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On the asymptotic density in a one-dimensional self-organized critical forest-fire model

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In the literature similar models have been studied for discrete time, finite (but large) volume and finite (but large) speed at which the fire spreads out. The most interesting behaviour seems to occur when the ignition rate goes to 0, as this allows clusters to grow very large before being hit by lightning. It has been stated by Drossel, Clar and Schwabl (1993) that then (in our notation) the density of vacant sites (in equilibrium) is of order $1/\log(1/\lambda)$. Their proof uses a 'scaling ansatz' and is not rigorous. We give, for our version of the model, a rigorous and mathematically more natural proof. Our proof shows that regardless of the initial configuration, already after time of order $\log(1/\lambda)$ the density is of the above mentioned order $1/\log(1/\lambda)$. We also point out how our proof can be modified for the model studied by Drossel et al.

2000 Mathematics Subject Classification: 60K35; 82C20

Keywords and Phrases: critical forest-fire; asymptotic density; self-organized criticality

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On the asymptotic density in a one-dimensional self-organized critical forest-fire model

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Consider the following forest-fire model where the possible locations of trees are the sites of $\mathbb{Z}$. Each site has two possible states: 'vacant' or 'occupied'. Vacant sites become occupied at rate 1. At each site ignition (by lightning) occurs at ignition rate λ, the parameter of the model. When a site is ignited, its occupied cluster becomes vacant instantaneously.

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1 Introduction

Suppose each site of the lattice $\mathbb{Z}^d$ is either vacant or occupied by a tree. Vacant sites become occupied according to independent rate 1 Poisson processes. Also, lightning strikes at any site according to independent rate λ Poisson processes. Here $\lambda > 0$ is the parameter of the model. When a site is hit by lightning, its entire occupied cluster burns down, that is, becomes vacant.

When $d = 1$, a process with the above description can be constructed in a standard way, by using a graphical representation; see e.g. Liggett (1985). For this, note that if we start with a configuration in which infinitely many sites on the negative and on the positive half-line are vacant, there are, with probability 1, at each time t infinitely many sites (on both half-lines) that have remained vacant througout the interval $[0, t]$. These sites 'break the infinite line into finite pieces', which enables a graphical representation mentioned above. When $d \geq 2$, the existence of the infinite-volume process is not clear. (In principle, the state of a given site can be influenced by infinitely many Poisson events in finite time).

In the physics literature usually a different but closely related forest-fire model is studied. In that model, time is discrete, space is large but finite, and the fire does not spread instantaneously but at a finite speed. The most interesting object of study seems to be the limiting behaviour as, roughly speaking, the ignition rate goes to 0 and the speed of fire and the volume go to infinity, jointly, in an appropriate way. It is believed that this behaviour resembles, in some sense, that of statistical mechanics systems at criticality. This belief is partly supported by heuristic arguments and computer simulations, but very little has been proved rigorously. See Jensen (1998) for a general overview of and introduction to these and other so-called self-organised critical systems, and Schenk, Drossel and Schwabl (2002) for current insights on the forest-fire model, in particular for $d = 2$. A paper by Malamud, Morein and Turcotte (1998) compares the model with data from real forest fires.

As to the one-dimensional case, one of the main statements in the paper by Drossel, Clar and Schwabl (1993) says that under the above mentioned asymptotics, the equilibrium probability that a site is vacant is (in our notation) of order $1/\log(1/\lambda)$. Their arguments leading to this statement are not rigorous (see our Remark at the end of Section 2.1).

We give, for our model, a rigorous and, in our opinion, more 'natural' proof of the above mentioned asymptotic behaviour of the density of vacancies. It turns out that, uniformly in the starting configuration, already after time of order $\log(1/\lambda)$ the density is of order $1/\log(1/\lambda)$. The ingredients of our proof are fairly elementary but the way one has to combine them is quite subtle. We also point out (see Section 3.2) how our arguments can be adapted to obtain the analogous results for the above mentioned model of Drossel et al.

We believe that for each $\lambda > 0$ there is a unique invariant distribution for the above dynamics. We have not been able to prove this, but hope that our results and ideas can also be used to make progress on that problem.

Notation and terminology. Let Ω denote the set of all configurations $\omega \in \{0,1\}^{\mathbb{Z}}$ for which there are infinitely many positive and negative i's with $\omega_i = 0$.

For each $x \in \mathbb{Z}$ let $B_x = \{b_1, b_2, \dots\} \subset (0,\infty)$ and $I_x = \{i_1, i_2, \dots\} \subset (0,\infty)$ denote the birth and ignition times (respectively) at x. As said before, these correspond to the points of independent Poisson point processes with intensities 1 and λ respectively. We let $\mathbb{P}_\lambda$ denote the probability measure governing B and I. Given B and I, and $\eta(0) \in \Omega$, let $\{\eta_x(t)\}_{(x,t)\in\mathbb{Z}\times[0,\infty)}$ denote the forest fire process with initial configuration $\{\eta_x(0)\}_{x\in\mathbb{Z}}$. We let $\mathbb{P}_\lambda^\xi$ denote the probability measure governing the forest fire process with lightning density λ and initial configuration $\eta(0) = \xi \in \Omega$. For arbitrary $J \subset \mathbb{Z}$, we let $\mathcal{F}_J(s,t)$ denote the information about the births and ignitions during the time interval $[s,t]$ in the set J. That is,

$$\mathcal{F}_J(s,t) = \sigma(B_x \cap [s,t] : x \in J) \vee \sigma(I_x \cap [s,t] : x \in J), \quad t \geq s \geq 0.$$

When s equals 0 or $J = \mathbb{Z}$, we omit these symbols from the notation. In particular, we have the notation $\mathcal{F}(t) = \mathcal{F}_{\mathbb{Z}}(0,t)$.

We denote the time shift operators on the underlying probability space by $(\theta_s)_{s\geq 0}$.

Finally, to avoid confusion we make the following remark about our terminology: Occasionally we make statements like 'i has an ignition at time t' and 'i burns at time t'. There is an essential difference between these two statements. The first means, formally, that $t \in I_i$, which informally says that site i is hit by lightning at time t. The site may be empty in which case the lightning has no effect. The second statement says that i is occupied just before time t and becomes vacant at time t (by lightning at i or somewhere else in its occupied cluster). Finally, we note that by 'i has a birth at time

t' we just mean that $t \in B_i$. If i was already occupied, this 'birth' has no effect.

2 Relevant scales and the blocking property

2.1 Relevant space and time scales

For a set of sites $J \subset \mathbb{Z}$ and $t > 0$, define the events

$$\begin{aligned} A_J(s,t) &= \{\forall x \in J : B(x) \cap [s,t] \neq \emptyset\} \\ &= \{\text{each } x \in J \text{ has a birth at some time in } [s,t]\} \\ B_J(s,t) &= \{\exists x \in J : I(x) \cap [s,t] \neq \emptyset\} \\ &= \{\text{ignition occurs at some } x \in J \text{ at a time in } [s,t]\}. \end{aligned}$$

We denote the complements of these events by $A_J^c(s,t)$ and $B_J^c(s,t)$, respectively. When $s = 0$, we simply write $A_J(t)$ and $B_J(t)$.

Definition. *Assume $\lambda < \lambda_0 = (3\log 3)^{-1}$. We define $n = n(\lambda) \geq 2$ as the positive integer satisfying*

$$n \log n \leq \frac{1}{\lambda} < (n+1)\log(n+1). \tag{1}$$

For convenience, we let $n(\lambda) = 2$ when $\lambda \geq \lambda_0$.

In the rest of the paper we assume that n and λ are related as in the definition.

It is easy to see that

$$\mathbb{P}_\lambda\left(A_{[0,n]}(\log n)\right) = (1 - e^{-\log n})^{n+1} = (1 - n^{-1})^{n+1} \in (C_1, 1 - C_1), \tag{2}$$

and that for $0 < \lambda < \lambda_0$

$$\mathbb{P}_\lambda\left(B_{[0,n]}(\log n)\right) = 1 - (e^{-\lambda \log n})^{n+1} = 1 - e^{-\lambda(n+1)\log n} \in (C_2, 1 - C_2), \tag{3}$$

for some constants $C_1, C_2 > 0$. This indicates that n and $\log n$ are the relevant space and time scales in the model. Even if we replace, in the above computation, the spatial scale by a constant multiple of n, the result is still bounded away from 0 and 1: for any $\alpha > 0$, there are constants $0 < C_1(\alpha) < 1$ and $0 < C_2(\alpha) < 1$ such that for all $0 < \lambda < \lambda_0$

$$\begin{aligned} &\mathbb{P}_\lambda\left(A_{[0,\alpha n]}(\log n)\right) \in (C_1(\alpha), 1 - C_1(\alpha)), \\ &\mathbb{P}_\lambda\left(B_{[0,\alpha n]}(\log n)\right) \in (C_2(\alpha), 1 - C_2(\alpha)). \end{aligned} \tag{4}$$

For the event B, we can also replace the time $\log n$ by $\beta \log n$ (and let C_2 depend not only on α but also on β). Note that we do not have similar flexibility in the time variable for the event A: for any $\beta > 1$ the probability of $A_{[0,n]}(\beta \log n)$ tends to 1 as $n \to \infty$.

Remark: In the paper by Drossel, Clar and Schwabl, an analog of (4) alone is taken as sufficient support to conclude that n is the 'relevant space scale'. This quantity is then explicitly inserted in the postulation of a 'scaling ansatz'. This ansatz, combined with other computations (concerning the conditional equilibrium probabilities of the local configuration near site 0, given that site 0 is vacant), leads to the earlier mentioned order $1/\log(1/\lambda)$ for the density of vacant sites. See the arguments between (7) and (9) in their paper. We do not see how their arguments can easily be made rigorous. Our rigorous proof for the asymptotic density is very different from their arguments and uses properties much more subtle than (4) (see Lemmas 1 and 2 in Section 2.2). On the other hand we avoid the computations regarding the equlibrium probabilities mentioned above (although these are interesting in themselves), so that our proof is in some sense more direct.

2.2 Blocking intervals

From (4) we see that if we start with a configuration in which $[0, n]$ is empty, then with probability bounded away from 0, there is at each time $s \in [0, \log n]$ at least one vacant site in $[0, n]$ (consider the event $A^c_{[0,n]}(\log n)$). Such events are useful, because they imply that the halflines to the left and to the right of $[0, n]$ 'do not communicate', that is, no fire can pass through in either direction.

Below we prove two technical lemmas concerning such and related 'blocking intervals'. For the proof of our main result, Theorem 4 in Section 3, we will need an initially vacant spatial interval of length of order n to maintain a certain blocking property during time $\beta \log n$ for some $\beta > 1$. However, as we said in the lines following (4), for each such β we have that $\mathbb{P}_\lambda(A^c_{[0,n]}(\beta \log n))$ tends to 0 as $\lambda \to 0$. Nevertheless, as the first lemma shows, we can indeed, by more subtle arguments involving suitable fires in space intervals of length n^α (with some $\alpha < 1$), reach our goal. It is important here that we establish the blocking property irrespective of what happens outside the initially vacant interval. In the second lemma, we show that a vacant interval of length n is created in time $O(\log n)$ with probability bounded away from 0, regardless of the initial configuration. The combination of these lemmas allows us to create suitable blocking intervals after time $2 \log n$ with reasonable probabilities, regardless of the initial configuration (Proposition 3 below). From this, Theorem 4 follows quite easily.

For $J \subset \mathbb{Z}$, let $N_J(t_1, t_2)$ denote the event that no fire propagates from J to J^c during the interval $[t_1, t_2]$. (Formally this is the event that there are no $s \in [t_1, t_2]$, $j \in J$, $k \in J^c$ and space interval I containing both j and k, with the properties that $\eta(s^-) \equiv 1$ on I, and j has an ignition at time s).

Definition. *For a segment $J \subset \mathbb{Z}$, we define the event*

$$H_J(s,t) = N_J(s,t) \cap \{\textit{for all } u \in [s,t] \textit{ there exists } x \in J \textit{ with } \eta_x(u) = 0\},$$

with $N_J(s,t)$ as above.

The complement of this event will be denoted by $H_J^c(s,t)$. Note that $H_J(s,t)$ implies that during $[s,t]$ no fire propagates from the half-line left of (and including) the rightmost point of J to the complement of this halfline. A similar statement holds with 'left' and 'right' interchanged. When $H_J(s,t)$ occurs, we say that the segment J *blocks* during $[s,t]$.

Lemma 1. *(a) For any $\alpha > 0$ there is a constant $C_3 = C_3(\alpha) > 0$, such that for all $0 < \lambda < \lambda_0$ and all intial configurations ξ with $\xi \equiv 0$ on $[0, \alpha n]$,*

$$\mathbb{P}_\lambda^\xi \left\{ H_{[0,\alpha n]}\left((3/2)\log n\right) \right\} > C_3.$$

(b) For all $\alpha > 0$ and $\beta > 0$ there is a constant $C_4 = C_4(\alpha, \beta) > 0$, such that for all $0 < \lambda < \lambda_0$ and all intial configurations ξ with $\xi \equiv 0$ on $[0, \alpha n]$,

$$\mathbb{P}_\lambda^\xi \left\{ H_{[0,\alpha n]}(\beta \log n) \right\} > C_4.$$

(c) Above we can even replace $H_{[0,\alpha n]}(\beta \log n)$ by an event that implies it, and is in $\mathcal{F}_{[0,\alpha n]}(\beta \log n)$. More precisely, for all $\alpha > 0$ and $\beta > 0$ there is a constant $C_4 = C_4(\alpha, \beta) > 0$, such that for all $0 < \lambda < \lambda_0$ there is an event $\hat{H}_{[0,\alpha n]}(\beta \log n) \in \mathcal{F}_{[0,\alpha n]}(\beta \log n)$ such that

$$\{\eta(0) \equiv 0 \textit{ on } [0, \alpha n]\} \cap \hat{H}_{[0,\alpha n]}(\beta \log n) \subset H_{[0,\alpha n]}(\beta \log n),$$

and

$$\mathbb{P}_\lambda\left\{\hat{H}_{[0,\alpha n]}(\beta\log n)\right\} > C_4.$$

Remark. Note that parts (b) and (c) with $\beta \leq 1$ are trivial; they follow immediately from (4) and the fact that $A^c_{[0,\alpha n]}(\log n)$ implies $H_{[0,\alpha n]}(\log n)$. The difficulty is to prove (b) and (c) for some $\beta > 1$. That is part (a) of the Lemma. We will see that once we have part (b) for some $\beta > 1$, it follows quite easily for $\beta + 3/4$, and hence for all positive β.

Proof. [Lemma 1] We first give the proof of part (a). Let $J_1 = [0, \alpha n/4)$, $J_2 = [\alpha n/4, \alpha 3n/4)$, $J_3 = [\alpha 3n/4, \alpha n]$. Here we assume, without loss of generality, that αn is sufficiently large, so that subdivision makes sense. Subdivide J_2 into $\alpha n^{1/4}$ segments of length $n^{3/4}$, denoted $K_1, K_2, \ldots$. (To reduce notation we write here, and in many other places in this paper, possibly non-integer numbers ($\alpha n^{1/4}$ and $n^{3/4}$ in this case) while we obviously mean their integer parts).

Consider the following events (i)–(v):

(i) There is no ignition in $J_1 \cup J_3$ before time $(3/2)\log n$. In formal notation, this event is:

$$B^c_{J_1\cup J_3}((3/2)\log n).$$

(ii) The intervals J_1 and J_3 do not try to fill before time $\log n$. More precisely,

$$A^c_{J_1}(\log n) \cap A^c_{J_3}(\log n).$$

(iii) There is no ignition in J_2 before time $(3/4)\log n$. That is,

$$B^c_{J_2}((3/4)\log n).$$

(iv) At least one of the blocks K_i has the following three properties: it tries to fill before time $(3/4)\log n$; it has an ignition between times $(3/4)\log n$ and $(7/8)\log n$; and it does not try to fill in the interval $((3/4)\log n, (3/2)\log n]$. More formally, this is the event

$$\bigcup_{i=1}^{\alpha n^{1/4}} A_{K_i}\left(\frac{3}{4}\log n\right) \cap B_{K_i}\left(\frac{3}{4}\log n, \frac{7}{8}\log n\right) \cap A^c_{K_i}\left(\frac{3}{4}\log n, \frac{3}{2}\log n\right).$$

(v) There is no ignition in J_2 between time $(7/8)\log n$ and time $(3/2)\log n$. That is,

$$B^c_{J_2}((7/8)\log n, (3/2)\log n).$$

Now we will show that if each of the events (i)–(v) occurs, then the event in part (a) of the Lemma occurs. First of all, events (i), (ii) and (v) guarantee that no fire propagates from $[0, \alpha n]$ to its complement during the time $[0, (3/2)\log n]$. Further, let K_i be a block with the three properties mentioned in (iv). Its first property, together with events (i), (ii) and (iii), ensure that K_i is indeed fully occupied at time $(3/4)\log n$. Its second property then ensures that at some time between $(3/4)\log n$ and $(7/8)\log n$ it becomes completely vacant. This, together with its third property then guarantees that some site in K_i remains vacant during $[(7/8)\log n, (3/2)\log n]$. Finally, this last property of K_i together with (ii) ensures that at each time in $[0, (3/2)\log n]$ some site in $[0, \alpha n]$ is vacant.

By independence of (i)–(v), it now suffices to show that for given $\alpha > 0$ each of the events (i)–(v) has probability bounded away from 0, uniformly in λ. For the events (i)–(iii) and (v), this follows easily from (4). The same computations which led to (4) show that for each i the probability that K_i has the first and third property in event (iv) is larger than some constant $c_1 > 0$. The probability that it has the second property is $1 - \exp(-\lambda n^{3/4}(1/8)\log n)$, which by (1) and some elementary computations is larger than or equal to $c_2 n^{-1/4}$, where c_2 is a positive constant. So, if X_i is the indicator of the event that K_i has the three properties mentioned above, then, since we have $\alpha n^{1/4}$ blocks, the expectation of the sum of the X_i's is at least $\alpha n^{1/4} c_1 c_2 n^{-1/4} = \alpha c_1 c_2$. Since the X_i's are independent, the probability that at least one X_i equals 1, and hence that event (iv) occurs, is therefore larger than some constant $c_3(\alpha)$. This completes the proof of part (a) of the Lemma.

Now we prove part (c), which clearly implies part (b). For $\beta = 3/2$, and hence for all $\beta \le 3/2$, we already know that part (c) holds: take for $\hat{H}_{[0,\alpha n]}(0, (3/2)\log n)$ the intersection of the events (i)–(v) in the proof of part (a). Now suppose part (c) holds for some $\beta \ge 3/2$. We will show that it then also holds for $\beta + 3/4$. Let, as above, $J_1 = [0, \alpha n/4)$, $J_2 = [\alpha n/4, \alpha 3n/4)$, $J_3 = [\alpha\, 3n/4, \alpha n]$. Consider the following events (I)–(V):

(I)

$$\hat{H}_{J_1}(\beta \log n) \cap \hat{H}_{J_3}(\beta \log n).$$

(II) Each site in the interval J_2 has a birth before time $\log n$, that is,

$$A_{J_2}(\log n).$$

(III) The interval J_2 has no ignition before time $(\beta - 1/4)\log n$, but does have an ignition between times $(\beta-1/4)\log n$ and $(\beta-1/8)\log n$. That is,

$$B^c_{J_2}((\beta - 1/4)\log n) \cap B_{J_2}((\beta - 1/4)\log n, (\beta - 1/8)\log n).$$

(IV) J_2 does not try to fill during $((\beta - 1/4)\log n, (\beta + 3/4)\log n)$. That is,

$$A^c_{J_2}((\beta - 1/4)\log n, (\beta + 3/4)\log n).$$

(V) There are no ignitions in $[0, \alpha n]$ during $(\beta \log n, (\beta + 3/4)\log n)$. That is,

$$A^c_{J_2}(\beta \log n, (\beta + 3/4)\log n).$$

With very similar (and even somewhat simpler) arguments as in the proof of part (a), one can show that the events (I)–(V) imply $H_{[0,\alpha n]}((\beta+3/4)\log n)$: event (I), together with (II) and (III) ensure that J_2 is completely occupied at time $\log n$ and becomes vacant at some time between $(\beta - 1/4)\log n$ and $(\beta - 1/8)\log n$. This, with (IV) implies that some site in J_2 remains vacant during $(\beta \log n, (\beta + 3/4)\log n)$. Finally, this, together with (I) and (V) implies that indeed $H_{[0,\alpha n]}((\beta + 3/4)\log n)$ occurs. Since the events (I)–(V) are $\mathcal{F}_{[0,\alpha n]}((\beta+3/4)\log n)$-measurable, we can define the desired $\hat{H}_{[0,\alpha n]}((\beta+3/4)\log n)$ as the intersection of these events.

Since the five events are independent (note that here we use that the events $\hat{H}_{J_1}(\beta \log n)$ and $\hat{H}_{J_3}(\beta \log n)$ in (I) are $\mathcal{F}_{J_1}(\beta \log n)$-measurable and $\mathcal{F}_{J_3}(\beta \log n)$-measurable respectively), it remains to show that each of the events (I)–(V) has a probability that is larger than some positive constant which depends on α and β but not on λ. The probability of (I), again using the above mentioned measurability properties, is at least $(C_4(\alpha/4, \beta))^2$. Suitable lower bounds for the other events follow easily from (4). This completes the proof of part (c) and hence of part (b). □

For the second lemma, define for $J \subset \mathbb{Z}$ the stopping time

$$T_J = \inf\{t > 0 : \eta_x(t) = 0 \text{ for all } x \in J\}. \quad (5)$$

Lemma 2. *Let*

$$T = T_{[0,n]} \wedge T_{[n,2n]} \wedge T_{[2n,3n]}.$$

(a) There exists $C_5 > 0$ *such that for all* $0 < \lambda < \lambda_0$ *and all initial configurations* ξ,

$$\mathbb{P}^{\xi}_{\lambda}(T \leq 2\log n) > C_5.$$

(b) We can even replace the event above by an $\mathcal{F}_{[0,3n]}(2\log n)$*-measurable event. More precisely, there is an* $\mathcal{F}_{[0,3n]}(2\log n)$*-measurable event* $A = A(\lambda)$ *with* $A \subset \{T \leq 2\log n\}$ *and*

$$\mathbb{P}_{\lambda}(A) > C_5.$$

Proof. Suppose each of the three events $B^c_{[0,3n]}(\log n)$, $A_{[n,2n]}(\log n)$ and $B_{[n,2n]}(\log n, 2\log n)$ occurs. We show this implies $T \leq 2\log n$. Since each of these events is clearly $\mathcal{F}_{[0,3n]}(2\log n)$-measurable and by (4) has a probability bounded away from 0, this will prove the Lemma.

Let

$$\sigma_L = \inf\{t > 0 : \text{site } n \text{ burns at time } t\},$$
$$\sigma_R = \inf\{t > 0 : \text{site } 2n \text{ burns at time } t\}.$$

If $\sigma_L \wedge \sigma_R > \log n$, then, by this and the first of the three events above, no site in $[n, 2n]$ burns before time $\log n$ and hence, by the second of the three events, this segment $[n, 2n]$ is filled at time $\log n$. Finally, by the third event, it will then burn completely down at some time between $\log n$ and $2\log n$, so that we have $T \leq T_{[n,2n]} \leq 2\log n$.

On the other hand, if $\sigma_L \wedge \sigma_R \leq \log n$, there must, by the first of the three events, have been a fire before or at time $\log n$ from outside $[0, 3n]$ which reached the site n or the site $2n$. So this fire has completely burnt the segment $[0, n]$ or the segment $[2n, 3n]$ and hence $T \leq \log n$. □

From Lemma 1 and Lemma 2 we obtain the following Proposition.

Proposition 3. *There is a constant* $C_6 > 0$ *such that the following holds. Let* m *be a positive integer, and* $K_1, \dots, K_m$ *disjoint segments* $\subset \mathbb{Z}$ *of length* $3n$ *each. For all* $0 < \lambda < \lambda_0$*, all* $t > 2\log n$*, and any initial configuration* ξ *we have*

$$\mathbb{P}^{\xi}_{\lambda}\left\{\cap_{i=1}^m H_{K_i}(t, t+\log n) \mid \mathcal{F}_{(\cup_{i=1}^m K_i)^c}\right\} > (C_6)^m, \tag{6}$$

and

$$\mathbb{P}^{\xi}_{\lambda}\left\{\cup_{i=1}^m H_{K_i}(t, t+\log n) \mid \mathcal{F}_{(\cup_{i=1}^m K_i)^c}\right\} > 1 - (C_6)^m. \tag{7}$$

Proof. For each i write K_i as the union of three segments $K_i(1)$, $K_i(2)$ and $K_i(3)$ of length n each. We look from time $t_0 = t - 2\log n > 0$. For $J \subset \mathbb{Z}$, let

$$\tau_J = \inf\{t > t_0 : \eta_t(x) = 0 \text{ for all } x \in J\} = \theta_{t_0}(T_J),$$

with T_J as in (5). Further, for $1 \le i \le m$ let

$$\tau(i) = \tau_{K_i(1)} \wedge \tau_{K_i(2)} \wedge \tau_{K_i(3)}. \tag{8}$$

We know from Lemma 2 that there is an $\mathcal{F}_{K_i}(t_0,t)-$measurable event $A(i) \subset \{\tau(i) \le t\}$ satisfying $\mathbb{P}_\lambda(A(i)) > C_5$. From Lemma 1 we know that there is an $\mathcal{F}_{K_i(1)}(3\log n)$-measurable event $\hat{H}_{K_i(1)}(3\log n)$ such that $\hat{H}_{K_i(1)}(3\log n) \cap \{\eta(0) \equiv 0 \text{ on } K_i(1)\} \subset H_{K_i(1)}(3\log n)$ and $\mathbb{P}_\lambda(\hat{H}_{K_i(1)}(3\log n)) > C_4$, and we have similar events $\hat{H}_{K_i(2)}(3\log n)$ and $\hat{H}_{K_i(3)}(3\log n)$ for $K_i(2)$ and $K_i(3)$ respectively. Let L_i be the minimizing segment in (8). Note that if $\tau(i) \le t$ and $\theta_{\tau(i)}\hat{H}_{L_i}(3\log n)$ occurs, then $H_{L_i}(t, t+\log n)$ occurs. Moreover, if also $B^c_{K_i \setminus L_i}(\tau(i), \tau(i) + 3\log n)$ occurs, then $H_{K_i}(t, t+\log n)$ occurs. The price to pay for the latter event is $(C_2(2))^3$, with C_2 as in (4).

If, for each i, $\tau(i)$ would be a stopping time with respect to the filtration $(\mathcal{F}_{K_i}(s))_{s\ge 0}$, the above observations would immediately give that the left hand side of (6) is at least $(C_4C_5(C_2(2))^3)^m$. However, $\tau(i)$ is not a stopping time with respect to that filtration but with respect to $(\mathcal{F}(s))_{s\ge 0}$. Nevertheless, with some more care one can, using fairly standard arguments, still obtain the above mentioned bound $(C_4C_5(C_2(2))^3)^m$. Similar arguments apply to (7). □

3 Main Theorem

3.1 Statement and proof of the main theorem

We are ready to prove our result on the asymptotic density of vacant sites. The theorem is formulated in a way that the former restriction $\lambda < \lambda_0$ can be dropped.

Theorem 4. *There exist constants $C_7, C_8 > 0$ such that for any initial configuration ξ, any $\lambda > 0$ and for all $t > 3\log n(\lambda) + 1$*

$$\frac{C_7}{\log(1/\lambda) \vee 1} \le \mathbb{P}^\xi_\lambda(\eta_0(t) = 0) \le \frac{C_8}{\log(1/\lambda) \vee 1}.$$

Proof. We start with the lower bound and with the more interesting case $0 < \lambda < \lambda_0$. Let $t_0 = t - \log n - 1 > 2 \log n$. It is clear that, to have $\eta_0(t) = 0$ it is sufficient that each of the following events occur: $H_{[-4n,-n]}(t_0, t-1)$, $H_{[n,4n]}(t_0, t-1)$, $A_{(-n,n)}(t_0, t-1)$, $B^c_{(-n,n)}(t_0, t-1)$, $B_{(-n,n)}(t-1, t)$ and $A^c_0(t-1, t)$. By Proposition 3 and (4), this has probability at least

$$(C_6)^2\, C_1(2)\, C_2(2)\, (1 - e^{-(2n-1)\lambda})\, e^{-1},$$

which, by (1), gives the desired lower bound.

For the case $\lambda \geq \lambda_0$, note that the event $\eta_0(t) = 0$ is implied by the event $A^c_0(t-1, t) \cap B_0(t-1, t)$. This has probability $e^{-1}(1 - e^{-\lambda}) \geq e^{-1}(1 - e^{-\lambda_0})$, completing the proof of the lower bound.

We continue with the proof of the upper bound. In the case $\lambda \geq \lambda_0$, the upper bound is trivial. For the case $0 < \lambda < \lambda_0$, we need the following claim.

Claim. There is a constant $c_1 > 0$ such that for all $0 < \lambda < \lambda_0$ and all $t > 2 \log n(\lambda)$,

$$\mathbb{P}^{\xi}_{\lambda}(O \text{ burns at some time in } [t, t+1]) \leq c_1 / \log n. \tag{9}$$

Proof of Claim. It is easy to check that, if the event in the claim happens, then there exists an integer $k \geq 0$ such that the following events (i) and (ii) occur:

(i) We have $\{S(k) \leq t + 1\}$, where

$$S(k) = \inf\{s \geq t : \eta(s) \equiv 1 \text{ on } [-4kn, 0] \text{ or } \eta(s) \equiv 1 \text{ on } [0, 4kn]\}.$$

(ii) An ignition occurs in $(-4(k+1)n, 4(k+1)n)$ at some time in $[S(k), t+1]$.

It is clear that given (i), the conditional probability that (ii) happens is bounded above by

$$\mathbb{P}_{\lambda}(B_{(-4(k+1)n, 4(k+1)n)}(1)) \leq 8\lambda(k+1)n \leq \frac{8(k+1)}{\log n},$$

where again we have used (1). Moreover, for fixed k the probability that (i) holds is at most $2(C_6)^k$ by (7) in Proposition 3. Combining these facts, the probability in the statement of the claim is at most

$$\frac{1}{\log n} \sum_{k=0}^{\infty} 16\, (k+1)(C_6)^k,$$

from which the claim follows.

We continue the proof of the upper bound in Theorem 4. If $\eta_0(t) = 0$, then either $\eta_0(s) = 0$ for all $s \in [t - (1/2)\log n, t]$ or there is an integer $k \in [0, (1/2)\log n]$ such that O burns at some time in the interval $[t-k-1, t-k]$ and has no birth attempt in $[t-k, t]$. Hence, using the above Claim,

$$\mathbb{P}_\lambda^\xi(\eta_0(t) = 0) \leq \exp(-(1/2)\log n) + \sum_{k:\, 0 \leq k \leq (1/2)\log n} \frac{c_1}{\log n} \exp(-k),$$

from which the upper bound in Theorem 4 follows immediately. □

3.2 Remarks and discussion

(i) Note that Theorem 4 immediately implies that if μ is a distribution that is invariant under the dynamics, then

$$\frac{C_7}{\log(1/\lambda) \vee 1} \leq \mu(\eta_0 = 0) \leq \frac{C_8}{\log(1/\lambda) \vee 1}. \tag{10}$$

For the special case where μ is also invariant under spatial translation, we have a considerably simpler proof of (10). (In particular, the proof of the lower bound in (10) then only needs a combination of the arguments in the proof of Lemma 2 and general stationarity arguments). Since we do not have a proof that all equlibrium distributions are translation invariant, our present argument is needed. Furthermore, Theorem 4 is much stronger than (10). Its major ingredient, Proposition 3, which in turn is based on Lemmas 1 and 2, gives strong properties of the spatial and temporal dependencies in the process. We believe these properties will also be useful for other purposes, for instance for the study of the question whether the model has a unique equlibrium distribution.

(ii) As we wrote in the Introduction, our model is somewhat different from the one studied by Drossel et al (1993). In that paper the fire propagates at a finite speed. In some more recent papers (see e.g. Schenk, Drossel and Schwabel (2002)) the speed is infinite, like in our model; that is, when a tree is hit by lightning, its occupied cluster becomes vacant instantaneously. Nevertheless we will point out in (a) and (b) below how a modification of our arguments also works for the original model of Drossel et al. (1993).

(a) Another look at the proofs shows that the arguments and estimates leading to Theorem 4 are, in some sense 'local': the births and ignitions outside the space interval $[-4n, 4n]$ 'do not matter'. In particular this means the following. Suppose that instead of the infinite line we have a finite forest, with locations $-N, \ldots, N$. The forest-fire process is then clearly a finite-state continuous-time Markov-chain. It is easy to see that it is irreducible and hence has a unique equilibrium (invariant distribution), which we denote by $\mu_{\lambda,N}$. For all $\lambda > 0$ and $N > 4n(\lambda)$ we have,

$$\frac{C_1}{\log(1/\lambda) \vee 1} \leq \mu_{\lambda,N}(\eta_0(t) = 0) \leq \frac{C_2}{\log(1/\lambda) \vee 1}.$$

(b) Apart from the above mentioned spatial locality, the arguments also have a locality in time. They essentially reduce to 'controlling' what happens in certain space-time blocks, with spatial length of order n and time length of order $\log n$. If we would modify our model and let the fire spread at some finite rate κ, a closer examination of our arguments show that they still work when the time it takes a fire to move through the segment $[0, n]$ is typically $o(\log n)$. That is, when $n/\kappa \ll \log n$. This in turn is guaranteed if $\kappa \gg 1/\lambda$, which corresponds with the condition $p \ll p/f$ in the paragraph preceding (2) in Drossel et al. (1993).

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