Existence of phase transition for heavy-tailed continuum percolation

Artëm Sapozhnikov*

We consider a continuum percolation model in $\mathbb{R}^d$, where $d \geq 2$. It is given by a homogeneous Poisson process of intensity λ and independent radii random variables of common distribution of a positive random variable r. Let λ_c be the critical intensity for the existence of infinite cluster. We provide conditions for positivity of λ_c. In case $\mathbf{E}r^{2d-1} = \infty$ our result is new.

Keywords: CONTINUUM PERCOLATION, PERCOLATION, LONG-RANGE PERCOLATION, CONNECTIVITY, CRITICAL DENSITY, MOMENTS

1 Introduction

Let $(X_n, r_n)_{n\geq 1}$ be a marked Poisson process with intensity λ in $\mathbb{R}^d$, $d \geq 2$. The marks (r_n) are radii of closed Euclidean balls centered at the points (X_n). Two points X_i and X_j of the Poisson process X are adjacent, $X_i \sim X_j$, if $D(X_i, r_i) \cap D(X_j, r_j) \neq \emptyset$, where $D(x, R) = \{y \in \mathbb{R}^d \ : \ ||x - y||_2 \leq R\}$. We say that $x, y \in \mathbb{R}^d$ are connected, $x \leftrightarrow y$, if there are $X_{i_1}, \ldots, X_{i_l} \in X$ such that $x \in D(X_{i_1}, r_{i_1})$, $y \in D(X_{i_l}, r_{i_l})$ and $X_{i_k} \sim X_{i_{k+1}}$ for all $1 \leq k < l$. For $x \in \mathbb{R}^d$, let $\mathcal{I} = \{i \ : \ x \leftrightarrow X_i\}$ and $C_x = \cup_{i\in\mathcal{I}} D(X_i, r_i)$. Set C_x is called the cluster at x. The number of elements in $\mathcal{I}$ is called the size of the cluster, and is denoted $|C_x|$. We write $\mathbb{P}_\lambda$ for the probability measure associated with X.

Continuum percolation was introduced by Gilbert [7] as a model of random network in communication theory. It has recently attracted a lot of attention because of its importance in various applications including wireless networks, sensor networks etc (see [6] and many references therein). For the physical applications of continuum percolation we refer the reader to [13]. The first rigorous analysis of the model is given in [9, 15, 16]. Basic methods for continuum percolation are developed in [2, 9, 14, 15, 16]. The uniqueness of unbounded occupied and vacant components is proved in [11]. The principal reference for continuum percolation is [10].

Similarities between continuum and lattice percolation were noted by Gilbert [7]. However the effect of unbounded radii on the properties of a cluster makes continuum percolation essentially different from the lattice one. The difference was noted in [9]. It is known that in the case of site or bond percolation on $\mathbb{Z}^d$, the critical probability at which percolation takes place is often the same as the probability at which mean cluster size becomes infinite [1, 12]. Hall [9] showed that for continuum percolation, the critical intensities at which cluster size and mean cluster size become infinite are not necessarily the same. More precisely (see [9, 10] for the proof and [5] for a more general result),

*Department of Mathematics / Boole Centre for Research in Informatics, University College Cork, Cork, Ireland; as2@proba.ucc.ie

Proposition 1.1. *1. For $\lambda > 0$ the origin is covered by finitely many balls with probability one (i.e. $\mathbb{P}_\lambda(|\{i \, : \, ||X_i||_2 \le r_i\}| < \infty) = 1$) if and only if $\mathbf{E} r_1^d < \infty$;*

2. $\mathbb{E}_\lambda |C_0| < \infty$ for some $\lambda > 0$ if and only if $\mathbf{E} r_1^{2d} < \infty$;

3. If $\mathbf{E} r_1^{2d-1} < \infty$ then $\mathbb{P}_\lambda(|C_0| < \infty) = 1$ for some $\lambda > 0$.

Remark 1. 1. If $\mathbf{E} r_1^d = \infty$ then, for any $\lambda > 0$, the space is completely covered by the balls centered at the points of X.
2. If $\mathbf{E} r_1^{2d-1} < \infty$ and $\mathbf{E} r_1^{2d} = \infty$ then there is $\lambda > 0$ such that the cluster at the origin is finite almost surely but the mean size of the cluster is infinite.

The proof of Proposition 1.1 is based on approximations of the size of the cluster at the origin by multi-type branching process. According to this approximation, roughly speaking, each ball of radius r_i generates $\sim r_i^d$ other balls of radius r_j, each of which independently generates $\sim r_j^d$ other balls of radius r_k and so on. Therefore the process blows up for any λ if $\mathbf{E} r_1^{2d} = \infty$. This explains the second part of Proposition 1.1. The third part is obtained by a slightly different choice of a branching process.

In this note we are interested in conditions under which the critical intensity $\lambda_c = \sup\{\lambda \, : \, \mathbb{P}_\lambda(|C_0| < \infty) = 1\}$ is positive. The main result is the following theorem.

Theorem 1.1. *If $\mathbf{E} r_1^s < \infty$ for some $s > d$ then $\lambda_c > 0$.*

The main difficulty in the proof of Theorem 1.1, as it was noticed before, comes from the existence of very big balls, which intersect a lot of other balls. Therefore the 'independent' branching process estimate is not useful any more. The proof of Theorem 1.1 consists of two main steps. On the first step we show that the presence of very big balls in a box is very unlikely. On the second step we show that, given there are no very big balls in a box, the probability of two points connectivity within the box decays exponentially with the distance between the points.

The proof of Theorem 1.1 is based on the analysis of some discrete model which we describe in Section 2. The discrete model is a generalization of the classical site percolation. Therefore an analogue of Theorem 1.1 for that model (see Theorem 2.1) is of independent interest.

We describe an auxiliary discrete model in Section 2. The core of the analysis of that model is the renormalization structure of [4] which we give in Section 3. In Section 4 we prove Theorem 2.1. We deduce Theorem 1.1 from Theorem 2.1 in Section 5.

2 Discrete model

We consider a random graph $\mathcal{G} = (\mathbb{Z}^d, E)$. The vertices of $\mathcal{G}$ are the sites of $\mathbb{Z}^d$. To define edges in $\mathcal{G}$, we introduce a set of independent identically distributed positive random variables $(r_x)_{x \in \mathbb{Z}^d}$. The set of edges of $\mathcal{G}$ is $E = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \, : \, |x - y| \le r_x + r_y\}$, where $|x| = \max_{1 \le i \le d} |x_i|$.

We study site percolation on $\mathcal{G}$. We say that a vertex is open with probability p and closed with probability $(1 - p)$ independently of other vertices. The case of constant radii ($r_x = c$ a.s. for all $x \in \mathbb{Z}^d$) corresponds to the $2c$ dependent site percolation on $\mathbb{Z}^d$. For every p, the product measure on $\mathbb{R}_+^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d}$ associated with the model is denoted by $\mathbb{P}_p$.

The ball of radius R centered at x will be denoted by $W(x, R) = \{y \in \mathbb{Z}^d \, : \, |x - y| \le R\}$.

For $A \subset \mathbb{Z}^d$, we say that two points x and y of G are connected within A, and we write $x \overset{A}{\leftrightarrow} y$, if there are open sites $v_1, \ldots, v_n$ in A such that $(v_i, v_{i+1}) \in E$ for all i, $x \in W(v_1, r_{v_1})$ and $y \in W(v_n, r_{v_n})$. If $A = \mathbb{Z}^d$ then we simply say that x and y are connected, and we write $x \leftrightarrow y$. Note that we do not require x and y to be open, however v_1 may coincide with x if x is open or v_n may coincide with y if y is open.

An open cluster at x, C_x is the set of all open sites connected to x. The size of C_x is the number of open cites in C_x. It is denoted by $|C_x|$.

We prove the following theorem.

Theorem 2.1. *If $\mathbf{E}r_0^s < \infty$ for some $s > d$ then there exists $p > 0$ such that $\mathbb{P}_p(|C_0| < \infty) = 1$.*

The proof of Theorem 2.1 is based on the renormalization introduced in [4] to study the chemical distance in sparse long-range percolation. We give a brief description of the renormalization scheme in the next section adapted to the model we consider.

3 Renormalization structure

Let $M > 100$ be an even integer. For $k \geq 0$, let

$$A_k = M(k!)^2.$$

A cube $Q_k(a) = a + [0, A_k)^d \subset \mathbb{Z}^d$ is called a **k-block**. The $(k-1)$-blocks $Q_{k-1}(a + A_{k-1}x)$, where $x \in [0, k^2)^d$, are **children** of $Q_k(a)$.

Definition 3.1. We say that a 0-block Q is **good** under the configuration r if $r_x \leq A_0/100$ for all $x \in Q$. A k-block Q is **good** under the configuration r if

1. $r_x \leq A_{k-1}/100$ for all $x \in Q$;
2. All but at most one children of Q are good;
3. Suppose that configuration r' agrees with r on Q and $r'_y = 1$ for $y \notin Q$, then all the translations of Q by elements of $\{-A_{k-1}/2, 0, A_{k-1}/2\}^d$ satisfy 1 and 2 under the configuration r'.

The proof of the next lemma word for word repeats the proof of [4, Lemma 1]. We omit it.

Lemma 3.1. *Let P_k be the probability that $Q_k(0)$ is not a good k-block. If M is large enough and $\mathbf{E}r^s < \infty$ for some $s > d$ then*

$$\sum_{k=1}^{\infty} P_k < \infty.$$

4 Proof of Theorem 2.1.

Let $Q_k = [-A_k/2, A_k/2)^d$. It is sufficient to show that $\mathbb{P}_p(0 \leftrightarrow Q_k^c) \to 0$ as $k \to \infty$, for some $p > 0$. Moreover, without loss of generality we can assume that the radii are positive integers.

The condition on the distribution of radii assures that the probability for a ball centred in the compliment of Q_k to intersect $\frac{1}{2}Q_k = [-A_k/4, A_k/4)^d$ vanishes

$$\mathbb{P}_p\left(\text{there exists } x \notin Q_k \text{ such that } W(x, r_x) \cap \frac{1}{2}Q_k \neq \emptyset\right) \to 0 \tag{4.1}$$

as $k \to \infty$. Indeed, the above probability is bounded by

$$\sum_{n=0}^{\infty} \sum_{x \,:\, |x| = n + Q_k} \mathbf{P}_p\left(r_x \geq n + \frac{1}{2}Q_k\right).$$

The cardinality of set $\{x : |x| = m\}$ is at most $K_d m^{d-1}$, for all m, where K_d only depends on d. Hence the probability in (4.1) is bounded by

$$K_d 2^{d-1} \sum_{n \geq Q_k/2} n^{d-1} \mathbf{P}_p (r_0 \geq n) \to 0$$

as $k \to \infty$. From Lemma 3.1,

$$\mathbb{P}_p(Q_k \text{ is a good } k\text{-block}) \to 1$$

as $k \to \infty$. Therefore to complete the proof of Theorem 2.1 it is sufficient to prove that

$$\mathbb{P}_p \left(0 \overset{Q_k}{\longleftrightarrow} \frac{1}{2} Q_k^c,\ Q_k \text{ is a good } k\text{-block} \right) \to 0,$$

for some $p > 0$, as $k \to \infty$. We prove a more general result.

Lemma 4.1. *There exist $p > 0$ and $\alpha > 0$ such that if Q is a block of size A_k and x and y in Q satisfy $|x - y| > A_k/8$ then*

$$\mathbb{P}_p \left(x \overset{Q}{\longleftrightarrow} y,\ Q \textit{ is a good } k\textit{-block} \right) \leq e^{-\alpha |x-y|}. \tag{4.2}$$

Proof of Lemma 4.1. Let $k_0 > 0$ be a large integer, δ is a positive constant. Let $\alpha = \inf_k \alpha_k$, where

$$\alpha_k = \delta \left(\prod_{i=k_0}^{k} \left(1 - \frac{16 \cdot 3^d}{i^2} \right) \right) \left(\prod_{i=k_0}^{k} \frac{A_{i-1} - 8A_{i-2}}{A_{i-1} + 8A_{i-2}} \right).$$

We assume that $\prod_{i=a}^{b} = 1$ for $a > b$. Note that $\alpha > 0$ when k_0 is large enough. The result immediately follows from Lemma 4.2. □

Remark 2. In the definition of α_k we split the product onto two parts to show that the first product results from random shortcuts of sites in $\mathbb{Z}^d$ in exactly the same way as it appears in [4], and the second product results from percolation.

Let $\mathcal{R}_Q$ be a set of all the configurations of radii of the balls centred in Q, $\mathbf{r} \in \mathbb{N}^Q$ for which Q is a good k-block. Since the radii are positive integers and Q is a good k-block, $\mathcal{R}_Q$ is a finite set. We prove

Lemma 4.2. *There exist $\delta > 0$, $k_0 > 0$ and $p > 0$ such that, for any $k \geq k_0$, if Q is a good k-block and $x, y \in Q$ are such that $|x - y| > A_k/8$ then*

$$\mathbb{P}_p \left(x \overset{Q}{\longleftrightarrow} y \mid r_Q = \mathbf{r} \right) \leq e^{-\alpha_k |x-y|}, \quad \textit{for all } \mathbf{r} \in \mathcal{R}_Q. \tag{4.3}$$

Remark 3. The left hand side of (4.3) is a probability of connectivity of x and y in site percolation model on the finite graph with vertex set Q and edge set induced by configuration $\mathbf{r}$.

The proof of Lemma 4.2 is based on Lemma 4.3 and Lemma 4.4. We firstly state and prove these lemmas. We will need some definitions. To shorten the presentation we write

$$\mathbb{P}_p^{\mathbf{r}}(\cdot) = \mathbb{P}_p(\cdot \mid r_Q = \mathbf{r}).$$

Definition 4.1. Let Q be a good k-block, and the configuration of balls centred in Q is $\mathbf{r} \in \mathcal{R}_Q$. Then for any $a \in \{-A_{k-1}/2, 0, A_{k-1}/2\}^d$, there exists at most one child of $a + Q$ that is not good. We denote these not good $(k-1)$-blocks by $B_1, \ldots, B_j$, and let $B = B_1 \cup \ldots \cup B_j$. The set B is deterministic once we fix the radii $\mathbf{r} \in \mathcal{R}_Q$. We also note that $j \leq 3^d$.

The R-neighbourhood of a set $A \subset \mathbb{Z}^d$ is denoted by $W(A, R) = \cup_{x\in A} W(x, R)$.

Definition 4.2. We introduce the sets $U_s = U_s(x, y, \mathbf{r}) \subset W(B, 3A_{k-1}/100)^{2s}$: $(x_1, x'_1, \ldots, x_s, x'_s) \in U_s$ if, for all $i, j \in \{1, \ldots, s\}$,

1. $x_i, x'_i \in W(B, 3A_{k-1}/100)\backslash B$;
2. $x_i \neq x_j, x'_i \neq x'_j$;
3. $|x_i - x'_i| > A_{k-1}/2$;
4. $\sum_{i=1}^{s} |x_i - x'_i| \geq \left(1 - \frac{16\cdot 3^d}{k^2}\right) |x - y|$.

Lemma 4.3. *If Q is a good k-block, $\mathbf{r} \in \mathcal{R}_Q$, and $x, y \in Q$ are such that $|x - y| > A_k/8$ then*

$$\mathbb{P}_p^{\mathbf{r}}(x \overset{Q}{\longleftrightarrow} y) \leq \sum_{s=1}^{3^d} \left(2A_{k-1}^d 3^d\right)^{2s} \max_{\{x_i,x'_i\}\in U_s} \mathbb{P}_p^{\mathbf{r}}(x_1 \overset{Q\backslash B}{\longleftrightarrow} x'_1) \ldots \mathbb{P}_p^{\mathbf{r}}(x_s \overset{Q\backslash B}{\longleftrightarrow} x'_s). \tag{4.4}$$

Proof of Lemma 4.3. Let $\gamma = (v_1, \ldots, v_l)$ be an open path connecting x and y within Q (see Section 2 for the definition). If there are several such paths then we choose the first one according to some order on the set of paths between x and y. It will be useful to define v_i for all $i \in \mathbb{Z}$. Let $v_i = v_1$ for $i < 1$ and $v_i = v_l$ for $i > l$. From Definition 3.1, radii $r_{v_i} \leq A_{k-1}/100$ for all $1 \leq i \leq l$.

Let a_1 be the smallest value i such that $v_i \in W(B, A_{k-1}/50)$ (the set B is defined in Definition 4.1), and $b_1 \in \{1, \ldots, j\}$ such that $v_{a_1} \in W(B_{b_1}, A_{k-1}/50)$. If v_{a_1} is in $A_{k-1}/50$-neighbourhood of several blocks, i.e. $v_{a_1} \in \cap_{i\in\mathcal{L}} W(B_i, A_{k-1}/50)$ for a subset $\mathcal{L}$ of $\{1, \ldots, j\}$, then we choose b_1 arbitrarily from the set $\mathcal{L}$. Let z_1 be the largest i such that $v_i \in W(B_{b_1}, A_{k-1}/50)$. For any $t \geq 1$, let a_{t+1} be the smallest $i > z_t$ such that $v_i \in W(B, A_{k-1}/50)$. If there is no such i we let $a_{t+1} = l$. If $a_{t+1} < l$, let b_{t+1} be such that $v_{a_{t+1}} \in W(B_{b_{t+1}}, A_{k-1}/50)$. If there is more than one i such that $v_{a_{t+1}} \in W(B_i, A_{k-1}/50)$ then choose b_{t+1} arbitrarily from that i's. Let z_{t+1} be the largest i such that $v_i \in W(B_{b_{t+1}}, A_{k-1}/50)$. If $a_{t+1} = l$ we let $z_{t+1} = l + 1$. Since the path γ is finite, after finite number of steps we get the set $v_{a_1}, v_{z_1}, \ldots, v_{a_n}, v_{z_n}$. For convenience we let $z_0 = 0$.

We make a further thinning of the set of vertices in γ. Let i_1 be the smallest $i \in \{1, \ldots, n\}$ such that $|v_{z_{i-1}+1} - v_{a_i-1}| > A_{k-1}/2$. For $t \geq 2$, let i_t be the smallest $i \in \{i_{t-1} + 1, \ldots, n\}$ such that $|v_{z_{i-1}+1} - v_{a_i-1}| > A_{k-1}/2$. If there is no such i_t we stop. The largest t for which there is i_t which satisfies the above property is denoted by ν. Note that $\nu \leq 3^d$. For convenience we write u_t for $v_{z_{i_t-1}+1}$ and u'_t for $v_{a_{i_t}-1}$ respectively. From the triangle inequality,

$$\begin{aligned} |x-y| &\leq |x - v_1| + |v_1 - v_{a_1-1}| + |v_{a_1-1} - v_{z_1+1}| + \ldots + |v_{z_n+1} - v_l| + |v_l - y| \\ &\leq \frac{2}{100} A_{k-1} + \left(A_{k-1} + 3\frac{A_{k-1}}{50}\right) 3^d + \frac{1}{2} A_{k-1} 3^d + \sum_{i=1}^{\nu} |u_i - u'_i| \\ &\leq \frac{16\cdot 3^d}{k^2} |x-y| + \sum_{i=1}^{\nu} |u_i - u'_i|. \end{aligned}$$

On the second line the first term bounds $|x - v_1| + |v_l - y|$, the second term is the upper bound for $\sum_i |v_{a_i-1} - v_{z_i+1}|$, and the last two terms estimate $\sum_i |v_{z_{i-1}+1} - v_{a_i+1}|$. The last inequality follows from the fact that $|x - y| > A_k/8 = (k^2 A_{k-1})/8$. Therefore

$$\sum_{i=1}^{\nu} |u_i - u'_i| \geq \left(1 - \frac{16\cdot 3^d}{k^2}\right) |x - y|. \tag{4.5}$$

Given $\mathbf{r} \in \mathcal{R}_Q$, the set $\{u_1, u_1', \ldots, u_\nu, u_\nu'\}$ is a random subset of $W(B, 3A_{k-1}/100)\backslash B$. If also $\nu = s$, the set $\{u_1, u_1', \ldots, u_\nu, u_\nu'\}$ is a random element of U_s.

To estimate $\mathbb{P}_p^{\mathbf{r}}(x \stackrel{Q}{\longleftrightarrow} y)$, we write the event $\{x \stackrel{Q}{\longleftrightarrow} y\}$ as a union over all possible configurations of $\{u_1, u_1', \ldots, u_\nu, u_\nu'\}$ in Q. For increasing events E and E' on $\mathbb{Z}^d$, event $E \circ E'$ is the set of configurations of sites in $\mathbb{Z}^d$ for which there exist disjoint sets of open sites with the property that the first such set guarantees the occurrence of E and the second guarantees the occurrence of E' (for more formal definition see [8, p.37]). Then we say that E and E' *occur disjointly.*

$$\begin{aligned} \mathbb{P}_p^{\mathbf{r}}(x \stackrel{Q}{\leftrightarrow} y) &\leq \sum_{s=1}^{3^d} \sum_{\{x_i, x_i'\} \in U_s} \mathbb{P}_p^{\mathbf{r}}\left(x \stackrel{Q}{\leftrightarrow} y; \nu = s, u_1 = x_1, u_1' = x_1', \ldots, u_\nu = x_s, u_\nu' = x_s'\right) && (4.6) \\ &\leq \sum_{s=1}^{3^d} \sum_{\{x_i, x_i'\} \in U_s} \mathbb{P}_p^{\mathbf{r}}\left(x_1 \stackrel{Q\backslash B}{\longleftrightarrow} x_1' \circ x_2 \stackrel{Q\backslash B}{\longleftrightarrow} x_2' \circ \ldots \circ x_s \stackrel{Q\backslash B}{\longleftrightarrow} x_s'\right) && (4.7) \\ &\leq \sum_{s=1}^{3^d} \left(2A_{k-1}^d 3^d\right)^{2s} \max_{\{x_i, x_i'\} \in U_s} \mathbb{P}_p^{\mathbf{r}}(x_1 \stackrel{Q\backslash B}{\longleftrightarrow} x_1' \circ \ldots \circ x_s \stackrel{Q\backslash B}{\longleftrightarrow} x_s'). && (4.8) \end{aligned}$$

The second inequality holds because the events $\{u_i \stackrel{Q\backslash B}{\longleftrightarrow} u_i'\}$ occur disjointly. The proof is completed by an application of the BK-inequality (see [3, 8]). ☐

Lemma 4.4. *Suppose that there exist $\delta > 0$, $k_0 > 0$ and $p > 0$ such that*

1. *for all $l \geq k_0$, $2d \log A_{l-1} \leq \alpha_{l-1} A_{l-2}$ and $\alpha(A_{l-1} - 8A_{l-2}) \geq 8 \log 2$;*
2. *(4.3) holds for $k_0, \ldots, (k-1)$, where $k > k_0$.*

Then, for $\mathbf{r} \in \mathcal{R}_Q$ and $x, x' \in Q\backslash B$ such that $|x - x'| > A_{k-1}/2$,

$$\mathbb{P}_p^{\mathbf{r}}(x \stackrel{Q\backslash B}{\longleftrightarrow} x') \leq 2 \exp\left[-\alpha_{k-1} \frac{A_{k-1} - 4A_{k-2}}{A_{k-1} + 8A_{k-2}} |x - x'|\right]. \tag{4.9}$$

Proof of Lemma 4.4. Let $\gamma = (g_1, \ldots, g_m)$ be an open path from x to x' within $Q\backslash B$. If there are several such paths then we choose the first one according to some order on the set of paths between x and x'. We define g_i for all $i \in \mathbb{Z}$ by setting $g_i = g_1$ for $i < 1$ and $g_i = g_m$ for $i > m$. Let $w_1 = g_1$, $i_1 = \min\{l > 1 \ : \ |g_l - w_1| > A_{k-1}/8\}$, $w_1' = g_{i_1}$, and $w_2 = g_{i_1+1}$. For $t \geq 2$, let $i_t = \min\{l > i_{t-1} + 1 \ : \ |g_l - w_t| > A_{k-1}/8\}$ if there is such l, otherwise i_t is undefined. If i_t is defined, let $w_t' = g_{i_t}$ and $w_{t+1} = g_{i_t+1}$. The last t for which i_t is defined is denoted by μ. Then $|w_l - w_l'| \in (A_{k-1}/8, A_{k-1}/8 + A_{k-2}/100]$ for $l \in \{1, \ldots, \mu\}$, and $|w_l' - w_{l+1}| \leq A_{k-2}/100$ for $l \in \{1, \ldots, \mu - 1\}$, since Q is a good k-block and $\gamma \subset Q\backslash B$. Therefore,

$$\mu + 1 \geq \frac{|x - x'|}{A_{k-1}/8 + A_{k-2}}.$$

We estimate $\mathbb{P}_p^{\mathbf{r}}(x \stackrel{Q\backslash B}{\longleftrightarrow} x')$ in the same way as (4.6) by writing the event $\{x \stackrel{Q\backslash B}{\longleftrightarrow} x'\}$ as a union over all possible configurations $\{y_1, y_1', \ldots, y_s, y_s'\}$ of $\{w_1, w_1' \ldots, w_\mu, w_\mu'\}$ in $Q\backslash B$, where $s \geq \frac{|x-x'|}{A_{k-1}/8 + A_{k-2}} - 1$. We denote the set of all possible realizations of $\{w_1, w_1' \ldots, w_\mu, w_\mu'\}$ of size $2s$ by V_s. The cardinality of V_s is at most $(A_{k-1}^d)^{2s}$. For $a, a' \in Q\backslash B$, let

$$E(a, a') = \left\{a \stackrel{S}{\longleftrightarrow} a'\right\}, \quad \text{where } S = W(a, A_{k-1}/8 + A_{k-2}/100) \cap (Q\backslash B).$$

We obtain

$$\begin{aligned}
\mathbb{P}_p^{\mathbf{r}}(x \overset{Q\setminus B}{\longleftrightarrow} x') &\leq \sum_{s \geq \frac{|x-x'|}{A_{k-1}/8+A_{k-2}}-1} (A_{k-1}^d)^{2s} \max_{\{y_i,y_i'\}\in V_s} \mathbb{P}_p^{\mathbf{r}}\left(E(y_1,y_1')\circ\ldots\circ E(y_s,y_s')\right) \\
&\leq \sum_{s \geq \frac{|x-x'|}{A_{k-1}/8+A_{k-2}}-1} (A_{k-1}^d)^{2s} \max_{\{y_i,y_i'\}\in V_s} \mathbb{P}_p^{\mathbf{r}}\left(E(y_1,y_1')\right)\ldots\mathbb{P}_p^{\mathbf{r}}\left(E(y_s,y_s')\right).
\end{aligned}$$

The last inequality follows from the BK-inequality.

Since Q is a good k-block, for any t there exists a good $(k-1)$-block containing the open path $(g_{i_{t-1}+1},\ldots,g_{i_t})$ between w_t and w_t'. Indeed, by the definition of a good k-block, there exists $a \in W(w_t, A_{k-1}/4)$ such that $Q_{k-1}(a)$ is a good $(k-1)$-block under configuration $\mathbf{r}'$ (recall from the definition of a good k-block that $\mathbf{r}'$ agrees with $\mathbf{r}$ on Q and $\mathbf{r}_y' = 1$ for $y \notin Q$). However $Q_{k-1}(a) \supset W(w_t, A_{k-1}/8 + A_{k-2}/100)$. Therefore, for $\{y_1, y_1', \ldots, y_s, y_s'\} \in V_s$, we can use (4.3) for $(k-1)$ to estimate

$$\mathbb{P}_p^{\mathbf{r}}\left(E(y_t,y_t')\right) \leq \exp\left[-\alpha_{k-1}|y_t-y_t'|\right] \leq \exp\left[-\alpha_{k-1}\frac{A_{k-1}}{8}\right].$$

for any t. In the last inequality we used the fact that $|y_t - y_t'| > A_{k-1}/8$ for all t. The resulting inequality is

$$\begin{aligned}
\mathbb{P}_p^{\mathbf{r}}(x \overset{Q\setminus B}{\longleftrightarrow} x') &\leq \sum_{s \geq \frac{|x-x'|}{A_{k-1}/8+A_{k-2}}-1} (A_{k-1}^d)^{2s} \exp\left[-\alpha_{k-1}\frac{A_{k-1}}{8}s\right] \\
&\leq 2\exp\left[-\alpha_{k-1}\frac{A_{k-1}-4A_{k-2}}{A_{k-1}+8A_{k-2}}\left|x-x'\right|\right].
\end{aligned}$$

We used inequalities $2d\log A_{k-1} \leq \alpha_{k-1}A_{k-2}$ and $\alpha(A_{k-1} - 8A_{k-2}) \geq 8\log 2$ to get the last bound. □

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. The proof of (4.3) is by induction. Note that for any fixed k and δ, the inequality (4.3) holds for sufficiently small positive p by continuity. Therefore we can start the induction from any large k_0. As soon as we choose k_0, we fix $p = p(k_0) > 0$ such that (4.3) holds for $k = k_0$ and p. We choose k_0 such that $2d\log A_{l-1} \leq \alpha_{l-1}A_{l-2}$ and $\alpha(A_{l-1} - 8A_{l-2}) \geq 8\log 2$ for all $l \geq k_0$.

Suppose that (4.3) holds for $k_0, \ldots, (k-1)$. We prove that (4.3) holds for k. Let Q be a good k-block, the configuration of the radii of the balls centred in Q is $\mathbf{r} \in \mathcal{R}_Q$, and let $x, y \in Q$ be such that $|x-y| > A_k/8$.

From Lemma 4.3 we obtain

$$\mathbb{P}_p^{\mathbf{r}}(x \overset{Q}{\longleftrightarrow} y) \leq \sum_{s=1}^{3^d}\left(2A_{k-1}^d 3^d\right)^{2s} \max_{\{x_i,x_i'\}\in U_s} \mathbb{P}_p^{\mathbf{r}}(x_1 \overset{Q\setminus B}{\longleftrightarrow} x_1')\ldots\mathbb{P}_p^{\mathbf{r}}(x_s \overset{Q\setminus B}{\longleftrightarrow} x_s'). \tag{4.10}$$

Every pair (x_i, x_i') satisfies conditions of Lemma 4.4. Hence (4.9) holds for every $\mathbb{P}_p^{\mathbf{r}}(x_i \overset{Q\setminus B}{\longleftrightarrow} x_i')$. We obtain

$$\begin{aligned}
\mathbb{P}_p^{\mathbf{r}}(x \overset{Q}{\leftrightarrow} y) &\leq \sum_{s=1}^{3^d}\left(2A_{k-1}^d 3^d\right)^{2s} 2^s \max_{\{x_i,x_i'\}\in U_s} \exp\left[-\alpha_{k-1}\frac{A_{k-1}-4A_{k-2}}{A_{k-1}+8A_{k-2}}\sum_{i=1}^{s}\left|x_i-x_i'\right|\right] \\
&\leq 3^d\left(2A_{k-1}^d 3^d\right)^{2\cdot 3^d} 2^{3^d} \exp\left[-\alpha_{k-1}\frac{A_{k-1}-4A_{k-2}}{A_{k-1}+8A_{k-2}}\beta_k\left|x-y\right|\right],
\end{aligned}$$

where $\beta_k = 1 - \frac{16\cdot 3^d}{k^2}$. The last inequality follows from the definition of U_s. Finally, we obtain

$$\begin{aligned}\mathbb{P}_p^{\mathbf{r}}(x \overset{Q}{\leftrightarrow} y) &\le \exp\left[-\alpha_{k-1}\frac{A_{k-1} - 8A_{k-2}}{A_{k-1} + 8A_{k-2}}\beta_k\,|x-y|\right] \\ &= \exp\left[-\alpha_k\,|x-y|\right].\end{aligned}$$

We used the estimate

$$3^d\left(2A_{k-1}^d 3^d\right)^{2\cdot 3^d} 2^{3^d} \le \exp\left[\alpha_{k-1}\frac{4A_{k-2}}{A_{k-1}+8A_{k-2}}\beta_k\,|x-y|\right],$$

which is true for large k, since $|x-y| > A_k/8$. □

5 Proof of Theorem 1.1.

We use the idea of [15, 16] (see also [8, page 373]). We introduce a discrete model on a graph $\mathcal{G}$ similar to the one studied in Section 2. The vertices of $\mathcal{G}$ are the sites of $\mathbb{Z}^d$. To define adjacency in $\mathcal{G}$, we partition $\mathbb{R}^d$ into cubes

$$B(x) = \prod_{i=1}^{d}\left[x_i - \frac{1}{2}, x_i + \frac{1}{2}\right), \quad \text{for } x \in \mathbb{Z}^d.$$

For each $x \in \mathbb{Z}^d$, let $\widetilde{r}_x = \frac{1}{2} + \max\{r_i \mid i \ : \ X_i \in B(x)\}$. Two points $x, y \in \mathbb{Z}^d$ are adjacent if $|x-y| \le \widetilde{r}_x + \widetilde{r}_y$, where $|x| = \max_{1\le i\le d}|x_i|$. The site $x \in \mathbb{Z}^d$ is open if there exist a point of the Poisson process within the cube $B(x)$, and closed otherwise. Note that

$$p = \mathbb{P}_\lambda(x \text{ is open}) = 1 - e^{-\lambda}.$$

Let $\mathbb{P}_p$ be a product measure on $\{0,1\}^{\mathbb{Z}^d}$. Let $\widetilde{C}_x$ be an open cluster at x in $\mathcal{G}$, and $|\widetilde{C}_x|$ the size of $\widetilde{C}_x$ (see Section 2 for the definition of two points connectivity). It is easy to show that, for any $s > 0$,

$$\mathbf{E}r_0^s < \infty \quad \text{if and only if} \quad \mathbb{E}_\lambda(\widetilde{r}_0)^s < \infty.$$

Therefore under the conditions of Theorem 1.1, $\mathcal{G}$ satisfies the conditions of Theorem 2.1. Hence

$$p_c(\mathcal{G}) = \sup\{p \ : \ \mathbb{P}_p(|\widetilde{C}_0| < \infty) = 1\} > 0.$$

By the definition of $\mathcal{G}$, if x and y from $\mathbb{Z}^d$ are connected in continuum model then they are connected in discrete model $\mathcal{G}$. Therefore if for some λ, $|C_0| = \infty$ then $|\widetilde{C}_0| = \infty$ for $p = 1 - e^{-\lambda}$. Therefore

$$\lambda_c \ge -\log(1 - p_c(\mathcal{G})).$$

In particular, $\lambda_c > 0$. □

Acknowledgment. I am grateful to the referee for pointing out inaccuracies in the proof of Theorem 2.1 and for very helpful comments. This paper has benefited from the discussions with Noam Berger, Marek Biskup and Mike Keane. The research of the author was supported by Science Foundation Ireland Grant No. SFI04/RP1/I512.

References

[1] M.Aizenman and D.J.Barsky (1987) Sharpness of the phase transition in percolation models, *Communications in Mathematical Physics* **108** 489–526.

[2] J. van den Berg (1996) A note on disjoint-occurrence inequalities for marked Poisson point processes. *J.Appl.Probab.* **33**, 420–426.

[3] J. van den Berg and H.Kesten (1985) Inequalities with applications to percolation and reliability, *J.Appl.Prob.* **22**, 556–569.

[4] N.Berger, A lower bound for the chemical distance in sparse long-range percolation models. Preprint. math.PR/0409021.

[5] B.Blaszczyszyn, Ch.Rau and V.Schmidt (1999) Bounds for clump size characteristics in the Boolean model. *Adv.Appl.Prob.* **31**, 910–928.

[6] O.Dousse (2005) *Asymptotic properties of wireless multi-hop networks.* PhD Thesis.

[7] E.N.Gilbert (1961) Random plane networks. *J.Soc.Indust.Appl.Math.* **9** 533–543.

[8] G.R.Grimmett (1999) *Percolation.* 2nd edition.

[9] P.Hall (1985) On continuum percolation. *Ann.Prob.* **13**, 1250–1266.

[10] R.Meester and R.Roy (1996) *Continuum percolation.* Cambridge University Press, Cambridge.

[11] R.Meester and R.Roy (1994) Uniqueness of unbounded occupied and vacant components in Boolean models. *Ann.Appl.Prob.* **4**, 933–951.

[12] M.V.Menshikov (1986) Coincidence of critical points in percolation problems, *Soviet Mathematics Doclady* **33** 856–859.

[13] G.E.Pike and C.H.Seager (1974) Percolation and conductivity: A computer study. I. *Phys. Rev. B* **10** 1421–1434.

[14] R.Roy (1988) *The Russo-Seymour-Welsh theorem and the equality of critical densities for continuum percolation of* $\mathbb{R}^2$. PhD Thesis.

[15] S.A.Zuyev and A.F.Sidorenko (1985) Continuous models of percolation theory I. *Theoretical and Mathematical Physics* (in Russian) **62**, 76–86 (51–58 in translation).

[16] S.A.Zuyev and A.F.Sidorenko (1985) Continuous models of percolation theory II. *Theoretical and Mathematical Physics* (in Russian) **62**, 253–262 (171–177 in translation).