An Osher-Type and Level-Set Scheme for Two-Fluid Flow Computations

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ABSTRACT. *A simple and efficient finite-volume method is presented for the computation of compressible flows of two immiscible fluids at very different densities. One novel ingredient in the method is a two-fluid Osher-type scheme, which is capable of computing the cell-face flux in case of two different fluids (e.g., water and air) left and right of the cell face. The other original property of the method is that a level-set term, for distinguishing between the two fluids, is consistently incorporated as one of the flux components. The level-set flux is properly treated by the Osher-type scheme.*

KEYWORDS: *compressible liquid-gas flows, interface capturing, Osher-type scheme, level-set method.*

1. Introduction

The present paper is directed towards an efficient, physically correct finite-volume computation of two-fluid flows consisting of, e.g., compressible water and air, at uniformly subsonic (preferably low-subsonic) speeds. Other premises are that the two fluids do not mix, that vaporization and condensation phenomena do not occur and that surface tension can be neglected. Last years, various

papers have been published in which specific finite-volume methods are presented for two-fluid flow computations. In most of these papers, a two-fluid flux formula is proposed, in which an approximate Riemann solver is applied. E.g., in [KAR 94], a Roe-type scheme is proposed, as is in [ABG 96]. In [SAU 99], Rusanov-type and Harten-Lax-van-Leer-type schemes are presented. Not yet seen in a two-fluid context is: an Osher-type scheme. (Note that the two-fluid method to which Osher himself contributed in [FED 99], the ghost fluid method, needs a single-fluid flux formula only.) In the present paper, we propose a two-fluid Osher-type flux formula. The formula is extremely simple and computationally very efficient. At the boundaries of the computational domain, it is completely consistent with that in the interior. Also of interest is that the Osher-type flux formula incorporates a level-set term for accurately capturing the two-fluid interface.

The 'pressure-oscillation problem', which is addressed and fixed in most of the aforementioned literature, is not considered in the present paper. Fixes that have already been proposed for it can be simply carried over to the present method.

2. Flow model

2.1. *Conservation equations*

In 1D, for a sufficiently small control volume Ω, conservation of mass and momentum read:

$$\int_\Omega \frac{d}{dt}\begin{pmatrix}\rho\\ \rho u\end{pmatrix} dx + \begin{pmatrix}\rho u\\ \rho u^2+p\end{pmatrix}_{\partial\Omega_{\text{right}}} - \begin{pmatrix}\rho u\\ \rho u^2+p\end{pmatrix}_{\partial\Omega_{\text{left}}} = 0, \qquad [1]$$

with ρ the bulk density

$$\rho = \alpha(x,t)\rho_w(p) + (1-\alpha(x,t))\rho_a(p), \qquad [2]$$

where α is, e.g., the volume-of-water fraction, and where $\rho_w(p)$ and $\rho_a(p)$ are the equations of state for water and air, respectively. To balance system [1]–[2], the latter equations and an equation for the location of the interface (determining α) still have to be chosen.

With the bulk-density formulation, in a finite-volume discretization, total mass of the fluid will be conserved, but not necessarily the masses of the two separate fluids. In case $\alpha(x,t)$ is poorly resolved, the two separate masses are poorly conserved as well. Hence, an accurate resolution of the interface location(s) is of paramount importance. For this purpose, we follow a level-set approach, to be discussed in the next section.

2.2. *Level-set equation*

To accurately resolve the interface location(s), a level-set approach [SET 96] is more appropriate than the classical volume-of-fluid (VOF) approach [HIR 81], because of its better smoothness properties at precisely the point of interest: the interface. Good smoothness of the level-set function is first taken care of in the level-set function's initialization. A common approach is to initialize the level-set function as the signed distance to the initial interface, with the distance positive in, e.g., water and negative in air. (Nonlinear initializations are possible [KOR 99].)

Denoting the level-set function by ϕ, it is advected by, in 1D:

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0. \qquad [3]$$

Combined with the bulk-mass conservation equation from [1], quasi-linear equation [3] may be written in the conservative control-volume form

$$\int_{\Omega} \frac{d(\rho\phi)}{dt} dx + (\rho u \phi)_{\partial\Omega_{\text{right}}} - (\rho u \phi)_{\partial\Omega_{\text{left}}} = 0. \qquad [4]$$

Conservation of $\rho\phi$ is not physical, there is no conservation law for it. The form [4] is simply practical because it is consistent with system [1]; it can be directly embedded into it. With $\phi(x,t)$ known, for any finite volume the VOF-function α can be computed. In Section 3.1, for an equidistant finite-volume grid, $\alpha = \alpha(\phi)$ is worked out in detail.

So, note that the VOF-*fraction* is used, but *not* the VOF-*method*. In the VOF-method, a transport equation for α is used. Instead, here we apply a transport equation for ϕ (in the consistent form of a conservation equation for $\rho\phi$).

2.3. *Equation of state*

In, e.g., water-air computations, for both fluids, elegant use can be made of a single equation of state, Tait's:

$$\frac{p + Bp_{\text{ref}}}{(1+B)p_{\text{ref}}} = \left(\frac{\rho}{\rho_{\text{ref}}}\right)^{\gamma}, \qquad [5]$$

where the subscript 'ref' indicates some reference state. For water, it holds: $\gamma = 7$, $B = 3000$ and for air: $\gamma = \frac{7}{5}$, $B = 0$. With [5], both the water and air density, $\rho_w(p)$ and $\rho_a(p)$, are convex functions of pressure. Likewise, the corresponding bulk density ρ according to [2] is. The physical consequences of this overall convexity are that neither locally low speeds of sound (lower than in pure water or pure air), nor entropy-condition-satisfying expansion shocks

can occur. A slight inconvenience of [2] in combination with [5] is that the calculation of p for known ρ and α ($\alpha \neq 0$ and $\alpha \neq 1$) needs to be done iteratively.

3. Discretization

3.1. *Finite volumes*

Summarizing, for a (sufficiently small) control volume Ω, the system of equations considered reads

$$\int_\Omega \frac{dq}{dt} dx + (f(q))_{\partial\Omega_{\text{right}}} - (f(q))_{\partial\Omega_{\text{left}}} = 0, \quad q = \begin{pmatrix} \rho \\ \rho u \\ \rho\phi \end{pmatrix}, \quad f(q) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u \phi \end{pmatrix}, \tag{6}$$

$$\rho = \alpha(\phi)\rho_w(p) + (1 - \alpha(\phi))\rho_a(p), \tag{7}$$

$$\rho_w(p) = \left(\frac{p + B_w p_{\text{ref}}}{(1 + B_w)p_{\text{ref}}}\right)^{\frac{1}{\gamma_w}} (\rho_w)_{\text{ref}}, \quad \rho_a(p) = \left(\frac{p + B_a p_{\text{ref}}}{(1 + B_a)p_{\text{ref}}}\right)^{\frac{1}{\gamma_a}} (\rho_a)_{\text{ref}}, \tag{8}$$

with $\alpha(\phi)$ the fraction of the size of Ω over which $\phi \geq 0$.

The natural space discretization for [6] is a finite-volume technique. We consider cell-centered finite volumes with, for convenience, constant mesh size. This choice directly allows us to work out the discretization of $\alpha(\phi)$. Consider finite volume Ω_i and its left and right neighbors, Ω_{i-1} and Ω_{i+1}, respectively, and define the level-set values at the cell faces $\partial\Omega_{i-\frac{1}{2}}$ and $\partial\Omega_{i+\frac{1}{2}}$ as

$$\phi_{i-\frac{1}{2}} = \frac{1}{2}(\phi_{i-1} + \phi_i), \quad \phi_{i+\frac{1}{2}} = \frac{1}{2}(\phi_i + \phi_{i+1}). \tag{9}$$

Then, for $\phi_i \geq 0$, we propose the following expression for α_i:

$$\phi_{i-\frac{1}{2}} \geq 0,\ \phi_{i+\frac{1}{2}} \geq 0: \quad \alpha_i = 1, \tag{10}$$

$$\phi_{i-\frac{1}{2}} < 0,\ \phi_{i+\frac{1}{2}} \geq 0: \quad \alpha_i = \frac{1}{2}\left(\frac{\phi_i}{\phi_i - \phi_{i-\frac{1}{2}}} + 1\right), \tag{11}$$

$$\phi_{i-\frac{1}{2}} \geq 0,\ \phi_{i+\frac{1}{2}} < 0: \quad \alpha_i = \frac{1}{2}\left(1 + \frac{\phi_i}{\phi_i - \phi_{i+\frac{1}{2}}}\right), \tag{12}$$

$$\phi_{i-\frac{1}{2}} < 0,\ \phi_{i+\frac{1}{2}} < 0: \quad \alpha_i = \frac{1}{2}\left(\frac{\phi_i}{\phi_i - \phi_{i-\frac{1}{2}}} + \frac{\phi_i}{\phi_i - \phi_{i+\frac{1}{2}}}\right). \tag{13}$$

So, in determining $\phi_{i-\frac{1}{2}}$ and $\phi_{i+\frac{1}{2}}$, as well as $x(\phi = 0)$, use is made of piecewise linear interpolation of ϕ. The linear interpolation is exact as long as the level-set function is the signed-distance function.

3.2. *Riemann-problem approach*

For the finite-volume formulation, we need a formula for the flux vector across a cell face. The formula must have built-in physics for accurately capturing the interface.

The exact solution of the 1D Riemann problem on each cell face, the well-known Godunov approach, requires the use of a numerical root finder. We avoid this by considering an approximate Riemann solver. For this, we prefer Osher's [OSH 82], particularly because of its consistent boundary-condition treatment. Denoting the left and right cell-face state by q_0 and q_1, and the flux formula by $F(q_0, q_1)$, the Osher scheme may be written as

$$F(q_0, q_1) = f(q_0) + \int_{q_0}^{q_1} \frac{df^-}{dq} dq, \qquad [14]$$

with $\frac{df^-}{dq}$ the negative eigenvalue part of $\frac{df}{dq}$. The eigenvalues of the present Jacobian are: $\lambda_1 = u - \sqrt{\frac{\partial p}{\partial \rho}}$, $\lambda_2 = u$, $\lambda_3 = u + \sqrt{\frac{\partial p}{\partial \rho}}$. ($\frac{\partial p}{\partial \phi}$ does not occur in the wave speeds!) The Riemann-invariant relations describing the two intermediate states $q_{\frac{1}{3}}$ and $q_{\frac{2}{3}}$ along the wave path in state space are for λ_1, λ_2 and λ_3 successively:

$$u_{\frac{1}{3}} + \int^{\rho_{\frac{1}{3}}} \frac{1}{\rho} \sqrt{\frac{\partial p}{\partial \rho}} d\rho = u_0 + \int^{\rho_0} \frac{1}{\rho} \sqrt{\frac{\partial p}{\partial \rho}} d\rho, \qquad \phi_{\frac{1}{3}} = \phi_0, \qquad [15]$$

$$u_{\frac{1}{3}} = u_{\frac{2}{3}} = u_{\frac{1}{2}}, \qquad p_{\frac{1}{3}} = p_{\frac{2}{3}} = p_{\frac{1}{2}}, \qquad [16]$$

$$u_{\frac{2}{3}} - \int^{\rho_{\frac{2}{3}}} \frac{1}{\rho} \sqrt{\frac{\partial p}{\partial \rho}} d\rho = u_1 - \int^{\rho_1} \frac{1}{\rho} \sqrt{\frac{\partial p}{\partial \rho}} d\rho, \qquad \phi_{\frac{2}{3}} = \phi_1. \qquad [17]$$

Hence, the level-set function ϕ can only change along the subpath corresponding with the eigenvalue λ_2, i.e., across the contact discontinuity. It is invariant along the outer subpaths; physically speaking: along the latter the distance to the two-fluid interface is constant. The integrals in [15] and [17] can be written out explicitly for the equations of state given in [8]. However, when, e.g., a water-air interface is captured, explicit calculation of $u_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ is hampered by nonlinearity; a transcendental equation needs to be solved.

3.3. *Linearized two-fluid Osher scheme*

Since ϕ is constant along the two outer subpaths of the wave path, along both subpaths the bulk density can only vary due to pressure changes. For flows that are low-subsonic, large density changes will not occur there and –

consequently – the integrals $\int^{\rho_{\frac{1}{3}}} \frac{1}{\rho}\sqrt{\frac{\partial p}{\partial \rho}}d\rho$ and $\int^{\rho_{\frac{2}{3}}} \frac{1}{\rho}\sqrt{\frac{\partial p}{\partial \rho}}d\rho$ can be linearized by good approximation around ρ_0 and ρ_1, respectively, yielding for [15]– [17]:

$$u_{\frac{1}{2}} = u_0 - (\rho_{\frac{1}{3}} - \rho_0)\frac{c_0}{\rho_0}, \qquad u_{\frac{1}{2}} = u_1 + (\rho_{\frac{2}{3}} - \rho_1)\frac{c_1}{\rho_1}. \qquad [18]$$

Likewise, $p_{\frac{1}{2}}$ can be linearized around ρ_0 and ρ_1:

$$p_{\frac{1}{2}} = p_0 + (\rho_{\frac{1}{3}} - \rho_0)c_0^2, \qquad p_{\frac{1}{2}} = p_1 + (\rho_{\frac{2}{3}} - \rho_1)c_1^2. \qquad [19]$$

Elimination of $\rho_{\frac{1}{3}} - \rho_0$ and $\rho_{\frac{2}{3}} - \rho_1$ from [18] and [19] gives $\frac{p_{\frac{1}{2}} - p_0}{u_{\frac{1}{2}} - u_0} = -\rho_0 c_0$ and $\frac{p_{\frac{1}{2}} - p_1}{u_{\frac{1}{2}} - u_1} = \rho_1 c_1$, i.e.,

$$\begin{pmatrix} u_{\frac{1}{2}} \\ p_{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \frac{C_0 u_0 + C_1 u_1 + (p_0 - p_1)}{C_0 + C_1} \\ \frac{C_1 p_0 + C_0 p_1 + C_0 C_1 (u_0 - u_1)}{C_0 + C_1} \end{pmatrix}, \quad C_0 \equiv \rho_0 c_0, \quad C_1 \equiv \rho_1 c_1. \qquad [20]$$

For the density and level-set function in the two intermediate points it holds:

$$\begin{pmatrix} \rho_{\frac{1}{3}} \\ \phi_{\frac{1}{3}} \end{pmatrix} = \begin{pmatrix} \rho_0 + \frac{p_{\frac{1}{2}} - p_0}{c_0^2} \\ \phi_0 \end{pmatrix}, \qquad \begin{pmatrix} \rho_{\frac{2}{3}} \\ \phi_{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} \rho_1 + \frac{p_{\frac{1}{2}} - p_1}{c_1^2} \\ \phi_1 \end{pmatrix}. \qquad [21]$$

Ignoring all supersonic possibilities among all possible combinations of signs of $u_0 - c_0$, $u_{\frac{1}{2}} - c_{\frac{1}{3}}$, $u_{\frac{1}{2}} + c_{\frac{2}{3}}$ and $u_1 + c_1$ (note the consequent, large improvement in efficiency), the linearized two-fluid scheme reads then:

$$F(q_0, q_1)_{u_{\frac{1}{2}} \geq 0} = \begin{pmatrix} \rho_{\frac{1}{3}} u_{\frac{1}{2}} \\ \rho_{\frac{1}{3}} u_{\frac{1}{2}}^2 + p_{\frac{1}{2}} \\ \rho_{\frac{1}{3}} u_{\frac{1}{2}} \phi_{\frac{1}{3}} \end{pmatrix}, \quad F(q_0, q_1)_{u_{\frac{1}{2}} \leq 0} = \begin{pmatrix} \rho_{\frac{2}{3}} u_{\frac{1}{2}} \\ \rho_{\frac{2}{3}} u_{\frac{1}{2}}^2 + p_{\frac{1}{2}} \\ \rho_{\frac{2}{3}} u_{\frac{1}{2}} \phi_{\frac{2}{3}} \end{pmatrix}. \qquad [22]$$

We remark that the real nonlinear flux functions $f(q_{\frac{1}{3}})$ and $f(q_{\frac{2}{3}})$ are applied, and not $F(q_0, q_1) = f(q_0) + (q_{\frac{1}{3}} - q_0)\frac{df(q_0)}{dq}$ if $u_{\frac{1}{2}} \geq 0$ and $F(q_0, q_1) = f(q_1) + (q_{\frac{2}{3}} - q_1)\frac{df(q_1)}{dq}$ if $u_{\frac{1}{2}} \leq 0$. There is no need for the latter linearized formulae. On the contrary, as opposed to [20]–[22], they may give rise to an erroneous, ambiguous flux at $u_{\frac{1}{2}} = 0$ (steady contact discontinuity); $f(q_0) + (q_{\frac{1}{3}} - q_0)\frac{df(q_0)}{dq}$ and $f(q_1) + (q_{\frac{2}{3}} - q_1)\frac{df(q_1)}{dq}$ may be different for $u_{\frac{1}{2}} = 0$.

3.4. *Boundary-condition treatment*

A very favorable property of the Osher scheme is that the fluxes across the boundary faces can be computed with the same formula as that for the interior

faces, i.e., with [22] as well. Denoting the state at the boundary by q_b, in case of a left boundary $q_0 = q_b$ and in case of a right $q_1 = q_b$. We work out the inflow and outflow boundary, and the non-permeable boundary as a limit case. For all three it holds, for boundary at the left and right, respectively:

$$\frac{p_b - p_1}{u_b - u_1} = C_1, \qquad \frac{p_b - p_0}{u_b - u_0} = -C_0. \qquad [23]$$

3.4.1. *Inflow boundary*

From [23], it follows that the two boundary conditions to be imposed here cannot be u_b and p_b simultaneously; when u_b is imposed, p_b follows – vice versa – when p_b is imposed, u_b follows. Hence, the second boundary condition must be one for ϕ_b. To compute the corresponding boundary flux $f(q_b) = \left(\rho_b u_b, \rho_b u_b^2 + p_b, \rho_b \phi_b\right)^T$, the '0D' bulk density still needs to be defined. (In 2D and 3D, the bulk density can be computed in a normal 1D and 2D way, respectively.) In 1D, an appropriate '0D' choice is

$$\rho_b = \rho_w(\phi_b), \quad \text{for } \phi_b \geq 0, \qquad \rho_b = \rho_a(\phi_b), \quad \text{for } \phi_b < 0. \qquad [24]$$

3.4.2. *Outflow boundary*

Here, in addition to [23], the equations

$$\phi_b = \phi_0, \qquad \phi_b = \phi_1 \qquad [25]$$

are available. So, the single boundary condition to be imposed must be u_b or p_b, or some combination of both. The bulk density ρ_b is defined as in the inflow case.

3.4.3. *Non-permeable boundary*

At a non-permeable boundary (at least) $u_b = 0$ must be imposed, which, given [23], already determines p_b. Considering a non-permeable boundary as the limit case of an inflow boundary, ϕ_b must still be imposed. Considering it as the limit of outflow, ϕ_b follows from the interior solution ($\phi_b = \phi_1$ for left boundary and $\phi_b = \phi_0$ for right). The outflow-limit case is to be preferred. As opposed to in the inflow-limit case, it allows the interface to freely move along the non-permeable boundary. Also here, the bulk density may be defined according to [24].

4. Conclusions

To accurately compute compressible, immiscible two-fluid flows with very large density differences (water-air flows, e.g.), we have proposed a method that uses a level-set technique to distinguish between the two fluids. The level-set equation has been incorporated consistently into the system of hyperbolic

conservation laws. The resulting equations have been discretized through a finite-volume method. To compute the fluxes across the finite-volume walls (the level-set flux being one of the flux-vector components), we have proposed a two-fluid, linearized Osher scheme (given by [22]). The scheme allows a physically correct capturing of the interface across a single cell face, as well as a neat boundary-condition treatment (no sticking of interfaces to solid walls, e.g.). The novel scheme combines good physical properties with great simplicity and efficiency. There are no principal difficulties in fixing the technique for the 'pressure-oscillation problem', and in extending it to multi-D.

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