)S(t) = S(t)T(1), \;\; t > 0\,. \tag{2.8}$$

From (2.8) we easily derive that

$$T(t)S(s) = S(s)T(t/s), \;\; s, t > 0\,. \tag{2.9}$$

Namely,

$$\begin{aligned} T(t)S(s) &= T(t)S(t)S(s/t) = S(t)T(1)S(s/t) \\ &= S(s)S(t/s)T(1)S(s/t) = S(s)T(t/s)S(t/s)S(s/t) \\ &= S(s)T(t/s)\,, \end{aligned}$$

where we have used property (ii) of Definition 2.1 along with (2.8). Putting

$$\psi = T(1)\,, \tag{2.10}$$

(2.8) can be written as

$$T_\psi(t) = S(t)\psi S(t)^{-1}, \;\; t > 0\,, \tag{2.11}$$

where we have replaced $T(t)$ by $T_\psi(t)$ to emphasise the dependence on ψ. We say that the operator ψ *induces* the $(S, \dot{+})$-scale-space $\{T_\psi(t)\}_{t>0}$. The construction in(2.11) means essentially that the same operator ψ is applied at different scales, ranging from small scales when t is small to large scales when t is large. It is easy to show that

$$T_\psi(st) = S(s)T_\psi(t)S(s)^{-1}\,, \tag{2.12}$$

and that we have the composition law

$$T_{\psi_2\psi_1}(t) = T_{\psi_2}(t)T_{\psi_1}(t), \quad t > 0\,, \tag{2.13}$$

for any two operators ψ_1, ψ_2 on $\mathcal{L}$.

From $T_\psi(1) = \psi$ and the semigroup property we find that

$$\psi^k = T_\psi(1 \dotplus 1 \dotplus \cdots \dotplus 1)\,, \tag{2.14}$$

where the argument of T_ψ contains k terms.

The semigroup property (2.7) imposes a strong condition on the image operator ψ, and will only be satisfied for very particular choices. It will also become clear that the characterisation of ψ for which (2.7) holds, depends heavily upon the underlying scaling $S(t)$ and addition $\dotplus$.

Our next result expresses that the intrinsic scale of the operator is not important.

2.10. Proposition. *If ψ induces an $(S, \dotplus)$-scale-space then $T_\psi(r)$ does so as well, for every $r > 0$.*

Proof. For $s, t > 0$ we have

$$\begin{aligned}
S(t)T_\psi(r)S(t)^{-1}S(s)T_\psi(r)S(s)^{-1} &= S(t)S(r)\psi S(r)^{-1}S(t)^{-1}S(s)S(r)\psi S(r)^{-1}S(s)^{-1} \\
&= S(r)S(t)\psi S(t)^{-1}S(s)\psi S(s)^{-1}S(r)^{-1} \\
&= S(r)T_\psi(t)T_\psi(s)S(r)^{-1} \\
&= S(r)T_\psi(t \dotplus s)S(r)^{-1} \\
&= S(r)S(t \dotplus s)\psi S(t \dotplus s)^{-1}S(r)^{-1} \\
&= S(t \dotplus s)T_\psi(r)S(t \dotplus s)^{-1}\,.
\end{aligned}$$

This proves the assertion. ∎

Thus the question whether a given image operator induces a scale-space is independent of the scale at which this operator is being applied.

If T has the semigroup property, that is, $T(t)T(s) = T(t \dotplus s)$, then

$$T(t^p)T(s^p) = T(t^p \dotplus s^p) = T((t \dotplus_p s)^p)\,.$$

In other words, if $t \mapsto T(t)$ has the semigroup property with respect to $(\mathcal{T}, \dotplus)$, then $t \mapsto T(t^p)$ has the semigroup property with respect to $(\mathcal{T}, \dotplus_p)$. Recalling the definition of S^p from (2.1), we arrive at the following result.

2.11. Proposition. *If ψ induces an $(S, \dotplus)$-scale-space, then ψ induces an $(S^p, \dotplus_p)$-scale-space for every $p > 0$. In particular, if ψ induces an $(S, \vee)$-scale-space, then ψ induces an $(S^p, \vee)$-scale-space for every $p > 0$.*

Basically, this result means that we can restrict ourselves to the cases $p = 1$ and $p = \infty$ (if the latter corresponds with a well-defined semigroup operation).

We conclude this section with some invariance properties.

2.12. Definition. Let λ, ψ be operators on $\mathcal{L}$, we say that ψ is λ-invariant if $\psi\lambda = \lambda\psi$. If Λ is a family of operators on $\mathcal{L}$, then ψ is said to be Λ-invariant if ψ is λ-invariant for every $\lambda \in \Lambda$. In particular, if S is a scaling on $\mathcal{L}$, then ψ is called S-invariant if $S(t)\psi = \psi S(t)$ for $t > 0$.

2.13. Proposition. *Assume that the operator ψ induces an $(S, \dot{+})$-scale-space and that λ, μ are S-invariant operators on $\mathcal{L}$ such that*

$$\lambda\mu = \mathsf{id}\,.$$

Then $\mu\psi\lambda$ induces an $(S, \dot{+})$-scale-space as well and

$$T_{\mu\psi\lambda}(t) = \mu T_\psi(t)\lambda\,.$$

PROOF. For $t > 0$

$$\begin{aligned} T_{\mu\psi\lambda}(t) &= S(t)\mu\psi\lambda S(t)^{-1} \\ &= \mu S(t)\psi S(t)^{-1}\lambda \\ &= \mu T_\psi(t)\lambda\,. \end{aligned}$$

Furthermore, using that $\lambda\mu = \mathsf{id}$:

$$\begin{aligned} T_{\mu\psi\lambda}(t)T_{\mu\psi\lambda}(s) &= \mu T_\psi(t)\lambda\mu T_\psi(s)\lambda \\ &= \mu T_\psi(t)T_\psi(s)\lambda \\ &= \mu T_\psi(t \dot{+} s)\lambda \\ &= T_{\mu\psi\lambda}(t \dot{+} s)\,. \end{aligned}$$

This concludes the proof. ∎

We present a nontrivial example, i.e. where λ and μ are not inverses of each other, for which the assumptions of Proposition 2.13 hold.

2.14. Example. Consider the grey-scale images $\mathcal{L} = \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}}_+)$. Let λ, μ be defined by

$$\begin{aligned} \lambda(f)(x) &= \begin{cases} 0, & f(x) < 1 \\ f(x) - 1, & f(x) \geq 1 \end{cases} \\ \mu(f)(x) &= f(x) + 1\,. \end{aligned}$$

It is obvious that $\lambda\mu = \mathsf{id}$ but $\mu\lambda \neq \mathsf{id}$. Both operators λ, μ are invariant with respect to the spatial scaling $S^{p,0}$.

2.15. Definition. The family Λ of operators on $\mathcal{L}$ is called *compatible* with the scaling S if $S(t)\lambda S(t)^{-1} \in \Lambda$ for $\lambda \in \Lambda$ and $t > 0$.

Let $\mathbb{T}_{\mathrm{hor}}$ be the family of horizontal translations $f \mapsto f_h$ on $\mathrm{Fun}(\mathbb{R}^d)$, where $f_h(x) = f(x - h)$, for $h \in \mathbb{R}^d$. Denote by $\mathbb{T}_{\mathrm{ver}}$ the vertical translations $f \mapsto f + v$, where $(f + v)(x) = f(x) + v$, for $v \in \mathbb{R}$. Finally, denote by $\mathbb{T}$ the family of all translations. In other words, $\mathbb{T}$ comprises all compositions of translations in $\mathbb{T}_{\mathrm{hor}}$ and $\mathbb{T}_{\mathrm{ver}}$. It is easy to verify that all three families are compatible with every scaling $S^{p,q}$ given by (2.3).

2.16. Proposition. *Assume that the operator family Λ on $\mathcal{L}$ is compatible with the scaling S, and let ψ be a Λ-invariant operator on $\mathcal{L}$ which induces an $(S, \dot{+})$-scale-space $T_\psi(t)$. Then every operator $T_\psi(t)$ is Λ-invariant.*

PROOF. The fact that λ is compatible with the scaling S means that for $\lambda \in \Lambda$ and $t > 0$ there exists a unique $\lambda_t \in \Lambda$ such that

$$S(t)\lambda = \lambda_t S(t)\,,$$

namely, $\lambda_t = S(t)\lambda S(t)^{-1}$. Thus

$$\begin{aligned} T_\psi(t)\lambda &= S(t)\psi S(t)^{-1}\lambda = S(t)\psi S(\frac{1}{t})\lambda \\ &= S(t)\psi\lambda_{1/t}S(t)^{-1} = S(t)\lambda_{1/t}\psi S(t)^{-1} \\ &= (\lambda_{1/t})_t S(t)\psi S(t)^{-1} = \lambda T_\psi(t)\,. \end{aligned}$$

Here we have used that $(\lambda_{1/t})_t = \lambda$. This proves the result. ∎

In the following sections we will look at specific choices for the image operator ψ in the construction of scale-space operators. In Section 3 we consider linear operators and in Sections 5-6 we look at morphological operators.

In this respect we make the following important observation. The family of scalings $S^{p,q}$ has the important property that every particular member $S^{p,q}(t)$ is a linear operator as well as an erosion (in mathematical morphology an operator is called an erosion if it distributes over infima [10]). This implies that the property of ψ being a linear operator or being an erosion is inherited by the induced family $T_\psi(t)$.

3. Linear scale-spaces

There exists an extensive literature dealing with various aspects of linear scale-spaces [2, 30, 31]. We refer in particular to a paper by Pauwels *et al.* [23] comprising an axiomatic approach involving a broad class of scalings, but restricting attention to the additive semigroup $+$ on $\mathcal{T}$. To a certain extent, the exposition in this section can be regarded as a slight extension of the work by Pauwels *et al.* [23], dealing with the general class of semigroup operations $\dot{+}_\nu$ for $0 < \nu \leq \infty$.

We shall deal exclusively with linear convolution operators on $L^2(\mathbb{R}^d)$ given by

$$\psi(f) = K \star f = \int_{\mathbb{R}^d} K(\cdot - y)f(y)dy\,, \tag{3.1}$$

i.e. we restrict ourselves to the linear, translation invariant operators. Here $K \in L^2(\mathbb{R}^d)$ is called the *convolution kernel.* In signal processing, K is called the *impulse response function* as it is the system's output due to a Dirac delta input. Throughout this section we assume that the kernel K is mass-preserving, that is,

$$\int_{\mathbb{R}^d} K(x)dx = 1\,. \tag{3.2}$$

Consider the scaling $S^{p,q}$ and the scale-space construction as detailed in the previous section. The scale-space operator $T_\psi(t) = S(t)\psi S(t)^{-1}$ is a linear translation invariant operator and thus we must have that $T_\psi(t)$ is a convolution as well:

$$T_\psi(t)f = K_t \star f\,.$$

A straightforward calculation shows that:

$$K_t(x) = t^{-pd}K(x/t^p)\,. \tag{3.3}$$

Note that, because of linearity, the factor t^q drops out. In other words, the choice of q does not play any role in the linear case. Also note that the scaling of K is such that K_t has the same 'energy' as the unit scale kernel, meaning that average grey value is preserved in a linear scale space.

The semigroup condition $T_\psi(t)T_\psi(s) = T_\psi(t \dot{+} s)$ amounts to the following condition on the kernels K_t:

$$K_t \star K_s = K_{t \dot{+} s}\,, \quad s,t > 0\,. \tag{3.4}$$

To study this equation, we compute the Fourier transform at both sides and find

$$(2\pi)^{\frac{d}{2}} \hat{K}_t \hat{K}_s = \hat{K}_{t \dot{+} s}, \quad s,t > 0\,. \tag{3.5}$$

Here $\hat{h}$ denotes the Fourier transform of h:

$$\hat{h}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} h(x)\, \exp(-i\langle x, \xi\rangle)dx\,.$$

Here $\langle x, y\rangle$ denotes the vector product of x and y. A straightforward calculation shows that

$$\hat{K}_t(\xi) = \hat{K}(t^p \xi)\,.$$

In combination with (3.5) this yields the following relation:

$$(2\pi)^{\frac{d}{2}} \hat{K}(t^p\xi)\hat{K}(s^p\xi) = \hat{K}((s \dot{+} t)^p \xi), \quad \xi \in \mathbb{R}^d,\ s,t > 0. \tag{3.6}$$

Consider first the additions $+_\nu$ where $0 < \nu < \infty$. Following the exposition in Pauwels *et al.* [23] we restrict ourselves to kernels for which

$$\hat{K}(\xi) = (2\pi)^{-\frac{d}{2}} \exp(-a|\xi|^k)\,, \tag{3.7}$$

where $a, k > 0$. Observe that the mass-preservingness of K expressed by (3.2) follows from the fact that $(2\pi)^{\frac{d}{2}}\hat{K}(0) = 1$. Substitution in (3.6) leads to the identity

$$t^{pk} + s^{pk} = (t +_\nu s)^{pk}, \quad s,t > 0\,.$$

This holds if

$$k = \nu/p\,. \tag{3.8}$$

Observe that this relation is in agreement with Proposition 2.11. We consider some examples.

3.1. Example: Gaussian scale-space. Let S be the quadratic scaling $S(t)f(x) = f(x/\sqrt{t})$, i.e., $p = 1/2$. Furthermore, take $\nu = 1$, then (3.8) yields that $k = 2$, i.e.

$$\hat{K}(\xi) = (2\pi)^{-\frac{d}{2}} \exp(-a|\xi|^2)\,.$$

The corresponding convolution kernel K is the Gaussian function

$$K(x) = (4\pi a)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{4a})\,.$$

If we choose $a = 1/2$, then the operator ψ is given by

$$\psi(f)(x) = (2\pi)^{-\frac{d}{2}} \int f(x-y)\exp(-\frac{|y|^2}{2})dy\,,$$

and the induced scale-space is given by

$$(T_\psi(t)f)(x) = (2\pi t)^{-\frac{d}{2}} \int f(x-y)\exp(-\frac{|y|^2}{2t})dy\,.$$

There exists a huge literature on Gaussian scale-spaces, and we do not feel any urge to add to this.

We point out that the Gaussian kernel K is found in all cases where $k = 2$, i.e., $\nu = 2p$. However, the kernel K_t in the corresponding scale-space is also dependent on p; see (3.3). For the scale-space $T_\psi(t)$ we find the expression

$$(T_\psi(t)f)(x) = (2\pi t^{2p})^{-\frac{d}{2}} \int f(x-y)\exp(-\frac{|y|^2}{2t^{2p}})dy\,.$$

Again, we have chosen $a = 1/2$.

3.2. Example: Cauchy scale-space. We choose $k = 1$ in (3.7) and take $a = 1$ for simplicity. Thus

$$\hat{K}(\xi) = (2\pi)^{-\frac{d}{2}}\exp(-|\xi|)\,.$$

This can be shown to correspond with the convolution kernel

$$K(x) = \pi^{-(\frac{d+1}{2})}\,,\,(\frac{d+1}{2})(1+|x|^2)^{-(\frac{d+1}{2})}\,,$$

where , denotes the gamma function; see [5]. The corresponding scale-space is given by

$$(T_\psi(t)f)(x) = \pi^{-(\frac{d+1}{2})}\,,\,(\frac{d+1}{2})t^p \int_{\mathbb{R}^d} \frac{f(x-y)}{[t^{2p}+|y|^2]^{\frac{d+1}{2}}}dy\,.$$

In the one-dimensional case ($d = 1$) we get

$$(T_\psi(t)f)(x) = \frac{t^p}{\pi}\int_{-\infty}^{\infty}\frac{f(x-y)}{t^{2p}+|y|^2}dy\,.$$

The convolution kernel $K(x) = \frac{1}{\pi}(1+|x|^2)^{-1}$ is called the *Cauchy kernel*, and for this reason we refer to the scale-space $T_\psi(t)$ as the *Cauchy scale-space*. For $p = 1$, this scale-space satisfies the additive semigroup relation $T_\psi(t)T_\psi(s) = T_\psi(t+s)$.

3.3. Example: supremal scale-space. In the case where $\dot{+}$ is the supremum, (3.6) amounts to

$$(2\pi)^{\frac{d}{2}}\hat{K}(t^p\xi)\hat{K}(s^p\xi) = \hat{K}(t^p\xi),\ \xi\in\mathbb{R}^d,\ t \geq s > 0\,. \tag{3.9}$$

A moment of reflection shows that $\hat{K}$ is a solution if $(2\pi)^{\frac{d}{2}}\hat{K}(\cdot)$ is the characteristic function† of a compact set $C \subseteq \mathbb{R}^d$ which is star-shaped with respect to the origin, i.e.,

$$\xi \in C \ \Rightarrow\ r\xi \in C \text{ for } 0 \leq r \leq 1\,.$$

† A characteristic function of a set takes value 1 on this set and 0 outside.

Since we want our convolution kernel to be real-valued we assume in addition that C is symmetric with respect to the origin, i.e., $\xi \in C$ iff $-\xi \in C$. The kernel K is then given by

$$\begin{aligned} K(x) &= (2\pi)^{-d} \int_C \exp(i\langle x, \xi\rangle) d\xi \\ &= (2\pi)^{-d} \int_C \cos(\langle x, \xi\rangle) d\xi\,. \end{aligned}$$

Let us consider the one-dimensional case. Then C is a closed interval of the form $[-\xi_0, \xi_0]$, and K is given by

$$K(x) = \frac{1}{\pi} \frac{\sin(x\xi_0)}{x} = \frac{\xi_0}{\pi} \mathrm{sinc}\,(x\xi_0)\,,$$

where $\mathrm{sinc}\,(x) = \sin x / x$. As in the previous examples, the expression for the scale-space $T_\psi(t)$ depends on the choice of p:

$$(T_\psi(t)f)(x) = \frac{\xi_0}{\pi t^p} \int_{-\infty}^{\infty} f(x-y) \mathrm{sinc}\,(\xi_0 y / t^p) dy\,.$$

Observe that the operators $T_\psi(t)$ are called *ideal low-pass filters* in the signal processing literature.

4. Morphological operators and convex analysis

§ 4.1. Basic morphological operators

We first recall some concepts from mathematical morphology that we use in the sequel. We refer to [10] for a comprehensive discussion. The main concept is that of an adjunction.

4.1. Definition. Consider a partially ordered set (poset) $\mathcal{L}$ and two operators ε: $\mathcal{L} \to \mathcal{L}$ and δ: $\mathcal{L} \to \mathcal{L}$. The pair (ε, δ) defines an adjunction on $\mathcal{L}$ if

$$\delta(y) \le x \iff y \le \varepsilon(x), \quad x, y \in \mathcal{L}\,. \tag{4.1}$$

It is easy to show that, in an adjunction, both operators ε and δ are increasing; i.e., $x_1 \le x_2$ implies that $\varepsilon(x_1) \le \varepsilon(x_2)$ (same for δ). Recall that a poset $\mathcal{L}$ is called a *lattice* if every finite subset in $\mathcal{L}$ has a supremum (least upper bound) and an infimum (greatest lower bound). The set $\mathcal{L}$ is called a *complete lattice* if *every* (finite or infinite) subset of $\mathcal{L}$ has an infimum and a supremum. If $\mathcal{K} \subseteq \mathcal{L}$, then we denote the supremum and infimum of $\mathcal{K}$ by $\bigvee \mathcal{K}$ and $\bigwedge \mathcal{K}$, respectively. Instead of $\bigvee\{x_1, x_1, ..., x_n\}$ we write $x_1 \vee x_2 \vee \cdots \vee x_n$ (same for the infimum). If (ε, δ) is an adjunction on the lattice $\mathcal{L}$, then

$$\varepsilon(x_1 \wedge x_2 \wedge \cdots \wedge x_n) = \varepsilon(x_1) \wedge \varepsilon(x_2) \wedge \cdots \wedge \varepsilon(x_n), \quad x_1, x_2, ..., x_n \in \mathcal{L}\,, \tag{4.2}$$

and, dually,

$$\delta(x_1 \vee x_2 \vee \cdots \vee x_n) = \delta(x_1) \vee \delta(x_2) \vee \cdots \vee \delta(x_n), \quad x_1, x_2, ..., x_n \in \mathcal{L}\,. \tag{4.3}$$

In a complete lattice, these relationships also hold for infinite infima and suprema, respectively. Operators ε and δ, with the properties stated above, are called *erosion* and *dilation*, respectively. In the following, id denotes the identity operator. The next two results can be easily proved.

4.2. Proposition. *Let (ε, δ) be an adjunction on the poset $\mathcal{L}$, then*

$$\varepsilon\delta\varepsilon = \varepsilon \quad \text{and} \quad \delta\varepsilon\delta = \delta$$
$$\varepsilon\delta \geq \mathsf{id} \quad \text{and} \quad \delta\varepsilon \leq \mathsf{id}\,.$$

4.3. Proposition. *Let (ε, δ) and (ε', δ') be adjunctions on the poset $\mathcal{L}$, then $(\varepsilon'\varepsilon, \delta\delta')$ is an adjunction as well.*

4.4. Definition. Let ψ be an operator from a poset $\mathcal{L}$ into itself.
(*a*) ψ is *idempotent*, if $\psi^2 = \psi$.
(*b*) If ψ is increasing and idempotent, then ψ is called a (*morphological*) *filter*.
(*c*) A filter ψ which satisfies $\psi \leq \mathsf{id}$ (ψ is anti-extensive) is called an *opening*.
(*d*) A filter ψ which satisfies $\psi \geq \mathsf{id}$ (ψ is extensive) is called a *closing*.

Adjunctions can be used as building blocks for openings and closings.

4.5. Proposition. *Let (ε, δ) be an adjunction on the poset $\mathcal{L}$, then $\varepsilon\delta$ and $\delta\varepsilon$ are a closing and opening on $\mathcal{L}$, respectively.*

Let us first consider the binary case. In this paper, as in most of the literature on mathematical morphology, binary images are modeled mathematically by $\mathcal{P}(\mathbb{R}^d)$, the power set of $\mathbb{R}^d$. This set, ordered by set inclusion, is a complete lattice.

An operator ψ on $\mathcal{P}(\mathbb{R}^d)$ is called translation invariant if $\psi(X_h) = (\psi(X))_h$ for $X \subseteq \mathbb{R}^d$ and $h \in \mathbb{R}^d$, where $X_h = \{x + h \mid x \in X\}$. The translation invariant adjunctions on $\mathcal{P}(\mathbb{R}^d)$ are of the form

$$\varepsilon_B(X) = X \ominus B = \bigcap_{h \in B} X_{-h}\,, \tag{4.4}$$

$$\delta_B(X) = X \oplus B = \bigcup_{h \in B} X_h\,, \tag{4.5}$$

where $B \subseteq \mathbb{R}^d$ is called a *structuring element*. The sets $X \oplus B$ and $X \ominus B$ are called the *Minkowski addition* and *subtraction*, respectively, of X and B.

Next, we consider morphological operators for grey-scale functions. The set $\mathrm{Fun}(\mathbb{R}^d)$ of functions $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is a complete lattice under the pointwise ordering, i.e., $f \leq g$ if $f(x) \leq g(x)$ for $x \in \mathbb{R}^d$. An operator ψ on $\mathrm{Fun}(\mathbb{R}^d)$ is translation invariant if ψ is $\mathbb{T}$-invariant (see § 2.3), that is, $\psi(f_h + v) = (\psi(f))_h + v$ for $f \in \mathrm{Fun}(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $v \in \mathbb{R}$. Every translation invariant adjunction on $\mathrm{Fun}(\mathbb{R}^d)$ is of the form $(\varepsilon_b, \delta_b)$ with

$$\varepsilon_b(f)(x) = \bigwedge_{h \in \mathbb{R}^d} [f(x - h) + b(h)]\,, \tag{4.6}$$

$$\delta_b(f)(x) = \bigvee_{h \in \mathbb{R}^d} [f(x + h) - b(h)]\,. \tag{4.7}$$

In these expressions, the function b is called the *structuring function*. The expression for the erosion ε_b is a well-known operation in convex function analysis, where it is known under the name *infimal convolution* and denoted by $f \boxminus b$, that is

$$(f \boxminus b)(x) = \bigwedge_{h \in \mathbb{R}^d} [f(x - h) + b(h)]\,. \tag{4.8}$$

Thus the erosion in (4.6) can be written as $\varepsilon_b(f) = f \boxminus b$. The dilation in (4.7) is denoted by $f \boxplus b$. Observe that the mapping $(f, b) \mapsto f \boxminus b$ is commutative and associative. This is not true for the dilation.

In most of the literature on mathematical morphology, dilation and erosion for grey-scale functions are expressed in terms of Minkowski addition and subtraction for functions. In that case, morphological erosion is given by $f \mapsto f \ominus a$, where $(f \ominus a)(x) = \bigwedge_{h \in \mathbb{R}^d} [f(x+h) - a(h)]$, a being the structuring function in this case. It is easy to see that this expression transforms to (4.6) by putting $a(h) = -b(-h)$. Since for our purposes, the connection with the literature on convex functions is very important, we have chosen to work with the expressions in (4.6)-(4.7).

§ 4.2. Convex analysis

Throughout the remainder of this paper, convex sets and functions play a major role. Therefore, we present a brief overview of some basic results in convex analysis that play a role in the sequel; refer to [12, 25] for a comprehensive account.

4.6. Definition. A set $X \subseteq \mathbb{R}^d$ is *convex* if $tx + (1-t)y \in X$ when $x, y \in X$ and $0 \le t \le 1$. If, in addition, $tx \in X$ for $x \in X$ and $t \ge 0$, then X is called a *convex cone.*

4.7. Definition.

(*a*) A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *convex* if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$, for $x, y \in \mathbb{R}^d$ and $0 \le t \le 1$.

(*b*) A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *lower semi-continuous* (*l.s.c.*) if for every $t \in \overline{\mathbb{R}}$ and $x \in E$ with $f(x) > t$ there exists a neighbourhood of x such that $f(y) > t$ for every y in that neighbourhood.

The family of convex functions is closed under (pointwise) suprema. If f is a convex function with values in $\mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$ is convex, then their composition $g(f(\cdot))$ is convex, too. Furthermore, it can easily be shown that the infimal convolution of two convex functions yields a convex function [12, Part I].

A function f is l.s.c. if the points on and above the graph of f, i.e., the points $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ with $t \ge f(x)$, form a closed set in $\mathbb{R}^d \times \mathbb{R}$. In this sense, lower semi-continuity is the function analogue of closedness of a set. It is obvious that every continuous function is l.s.c.

4.8. Examples.

(*a*) If X is a convex subset of $\mathbb{R}^d$, then the indicator function I_X which is 0 for points in X and ∞ outside, is convex.

(*b*) The function $b_Q(x) = \frac{1}{2}\langle Qx, x\rangle$, where Q is a symmetric positive semi-definite matrix and where $\langle x, y\rangle$ denotes the inner product, is a convex function. One can show that [12, Part I]

$$b_Q \boxminus b_R = b_{Q \times R}, \text{ where } Q \times R = (Q^{-1} + R^{-1})^{-1}. \tag{4.9}$$

The function b_Q is sometimes referred to as the *quadratic seminorm.*

4.9. Definition. A function f is called (*positively*) *homogeneous of degree k*, where $k \ge 1$, if

$$f(tx) = t^k f(x), \quad x \in \mathbb{R}^d, \ t > 0.$$

It is called *subpolynomial of degree k* if it is also convex. A function which is subpolynomial of degree 1 is called *sublinear.* We say that a function is *subpolynomial of degree* ∞ if it is an indicator function. We denote by $SP(k)$ the set of all convex functions that are subpolynomial of degree k.

We present some elementary results concerning subpolynomial functions.

4.10. Proposition.

(a) *If* $f, g \in SP(k)$ *then* $f + g \in SP(k)$ *as well.*
(b) *If* $f_i \in SP(k)$ *for* $i \in I$, *then then* $\bigvee_{i \in I} f_i \in SP(k)$.
(c) *If* $f, g \in SP(k)$ *then* $f \boxminus g \in SP(k)$ *as well.*
(d) *If* f *taking values in* $\mathbb{R}$ *is an element of* $SP(k)$ *and if* $g : \mathbb{R} \to \overline{\mathbb{R}}$ *is an element of* $SP(l)$, *then their composition* $g(f(\cdot))$ *is an element of* $SP(kl)$.

Note that the last result implies the following: if f is a sublinear function, then its power f^k is subpolynomial of degree k.

Observe that $(a) - (c)$ also hold for $k = \infty$. If $f = I_X$ and $g = I_Y$, then $I_X + I_Y = I_X \vee I_Y = I_{X \cap Y}$, and $X \cap Y$ is convex if both X and Y are convex sets. Furthermore, $I_X \boxminus I_Y = I_{X \oplus Y}$, where $X \oplus Y$ is the Minkowski addition of X and Y, which is also a convex set.

4.11. Examples of sublinear functions.

(a) The indicator function I_K of a convex cone K is subpolynomial of degree k for every $k \geq 1$. Recall that a convex set is called a cone if $x \in K$ and $t > 0$ implies that $tx \in K$.
(b) The distance function $d_K(x) = \inf\{\|x - k\| \mid k \in K\}$, where K is a convex cone, is sublinear.
(c) Let B be a closed convex set that contains the origin. Then its *gauge function*

$$G_B(x) = \inf\{r > 0 \mid x \in rB\}$$

is sublinear [12, Part I].
(d) Let Q be a symmetric and positive semi-definite matrix; then the quadratic seminorm $\|x\|_Q = \langle Qx, x\rangle^{1/2}$ is sublinear. If Q is positive definite, then $\|\cdot\|_Q$ is a norm; see also Example 4.12.
(e) The *support function* of a set B is defined as

$$H_B(x) = \sup_{y \in B} \langle x, y\rangle, \quad x \in \mathbb{R}^d .$$

The support function H_B of a set B is l.s.c. and sublinear. Conversely, every l.s.c. sublinear function f is the support function of the closed convex set

$$B = \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq f(y) \text{ for all } y \in \mathbb{R}^d\} .$$

It is not difficult to show that $0 \in B$ is equivalent with $H_B(x) \geq 0$ for all $x \in \mathbb{R}^d$.

4.12. Example: norms, gauge functions, and polarity. Every norm $\|\cdot\|$ on $\mathbb{R}^d$ defines a sublinear function. The set $B = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$ is called the *unit ball* with respect to $\|\cdot\|$. It is easy to verify that B is a symmetric, convex, compact set containing the origin as interior point. Furthermore, the gauge function of B is given by $G_B(x) = \|x\|$. The *polar set* $B^\star$ defined by

$$B^\star = \{x \in \mathbb{R}^d \mid \langle x, b\rangle \leq 1 \text{ for all } b \in B\} ,$$

is also a symmetric, convex, compact set containing the origin as interior point. One can show that

$$G_{B^\star} = H_B ,$$

and that

$$(B^\star)^\star = B .$$

The norm $\|\cdot\|^\star = G_{B^\star}(\cdot)$ is called the *dual norm* of $\|\cdot\|$. It is given by

$$\|x\|^\star = \sup\{\langle x, y\rangle \mid \|y\| \le 1\}\,.$$

We give two concrete examples. The dual of the quadratic norm

$$\|x\|_Q = \langle Qx, x\rangle^{\frac{1}{2}}\,,$$

where Q is a symmetric positive definite matrix, is $\|\cdot\|_{Q^{-1}}$. The dual of the p-norm $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ is the q-norm $\|\cdot\|_q$, where $p^{-1} + q^{-1} = 1$.

If B is a closed convex cone, then the *polar cone* $B^\star$ can also be defined as

$$B^\star = \{x \in \mathbb{R}^d \mid \langle x, b\rangle \le 0 \text{ for all } b \in B\}\,.$$

It is easy to verify that in this case

$$H_B = I_{B^\star}\,;$$

see also [12, Part I, p.215].

4.13. Example: convex ray set. A set $B \subseteq \mathbb{R}^d$ is called a *ray set* if $b \in B$ implies that $tb \in B$ for $t \ge 1$. If B is also convex, it is called a *convex ray set*. Some examples are depicted in Figure 4.1.

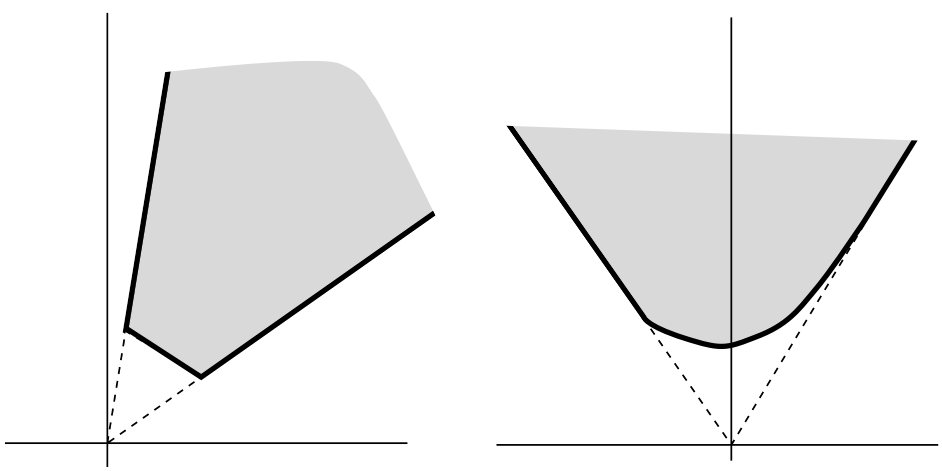

Fig. 4.1. Convex ray sets.

Obviously, every convex cone is a convex ray set. On the other hand, every convex ray set that contains the origin is a convex cone. We show that B is a convex ray set if and only if the support function H_B satisfies the condition

$$H_B(x) \le 0 \text{ or } H_B(x) = +\infty, \text{ for every } x \in \mathbb{R}^d\,.$$

To prove this, assume first that B is a convex ray set. If $H_B(x) > 0$, then $\langle x, y_0\rangle > 0$ for some $y_0 \in B$. But since $ty_0 \in B$ for $t \ge 1$, we get that $H_B(x) = \sup_{y\in B}\langle x, y\rangle \ge \langle x, ty_0\rangle = t\langle x, y_0\rangle$, for every $t \ge 1$. We find that $H_B(x) = +\infty$. Conversely, assume that H_B satisfies the condition above. We know that $B = \{x \in \mathbb{R}^d \mid \langle x, y\rangle \le H_B(y) \text{ for every } y \in \mathbb{R}^d\}$; see Example 4.11(e). Assume $x \in B$ and $t \ge 1$; we must show that $tx \in B$, i.e., that $\langle tx, y\rangle \le H_B(y)$ for all $y \in \mathbb{R}^d$. There are two possibilities: $H_B(y) = +\infty$ or $H_B(y) \le 0$. In the first case we get $\langle tx, y\rangle \le H_B(y)$, and in the second case $\langle tx, y\rangle = t\langle x, y\rangle \le tH_B(y) \le H_B(y)$. This shows the result.

The *conjugate* of a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is defined by

$$f^\star(\xi) = \sup\{\langle x, \xi\rangle - f(x) \mid x \in \mathbb{R}^d\}. \tag{4.10}$$

Conjugation is a well-known operation in convex analysis [12, Part I] where it goes under different names, such as *Fenchel conjugate*, *Young-Fenchel conjugate*, or *Legendre transform*. We list some of its basic properties.

4.14. Proposition.
(a) *For every $f \in \mathrm{Fun}(\mathbb{R}^d)$ its conjugate $f^\star$ is an l.s.c. convex function.*
(b) *If f is convex then $f^{**} = \overline{f}$, the l.s.c. closure of f.*

In recent years, the importance of conjugation in the context of mathematical morphology has been emphasized by Dorst and van den Boomgaard [6] and independently by Maragos [20, 21], who call this operation the *morphological slope transform*. To a certain extent the slope transform plays a role in mathematical morphology which is comparable to that of the Fourier transform in linear signal processing. This is mainly because of the following result.

4.15. Proposition. *Let f, g be convex functions, then*

$$(f \boxminus g)^\star = f^\star + g^\star. \tag{4.11}$$

We point out, however, that in the morphological context one works with concave functions rather than with convex ones. Refer to [11] for a systematic treatment of the slope transform in the complete lattice framework.

We present some examples.

4.16. Example.
(*a*) Let $b_Q(x) = \frac{1}{2}\langle Qx, x\rangle$, where Q is a symmetric positive definite matrix, then $b_Q^\star(\xi) = \frac{1}{2}\langle Q^{-1}\xi, \xi\rangle$. Refer to [12, Part II, Sect X.1] for more details.
(*b*) If B is a nonempty closed convex set, then $I_B^\star = H_B$ and $H_B^\star = I_B$.

4.17. Definition. For a number $k \in [1, \infty]$ we define its reciprocal $k^\star$ through the relation: $1/k + 1/k^\star = 1$.

Thus the reciprocal of 1 is ∞ and vice versa. Furthermore, $(k^\star)^\star = k$ by definition.

4.18. Proposition. *Let f be an l.s.c. function, then $f \in SP(k)$ if and only if $f^\star \in SP(k^\star)$.*

PROOF. From the fact that $f^{\star\star} = f$ for an l.s.c. convex function (Proposition 4.14(b)) it follows that it suffices to prove the 'only if' part.

Assume first that $1 < k < \infty$. For $t > 0$ and $\xi \in \mathbb{R}^d$ we have

$$f^\star(t\xi) = \sup_{x \in \mathbb{R}^d} \big(\langle x, t\xi\rangle - f(x)\big).$$

Substituting $y = t^{-\frac{1}{k-1}}x$ and using that $k^\star = \frac{k}{k-1}$, we get

$$\begin{aligned}
f^\star(t\xi) &= \sup_{y \in \mathbb{R}^d} \Big(\langle t^{\frac{1}{k-1}} y, t\xi\rangle - f(t^{\frac{1}{k-1}} y)\Big) \\
&= \sup_{y \in \mathbb{R}^d} \Big(\langle t^{\frac{k}{k-1}} y, \xi\rangle - t^{\frac{k}{k-1}} f(y)\Big) \\
&= t^{k^\star} \cdot \sup_{y \in \mathbb{R}^d} \Big(\langle y, \xi\rangle - f(y)\Big) = t^{k^\star} \cdot f^\star(\xi),
\end{aligned}$$

which proves the result.

If $k = 1$ then f is sublinear, and from Example 4.11(e) we know that $f = H_B$ for some closed convex set B. In Example 4.16(b) we have seen that in this case $f^\star$ is the indicator function of B, and hence $f^\star$ is subpolynomial of degree ∞ by definition.

The result for $k = \infty$ follows by a dual argument. ∎

5. Erosion scale-space

This section is entirely devoted to scale-spaces induced by the morphological erosion given by the infimal convolution in (4.8). We distinguish two semigroup operations: addition and supremum. These two cases will be treated in § 5.3 and § 5.4, respectively.

§ 5.1. Generalities

Before restricting ourselves to scale-spaces induced by infimal convolution, we make a few general observations on morphological scale-spaces. Let $\mathcal{L}$ be a complete lattice and let $\nu : \mathcal{L} \to \mathcal{L}$ be a negation on $\mathcal{L}$ [10], that is, ν is a bijection that reverses the ordering ($f \leq g$ implies $\nu(f) \geq \nu(g)$ for $f, g \in \mathcal{L}$) and satisfies $\nu^2 = \mathsf{id}$, where id is the identity operator. The most well known negation on $\mathcal{P}(\mathbb{R}^d)$ is the complement operator $\nu(X) = X^c$. On $\mathrm{Fun}(\mathbb{R}^d)$ one usually considers $\nu(f)(x) = -f(x)$. With every operator ψ on $\mathcal{L}$ we can associate its *dual* or *negative* ψ^* given by $\psi^* = \nu\psi\nu$. The operator ψ is called *self-dual* if $\psi^* = \psi$. It is obvious that the family $S^* = \{S(t)^* \mid t > 0\}$ is a scaling if S is a scaling. From this observation the following result is easily proved.

5.1. Proposition. *Let $\mathcal{L}$ be a complete lattice with negation ν. Assume that ψ induces an $(S, \dot{+})$-scale-space on $\mathcal{L}$, then the negative operator $\psi^* = \nu\psi\nu$ induces an $(S^*, \dot{+})$-scale-space.*

In many practical cases, the scaling S is self-dual in the sense that $S^*(t) = S(t)$ for every $t > 0$. This holds in particular for the scalings $S^{p,q}$ on $\mathrm{Fun}(\mathbb{R}^d)$ given by (2.3). In that case, Proposition 5.1 says that ψ induces a $(S, \dot{+})$-scale-space if and only if ψ^* does such as well, and we have $(T_\psi(t))^* = T_{\psi^*}(t)$ for $t > 0$.

The following result is concerned with adjunctions.

5.2. Proposition. *Let $\mathcal{L}$ be a complete lattice and assume that every scaling operator $S(t)$ is increasing. If (ε, δ) is an adjunction on $\mathcal{L}$, then $(T_\varepsilon(t), T_\delta(t))$ is an adjunction, too, for every $t > 0$. Assume moreover that $T_\varepsilon(t)$ defines an $(S, \dot{+})$-scale-space on $\mathcal{L}$, then $T_\delta(t)$ defines an $(S, \dot{+})$-scale-space as well.*

Proof. The first result is straightforward. We prove the second. From the theory of adjunctions [10] it follows that $T_\delta(t)T_\delta(s)$ is the adjoint of $T_\varepsilon(s)T_\varepsilon(t) = T_\varepsilon(t \dot{+} s)$. On the other hand, the adjoint of $T_\varepsilon(t \dot{+} s)$ is $T_\delta(t \dot{+} s)$. By the uniqueness of adjoints we find that $T_\delta(t)T_\delta(s) = T_\delta(t \dot{+} s)$. Thus T_δ satisfies the semigroup property. ∎

An important implication of this proposition is that the results below concerning erosion scale-spaces have a straightforward analogue for dilation scale-spaces.

5.3. Example: binary erosion. Let $S(t)(X) = tX$ and $\varepsilon(X) = X \ominus B$, where B is a compact structuring element; see (4.4)-(4.5). An easy computation gives $T_\varepsilon(t)X = X \ominus tB$. Thus

$$T_\varepsilon(t)T_\varepsilon(s)X = (X \ominus sB) \ominus tB = X \ominus (sB \oplus tB).$$

It is not difficult to prove that

$$sB \oplus tB = (s+t)B, \quad s,t \geq 0, \tag{5.1}$$

if and only if B is convex [10, Prop. 9.2]. This yields that $T_\varepsilon(t)$ is an *additive* scale-space iff B is convex.

Furthermore, $T_\varepsilon(t)$ is a *supremal* scale-space if

$$sB \oplus tB = tB, \quad \text{for } t \geq s > 0. \tag{5.2}$$

We show that this condition holds if and only if

$$B \text{ is convex and } B = tB,\ t > 0. \tag{5.3}$$

First we show that (5.2) implies $B = tB$ for $t \geq 1$. Namely, if $t \geq 1$, we write t as $t = n + r$ where n is the largest integer $\leq t$ and $0 \leq r < 1$. From (5.2) we get

$$tB = B \oplus B \oplus \cdots \oplus B \oplus rB = B.$$

But if $tB = B$ for $t \geq 1$, then $B = \frac{1}{t}B$ for $t > 1$ which means that $tB = B$ for $t > 0$. This means in particular that (5.1) is satisfied, hence that B is convex. Thus (5.2) holds only if (5.3) is satisfied. The converse, namely that (5.3) implies (5.2), is obvious.

Note that the condition (5.3) is very similar to the condition that B is a convex cone, the only difference being that a convex cone contains the origin by definition, and this property is not guaranteed by (5.3).

§ 5.2. Scale-spaces based on infimal convolution

In the remainder of this section we are exclusively concerned with the class of scale-spaces on $\mathrm{Fun}(\mathbb{R}^d)$ induced by the morphological erosion

$$\varepsilon(f)(x) = (f \boxminus b)(x) = \bigwedge_{y \in \mathbb{R}^d} [f(x-y) + b(y)].$$

During the rest of this section we make the following assumption.

5.4. Assumption. The function b is l.s.c. and convex and satisfies $b(x) > -\infty$ for every x.

Let us, before investigating the scale-space property, make the following observation.

5.5. Remark. The erosion $\varepsilon(f) = f \boxminus b$ satisfies $\varepsilon^2 = \varepsilon$, or equivalently, $b \boxminus b = b$, if and only if b is homogeneous of degree 1. For, Proposition 4.15 yields that $b \boxminus b = b$ is equivalent to $b^\star + b^\star = b^\star$, hence $b^\star(\xi) = 0$ or $+\infty$ for every ξ. In other words, $b^\star$ is an indicator function, hence an element of $SP(\infty)$. Now Proposition 4.18 yields that $b \in SP(1)$. This gives the result.

Furthermore, we limit ourselves to scalings $S^{p,q}$ given by (2.3); for the sake of brevity we write $S = S^{p,q}$. A straightforward calculation shows that

$$T_\varepsilon(t)f = f \boxminus S(t)b. \tag{5.4}$$

Note in particular that $T_\varepsilon(t)$ defines an erosion for every $t > 0$. The next result gives necessary and sufficient conditions which guarantee that the operator $T_\varepsilon(t)$ does not depend on t. Note that we have not yet assumed that the semigroup condition holds.

5.6. Proposition. *The family $T_\varepsilon(t)$ is constant, i.e. $T_\varepsilon(t) = \varepsilon$ for $t > 0$, if and only if b is homogeneous of degree q/p.*

PROOF. 'if': we have

$$(S(t)b)(x) = t^q b(x/t^p) = t^{q-\frac{q}{p}p} b(x) = b(x)\,,$$

if b is homogeneous of degree q/p.

'only if': evidently, $T_\varepsilon(t) = \varepsilon$ for $t > 0$ only if $S(t)b = b$ for $t > 0$. The latter means that $b(x) = t^q b(x/t^p)$ for $t > 0$. Substituting $s = t^{-p}$, this yields $b(sx) = s^{q/p} b(x)$, $s > 0$, $x \in \mathbb{R}^d$. In other words, b has to be homogeneous of degree q/p. ∎

For $p = 0$, this proposition says that $T_\varepsilon(t)$ is independent of t iff b is an indicator function, say $b = I_B$. In that case

$$\varepsilon(f)(x) = \bigwedge_{h \in B} f(x-h)\,,$$

which is called a *flat erosion* in the morphological literature [10].

The semigroup condition $T_\varepsilon(t) T_\varepsilon(s) = T_\varepsilon(t \dot{+} s)$ amounts to the following condition on b:

$$S(t)b \boxminus S(s)b = S(t \dot{+} s)b\,. \tag{5.5}$$

Again, note the similarity with the linear case as expressed by formula (3.4). Taking conjugates at both sides of (5.5) and using (4.11) we get

$$(S(t)b)^\star + (S(s)b)^\star = (S(t \dot{+} s)b)^\star\,. \tag{5.6}$$

The conjugate of $S(t)b$ is given by

$$\begin{aligned}
(S(t)b)^\star(\xi) &= \sup_{x \in \mathbb{R}^d} \Big(\langle \xi, x \rangle - (S(t)b)(x) \Big) \\
&= \sup_{x \in \mathbb{R}^d} \Big(\langle \xi, x \rangle - t^q b(x/t^p) \Big) \\
&= \sup_{y \in \mathbb{R}^d} \Big(\langle \xi, y t^p \rangle - t^q b(y) \Big) \\
&= t^q \sup_{y \in \mathbb{R}^d} \Big(\langle t^{p-q} \xi, y \rangle - b(y) \Big)\,,
\end{aligned}$$

where we have substituted $y = x/t^p$. Thus we find

$$(S(t)b)^\star(\xi) = t^q b^\star(t^{p-q}\xi)\,.$$

Substitution in (5.6) yields

$$t^q b^\star(t^{p-q}\xi) + s^q b^\star(s^{p-q}\xi) = (t \dot{+} s)^q b^\star((t \dot{+} s)^{p-q}\xi)\,. \tag{5.7}$$

Proposition 2.11 in combination with the fact that

$$(S^{p,q})^r = S^{pr,qr}\,,$$

says that, as far as the semigroup operations $+_\nu$ are concerned, we only need to consider two cases: additive semigroups corresponding with $\nu = 1$ and supremal semigroups corresponding with $\nu = \infty$. These two cases will be treated in two separate sections. Before doing so, we give conditions under which the family $T_\varepsilon(t)$, deriving from a scaling $S = S^{p,q}$, is compatible with respect to another fixed scaling $\sigma(f)(x) = v f(x/u)$.

5.7. Proposition. *Let $T_\varepsilon(t)f = f \boxminus S(t)b$, where $S = S^{p,q}$, and assume that b is homogeneous of degree k, where k, p, q are such that $kp - q \neq 0$. The scaling operator $\sigma(f)(x) = vf(x/u)$ is compatible with $T_\varepsilon(t)$ in the sense that*

$$T_\varepsilon(t)\sigma = \sigma T_\varepsilon(at), \quad t > 0, \tag{5.8}$$

where a is the positive constant given by

$$a = \begin{cases} (u^k/v)^{\frac{1}{q-kp}} & \text{if } k \neq \infty \\ u^{-\frac{1}{p}} & \text{if } k = \infty \text{ and } p \neq 0 \end{cases}.$$

Proof. Assume first that $k \neq \infty$. Fix $t > 0$. A straightforward calculation shows that $\sigma^{-1}T_\varepsilon(t)\sigma$ is an infimal convolution with kernel

$$\frac{u^k}{v} t^{q-kp} b(x) = (at)^{q-kp} b(x).$$

The operator $T_\varepsilon(at)$ is an infimal convolution with the kernel at the right hand-side. Therefore, both operators coincide.

If $k = \infty$, then $b = I_B$ for some set B, hence

$$T_\varepsilon(t)f = f \boxminus I_{t^p B}.$$

An easy computation shows that

$$(\sigma^{-1}T_\varepsilon(t)\sigma)(f) = f \boxminus I_{\frac{t^p}{u}B} = f \boxminus I_{(at)^p B},$$

where $a = u^{-\frac{1}{p}}$. The operator corresponding with the right hand-side is $T_\varepsilon(at)$. This concludes the proof. ∎

§ 5.3. Additive semigroup property

In (5.7) we replace $\dot{+}$ by $+$ and find

$$t^q b^\star(t^{p-q}\xi) + s^q b^\star(s^{p-q}\xi) = (t+s)^q b^\star((t+s)^{p-q}\xi). \tag{5.9}$$

Below, we discuss three different solutions to this relation.

First, if $p = q = 1$, then (5.9) is trivially satisfied. This leads to the following result; see also [24].

5.8. Proposition ($p = q = 1$). *If $S = S^{1,1}$ is the umbral scaling, then $\varepsilon(f) = f \boxminus b$ induces an $(S, +)$-scale-space for every convex function b.*

The umbral scaling for grey-level functions is the geometrical analogue of the usual scaling of sets, and as such, Proposition 5.8 is well-known in the morphological literature.

For more general scalings $S^{p,q}$, relation (5.9) can be solved relatively easy if it is assumed that b, and hence $b^\star$, is subpolynomial. In fact, we believe that it should be possible to *prove* this fact, rather than assume it, but we have not succeeded in doing so. Therefore, we assume from this point onward that $b \in SP(k)$. In view of Proposition 4.18, this means that $b^\star \in SP(k^\star)$, where k and $k^\star$ are reciprocal numbers. We distinguish two cases: $k = 1$ and $k > 1$. In the next proposition we consider the case $k = 1$, hence $k^\star = \infty$.

5.9. Proposition ($k = 1$). *Assume that b is an l.s.c. sublinear function with $b(x) > -\infty$ for every $x \in \mathbb{R}^d$. The erosion $\varepsilon(f) = f \boxminus b$ induces a $(S^{p,q}, +)$-scale-space $T_\varepsilon(t)$ if and only if at least one of the following conditions is satisfied:*
(i) $p = q$;
(ii) $b = I_B$, *where B is a closed convex cone.*
In either of these two cases $T_\varepsilon(t) = \varepsilon$ for $t > 0$ and $\varepsilon^2 = \varepsilon$.

PROOF. The 'if' part of this result is left to the reader; here we only prove the 'only if' part. Towards that goal, let us assume that (5.9) holds. In virtue of Example 4.11(e), b is the support function of a closed convex set B, i.e., $b = H_B$. In Example 4.16(b) we have seen that $b^\star = I_B$ in this case. For this $b^\star$, equation (5.9) translates into

$$I_B(t^{p-q}\xi) + I_B(s^{p-q}\xi) = I_B((t+s)^{p-q}\xi)\,,$$

for every $\xi \in \mathbb{R}^d$ and $s, t > 0$. If $p = q$, this equation is trivially satisfied. If $p \neq q$, it is equivalent to

$$t^r\xi \in B \text{ and } s^r\xi \in B \iff (t+s)^r\xi \in B,\ s,t > 0\,,$$

where $r = p - q$. In other words,

$$t^{-r}B \cap s^{-r}B = (t+s)^{-r}B,\ s,t > 0\,.$$

It is easy to see that this relation holds if and only if B is a convex cone.

If $p = q$, then $T_\varepsilon(t) = \varepsilon$ by Proposition 5.6. If $p \neq q$ and $b^\star = I_B$, then $b = H_B = I_{B^\star}$, where $B^\star$ is the polar cone of B; see Example 4.12. Now $S(t)b = b$ for $t > 0$, hence $T_\varepsilon(t) = \varepsilon$ for $t > 0$. ∎

If b is sublinear and $b = H_B$ for a closed convex set B, then the erosion $\varepsilon(f) = f \boxminus H_B$ has a resemblance with the band-pass filter known from linear signal processing. For, the conjugate of $f \boxminus H_B$ equals $f^\star + H_B^\star = f^\star + I_B$, which equals $f^\star(\xi)$ if $\xi \in B$ and ∞ if $\xi \notin B$. Thus the erosion ε sort of 'filters away all slopes that lie outside B'.

Next, we consider the case that $k > 1$, hence $k^\star < \infty$. Relation (5.9) in combination with the fact that $b^\star$ is homogeneous of degree $k^\star$, yields

$$t^N + s^N = (t+s)^N \quad \text{with} \quad N = q + k^\star(p-q)\,.$$

This equation holds for all $s, t > 0$ if and only if $N = 1$, which leads us to the following result.

5.10. Proposition ($k > 1$). *Assume that $b \in SP(k)$ is an l.s.c. function which satisfies $b(x) > -\infty$ for every $x \in \mathbb{R}^d$. The erosion $\varepsilon(f) = f \boxminus b$ induces a $(S^{p,q}, +)$-scale-space $T_\varepsilon(t)$ if and only if*

$$q + k^\star(p-q) = 1\,, \tag{5.10}$$

where $k^\star$ is the reciprocal number of k.

A first solution to (5.10) is given by $p = q = 1$ and $1 \le k^\star < \infty$ arbitrary, that is, $1 < k \le \infty$. This case has been treated in Proposition 5.8. If $p \neq q$, then $k^\star = (1-q)/(1-p)$. The fact that $k^\star \ge 1$ implies that either $q < p < 1$ or $1 < p < q$. Finally we observe that condition $kp - q \neq 0$ in Proposition 5.7 is satisfied for all solutions of (5.10) with $k^\star < \infty$. We present two examples.

5.11. Example: quadratic scaling. Consider the quadratic scaling given by $p = 1/2$ and $q = 0$. Proposition 5.10 yields that $\varepsilon(f) = f \boxminus b$ induces an additive scale-space if $b \in SP(2)$. This scale-space $T_\varepsilon(t)$ is given by

$$(T_\varepsilon(t)f)(x) = \bigwedge_{y \in \mathbb{R}^d} [f(x-y) + b(y/\sqrt{t})] \,.$$

A typical example of a function which is subpolynomial of degree 2 is (cf. Example 4.8(b)):

$$b_Q(x) = \frac{1}{2}\langle Qx, x\rangle \,,$$

where Q is a symmetric positive semi-definite matrix. From (4.9) we know that

$$b_Q \boxminus b_R = b_{Q \times R} \,.$$

The erosion $T_\varepsilon(t)$ is given by $T_\varepsilon(t)f = f \boxminus b_{t^{-1}Q}$ and the semigroup property $T_\varepsilon(t)T_\varepsilon(s) = T_\varepsilon(t+s)$ can also be derived from the fact that

$$b_{t^{-1}Q} \boxminus b_{s^{-1}Q} = b_{t^{-1}Q \times s^{-1}Q} = b_{(t+s)^{-1}Q} \,.$$

This scale-space has been called the *parabolic morphological scale-space*; see [28, 29] as well as [15, 16, 17]. An illustration, with $Q = I$, is given in Figure 5.1.

5.12. Example: spatial scaling. Consider the spatial scaling given by $p = 1$ and $q = 0$. From Proposition 5.10 we get that $k = \infty$, hence $b = I_B$, where B is a closed convex set. The scale-space induced by $\varepsilon(f) = f \boxminus b$ is given by

$$(T_\varepsilon(t)f)(x) = \bigwedge_{y \in tB} f(x-y) \,,$$

the flat erosions with structuring element tB. See Figure 5.1 for an illustration, B being the disk.

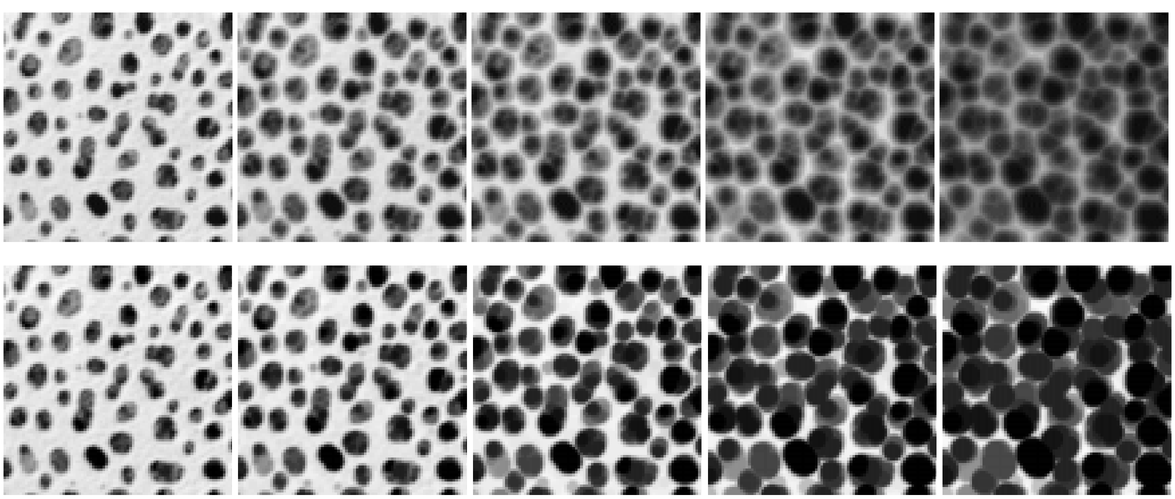

Fig. 5.1. The top row shows the parabolic scale space (see Example 5.11), the bottom row shows a flat disk scale-space (see Example 5.12). The images at the left in both rows represent the 'zero scale image' f. The four other images on the top row show the erosion $f \boxminus S(t)b$ where b is the parabolic function $b(x) = \frac{1}{2}\|x\|^2$ for the top row and $b = I_B$ with B a disk of radius 1 for the bottom row.

§ 5.4. SUPREMAL SEMIGROUP PROPERTY

In (5.7) we replace $\dot{+}$ by $\vee$ and find

$$t^q b^\star(t^{p-q}\xi) + s^q b^\star(s^{p-q}\xi) = t^q b^\star(t^{p-q}\xi), \quad t \geq s,\ \xi \in \mathbb{R}^d. \tag{5.11}$$

Substituting $s = t = 1$ we find that

$$b^\star(\xi) + b^\star(\xi) = b^\star(\xi),\ \xi \in \mathbb{R}^d,$$

whence we conclude that $b^\star(\xi) = 0$ or $+\infty$ for every $\xi \in \mathbb{R}^d$. Therefore

$$b^\star = I_B,$$

for some closed convex set B, is a necessary condition for (5.11) to hold. However, this condition is not sufficient in general. Towards that end we must also have that $t^{p-q}\xi \notin B$ if $s^{p-q}\xi \notin B$, in other words

$$t^{q-p}B \subseteq s^{q-p}B, \text{ for } t \geq s. \tag{5.12}$$

We distinguish three cases:

- $p = q$: then (5.12) is trivially satisfied.
- $p > q$: now (5.12) can be reformulated as follows:

  $$\lambda B \subseteq B \text{ for } 0 < \lambda < 1.$$

  For this condition to be satisfied it is necessary and sufficient that $0 \in B$. That it is sufficient is obvious. To prove that it is necessary, assume that $0 \notin B$. Suppose $\xi \in \mathbb{R}^d \backslash \{0\}$ were an element of B, then $\xi/n \in B$ for $n \geq 1$. Taking the limit for $n \to \infty$ and using that B is closed, we would find that $0 \in B$, which is a contradiction. We conclude that $b = I_B^\star = H_B$. We have seen in Example 4.11(e) that $0 \in B$ is equivalent with $H_B(x) \geq 0$ for all $x \in \mathbb{R}^d$.
- $p < q$: in this case, (5.12) leads to the condition

  $$\lambda B \subseteq B \text{ for every } \lambda \geq 1.$$

  In other words, B is a convex ray set. Thus we get from Example 4.13 that the sublinear function $b = I_{B^\star} = H_B$ satisfies

  $$b(x) \leq 0 \text{ or } b(x) = +\infty \text{ for } x \in \mathbb{R}^d.$$

  We have established the following result.

5.13. Proposition. *Let b be an l.s.c. convex function and let ε be the erosion $\varepsilon(f) = f \boxminus b$. Then ε induces an $(S^{p,q}, \vee)$-scale-space if and only if b is sublinear, i.e., $b = H_B$ for some closed convex set B, and one of the following conditions holds:*

(i) $p = q$;
(ii) $p > q$ and $0 \in B$;
(iii) $p < q$ and B is a convex ray set.

In case (i) we have $T_\varepsilon(t) = \varepsilon$ in virtue of Proposition 5.6. In cases (ii)-(iii), the scale-space $T_\varepsilon(t)$ is given by

$$\begin{aligned} T_\varepsilon(t)f(x) &= \bigwedge_{y \in \mathbb{R}^d} \left[f(x-y) + t^{q-p}H_B(y)\right] \\ &= \bigwedge_{y \in \mathbb{R}^d} \left[f(x-y) + H_{t^{q-p}B}(y)\right]. \end{aligned}$$

Finally, if the conditions $0 \in B$ and B a convex ray set are both satisfied, that is, if B is a convex cone, then $T_\varepsilon(t) = \varepsilon$ for $t > 0$ and $\varepsilon^2 = \varepsilon$ (see Proposition 5.9). Thus $T_\varepsilon(t)$ defines an $(S^{p,q}, \vee)$-scale-space for all p and q.

We conclude this section with Table 5.1 which summarizes the results of this section.

p and q	properties of b	$T_\varepsilon(t)$ (or ε)
$p = q$	$b \in SP(1)$ i.e. $b = H_B$ and B closed convex set	$T_\varepsilon(t) = \varepsilon$ for $t > 0$ and ε idempotent
any p and q	$b = I_B$ and B closed convex cone (in particular, $b \in SP(1)$)	$T_\varepsilon(t) = \varepsilon$ for $t > 0$ and ε idempotent
$p = q = 1$	b convex	$T_\varepsilon(t)$ additive scale-space
$q < p < 1$ or $1 < p < q$	$b \in SP(k)$ where $k = (1-q)/(p-q)$	$T_\varepsilon(t)$ additive scale-space
$q < p$	$b = H_B$, B closed convex set, $0 \in B$ in particular, $b \in SP(1)$	$T_\varepsilon(t)$ supremal scale-space
$p < q$	$b = H_B$, B closed convex ray set in particular, $b \in SP(1)$	$T_\varepsilon(t)$ supremal scale-space

Table 5.1. Table expressing for which p, q and which b the infimal convolution $\varepsilon(f) = f \boxminus b$ induces an additive or supremal scale-space.

6. Opening scale-space

In this section we examine scale-spaces induced by morphological openings. In the first subsection, we prove some general statements and discuss the relation between granulometries and scale-spaces induced by openings. In § 6.2 we restrict ourselves to supremal scale-spaces (i.e., $(S, \vee)$-scale-spaces) induced by the so-called structural opening.

§ 6.1. Generalities

We start with some simple observations.

6.1. Proposition. *Let S be a scaling on the set $\mathcal{L}$.*
(a) *If ψ is an idempotent operator on $\mathcal{L}$, then $T_\psi(t)$ is idempotent too, for every $t > 0$.*
(b) *If T is an $(S, \vee)$-scale-space, then every operator $T(t)$ is idempotent.*
(c) *Assume that $(0, \infty)$ is naturally linearly ordered under $\dot{+}$. If the idempotent operator ψ induces an $(S, \dot{+})$-scale-space, then $T_\psi(t) = \psi$ for all $t > 0$.*

Proof. (a) and (b) are straightforward. To prove (c), assume that the idempotent operator ψ induces the $(S, \dot{+})$-scale-space $T(t)$. Let $s < t$; we show that $T(t) = T(s)$. Obviously, this establishes the result.

Because of Hölders theorem (Proposition 2.7) we can write $t = ns \dot{+} r$ with $ns = s \dot{+} \cdots \dot{+} s$ (n terms), and n is the smallest integer for which $(n+1)s \geq t$ and where $r \leq s$. Then

$T(t) = T(ns)T(r) = T(s)^n T(r) = T(s)T(r)$, where we have used that $T(s)$ is idempotent (Proposition 6.1(a)). If $r = s$ we find $T(t) = T(s)^2 = T(s)$. If $r < s$ we can write $s = r + r'$, hence

$$T(s)T(r) = T(r')T(r)^2 = T(r')T(r) = T(r \dot{+} r') = T(s)\,.$$

Indeed, we find that $T(t) = T(s)$, which concludes the proof. ∎

A straightforward consequence of this proposition is that every operator $T(t)$ of a supremal scale-space is idempotent.

In the remainder of this subsection we explain the relation between the well-known morphological concept of a *granulometry* and the opening scale-space. The following result is straightforward.

6.2. Proposition. *Assume that every scaling operator $S(t)$ is increasing. If α is an opening (closing), then $T_\alpha(t)$ is an opening (closing), too, for every $t > 0$.*

It follows that $\alpha_t := \bigvee_{s \geq t} T_\alpha(s)$, being a supremum of openings, is an opening (see [10]) and that

$$\alpha_t \leq \alpha_s \quad \text{if} \quad t \geq s\,. \tag{6.1}$$

This is equivalent to

$$\alpha_t \alpha_s = \alpha_s \alpha_t = \alpha_t, \quad t \geq s\,. \tag{6.2}$$

In the morphological literature, a family of openings $\{\alpha_t \mid t > 0\}$ that satisfies (6.2) is called a *granulometry*. By definition,

$$\alpha_t \geq T_\alpha(t), \quad t > 0\,.$$

If $T_\alpha(t)$ is an $(S, \dot{+})$-scale space, where $\dot{+}$ is a naturally linearly ordered semigroup operation, then, by Proposition 6.1(c), $T_\alpha(t) = \alpha$ for $t > 0$. Therefore we will, from now on, restrict ourselves to the supremal case, where there do exist non-constant scale-spaces induced by openings.

It is not difficult to verify that the following conditions are equivalent:

- α induces an $(S, \vee)$-scale-space $T_\alpha(t)$;
- $T_\alpha(t) \leq T_\alpha(s)$ for $t \geq s$;
- $T_\alpha(t)$ is a granulometry.

Obviously, $\alpha_t = T_\alpha(t)$ if and only if these equivalent conditions are satisfied.

6.3. Example. Consider the space of binary images $\mathcal{L} = \mathcal{P}(\mathbb{R}^d)$. Suppose that α is the structural opening $\alpha(X) = X \circ B$, and that S is the scaling $S(t)X = t^p X$. Then $T_\alpha(t)(X) = X \circ t^p B$. This family defines a granulometry if and only if B is convex [22]. More generally, $T_\alpha(t)$ is an $(S, \vee)$-scale-space if we take $\alpha(X) = \bigcup_{i \in I} X \circ B_i$, where every B_i is convex. Here we use that $S(t)$ distributes over unions.

§ 6.2. Scale-spaces induced by structural openings

The first result of this section shows how to build opening scale-spaces from adjunctions.

6.4. Proposition. *Assume that every operator $S(t)$ is increasing on the poset $\mathcal{L}$, and let (ε, δ) be an adjunction on $\mathcal{L}$. Assume moreover that one of the following conditions is satisfied:*

(i) $\dot{+}$ is a naturally linearly ordered semigroup and ε induces an $(S, \dot{+})$-scale-space on $\mathcal{L}$;

(ii) ε induces an $(S, \vee)$-scale-space.

Then $\delta\varepsilon$ induces an $(S, \vee)$-scale-space and $T_{\delta\varepsilon}(t) = T_\delta(t) T_\varepsilon(t)$. Similarly, $\varepsilon\delta$ induces an $(S, \vee)$-scale-space and $T_{\varepsilon\delta}(t) = T_\varepsilon(t) T_\delta(t)$.

PROOF. We give a proof for the case where (i) holds. For (ii) the result is straightforward.

The relation $T_{\delta\varepsilon}(t) = T_\delta(t)T_\varepsilon(t)$ is a special case of (2.13). We show that $\delta\varepsilon$ induces an $(S,\vee)$-scale-space. Let $t \geq s$:

$$\begin{aligned} T_{\delta\varepsilon}(t)T_{\delta\varepsilon}(s) &= T_\delta(t)T_\varepsilon(t)T_\delta(s)T_\varepsilon(s) \\ &= T_\delta(t)T_\varepsilon(t \dot{-} s)T_\varepsilon(s)T_\delta(s)T_\varepsilon(s) \\ &= T_\delta(t)T_\varepsilon(t \dot{-} s)T_\varepsilon(s) \\ &= T_\delta(t)T_\varepsilon(t) = T_{\delta\varepsilon}(t)\,. \end{aligned}$$

Here we have used that $(T_\varepsilon(s), T_\delta(s))$ is an adjunction (Proposition 5.2) hence $T_\varepsilon(s)T_\delta(s)T_\varepsilon(s) = T_\varepsilon(s)$.

The proof for $t < s$ is analogous. ∎

In this section we consider one-parameter families $T_\alpha(t),\ t > 0$, where α is the structural opening given by

$$\alpha(f) = f \,\square\, b = (f \boxminus b) \boxplus b\,.$$

Thus $\alpha = \delta\varepsilon$, where δ is the dilation $\delta(f) = f \boxplus b$ and ε is the adjoint erosion $\varepsilon(f) = f \boxminus b$. Throughout this section we suppose that Assumption 5.4 is satisfied. From Proposition 6.4 we know that $T_\alpha(t) = T_\delta(t)T_\varepsilon(t)$. Let us assume that $S = S^{p,q}$ for some $p, q \in I\!R$. It is easy to verify that (cf. (5.4)):

$$T_\alpha(t)f = f \,\square\, S(t)b, \quad t > 0\,. \tag{6.3}$$

Proposition 6.4 in combination with Proposition 2.11 gives us that α induces an $(S^\nu,\vee)$-scale-space if ε induces an $(S,\dot{+})$-scale-space for a given naturally linearly ordered semigroup operation $\dot{+}$ or if $\dot{+}$ equals $\vee$.

6.5. Proposition. *Let $S = S^{p,q}$, let $\nu > 0$, and let b be an l.s.c. convex function with $b(x) > -\infty$ for every $x \in I\!R^d$. The structural opening $\alpha(f) = f \,\square\, b$ generates an $(S^\nu,\vee)$-scale-space for $\nu > 0$ in each of the following cases:*

- (i) *$p = q > 0$;*
- (ii) *$b = I_B$, where B is a closed convex cone (in this case $T_\alpha(t) = \alpha$ for every $t > 0$);*
- (iii) *$p > 0$ and $b = I_B$, where B is a closed convex set;*
- (iv) *$b \in SP(k)$ with $1 < k < \infty$ and $kp - q > 0$.*
- (v) *$p = q$ and $b = H_B$, where B is a closed convex set.*
- (vi) *$p > q$ and $b = H_B$, where B is a closed convex set containing the origin.*
- (vii) *$p < q$ and $b = H_B$, where B is a convex ray set.*

PROOF.

- (i) From Proposition 5.8 we know that $\varepsilon(f) = f \boxminus b$ induces an $(S,+)$-scale-space if $p = q = 1$. Thus α induces an $(S^\nu,\vee)$-scale-space for every $\nu > 0$.
- (ii) Straightforward consequence of Proposition 5.9(ii).
- (iii) Proposition 5.10 yields that ε induces an $(S^{1,q},+)$-scale-space if $k^\star = 1$. But $k = \infty$ iff $k^\star = 1$, and $b = I_B$ for some closed convex set in this case. We conclude that α induces an $(S^{\nu,\nu q},\vee)$-scale-space for every $\nu > 0$ in this case.
- (iv) Applying Proposition 5.10 with $1 < k^\star < \infty$, we obtain that ε induces an $(S^{p,q},+)$-scale-space if $b \in SP(k)$ and $q + k^\star(p-q) = 1$. Suppose that $1 < k^\star < \infty$ and that p, q are such that $q + k^\star(p-q) > 0$. Choosing $\nu = q + k^\star(p-q)$, we obtain that ε induces an $(S^{p/\nu,q/\nu},+)$-scale-space, hence α induces an $(S^{p,q},\vee)$-scale-space. Furthermore, the relation $q + k^\star(p-q) > 0$ is equivalent to $kp - q > 0$, and this yields the result.

(v)-(vii) follow from Proposition 6.4 in combination with Proposition 5.13$(i) - (iii)$. ∎

In the proof above, we have come across the condition $q+k^\star(p-q) > 0$. It is not difficult to derive this condition by direct means, that is, without taking recourse to Proposition 6.4. The fact that $T_\alpha(t)$ is an $(S,\vee)$-scale-space, or alternatively, a granulometry, means that $T_\alpha(t) \le T_\alpha(s)$ for $t \ge s$. This is equivalent to saying that $S(t)b$ is $S(s)b$-open, that is

$$S(t)b \,\square\, S(s)b = S(t)b, \quad t \ge s\,.$$

It is not difficult to show that this identity holds if and only if [10, Props. 4.37–4.38]

$$(-S(t))b = (-S(s)b) \boxminus f_{t,s}, \quad t \ge s\,,$$

for some function $f_{t,s}$. Taking conjugates, we arrive at the equation

$$(-S(t)b)^\star = (-S(s)b)^\star + f^\star_{t,s}, \quad t \ge s\,,$$

where $f^\star_{t,s}$ is a closed convex function. Assume that b is homogeneous of degree k, hence that $b^\star$ is homogeneous of degree $k^\star$, we find that

$$f^\star_{t,s}(\xi) = (t^N - s^N)(-b)^\star(\xi), \quad t \ge s\,,$$

with $N = q+k^\star(p-q)$. It is obvious that the constraint that $f_{t,s}$ is closed and convex is satisfied iff $N \ge 0$, i.e.

$$q + k^\star(p-q) \ge 0\,.$$

Finally, we point out that we can prove compatibility of the family $T_\alpha(t)$ with respect to another fixed scaling $\sigma(f)(x) = vf(x/u)$, analogous to the result in Proposition 5.7.

We conclude this section with Table 6.1 which summarizes the results of this section.

7. Conclusions

In this paper we proposed an algebraic construction scheme for scale-space operators. This construction scheme is based on an explicit definition of the scale-space operators, where we first downscale the image using a scaling operator $S^{-1}(t)$, then apply an image operator ψ, and finally upscale the outcome using $S(t)$. If the resulting one-parameter family has the semigroup property, it is called a scale-space. This construction scheme guarantuees that the resulting scale-space operator is a 'macroscopic' operator in the sense that the observation at scale t can be made without the need to generate all observations at smaller scales. We thus choose a distinctly different view on scale-space as the evolution processes guided by partial differential equations.

In this paper we have looked at two classes of image operators: (i) linear convolution operators and (ii) morphological operators, to be precise, translation invariant erosions given by infimal convolutions and translation invariant structural openings. The literature on linear scale-spaces is overwhelming and we have only sketched a brief overview in Section 3. Well-known in the linear context are the naturally linearly ordered semigroups, corresponding with scale-space operators where the 'scales add up' in a sequential application of scale-space operators. We have shown that our construction technique may also results in a supremal semigroup scale-space the operators of which are ideal low-pass filters.

p and q	properties of b	$T_\alpha(t)$
$p > 0$	$b \in SP(1)$ i.e. $b = H_B$ and B closed convex set	$T_\alpha(t)$ supremal scale-space
$p = q$	$b \in SP(1)$ i.e. $b = H_B$ and B closed convex set	$T_\alpha(t) = \alpha$, $t > 0$
any p and q	$b = I_B$ and B closed convex cone (in particular, $b \in SP(1)$)	$T_\alpha(t) = \alpha$
$p = q > 0$	b convex	$T_\alpha(t)$ supremal scale-space
$kp - q > 0$	$b \in SP(k)$ and $1 < k < \infty$	$T_\alpha(t)$ supremal scale-space
$q < p$	$b = H_B$, B closed convex set, $0 \in B$ in particular, $b \in SP(1)$	$T_\alpha(t)$ supremal scale-space
$p < q$	$b = H_B$, B closed convex ray set in particular, $b \in SP(1)$	$T_\alpha(t)$ supremal scale-space

Table 6.1. Table expressing for which p, q and which b the structural opening $\alpha(f) = f \,\square\, b$ induces a scale-space.

The derivations in the linear domain make heavily use of the Fourier transform, as this enables us to represent the image operators in a domain which is more suited for the problem at hand. Morphological operators allow for a similar change in representation to facilitate the analysis of scale-space operators: the slope transform. As we are only dealing with convex structuring functions in this paper, we may use Fenchel conjugation as the morphological equivalent of the Fourier transform. Using classical results from convex analysis, we prove the existence of various morphological scale-spaces. These scale-spaces can be subdivided into two classes: scale-spaces based on erosions and scale-spaces based on openings. This second class only comprises scale-spaces with a supremal semigroup operation, and therefore they do not have an infinitesimal generator. The first erosion-based class contains scale-spaces based on naturally ordered semigroups, such as the addition, as well as scale-spaces based on supremal-type semigroups; this depends not only on the properties of the structuring function but also on the underlying scaling.

Although we were not able to prove this we believe that (within the domain of the proposed construction scheme) morphological scale-spaces based on scalings other than of the umbral type, are necessarily based on subpolynomial structuring functions (see Section 5). In fact, this class of functions encompasses all structuring functions that have been used in the literature for the construction of morphological scale-spaces, ranging from the flat convex structuring functions ('sets') to the parabolic and quadratic structuring functions being the morphological equivalent of the Gaussian function.

We conclude this paper with the description of a problem that we have not been able to solve. Suppose that T is an $(S, \dot{+})$-scale-space, is it possible to endow the collection of operators $\{T(t) \mid t > 0\}$ with a total ordering $\preccurlyeq$ which is compatible with the ordering of $(0, \infty)$? This means that the following property must hold:

$$s \leq t \;\text{ iff }\; T(s) \preccurlyeq T(t)\,.$$

It is evident that the total ordering of $(0, \infty)$ induces a total ordering $\preccurlyeq$ on T if and only if $T(t_1) = T(t_2)$ for $t_1 < t_2$ implies that $T(s) = T(t_1)$ for all $s \in [t_1, t_2]$. If $\dot{+}$ is the supremum, then this is easy to show:

$$\begin{aligned} T(s) &= T(t_1 \vee s) = T(t_1)T(s) \\ &= T(t_2)T(s) = T(t_2 \vee s) = T(t_2) \end{aligned}$$

For the case where $\dot{+}$ is the addition, we have not been able to find a proof and in fact, we are not sure if the desired ordering $\preccurlyeq$ does exist under all circumstances. In view of Example 2.4, where we discussed a scaling $S(t)$ that is periodic in t, one might suspect that there exist additive scale-spaces which are periodic. Obviously, any such periodic scale-space would provide a counterexample against the supposition above.

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