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## ABSTRACT

A complete ternary tree is a tree in which every non-leaf vertex has exactly three children. We prove that a complete ternary tree of height  $h$ ,  $TTh$ , is embeddable in a hypercube of dimension  $3h$ . This result coincides with the result of [2]. However, in this paper, the embedding utilizes the knowledge of AVL trees. We prove that a subclass of AVL trees is a subgraph of hypercube. The problem of embedding AVL trees in hypercube is an independent emerging problem.

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*Keywords and Phrases:* Parallel algorithm, hypercube, complete ternary tree, AVL tree, embedding, dilation

*Note:* A part of second author's work was carried out during her three months visit (May-July 2008) to Centrum Wiskunde en Informatica, PNA1 Group, 1098 SJ, Amsterdam, The Netherlands.



# Embedding complete ternary tree in hypercubes using AVL trees

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## Abstract

A complete ternary tree is a tree in which every non-leaf vertex has exactly three children. We prove that a complete ternary tree of height  $h$ ,  $TT_h$ , is embeddable in a hypercube of dimension  $\lceil 1.5h \rceil + 1$ . This result coincides with the result of [2]. However, in this paper, the embedding utilizes the knowledge of AVL trees. We prove that a subclass of AVL trees is a subgraph of hypercube. The problem of embedding AVL trees in hypercube is an independent emerging problem.

**Keywords:** Parallel algorithm, hypercube, complete ternary tree, AVL tree, embedding, dilation

## 1 Introduction

An embedding  $\varphi$  of a (guest) graph  $G = (V_G, E_G)$  into a (host) graph  $H = (V_H, E_H)$  is a map  $\varphi : V_G \rightarrow V_H$  (not necessarily a bijection) such that every edge  $(u, v)$  of  $G$  is mapped onto a path  $P(\varphi(u), \varphi(v))$  which connects  $\varphi(u)$  and  $\varphi(v)$  in  $H$ . The dilation of an edge  $(u, v)$  in  $G$  is defined to be the length of the path  $P(\varphi(u), \varphi(v))$ . The dilation of  $\varphi$ ,  $dil(\varphi)$  is then defined to be the integer  $\max \{ \text{length}(P(\varphi(u), \varphi(v))) : (u, v) \text{ is an edge in } G \}$ . The expansion of  $\varphi$  is defined to be the ratio of  $V_H$  and  $V_G$ .

One of the motivations for these concepts is in the area of parallel algorithms and parallel computers. Here, the computational structure of a parallel algorithm  $A$  is represented by a graph  $G(A)$  and the interconnection network of a parallel computer  $N$  is represented by a graph  $H(N)$ . An embedding of  $G(A)$  into  $H(N)$  describes the working of the parallel algorithm  $A$  when implemented on  $N$ . The  $dil(\varphi)$  is used to estimate the computational running time and the expansion of  $\varphi$  is used to estimate the number of unutilized processors in  $N$ . Leighton [9] gives an extensive survey of embedding various graphs into interconnection networks like hypercubes, meshes and tori. Note, that the dilation of any embedding is atleast 1, and if  $dil(\varphi) = 1$ , then  $G$  is (isomorphic with) a subgraph of  $H$ . In the following all our embeddings are injections, so the expansion is atleast 1. Clearly, an injective embedding with dilation 1 and expansion 1 is the one with least communication delay and the most cost effective. In this note, we first embed a subfamily of AVL trees into hypercubes, then use this result to give an alternate proof of a result of [2] which embeds complete ternary trees into hypercubes.

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## 2 Preliminaries

Before we proceed to the embedding, we give a brief description of hypercubes and AVL trees.

### 2.1 Hypercubes

An  $n$ -dimension hypercube,  $Q_n$ , has  $2^n$  vertices each labelled with a binary string of length  $n$ . Two vertices are adjacent if and only if their labels differ in exactly one position. We use the following properties of hypercubes for the embedding.

- $Q_n$  is  $n$ -regular and its diameter is  $n$ .
- $Q_n$  possesses exactly  $2^n n!$  automorphisms which can be described as follows. For each subset  $S$  of  $\{1, 2, \dots, n\}$  and each permutation  $\pi$  of  $\{1, 2, \dots, n\}$ , the map  $f(S, \pi) : V(Q_n) \rightarrow V(Q_n)$  defined by  $f(S, \pi)(x_1 x_2 \dots x_n) = (y_1 y_2 \dots y_n)$  is an automorphism of  $Q_n$  where
 
$$y_i = \begin{cases} \bar{x}_{\pi(i)}, & \text{if } \pi(i) \text{ is in } S \\ x_{\pi(i)}, & \text{if otherwise} \end{cases}$$
- $Q_n$  is a vertex-symmetric, edge-symmetric and  $P_3$ -symmetric graph. (Here,  $P_3$  is a path on 3 vertices)

A graph  $G$  is called a  $P_k$ -symmetric graph if for any two paths  $P = (u_1, u_2, \dots, u_k)$  and  $Q = (v_1, v_2, \dots, v_k)$ , there exists an automorphism  $\alpha$  of  $G$  such that  $\alpha(u_i) = v_i$ , for every  $i$ ,  $1 \leq i \leq k$ .  $P_1$ -symmetric graphs and  $P_2$ -symmetric graphs are referred to as vertex-symmetric graphs and edge-symmetric graphs respectively.

### 2.2 AVL trees

An AVL tree is a rooted binary tree  $T$  in which for every vertex  $v$  in  $V(T)$ , the heights of the subtrees, rooted at the left and right child of  $v$ , differ by at most one. Adelson-Velskii and Landis [1] defined these trees to provide most efficient data structures for computational routines like searching and sorting; see [3, 4, 8, 12, 13]. It is also known as height-balanced tree. Subsequently, there have been several variations of AVL trees all of which maintain some balance in the heights of the subtrees rooted at every internal node. All these variations are studied under a common title called the height balanced trees; see [5, 7, 11]. The importance of AVL trees is due to the fact that their height is logarithmic to their size. Therefore, operations like searching can be performed in logarithmic time on an AVL data structure, whereas the same operations can take linear time (in the worst case) on an arbitrary binary data structure. We are concerned with a subclass of AVL trees,  $T_h$  of height  $h$  which is defined as follows:  $T_0 = K_1$ ,  $T_1 = K_{1,2}$  and  $T_h$  ( $h \geq 2$ ) is obtained by taking three copies of  $T_{h-2}$  with roots  $r_1, r_2, r_3$  (say), and two new vertices  $R, S$  and adding the edges  $(R, S)$ ,  $(S, r_1)$ ,  $(S, r_2)$  and  $(R, r_3)$ ; see Figure 1 and  $R$  is designated as the root of  $T_h$ . The number of vertices ( $t_h$ ) of  $T_h$  can be computed using the recurrence relation  $t_h = 3t_{h-2} + 2$ , for  $h \geq 2$  and the initial conditions  $t_0 = 1$  and  $t_1 = 3$ .

On solving this relation, we get  $t_h = \begin{cases} 2(3)^{\frac{h}{2}} - 1, & \text{if } h \text{ is even} \\ 4(3)^{\frac{h-1}{2}} - 1, & \text{if } h \text{ is odd} \end{cases}$

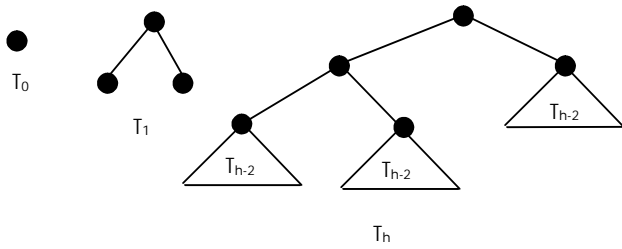


Figure 1: Structure of  $T_h$

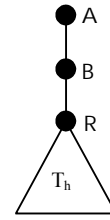


Figure 2: Structure of  $T_h^*$

### 3 Embedding

#### 3.1 Embedding AVL tree into hypercube

We are now ready to embed the AVL tree  $T_h$  (defined in section 2.2) into  $Q_{d(h)}$  with expansion  $2^{d(h)} / t_h$  which is close to 1. Thus the embedding is nearly optimal. The embedding of the AVL tree  $T_h$  is given by the following theorem.

**Theorem 3.1:** For every  $h \geq 2$ ,  $T_h$  is a subtree of  $Q_{d(h)}$  where  $d(h) = \begin{cases} \left\lceil \frac{4h}{5} \right\rceil + 1, & \text{if } h \text{ is even} \\ \left\lceil \frac{4(h-1)}{5} \right\rceil + 2, & \text{if } h \text{ is odd} \end{cases}$

**Proof:** To achieve such an embedding, we label  $T_h$  using  $d(h)$ -bit binary strings by recursion on  $h$  and depending on the value of  $h \pmod{10}$ . Since  $Q_{d(h)} = Q_{d(h-2)} \times Q_t$  where  $t = d(h) - d(h-2)$ , we prefix every label of a copy of  $T_{h-2}$  by one of  $2^t$ -bit string. We also make use of a small extension in the structure of the tree for the proof to work. Given a tree  $T_h$  with root  $R$ , we denote by  $T_h^*$ , a supertree of  $T_h$  formed by adding two new vertices  $A$  and  $B$  and two new edges  $(A,B)$  and  $(B,R)$ . We call  $T_h^*$ , the auxiliary tree of  $T_h$  and the path  $(A,B,R)$  the auxiliary path of  $T_h^*$ ; see Figure 2.

#### 3.2 Embedding complete ternary tree into hypercube

A complete ternary tree is a tree in which every non-leaf vertex has exactly three children. The structure of ternary trees has found application in parallel computing. In [10], the authors have developed and implemented software whose strategy was implemented using the CS tool software and a ternary tree network topology. A balanced ternary tree has been used to represent the widely used binary trees; see [12]. Hence, the problem of embedding ternary trees in hypercubes is of interest. In [6], Havel has conjectured the following:

**Conjecture:** The complete ternary tree can be embeddable into its optimal hypercube with dilation 2.

In this section, we first provide an embedding of a complete ternary tree of height  $h$ ,  $TT_h$  in  $T_{2h}$  (discussed in section 2). We next provide an embedding of  $TT_h$  into the hypercube using the result of the previous section. The dimension of optimal hypercube for complete ternary tree is

$\lceil 1.585h \rceil + 1$ . However in this paper, the expansion of our embedding is  $\lceil 1.6h \rceil + 1$ . Hence, there is a slight relaxation in the expansion factor as compared to the statement of the conjecture.

**Theorem 3.2:** A complete ternary tree  $TT_h$  of height  $h$  is embeddable into the AVL tree  $T_{2h}$  with dilation 2.

**Proof:** We prove by induction on  $h$ . Figure 3 gives a pictorial depiction of the embedding  $TT_h$  in  $T_{2h}$  with dilation 2.

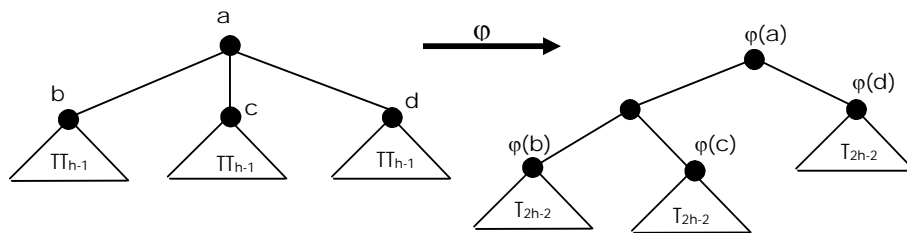


Figure 3: Embedding  $TT_h$  into  $T_{2h}$

Combining Theorem 3.1 and Theorem 3.2, we have the following result.

**Corollary 3.3:** A complete ternary tree  $TT_h$  of height  $h$  is embeddable into a hypercube of dimension  $\lceil 1.6h \rceil + 1$  with dilation 2.

The result has been proved by S.A. Choudum et. al., in [2]. The interesting aspect of the proof provided in this paper is the use of an AVL tree as an intermediate tree. The problem of embedding AVL trees in hypercubes is an independent research problem and is open.

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