



# Acyclic edge-colouring of planar graphs. Extended abstract

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## Abstract

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph  $G$ , denoted  $\chi'_a(G)$  is the minimum  $k$  such that  $G$  admits an *acyclic edge-colouring* with  $k$  colours. We conjecture that if  $G$  is planar and  $\Delta(G)$  is large enough then  $\chi'_a(G) = \Delta(G)$ . We settle this conjecture for planar graphs with girth at least 5 and outerplanar graphs. We also show that if  $G$  is planar then  $\chi'_a(G) \leq \Delta(G) + 25$ .

*Keywords:* edge-colouring, graphs of bounded density, planar graph, outerplanar graph

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<sup>1</sup> Partially supported by the european project FET-AEOLUS

<sup>2</sup> The authors would like to thank Jan van den Heuvel for helpful discussions.

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## 1 Introduction

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph  $G$ , denoted  $\chi'_a(G)$  is the minimum  $k$  such that  $G$  admits an *acyclic edge-colouring* with  $k$  colours.

**Conjecture 1.1 (Alon et al. [1])** *For every graph  $G$ ,  $\chi'_a(G) \leq \Delta(G) + 2$ .*

This conjecture would be tight as there are cases where more than  $\Delta + 1$  colours are needed. However, the only known such graphs  $G$  are subgraphs of  $K_{2n}$ . Therefore the following conjecture might even be true:

**Conjecture 1.2 (Alon et al. [1])** *If  $G$  is a  $\Delta$ -regular graph then  $\chi'_a(G) = \Delta(G) + 1$  unless  $G = K_{2n}$ .*

Molloy and Reed [4] showed  $\chi'_a(G) \leq 16\Delta(G)$ , which is the best general upper bound so far. For graph with large girth, better upper bounds are known. For example, Alon et al. also showed that Conjecture 1.1 is true for graphs with girth at least  $C\Delta \log(\Delta)$  for some fixed constant  $C$ .

Muthu et al [5] proved that  $\chi'_a(G) \leq 2\Delta(G) + 29$  if  $G$  is planar and  $\chi'_a(G) \leq \Delta(G) + 6$  if  $G$  is planar and triangle-free. In this paper, we improve the first of these two results.

**Theorem 1.3**  $\chi'_a(G) \leq \Delta(G) + 25$  for all planar graphs  $G$ .

It is known (see [7]) that a planar graph  $G$  is  $\Delta(G)$ -edge-colourable if  $\Delta(G)$  is large. We conjecture that the same is true for acyclic edge-colouring.

**Conjecture 1.4** There exists  $\Delta_0$  such that every planar graph with maximum degree  $\Delta \geq \Delta_0$  has an acyclic edge-colouring with  $\Delta$  colours.

As evidences to this conjecture, we show that it holds for planar graphs of girth at least 5. In fact we prove the following more general result regarding graphs with bounded maximum average degree. Recall that the *maximum average degree* of  $G$  is  $Mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \text{ is a subgraph of } G\}$ . It is well known that a planar graph of girth  $g$  has maximum average degree less than  $2 + \frac{4}{g-2}$ .

**Theorem 1.5** *For any  $\epsilon > 0$ , there exists an integer  $\Delta_\epsilon$  such that if  $\Delta(G) \geq \Delta_\epsilon$  and  $Mad(G) \leq 4 - \epsilon$  then  $\chi'_a(G) = \Delta(G)$ .*

Moreover, we show that Conjecture 1.4 holds for outerplanar graphs. This improve the upper bound  $\Delta + 1$  for the acyclic edge-chromatic number of such graphs obtained by Muthu et al. [5]. Note that  $\sup\{Mad(G) \mid G \text{ is outerplanar}\} = 4$ .

The detailed proofs of all these results may be found in [3]. We just give here sketches of the proofs of Theorem 1.3 and 1.5.

## 2 Sketch of proofs

The proof of Theorem 1.3 relies heavily on a structural result that can be derived from a theorem of Borodin et al. [2]. Before stating this structural result we need to introduce some notation and terminology. Let  $G$  be a graph. A vertex  $v$  is said to be *good* if its degree is at most 5 and it has a neighbour  $v^*$  such that  $\sum_{u \in N(v) \setminus v^*} d(u) \leq 38$  and for all  $u \in N(v) \setminus v^*$ ,  $d(u) \leq 25$ . We say that  $G$  has a bunch of length  $m \geq 3$  with *poles* the vertices  $p$  and  $q$  (where  $p \neq q$ ), if  $G$  contains a sequence of paths  $(P_1, P_2, \dots, P_m)$  with the following properties. Each  $P_i$  has length 1 or 2 and joins  $p$  with  $q$ . Furthermore, for each  $i = 1, \dots, m - 1$ , the cycle formed by  $P_i$  and  $P_{i+1}$  is not separating in  $G$  (i.e., has no vertex of  $G$  inside). If a path  $P_i$  in the bunch has length 2, i.e.,  $P_i = pv_iq$ , then the vertex  $v_i$  will be called a *bunch vertex*. A path  $P_i = pq$  of length 1 in the bunch will be referred to as a *parental edge*.

Theorem 2.1 below is a relatively straightforward corollary to a theorem in [2], and it will play an important role in our proof of Theorem 1.3 below.

**Theorem 2.1** *For every plane graph at least one of the following holds:*

- (i)  $G$  has a good vertex  $v$ ;
- (ii)  $G$  has a vertex  $v$  that is a pole for  $1 \leq k \leq 6$  bunches of length at least 6, and has at most  $24 - 4k$  neighbours that are not part of these bunches.

*Proof of Theorem 1.3 (Sketch)* For the sake of presentation we sketch the proof that  $\chi'_a(G) \leq \Delta + 38$  for all planar  $G$ . The proof of  $\Delta + 25$  is very similar but slightly more involved. We consider a minimal counterexample  $G$ . We first show that  $G$  has the following property:

- If  $(P_1, \dots, P_m)$  is a bunch in  $G$  with  $m \geq 6$  and if for some  $2 \leq i \leq m - 2$  neither of the paths  $P_i, P_{i+1}$  is a parental edge, then  $v_i v_{i+1}$  is not an edge.

This is shown by extending an acyclic edge colouring of  $G \setminus v_i v_{i+1}$  with  $\Delta + 38$  colours (which exists by minimality), as follows. If  $3 \leq i \leq m - 3$  then we can find a colour for  $v_i v_{i+1}$  that is distinct from the colours used on edges incident with either  $p, v_{i-1}, v_i, v_{i+1}$  or  $v_{i+2}$  (there are at most  $\Delta + 8$  such edges) to obtain an acyclic edge colouring of  $G$ . Now consider  $i = 2$ . As  $v_3 v_4$  is not an edge, if  $v_1 v_2$  has a colour different from those of  $pv_3, qv_3$  any colour different from the colours of edges incident with  $p, v_2$  or  $v_3$  (and there are at most  $\Delta + 3$  of those) will result in an acyclic edge colouring of  $G$  with  $\Delta + 38$  colours. If  $v_1 v_2$  has the same colour as  $pv_3$  we also pick a colour for  $v_2 v_3$  that differs from the colours of edges incident

with  $p, v_2$  or  $v_3$ , and if  $v_1v_2$  has the same colour as  $qv_3$  then we pick a colour for  $v_2v_3$  that differs from the colours of edges incident with  $q, v_2$  or  $v_3$ . The case  $i = m - 2$  is analogous to the case  $i = 2$ .

Next, we show:

- $G$  does not have a good vertex.

To prove this, suppose  $v$  is a good vertex. We extend an acyclic edge colouring of  $G \setminus v$  with  $\Delta + 38$  colours as follows. Assume  $d(v) = 5$  (the case when  $d(v) < 5$  is similar) and write  $N(v) = \{v^*, u_1, u_2, u_3, u_4\}$  where  $d(v^*) \geq d(u_i)$  for  $i = 1, \dots, 4$ . We colour  $vv^*$  with a colour different from those of edges incident with  $v^*, u_1, \dots, u_4$  (there are at most  $\Delta - 1 + 38 - 4$  such edges in  $G \setminus v$  by definition of a good vertex), and we colour the  $vu_i$  with colours different from those of edges incident with  $u_1, \dots, u_4$ .

Since there is no good vertex, by Theorem 2.1, there exists a vertex  $p$  that is a pole for  $1 \leq k \leq 6$  bunches of length at least 6 and has at most  $24 - 4k$  neighbours not in those bunches. Let us denote those bunches by  $B_1 = (P_1^1, \dots, P_{m_1}^1), \dots, B_k = (P_1^k, \dots, P_{m_k}^k)$ . Let us denote the other pole of  $B_i$  by  $q_i$  and we will denote the middle vertex of  $P_j^i$  by  $v_j^i$  if it exists. We have  $d(v_j^i) = 2$  for all  $3 \leq j \leq m_i - 2$  and  $1 \leq i \leq k$ . We can assume (w.l.o.g.) that  $P_3^1$  is not a parental edge. By minimality of  $G$  there is an acyclic edge colouring of  $G \setminus v_3^1$  with  $\Delta + 38$  colours. We extend it as follows. For  $pv_3^1$  we choose a colour different from those used on the edges  $v_1^i v_2^i, v_{m_i-1}^i v_{m_i}^i$  and the edges incident with  $p$  (there are at most  $\Delta - 1 + 12$  such edges). Next we colour  $q_1 v_3^1$  with a colour different from those used on  $pv_3^1$ , the edges incident with  $q_1$ , the edges  $pv_1^i, pv_{m_i}^i, pq_i$ , the edges incident with  $p$  that do not belong to the bunches and those edges  $pv_j^i$  for which  $q_i v_j^i$  has the same colour as  $pv_3^1$  (there are at most  $1 + \Delta - 1 + 2k + 24 - 4k + k + k = \Delta + 24$  such edges). It can be checked that this indeed gives an acyclic edge colouring of  $G$  with at most  $\Delta + 38$  colours. □

*Proof of Theorem 1.5 (Sketch)* By considering a minimal counterexample  $G$ .

A *thread* is a path of length two whose internal vertex has degree 2. We first show that  $G$  has the following properties:

- $G$  is 2-connected. In particular,  $\delta(G) \geq 2$ .
- For every vertex  $v \in V(G)$ ,  $\sum_{u \in N(v)} d(u) \geq \Delta + 1$ .
- A  $\Delta$ -vertex is the end of at most  $k$  threads whose other endvertex has degree at most  $k$ .
- A  $(\Delta - l)$ -vertex is the end of at most  $k - 1 - l$  threads whose other endvertex has degree at most  $k$ .

We then use the discharging method. We assign an initial charge of  $d(v)$  to each vertex  $v$  and discharge according to the following rules where  $d_\epsilon = \lceil \frac{8}{\epsilon} - 2 \rceil$ .

- R1:** for  $4 \leq d < d_\epsilon$ , every  $d$ -vertex sends  $a(d) = 1 - \frac{4-\epsilon}{d}$  to each neighbour.
- R2:** for  $d_\epsilon \leq d \leq \Delta + 1 - d_\epsilon$  then every  $d$ -vertex sends  $1 - \frac{\epsilon}{2}$  to each neighbour.
- R3:** for  $\Delta + 2 - d_\epsilon \leq d \leq \Delta$  then every  $d$ -vertex sends
- $1 - \epsilon$  to each 3-neighbour;
  - $2 - \epsilon$  to each 2-neighbour whose second neighbour has degree 2 or 3;
  - $b(d) = 2 - \epsilon - a(d)$  to each 2-neighbour whose second neighbour has degree  $d$  with  $4 \leq d < d_\epsilon$ ;
  - $1 - \frac{\epsilon}{2}$  to each 2-neighbour whose second neighbour has degree  $d \geq d_\epsilon$ .

We then check that every vertex  $v$  has final charge  $f(v)$  at least  $4 - \epsilon$  if  $\Delta \geq \Delta_\epsilon = \left\lceil \frac{2}{\epsilon} \left( \left[ d_\epsilon - 2 + 3(2 - \epsilon) + \sum_{d=4}^{d_\epsilon-1} b(d) - (1 - \frac{\epsilon}{2})(d_\epsilon - 1) \right] + 4 - \epsilon \right) \right\rceil$ . Hence  $G$  has average degree at least  $4 - \epsilon$ , which is a contradiction.  $\square$

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