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A Series Expansion of Fractional Brownian Motion

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ABSTRACT
Let B be a fractional Brownian motion with Hurst index $H \in (0,1)$. Denote by $x_1 < x_2 < \cdots$ the positive, real zeros of the Bessel function J_{-H} of the first kind of order $-H$, and let $y_1 < y_2 < \cdots$ be the positive zeros of J_{1-H}. We prove the series representation

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos y_n t}{y_n} Y_n,$$

where $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent, Gaussian random variables with mean zero and $\mathbb{V}\mathrm{ar} X_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n)$, $\mathbb{V}\mathrm{ar} Y_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n)$, where the constant c_H^2 is defined by $c_H^2 = \pi^{-1}\Gamma(1+2H)\sin \pi H$. With probability 1, both random series converge absolutely and uniformly in $t \in [0,1]$.

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1 Introduction

Let $B = (B_t)_{t \geq 0}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, that is a centered Gaussian process with continuous sample paths and covariance function

$$\mathbb{E} B_s B_t = \tfrac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right). \tag{1.1}$$

The fBm with Hurst index H is a process with stationary increments and self-similarity index H, meaning that $(B_{at})_{t\geq 0} \overset{\mathrm{d}}{=} (a^H B_t)_{t \geq 0}$ for every $a > 0$. When the Hurst index equals $1/2$, the fBm is simply the ordinary standard Brownian motion. The study of fBm goes back to Kolmogorov

(1940), who showed in particular that the expression on the right-hand side of (1.1) defines a covariance function. Mandelbrot and Van Ness (1968) gave the fBm its present name.

The increments of fBm are negatively correlated for $H < 1/2$, and positively for $H > 1/2$. The fBm with Hurst index $H > 1/2$ is often used to incorporate long-range dependence in stochastic models. One area where fBm has been widely used in recent years is telecommunications (see e.g. Leland et al. (1994), Norros (1995)). Another example is continuous-time mathematical finance, where the fBm is sometimes considered as an alternative for ordinary Brownian motion (see e.g. Cutland et al. (1995), Salopek (1998), Sottinen (2001)). This approach is however subject to some controversy, since the fBm introduces arbitrage opportunities into the models (cf. Rogers (1997), Sottinen and Valkeila (2001)).

Motivated by the applications, considerable progress has recently been achieved in the theoretical study of fractional Brownian motion. Let us mention in particular the development of stochastic integration with respect to fBm (see for instance Decreusefond and Üstünel (1999), Alòs et al. (2000), Pipiras and Taqqu (2000), Coutin et al. (2001), Pipiras and Taqqu (2001)) and the rediscovery of certain relations between fBm and continuous, Gaussian martingales (see e.g. Norros et al. (1999), Nuzman and Poor (2000)).

For standard Brownian motion (the case $H = 1/2$), there exist various explicit, almost sure series expansions. These represent the Brownian motion W as a sum of the type $\sum_n \psi_n(t)X_n$, where $X_1, X_2, \ldots$ are i.i.d., standard Gaussian random variables and $\psi_1, \psi_2, \ldots$ are certain functions. A well-known example is the Karhunen-Loève expansion

$$W_t = \sqrt{2}\sum_{n=1}^{\infty} \frac{\sin(n-\frac{1}{2})\pi t}{(n-\frac{1}{2})\pi} X_n, \quad t \in [0,1] \tag{1.2}$$

(cf. e.g. Yaglom (1987), p. 451). To obtain alternative expansions for the case $H = 1/2$, simply note that if we restrict the time parameter to the interval $[0,1]$, the covariance $\mathbb{E}W_sW_t = s \wedge t$ is the inner product in $L^2[0,1]$ of the indicator functions $1_{(0,s)}$ and $1_{(0,t)}$. If we expand these indicators with respect to an arbitrary complete, orthonormal system of functions in $L^2[0,1]$, we obtain a series expansion for the Brownian motion. For example, the orthonormal system $\sqrt{2}\sin n\pi x$ yields the expansion

$$W_t = \sqrt{2}\sum_{n=1}^{\infty} \frac{1-\cos n\pi t}{n\pi} Y_n, \quad t \in [0,1], \tag{1.3}$$

where $Y_1, Y_2, \ldots,$ are i.i.d., standard Gaussian random variables. We can of course also combine expansions (1.2) and (1.3). This yields the representation

$$W_t = \sum_{n=1}^{\infty} \frac{\sin(n-\frac{1}{2})\pi t}{(n-\frac{1}{2})\pi} X_n + \sum_{n=1}^{\infty} \frac{1-\cos n\pi t}{n\pi} Y_n, \quad t \in [0,1], \tag{1.4}$$

where $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots,$ are two independent sequences of i.i.d., standard Gaussian random variables.

To the best of our knowledge, explicit series representations like (1.2), (1.3) or (1.4) have never been obtained up to now for fBm with Hurst index $H \neq 1/2$. In this paper, we extend the expansion (1.4) to the fractional Brownian motion with arbitrary Hurst index $H \in (0,1)$. It turns out that for general H, the numbers $(n-1/2)\pi$ and $n\pi$ appearing in (1.4) have to be replaced by the zeros of certain Bessel functions. Recall that for $\nu \neq -1,-2,\dots$ the Bessel function J_ν of the first kind of order ν is defined on the region $\{z \in \mathbb{C} : |\arg z| < \pi\}$ as the absolutely convergent sum

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}.$$

It is well-known that for $\nu > -1$, the function J_ν has a countable number of real, positive, simple zeros (see e.g. Watson (1944), Chapter 15). These zeros can be arranged in ascending order of magnitude and they become arbitrarily large. Now let the Hurst index $H \in (0,1)$ be fixed, let $x_1 < x_2 < \cdots$ be the positive zeros of J_{-H} and let $y_1 < y_2 < \cdots$ be the positive zeros of J_{1-H}. In this notation, our general expansion reads

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos y_n t}{y_n} Y_n, \quad t \in [0,1], \tag{1.5}$$

where $X_1, X_2, \dots$ and $Y_1, Y_2, \dots,$ are two independent sequences of independent Gaussian random variables, with $\mathbb{E}X_n = \mathbb{E}Y_n = 0$ and $\mathbb{V}\mathrm{ar}\,X_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n)$, $\mathbb{V}\mathrm{ar}\,Y_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n)$, where the constant c_H^2 is defined by $c_H^2 = \pi^{-1}\Gamma(1+2H)\sin \pi H$. To see that (1.5) indeed extends the expansion (1.4) of standard Brownian motion, note that for $H = 1/2$ we have $c_{1/2}^2 = 1/\pi$ and

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z,$$

so that $x_n = (n-1/2)\pi$, $y_n = n\pi$, $\mathbb{V}\mathrm{ar}\,X_n = 1$ and $\mathbb{V}\mathrm{ar}\,Y_n = 1$.

Several proofs in the paper rely heavily on special function theory. We use properties of the zeros of Bessel functions, and work with Hankel transforms and Fourier-Bessel expansions. For background information on these topics we refer the reader to the classical treatise of Watson (1944). A more concise treatment can be found for instance in Erdélyi et al. (1953) or Hochstadt (1971). Another technical tool that we use is the Erdélyi-Kober version of the fractional calculus, see the appendix to this paper, where the basic definitions of Samko et al. (1993), pp. 322–324, are recalled.

2 Spectral representations

In this section, it is useful to consider a two-sided fBm. So we assume that $B = (B_t)_{t\in\mathbb{R}}$ and

$$\mathbb{E}B_s B_t = \tfrac{1}{2}\left(|t|^{2H} + |s|^{2H} - |s-t|^{2H}\right)$$

for all $s, t \in \mathbb{R}$. It is well-known that the covariance function of fBm is harmonizable. Up to a constant that depends on H, the spectral measure μ of fBm has density $|\lambda|^{1-2H}$ with respect to Lebesgue measure and the covariance $\mathbb{E}B_s B_t$ can be written as the inner product in $L^2(\mu)$ of the Fourier transforms $(\exp i\lambda s - 1)/i\lambda$ and $(\exp i\lambda t - 1)/i\lambda$ of the indicator functions $1_{(0,s)}$ and $1_{(0,t)}$. The precise statement is as follows (see for instance Yaglom (1987), p. 407 or Samorodnitsky and Taqqu (1994), p. 328).

Theorem 2.1. *For all $H \in (0,1)$ and $s, t \in \mathbb{R}$*

$$\mathbb{E}B_s B_t = \frac{c_H^2}{2} \int_{\mathbb{R}} \frac{(e^{i\lambda t} - 1)(e^{-i\lambda s} - 1)}{\lambda^2} |\lambda|^{1-2H} \, d\lambda, \tag{2.1}$$

where

$$c_H^2 = \frac{\Gamma(1+2H) \sin \pi H}{\pi}. \tag{2.2}$$

The left-hand side of (2.1) is obviously real-valued. Taking the real part of both sides of (2.1) and using the symmetry of the integrand around 0, we obtain the following 'real-valued version' of Theorem 2.1, see also Samorodnitsky and Taqqu (1994), p. 329.

Corollary 2.2. *For all $H \in (0,1)$ and $s, t \in \mathbb{R}$ we have*

$$\mathbb{E}B_s B_t = c_H^2 \int_0^\infty \frac{\sin \lambda s \sin \lambda t + (1 - \cos \lambda s)(1 - \cos \lambda t)}{\lambda^{1+2H}} \, d\lambda, \tag{2.3}$$

where c_H^2 is given by (2.2).

Let us note that the two terms on the right-hand side of (2.3) correspond to the 'odd' and 'even' parts of the fBm. Indeed, let the odd and even parts be defined by $B_t^o = \frac{1}{2}(B_t - B_{-t})$ and $B_t^e = \frac{1}{2}(B_t + B_{-t})$. Clearly, the sample paths of B^o (resp. B^e) are odd (resp. even) functions and $B = B^o + B^e$. Moreover, since $\mathbb{E}B_{-t}B_s = \mathbb{E}B_t B_{-s}$ for all $s, t \in \mathbb{R}$, the Gaussian processes B^o and B^e are independent. In particular, we have

$$\mathbb{E}B_s B_t = \mathbb{E}B_s^o B_t^o + \mathbb{E}B_s^e B_t^e. \tag{2.4}$$

It is easily verified that the odd part has covariance function

$$\mathbb{E}B_s^o B_t^o = \tfrac{1}{4} \left(|s+t|^{2H} - |s-t|^{2H} \right).$$

Hence, by formula 2.6 (3) on p. 78 of Erdélyi et al. (1954a), we have

$$\mathbb{E}B_s^o B_t^o = c_H^2 \int_0^\infty \frac{\sin \lambda s \sin \lambda t}{\lambda^{1+2H}} \, d\lambda. \tag{2.5}$$

By relations (2.3) and (2.4), it follows that

$$\mathbb{E}B_s^e B_t^e = c_H^2 \int_0^\infty \frac{(1 - \cos \lambda s)(1 - \cos \lambda t)}{\lambda^{1+2H}} \, d\lambda. \tag{2.6}$$

3 Integral representations

Relations (2.3), (2.5) and (2.6) represent the covariance functions of the fBm and its odd and even parts in terms of inner products in the frequency domain. In the present section we write the covariances as inner products in the time domain. The proofs rely on the fact that we can find explicit expressions for the Hankel transforms (of appropriate order) of the functions $\lambda \mapsto (\sin \lambda t)/\lambda^{H+1/2}$ and $\lambda \mapsto (1 - \cos \lambda t)/\lambda^{H+1/2}$ appearing in (2.5) and (2.6).

We begin with the odd part of the fBm. For $t \geq 0$ we define the kernel k_t^o by

$$k_t^o(u) = \frac{\sqrt{\pi}}{2^H \Gamma(\frac{1}{2} + H)} u^{\frac{1}{2} - H} (t^2 - u^2)^{H - \frac{1}{2}} 1_{(0,t)}(u). \tag{3.1}$$

Theorem 3.1. *For all $H \in (0, 1)$ and $s, t \geq 0$ we have*

$$\mathbb{E} B_s^o B_t^o = c_H^2 \int_0^{s \wedge t} k_s^o(u) k_t^o(u)\, du, \tag{3.2}$$

where c_H^2 is defined by (2.2) and k_t^o by (3.1).

Proof. We use the fact that for every $t \geq 0$ the function k_t^o is the Hankel transform of order $-H$ of the function $\lambda \mapsto (\sin \lambda t)/\lambda^{H+1/2}$ and vice versa. Indeed, by formulas 8.7 (4) on p. 32 and 8.5 (33) on p. 26 of Erdélyi et al. (1954b), we have

$$k_t^o(u) = \int_0^\infty \frac{\sin \lambda t}{\lambda^{H + \frac{1}{2}}} J_{-H}(\lambda u) \sqrt{\lambda u}\, d\lambda \tag{3.3}$$

and

$$\frac{\sin \lambda t}{\lambda^{H + \frac{1}{2}}} = \int_0^\infty k_t^o(u) J_{-H}(\lambda u) \sqrt{\lambda u}\, du. \tag{3.4}$$

Both functions are easily seen to belong to $L^2[0, \infty)$, so by Parseval's relation for Hankel transforms (see Macaulay-Owen (1939)), we have

$$\int_0^\infty k_s^o(u) k_t^o(u)\, du = \int_0^\infty \frac{\sin \lambda t \sin \lambda s}{\lambda^2} \lambda^{1-2H}\, d\lambda. \tag{3.5}$$

If we multiply this by c_H^2 and use relation (2.5), we obtain (3.2).

We note that usually, the Parseval relation is only proved for Hankel transforms of order $\nu \geq -1/2$, which corresponds in our case to $H \leq 1/2$. It is well-known however that for $-1 < \nu < -1/2$, the L^2-theory of Hankel transforms still goes through in great generality (see e.g. Titchmarsh (1937), Theorem 129, p. 221). In our particular case, it is quite easy to give a direct proof of relation (3.5) for $H > 1/2$. First we use (3.3) to write the left-hand side of (3.5) as

$$\int_0^\infty k_s^o(u) \left(\int_0^\infty \frac{\sin \lambda t}{\lambda^{H + \frac{1}{2}}} J_{-H}(\lambda u) \sqrt{\lambda u}\, d\lambda \right) du.$$

Since the function $x \mapsto J_{-H}(x)\sqrt{x}$ is bounded, k_t^o is integrable and $\lambda \mapsto 1/\lambda^{H+1/2}$ is integrable for $H > 1/2$, we are allowed to interchange the order of integration. Hence, the integral is equal to

$$\int_0^\infty \frac{\sin \lambda t}{\lambda^{H+\frac{1}{2}}} \left(\int_0^\infty k_s^o(u) J_{-H}(\lambda u) \sqrt{\lambda u}\, du \right) d\lambda.$$

In view of (3.4), the inner integral now equals $(\sin \lambda s)/\lambda^{H+1/2}$ and we arrive at the desired relation (3.5). □

For the even part of the fBm, we need a more complicated kernel function. For $t \geq 0$, we define

$$\begin{aligned} k_t^e(u) &= -\frac{\sqrt{\pi} u^{\frac{1}{2}-H}}{2^H \Gamma(H+\frac{1}{2})} \frac{d}{du} \left(\int_u^t (x^2 - u^2)^{H-\frac{1}{2}}\, dx \right) 1_{(0,t)}(u) \\ &= \frac{\sqrt{\pi} u^{\frac{3}{2}-H}}{2^H \Gamma(H+\frac{1}{2})} \left(\frac{(t^2-u^2)^{H-\frac{1}{2}}}{t} + \int_u^t \frac{(x^2-u^2)^{H-\frac{1}{2}}}{x^2}\, dx \right) 1_{(0,t)}(u). \end{aligned} \tag{3.6}$$

To see that the two expressions are indeed equal, use integration by parts to rewrite the integral in the first expression as

$$\frac{(t^2-u^2)^{H+\frac{1}{2}}}{(2H+1)t} + \int_u^t \frac{(x^2-u^2)^{H+\frac{1}{2}}}{(2H+1)x^2}\, dx.$$

It is then straightforward to obtain the second expression. Also observe that for $H > 1/2$, the kernel k_t^e can be written in a less complicated form. We have

$$k_t^e(u) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} u^{\frac{3}{2}-H} \left(\int_u^t (x^2-u^2)^{H-\frac{3}{2}}\, dx \right) 1_{(0,t)}(u) \tag{3.7}$$

in that case.

Theorem 3.2. *For all $H \in (0,1)$ and $s, t \geq 0$ we have*

$$\mathbb{E} B_s^e B_t^e = c_H^2 \int_0^{s\wedge t} k_s^e(u) k_t^e(u)\, du, \tag{3.8}$$

where c_H^2 is defined by (2.2) and k_t^e by (3.6).

Proof. We will show that for every $t \geq 0$, the function k_t^e is the Hankel transform of order $1-H$ of the function $\lambda \mapsto (1-\cos \lambda t)/\lambda^{H+1/2}$ and vice versa, i.e.

$$k_t^e(u) = \int_0^\infty \frac{1-\cos \lambda t}{\lambda^{H+\frac{1}{2}}} J_{1-H}(\lambda u) \sqrt{\lambda u}\, d\lambda \tag{3.9}$$

and

$$\frac{1-\cos\lambda t}{\lambda^{H+\frac{1}{2}}} = \int_0^\infty k_t^e(u) J_{1-H}(\lambda u)\sqrt{\lambda u}\, du. \tag{3.10}$$

First we note that the kernel k_t^e is an Erdélyi-Kober-type fractional integral of the function

$$f(u) = \frac{\sqrt{\pi}}{2^H} u^{H-\frac{1}{2}}.$$

With $\sigma = 2$ and $\eta = 3/4 - H/2$ we have, in the notation of Samko et al. (1993), p. 322,

$$k_t^e(u) = \left(I^{H-\frac{1}{2}}_{t-;\sigma,\eta} f(u)\right) 1_{(0,t)}(u).$$

This can easily be verified by comparing the definition of the operator $I^{H-\frac{1}{2}}_{t-;\sigma,\eta}$ (see the appendix) with the first expression in (3.6). Hence, by fractional integration by parts (see (A.5)), the Hankel transform of order $1-H$ of the kernel k_t^e satisfies

$$\begin{aligned}&\int_0^\infty k_t^e(u) J_{1-H}(\lambda u)\sqrt{\lambda u}\, du = \\ &\int_0^t I^{H-\frac{1}{2}}_{t-;\sigma,\eta} f(u) J_{1-H}(\lambda u)\sqrt{\lambda u}\, du = \int_0^t u f(u) I^{H-\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u)\, du,\end{aligned} \tag{3.11}$$

where $g_\lambda(u) = \sqrt{\lambda/u} J_{1-H}(\lambda u)$. The fractional integral of g_λ appearing in (3.11) can be calculated explicitly. Note that the order $H - 1/2$ of the fractional integral may be negative, but always exceeds $-1/2$. Therefore, by formula (A.2), we have

$$\begin{aligned} I^{H-\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u) &= \frac{u^{-\sigma(H-\frac{1}{2}+\eta)}}{2u} \frac{d}{du}\left(u^{\sigma(H+\frac{1}{2}+\eta)} I^{H+\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u)\right) \\ &= \tfrac{1}{2} u^{-H-\frac{3}{2}} \frac{d}{du}\left(u^{H+\frac{5}{2}} I^{H+\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u)\right). \end{aligned} \tag{3.12}$$

Since $H + 1/2 > 0$, formula (A.1) implies that

$$I^{H+\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u) = \frac{2u^{-H-\frac{5}{2}}}{\Gamma(H+\frac{1}{2})} \int_0^u (u^2 - y^2)^{H-\frac{1}{2}} y^{\frac{3}{2}-H} J_{1-H}(y\lambda)\sqrt{y\lambda}\, dy.$$

The latter integral may be evaluated with the help of formula 8.5 (33) on p. 26 of Erdélyi et al. (1954b). We find that

$$I^{H+\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u) = 2^{H+\frac{1}{2}} u^{-H-1} \lambda^{-H} J_{\frac{3}{2}}(\lambda u).$$

Plug this into (3.12) and use the fact that $(d/dz)(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z)$ to obtain

$$I^{H-\frac{1}{2}}_{0+;\sigma,\eta} g_\lambda(u) = \frac{2^H}{\sqrt{\pi}} u^{-H-\frac{1}{2}} \lambda^{\frac{1}{2}-H} \sin\lambda u.$$

In view of (3.11), we conclude that

$$\int_0^\infty k_t^e(u) J_{1-H}(\lambda u)\sqrt{\lambda u}\, du = \int_0^t \lambda^{\frac{1}{2}-H} \sin \lambda u\, du = \frac{1-\cos \lambda t}{\lambda^{\frac{1}{2}+H}}.$$

So indeed, the kernel k_t^e and the function $\lambda \mapsto (1-\cos\lambda t)/\lambda^{H+1/2}$ form a Hankel transform pair of order $1-H$, which proves (3.9) and (3.10). Apply the Parseval relation for Hankel transforms to complete the proof. □

Remark 3.3. In the preceding proof we have used the fact that the kernel k_t^e defined by (3.6) can be expressed as an Erdélyi-Kober-type fractional integral of the function $f(u) = \pi^{1/2} 2^{-H} u^{H-1/2}$, we have

$$k_t^e(u) = I^{H-\frac{1}{2}}_{t-;2,\frac{3}{4}-\frac{1}{2}H} f(u), \quad 0 < u < t.$$

Let us note that the kernel k_t^o defined by (3.1) admits a similar expression. It is easily verified that

$$k_t^o(u) = I^{H-\frac{1}{2}}_{t-;2,\frac{1}{4}-\frac{1}{2}H} f(u), \quad 0 < u < t.$$

If we combine Theorems 3.1 and 3.2 and use relation (2.4), we obtain the following representation of the covariance function of the fBm.

Theorem 3.4. *For all $H \in (0,1)$ and $s, t \geq 0$ we have*

$$\mathbb{E}B_s B_t = c_H^2 \int_0^{s\wedge t} \Big(k_s^o(u) k_t^o(u) + k_s^e(u) k_t^e(u) \Big)\, du, \tag{3.13}$$

where c_H^2 is defined by (2.2) and k_t^o and k_t^e by (3.1) and (3.6).

Note that Theorem 3.4 can be rephrased as a (finite past) moving average-type result. It states that

$$B_t \stackrel{\mathrm{d}}{=} c_H \int_0^t k_t^o(u)\, dW_u^o + c_H \int_0^t k_t^e(u)\, dW_u^e,$$

where W^o and W^e are two independent, standard Brownian motions. Compare this for instance with Theorem 5.2 of Norros et al. (1999), which gives a moving average representation of the fBm in terms of a single standard Brownian motion.

4 Series expansions

Now let $H \in (0,1)$ be fixed. Then for every $t \in [0,1]$, the kernels k_t^o and k_t^e defined by (3.1) and (3.6) belong to $L^2[0,1]$. So if $\varphi_1, \varphi_2, \ldots$ is a complete, orthonormal system of functions in $L^2[0,1]$, we have $k_t^o(u) = \sum_{n=1}^\infty a_n^o(t)\varphi_n(u)$ and $k_t^e(u) = \sum_{n=1}^\infty a_n^e(t)\varphi_n(u)$ in $L^2[0,1]$, where

$$a_n^o(t) = \int_0^1 k_t^o(v)\varphi_n(v)\, dv, \quad a_n^e(t) = \int_0^1 k_t^e(v)\varphi_n(v)\, dv. \tag{4.1}$$

Theorem 3.4 then implies that $\mathbb{E}B_sB_t = c_H^2 \sum_{n=1}^\infty (a_n^o(s)a_n^o(t) + a_n^e(s)a_n^e(t))$. To obtain an explicit series representation of the covariance function of fBm, we are now going to choose a complete, orthonormal system of functions φ_n for which we can calculate the coefficients in (4.1) explicitly. The so-called Fourier-Bessel functions constitute such a system. The corresponding coefficients can be expressed in terms of the Hankel transforms of the kernels k_t^o and k_t^e for which, as we saw in the proofs of Theorems 3.1 and 3.2, we have an explicit expression.

To prove that the expansions that we obtain in this section are uniform in the time parameter, we need the following lemma.

Lemma 4.1. *Let $\nu > -1$ be arbitrary and let $z_1 < z_2 < \cdots$ be the positive zeros of J_ν. Then for all $p > 0$*

$$\sum_{n=1}^\infty \frac{1}{z_n^{p+2} J_{1+\nu}^2(z_n)} < \infty.$$

Proof. For the Bessel function J_ν, we have the asymptotic relation

$$J_\nu^2(z) + J_{\nu+1}^2(z) \sim \frac{2}{\pi z} \tag{4.2}$$

for large $|z|$ (cf. Watson (1944), p. 200). Since the zeros z_n of J_ν tend to infinity, we have $J_{1+\nu}^2(z_n) \sim 2/\pi z_n$ for $n \to \infty$. Hence, it suffices to show the convergence

$$\sum_{n=1}^\infty \frac{1}{z_n^{p+1}} < \infty.$$

The proof is completed by evoking the last formula on p. 506 of Watson (1944), according to which the n-th positive zero z_n of J_ν is asymptotically of order $n\pi$. □

For the covariance function of the odd part of the fBm we obtain the following series expansion.

Theorem 4.2. *Let $H \in (0,1)$ be arbitrary. Let $x_1 < x_2 < \cdots$ be the positive, real zeros of J_{-H}. For $n \in \mathbb{N}$, define*

$$\sigma_n^2 = \frac{2c_H^2}{x_n^{2H} J_{1-H}^2(x_n)}, \tag{4.3}$$

where c_H^2 is given by (2.2). Then for all $s, t \in [0,1]$ we have

$$\mathbb{E} B_s^o B_t^o = \sum_{n=1}^{\infty} \frac{\sin x_n s \, \sin x_n t}{x_n^2} \, \sigma_n^2,$$

where the series converges absolutely and uniformly in $(s,t) \in [0,1] \times [0,1]$.

Proof. If we apply Lemma 4.1 with $\nu = -H$ and $p = 2H$, we see that the series converges absolutely and uniformly on the unit square to some limit. Hence, it remains to prove the expansion for fixed $s, t \in [0,1]$. For $n \in \mathbb{N}$, let φ_n be the n-th Fourier-Bessel function of order $-H$, i.e.

$$\varphi_n(z) = \frac{\sqrt{2}}{|J_{1-H}(x_n)|} J_{-H}(x_n z)\sqrt{z},$$

where $x_1 < x_2 < \cdots$ are the positive zeros of J_{-H}. Recall that the functions φ_n form a complete, orthonormal system in $L^2[0,1]$ (see e.g. Hochstadt (1971), p. 264). Hence, arguing as in the beginning of this section, we find that $\mathbb{E} B_s^o B_t^o = c_H^2 \sum_{n=1}^{\infty} a_n^o(s) a_n^o(t)$, with $a_n^o(t)$ as in (4.1). Since φ_n is now the n-th Fourier-Bessel function of order $-H$, the coefficient $a_n^o(t)$ is the Hankel integral that we already encountered in the proof of Theorem 3.1. By formula (3.4) we have

$$a_n^o(t) = \frac{\sqrt{2}}{|J_{1-H}(x_n)|} \frac{\sin x_n t}{x_n^{H+1}}.$$

This completes the proof of the theorem. □

Similarly, we get the following result for the even part of the fBm.

Theorem 4.3. *Let $H \in (0,1)$ be arbitrary. Let $y_1 < y_2 < \cdots$ be the positive, real zeros of J_{1-H}. For $n \in \mathbb{N}$, define*

$$\tau_n^2 = \frac{2c_H^2}{y_n^{2H} J_{-H}^2(y_n)}, \tag{4.4}$$

where c_H^2 is given by (2.2). Then for all $s, t \in [0,1]$ we have

$$\mathbb{E} B_s^e B_t^e = \sum_{n=1}^{\infty} \frac{(1 - \cos y_n s)(1 - \cos y_n t)}{y_n^2} \, \tau_n^2,$$

where the series converges absolutely and uniformly in $(s,t) \in [0,1] \times [0,1]$.

Proof. The uniform and absolute convergence follows from Lemma 4.1 again, but now applied with $\nu = 1-H$ and $p = 2H$. The remainder of the proof is also analogous to the proof of Theorem 4.2. Simply expand the kernel k_t^e with respect to the Fourier-Bessel functions of order $1-H$ and use formula (3.10) to calculate the coefficients. Finally, use the fact that $J^2_{2-H}(y_n) = J^2_{-H}(y_n)$ (see the first display on p. 480 of Watson (1944)). □

In view of relation (2.4), a combination of Theorems 4.2 and 4.3 yields the following series expansion for the covariance function of the fBm itself.

Theorem 4.4. *Let $H \in (0,1)$ be arbitrary. Let $x_1 < x_2 < \cdots$ be the positive, real zeros of J_{-H} and let $y_1 < y_2 < \cdots$ be the positive, real zeros of J_{1-H}. For $n \in \mathbb{N}$, define σ_n^2 and τ_n^2 by (4.3) and (4.4). Then for all $s,t \in [0,1]$ we have*

$$\mathbb{E}B_sB_t = \sum_{n=1}^{\infty} \frac{\sin x_n s \, \sin x_n t}{x_n^2} \sigma_n^2 + \sum_{n=1}^{\infty} \frac{(1-\cos y_n s)(1-\cos y_n t)}{y_n^2} \tau_n^2,$$

where both series converge absolutely and uniformly in $(s,t) \in [0,1] \times [0,1]$.

Theorem 4.4 implies that we have a series expansion of the fBm in mean square sense. Using Lemma 4.1 again, this can easily be strengthened to an almost sure series expansion.

Theorem 4.5. *Let $H \in (0,1)$ be arbitrary. Let c_H^2 be given by (2.2), let $x_1 < x_2 < \cdots$ be the positive, real zeros of the Bessel function J_{-H} and let $y_1 < y_2 < \cdots$ be the positive, real zeros of J_{1-H}. For $n \in \mathbb{N}$, define σ_n^2 by (4.3) and τ_n^2 by (4.4). Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent sequences of independent, centered Gaussian random variables on a common probability space, with $\mathbb{V}\mathrm{ar} X_n = \sigma_n^2$ and $\mathbb{V}\mathrm{ar} Y_n = \tau_n^2$. Then the random process $B = (B_t)_{t\in[0,1]}$ given by*

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos y_n t}{y_n} Y_n$$

is well-defined and with probability 1, both series converge absolutely and uniformly in $t \in [0,1]$. The process B is a fBm with Hurst index H.

Proof. Theorem 4.4 already shows that we have equality in mean square sense, so it remains to show that with probability 1, both series converge absolutely and uniformly. The limit B is then automatically continuous. First consider the partial sums

$$S_t^N = \sum_{n=1}^{N} \frac{\sin x_n t}{x_n} X_n.$$

We want to show that with probability 1, the processes $S^N = (S_t^N)_{t\in[0,1]}$ form a Cauchy sequence in the space $C[0,1]$ of continuous functions on the interval $[0,1]$, endowed with the supremum metric. For $N < M$ we have

$$\sup_{t\in[0,1]} \left|S_t^M - S_t^N\right| \le \sum_{n=N+1}^{M} \frac{|X_n|}{x_n}.$$

Hence, it suffices to show that with probability 1, the random series $\sum |X_n|/x_n$ converges to a finite limit. By Kolmogorov's three-series theorem, a sufficient condition for this convergence is that $\sum \sigma_n^2/x_n^2 < \infty$. This is precisely the content of Lemma 4.1, with $\nu = -H$ and $p = 2H$. The absolute and uniform convergence of the second series can be shown in exactly the same manner. □

A Appendix

In this appendix we recall the basic definitions of the Erdélyi-Kober-type fractional integrals, see Samko et al. (1993), p. 322. Let f be a function defined on the interval $[a,b]$. Then for $\alpha > 0$, the so-called left-sided integral is defined by

$$I^{\alpha}_{a+;\sigma,\eta} f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma\eta+\sigma-1} f(t)\, dt \tag{A.1}$$

for every $x \in [a,b]$, while for $\alpha > -n$

$$I^{\alpha}_{a+;\sigma,\eta} f(x) = x^{-\sigma(\alpha+\eta)} \left(\frac{d}{\sigma x^{\sigma-1}\, dx}\right)^n x^{\sigma(\alpha+n+\eta)} I^{\alpha+n}_{a+;\sigma,\eta} f(x). \tag{A.2}$$

Similarly, right-sided integral is defined by

$$I^{\alpha}_{b-;\sigma,\eta} f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt \tag{A.3}$$

for $\alpha > 0$ and

$$I^{\alpha}_{b-;\sigma,\eta} f(x) = x^{\sigma\eta} \left(-\frac{d}{\sigma x^{\sigma-1}\, dx}\right)^n x^{\sigma(n-\eta)} I^{\alpha+n}_{b-;\sigma,\eta-n} f(x) \tag{A.4}$$

for $\alpha > -n$. Throughout the present paper, we put $\sigma = 2$ and $\eta = 3/4 - H/2$ or $\eta = 1/4 - H/2$. In the proof of Theorem 3.2 we apply the formula of fractional integration by parts for Erdélyi-Kober integrals. It states that

$$\int_a^b x^{\sigma-1} f(x) I^{\alpha}_{a+;\sigma,\eta} g(x)\, dx = \int_a^b x^{\sigma-1} g(x) I^{\alpha}_{b-;\sigma,\eta} f(x)\, dx, \tag{A.5}$$

see formula (18.18) on p. 324 of Samko et al. (1993).

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