Centrum voor Wiskunde en Informatica
REPORT RAPPORT

A Reduced-load Equivalence for Generalised Processor Sharing Networks with Heavy-tailed Input Flows

M.J.G. van Uitert, S.C. Borst

Probability, Networks and Algorithms (PNA)

PNA-R0007 August 31, 2000

Report PNA-R0007
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

A Reduced-load Equivalence for Generalised Processor Sharing Networks with Heavy-tailed Input Flows

Miranda van Uitert*, Sem Borst*,†,‡,

*CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

†Department of Mathematics & Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

‡Bell Laboratories, Lucent Technologies
P.O. Box 636, Murray Hill, NJ 07974, USA

ABSTRACT

We consider networks where traffic is served according to the Generalised Processor Sharing (GPS) principle. GPS-based scheduling algorithms are considered important for providing differentiated quality of service in integrated-services networks. We are interested in the workload of a particular flow i at the bottleneck node on its path. Flow i is assumed to have long-tailed traffic characteristics. We distinguish between two traffic scenarios, (i) flow i generates instantaneous traffic bursts and (ii) flow i generates traffic according to an on/off process. In addition, we consider two configurations of feed-forward networks. First we focus on the situation where other flows join the path of flow i. Then we extend the model by adding flows which can branch off at any node, with cross traffic as a special case. We prove that under certain conditions the tail behaviour of the workload distribution of flow i is equivalent to that in a *two*-node tandem network where flow i is served in isolation at *constant* rates. These rates only depend on the traffic characteristics of the other flows through their average rates. This means that the results do not rely on any specific assumptions regarding the traffic processes of the other flows. In particular, flow i is not affected by excessive activity of flows with 'heavier-tailed' traffic characteristics. This confirms that GPS has the potential to protect individual flows against extreme behaviour of other flows, while obtaining substantial multiplexing gains.

2000 Mathematics Subject Classification: 60K25 (primary), 68M20, 90B18, 90B22 (secondary).

Keywords and Phrases: Generalised Processor Sharing (GPS), heavy-tailed traffic, regular variation, Weighted Fair Queueing (WFQ).

Note: Work carried out under the project PNA2.1 "Communication and Computer Networks", with financial support from the Telematics Institute.

1 Introduction

Integrated-services networks carry a large amount of different services. Each of these services has its own traffic characteristics and requires its own quality of service (QoS) guarantees. This heterogeneity in traffic characteristics and QoS guarantees requires traffic control mechanisms to regulate the usage of network resources. In particular, scheduling mechanisms play an important role in achieving differentiated QoS. One of the most important scheduling algorithms is the Generalised Processor Sharing (GPS) mechanism, which was first studied by Parekh and Gallager [12, 13]. GPS is characterised by two attractive properties, (i) each backlogged flow is guaranteed a minimum service rate and (ii) the excess service rate is redistributed among the backlogged flows in proportion to their minimum service rates. Because of the second property GPS is work-conserving. Commonly-used scheduling mechanisms in packet-switched networks, such as Weighted Fair Queueing (WFQ) and other algorithms [16], are based on GPS.

Achieving differentiated QoS is a challenging task due to the highly bursty traffic characteristics in high-speed communication networks. In contrast to traditional assumptions, the burstiness extends over a wide range of time scales. Statistical data analysis [14, 17] has in fact shown that traffic patterns may look similar when observed on various time scales. This behaviour is usually referred to as self-similarity. Several studies, e.g. [10], further offered evidence of a closely related property called long-range dependence, which means that correlations in the traffic activity decay slowly over time. These findings caused a fundamental shift in modelling traffic behaviour. Classical models mostly assume traffic processes with a Markovian structure, which are inherently short-range dependent. Recently though, the focus has shifted to traffic processes with long-tailed characteristics, which provide a useful paradigm for modelling long-range dependence and self-similarity. An example of such a model is an on/off process where the on periods are regularly varying with index $-\nu$, $\nu \in (1, 2)$.

It is not clear to what extent long-tailed traffic may impact the potential for scheduling mechanisms to help achieve differentiated QoS. To be able to guarantee end-to-end QoS, it is particularly relevant to understand to what degree traffic flows are negatively affected as they traverse the network. Anantharam [1] was one of the first to study the influence of scheduling strategies on the extent to which long-tailed traffic affects network performance. He showed the influence can be significant, depending on whether or not preemption is admissible.

In this study we investigate the impact of long-tailed traffic on performance in GPS networks. Existing work on GPS networks is largely restricted to a deterministic setting. In [13] Parekh and Gallager show that the first GPS property, minimum guaranteed rates, translates into worst-case bounds on delay and workload for leaky bucket controlled traffic flows. It is clear that the second GPS property, work conservation, yields statistical multiplexing gains. In order to quantify these gains however, and to examine how they are possibly influenced by the occurrence of long-tailed traffic, a stochastic analysis of GPS networks is required.

Networks of fluid flows seem to defy exact analysis for all but a few specific cases, and in particular we are not familiar with any stochastic analysis of GPS networks. In [15] Ramanan and Dupuis study a FIFO network fed by fluid flows defined in terms of finite-state Markov processes. Aalto and Scheinhardt [3] determine the buffer content distribution in a tandem queue fed by independent on/off flows with exponential off periods and generally distributed on periods. In the present paper we specifically focus on GPS networks fed by several traffic flows, of which at least one has long-tailed traffic characteristics. Under certain conditions we show that the tail distribution of the workload of the long-tailed flow at the bottleneck node on its path is equivalent to that in a *two*-node tandem network where it is served in isolation at *constant* rates. These rates are the service rates of the two bottleneck nodes for the long-tailed flow in the original network, reduced by the average traffic intensities of the other flows. Hence, the long-tailed flow is only affected by the traffic characteristics of the other flows through their average rates and is not influenced by excessive behaviour of any of the other flows. This result extends the results in Borst, Boxma and Jelenković [4, 5] for a single GPS node fed by traffic with long-tailed characteristics. Agrawal, Makowski and Nain [2] establish a similar reduced-load equivalence result for a fluid queue fed by a flow with subexponentially distributed on periods and a general light-tailed flow.

The remainder of this paper is organised as follows. In the next section we consider a simple two-node tandem network, which is fed by a single flow. As alluded to above, this model will play a key role in analysing more complex network configurations. We relate the tail behaviour of the busy-period distribution at node 1 to the arrival process. Then we determine the tail behaviour of the workload distribution at the second node in terms of the residual busy-period distribution at node 1. Two traffic processes are considered, (i) a traffic flow generating instantaneous bursts and (ii) a traffic flow behaving according to an on/off process. We describe the GPS mechanism in more detail in Section 3. In Sections 4 and 6 we extend the model of Section 2 to a GPS tandem network that is fed by multiple flows. We consider two network configurations: in Section 4 we assume that all flows which are served at node 1 proceed to node 2, while in Section 6 we allow for flows which are only served at node 1. In both sections we determine an upper and a lower bound for the workload distribution of the long-tailed flow at node 2. In Section 5 we prove a general lemma which shows that the lower and upper bounds for the workload distribution asymptotically coincide. We use this lemma to derive the asymptotics for the other models in this paper as well. In the subsequent sections we extend the analysis to more general GPS networks with the long-tailed flow traversing more than two nodes. In particular, in Sections 8 and 9 we consider an extension of the GPS network in Sections 4 and 6 respectively. We determine for both network configurations an upper and a lower bound for the workload distribution of the long-tailed flow at the bottleneck node on its path in order to obtain the tail behaviour.

2 Two-node tandem network fed by a single flow

In this section we consider a simple tandem network, which is fed by a single flow. We analyse the tail behaviour of the workload distribution at the first and second node. Admittedly, this model represents the simplest possible network scenario, but it plays a central role in the further analysis. We need the results concerning the tail behaviour of the workload distribution in this tandem network to analyse more general networks, where multiple flows share the capacity according to the GPS principle. Surprisingly, it turns out that in the GPS networks that we consider, the tail behaviour of the workload distribution of an individual flow is equivalent to that in a tandem network where the flow is served in isolation at constant rates. We consider two traffic scenarios, (i) the flow generates instantaneous traffic bursts and (ii) the flow behaves according to an on/off process. In Subsections 2.1 and 2.2 we give for both traffic scenarios the tail behaviour of the busy-period distribution at node 1. In Subsection 2.3 we derive the tail behaviour of the workload distribution at node 2 for both traffic scenarios using the busy-period characteristics at node 1.

First we introduce some notation. Denote by d_1 and d_2 the constant service rates at node 1 and node 2, respectively. We assume $d_1 > d_2$ to exclude the trivial case where the workload at node 2 is always zero. We define ρ to be the traffic intensity, i.e., the mean amount of traffic offered to the network per unit of time. For stability we assume $\rho < d_2$. Denote by $A(s,t)$ the amount of traffic generated during the time interval $(s,t]$. We define $W^c(t)$ to be the workload at time t if the flow were fed into a queue of rate c,

$$W^c(t) := \sup_{0\leq s\leq t}\{A(s,t) - c(t-s)\},$$

assuming $W^c(0) = 0$. For $c > \rho$, W^c is a stochastic variable with the limiting distribution of $W^c(t)$ for $t \to \infty$. We define P to be the busy period in this queue. Observe that the total workload in the tandem network at time t is $W^{d_2}(t)$, while the workload at node 1 is $W^{d_1}(t)$. Thus the workload at node 2 at time t is

$$\begin{aligned} W^{d_1,d_2}(t) &:= W^{d_2}(t) - W^{d_1}(t) \\ &= \sup_{0\leq s\leq t}\{A(s,t) - d_2(t-s)\} - \sup_{0\leq s\leq t}\{A(s,t) - d_1(t-s)\}, \end{aligned} \quad (1)$$

assuming the system is empty at time 0. For $d_2 > \rho$, let W^{d_1,d_2} be a stochastic variable with the limiting distribution of $W^{d_1,d_2}(t)$ for $t \to \infty$.

For any two real functions $f(\cdot)$ and $g(\cdot)$, we use the notational convention $f(x) \sim g(x)$ to denote $\lim_{x\to\infty} f(x)/g(x) = 1$, or equivalently, $f(x) = g(x)(1 + o(1))$ as $x \to \infty$. For any stochastic variable X with distribution function $F(\cdot)$ and $\mathbb{E}X < \infty$, denote by $F^r(\cdot)$ the distribution function of the residual lifetime of X, i.e., $F^r(x) = \frac{1}{\mathbb{E}X}\int_0^x (1 - F(y))\mathrm{d}y$, and by X^r a stochastic variable with that distribution.

The classes of *long-tailed, subexponential, regularly varying*, and *intermediately regularly varying* distributions are denoted with the symbols $\mathcal{L}, \mathcal{S}, \mathcal{R}$ and $\mathcal{IR}$, respectively. The definitions of these classes are given in Appendix A.

We now state some results for the distribution of the workload and the busy period at a single node. We need these results to determine the asymptotic behaviour of W^{d_1,d_2}, and later that of the workload in more general networks.

2.1 Instantaneous arrivals

Suppose the flow generates instantaneous traffic bursts according to a Poisson process with rate λ. Let K be the stochastic variable representing the burst size. We assume that the burst size distribution $K(\cdot)$ is intermediately regularly varying with mean κ. The traffic intensity is $\rho = \lambda\kappa$. The following three results play a crucial role in the analysis in subsequent sections.

Theorem 2.1 (Pakes [11]) *If $K^r(\cdot) \in \mathcal{S}$ and $\rho < c$, then*

$$\mathbb{P}(W^c > x) \sim \frac{\rho}{c-\rho}\mathbb{P}(K^r > x).$$

Theorem 2.2 (Zwart [18]) *If $K(\cdot) \in \mathcal{IR}$ and $\rho < c$, then*

$$\mathbb{P}(P > x) \sim \frac{c}{c-\rho}\mathbb{P}(K > x(c-\rho)).$$

The above theorem immediately gives the tail distribution of the residual busy period.

Theorem 2.3 (residual busy period) *If $K(\cdot) \in \mathcal{IR}$ and $\rho < c$, then*

$$\mathbb{P}(P^r > x) \sim \frac{c}{c-\rho}\mathbb{P}(K^r > x(c-\rho)).$$

Remark 2.1 The assumption that $K(\cdot) \in \mathcal{IR}$ is in fact not necessary for Theorem 2.3 to hold. In [5] it is shown that the weaker condition $K^r(\cdot) \in \mathcal{IR}$ is also sufficient.

2.2 On/off processes

Suppose the flow generates traffic according to an on/off process. We assume the off periods to be exponentially distributed with mean $1/\lambda$. While on, the flow produces traffic at a constant rate r. Assume the stochastic variable representing the on period K to have an intermediately regularly varying distribution with mean κ. Because the fraction of off time is equal to $p = \frac{1}{1+\lambda\kappa}$, the traffic intensity is equal to $\rho = \frac{\lambda\kappa r}{1+\lambda\kappa}$.

The following three results are the analogues of Theorems 2.1, 2.2 and 2.3, respectively.

Theorem 2.4 (Jelenković and Lazar [8]) *If $K^r(\cdot) \in \mathcal{S}$ and $\rho < c < r$, then*

$$\mathbb{P}(W^c > x) \sim p\frac{\rho}{c-\rho}\mathbb{P}(K^r > \frac{x}{r-c}).$$

Theorem 2.5 (Boxma and Dumas [7], Zwart [18]) *If $K(\cdot) \in \mathcal{IR}$ and $\rho < c < r$, then*

$$\mathbb{P}(P > x) \sim p\frac{c}{c-\rho}\mathbb{P}(K > \frac{x(c-\rho)}{r-\rho}).$$

The following theorem immediately follows from Theorem 2.5.

Theorem 2.6 (residual busy period) *If $K(\cdot) \in \mathcal{IR}$ and $\rho < c < r$, then*

$$\mathbb{P}(P^r > x) \sim p\frac{c}{c-\rho}\mathbb{P}(K^r > \frac{x(c-\rho)}{r-c}).$$

Remark 2.2 Again the assumption $K(\cdot) \in \mathcal{IR}$ is sufficient but not necessary for the above theorem to hold. In [5] it is shown that the weaker condition $K^r(\cdot) \in \mathcal{IR}$ is also sufficient.

2.3 Workload distribution

The above results completely specify the tail behaviour of the workload distribution at node 1. Moreover, we can use them to analyse the workload distribution at node 2. Observe that the input process at node 2 is an on/off process with as on periods the busy periods at node 1. The on rate is equal to the service rate at node 1, d_1. The off periods correspond to the idle periods at node 1, which are exponentially distributed. In addition, the on and off periods at node 2 are independent.

For both traffic scenarios the tail distribution of the residual busy period at node 1 is intermediately regularly varying. Hence, we can apply Theorem 2.4 to determine the tail behaviour of the workload distribution at node 2, which is given in the following lemma.

Lemma 2.1 (workload second node) *If $K(\cdot) \in \mathcal{IR}$, then*

$$\mathbb{P}(W^{d_1,d_2} > x) \sim p' \frac{\rho}{d_2 - \rho} \mathbb{P}(P^r > \frac{x}{d_1 - d_2}),$$

with the fraction of off time $p' = \frac{d_1 - \rho}{d_1}$.

In Section 5 we give our main theorem concerning the tail behaviour of the workload distribution. In the proof of that theorem we need three properties which are satisfied for the two traffic scenarios that we described in the previous subsections. In the following lemma these properties are given.

Lemma 2.2 (properties traffic scenarios) *For the traffic scenarios described in Subsections 2.1 and 2.2 the following three properties hold:*

(i) *for α, β sufficiently small,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(W^{d_1+\alpha, d_2+\beta} > x)}{\mathbb{P}(W^{d_1,d_2} > x)} = G(\alpha, \beta), \text{ with } \lim_{\alpha,\beta\to 0} G(\alpha,\beta) = 1; \tag{2}$$

(ii) *for any real y,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(W^{d_1,d_2} > x - y)}{\mathbb{P}(W^{d_1,d_2} > x)} = 1; \tag{3}$$

(iii) *for each $c > \rho$ there exists a finite constant C such that,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(W^c > x)}{\mathbb{P}(W^{d_1,d_2} > x)} = C < \infty. \tag{4}$$

Proof. Theorems 2.3 (instantaneous arrivals), 2.6 (on/off processes) and Lemma 2.1 have to be used for all properties. In addition, we use for (ii) that $P^r(\cdot) \in \mathcal{IR} \subset \mathcal{L}$ for both traffic scenarios. Finally, for (iii) we obtain, using Theorems 2.1 (instantaneous arrivals), 2.4 (on/off processes) and Lemma 2.1,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(W^c > x)}{\mathbb{P}(W^{d_1,d_2} > x)} = \frac{g\frac{\rho}{c-\rho}}{\frac{d_1-\rho}{d_1}\frac{\rho}{d_2-\rho}} \limsup_{x\to\infty} \frac{\mathbb{P}(K^r > \frac{x}{h})}{\mathbb{P}(P^r > \frac{x}{d_1-d_2})}$$

with $g = 1$, $h = 1$ and K^r denoting the residual burst size for instantaneous arrivals, and $g = \frac{1}{1+\lambda\kappa}$, $h = r - c$ and K^r denoting the residual on period for on/off processes. Because $K^r(\cdot) \in \mathcal{IR}$, (4) follows. $\square$

3 Preliminaries

In the next sections we extend the model which we described in the previous section. We consider again a two-node tandem network, but now fed by multiple flows, where traffic is scheduled according to the GPS mechanism. We focus on the workload distribution of a particular flow i which passes through both nodes. In this section we introduce the notation which we use throughout the paper and we explain how the GPS mechanism operates. Although the network that we consider in Sections 4 and 6 has only two nodes, we introduce notation for networks where flow i traverses N nodes. We conclude with a number of lemmas which we use in our analysis.

At each node of the network, traffic is served according to the GPS mechanism which operates as follows. Define c_n to be the service rate of node n and $S^{(n)}$ to be the set of all flows that receive service at node n, $n = 1, \ldots, N$. Each flow $q \in S^{(n)}$ is assigned a weight $\hat{\phi}_{q,n}$. If every flow at node n is backlogged at time t, then flow $q \in S^{(n)}$ is served at node n at rate

$$\phi_{q,n} := \frac{\hat{\phi}_{q,n}}{\sum_{q \in S^{(n)}} \hat{\phi}_{q,n}} c_n.$$

If some of the flows that are served at node n are not backlogged at time t, then the excess service rate is redistributed among the backlogged flows at node n in proportion to their respective weights. This means that the server always operates at the full service rate when there is work and thus GPS is work-conserving.

Denote by $A_Q(s,t) := \sum_{q \in Q} A_q(s,t)$ the amount of traffic generated by flows $q \in Q$ in the time interval $(s,t]$, and denote by $A_{q,n}(s,t)$ the amount of traffic that arrives at node n originating from flow q during $(s,t]$. In particular, $A_{q,n}(s,t) = A_q(s,t)$ if node n is the first node flow q feeds into and we define $A_{Q,n}(s,t) := \sum_{q \in Q} A_{q,n}(s,t)$. Let $B_{q,n}(s,t)$ be the amount of traffic from flow q that is served at node n during the time interval $(s,t]$. Define $V_{q,n}(t)$ as the workload of flow q at node n at time t, and $V_{q,n}$ as a stochastic variable with the limiting distribution of $V_{q,n}(t)$ for $t \to \infty$ (assuming it exists). Similarly, we define $V_{Q,n}(t) := \sum_{q \in Q} V_{q,n}(t)$ and we denote by $V_n(t) := \sum_{q \in S^{(n)}} V_{q,n}(t)$ the total workload at node n at time t.

Using the above definitions, the following identity relation holds,

$$V_{q,n}(t) = A_{q,n}(s,t) + V_{q,n}(s) - B_{q,n}(s,t), \text{ for } 0 \leq s \leq t. \tag{5}$$

Using (5), the following relation exists between the arrival processes at two successive nodes,

$$A_{q,n+1}(s,t) = B_{q,n}(s,t) = A_{q,n}(s,t) + V_{q,n}(s) - V_{q,n}(t). \tag{6}$$

The total workload at node n at time t is given by,

$$V_n(t) = \sup_{0 \leq s \leq t} \{A_{S^{(n)},n}(s,t) - c_n(t-s)\}. \tag{7}$$

We define ρ_q to be the average rate of flow q and $\rho_Q := \sum_{q \in Q} \rho_q$ to be the aggregate average rate of all flows $q \in Q$. Let $W_Q^c(t)$ be the workload at time t in a queue with service rate $c \geq 0$

which is fed by flows $q \in Q$. Then, for $c > \rho_Q$, W_Q^c is a stochastic variable with the limiting distribution of $W_Q^c(t)$ for $t \to \infty$. Analogously we denote by $W_Q^{d_1,d_2}(t)$ the workload at time t at node 2 of a tandem network fed by the flows $q \in Q$. For $d_2 > \rho_Q$ $W_Q^{d_1,d_2}$ is a stochastic variable with the limiting distribution of $W_Q^{d_1,d_2}(t)$ for $t \to \infty$.

We make the following crucial assumptions throughout the remainder of this paper.

Assumption 3.1. For stability, we assume for each flow q, $\phi_{q,n} > \rho_q$ for all $n = 1, \dots, N$.

This way each flow is guaranteed a higher rate than its average rate. Define $\tilde{c}_n := c_n - \rho_{S^{(n)}\setminus\{i\}}$ as the average service rate available at node n for flow i, i.e., the service rate at node n minus the aggregate average rate of all flows in $S^{(n)}$ other than i.

Assumption 3.2. We assume $\tilde{c}_N < \tilde{c}_n$ for all $n = 1, \dots, N-1$.

The above assumption implies that node N can be viewed as the bottleneck node for flow i. In the following lemma we express the workload of the set of flows Q at node n in terms of the amount of traffic served of the other flows. The proof can be found in Appendix B.

Lemma 3.1 (workload at time t) *Assuming* $V_{Q,n}(0) = 0$,

$$V_{Q,n}(t) = \sup_{0\le s\le t}\{A_{Q,n}(s,t) - B_{Q,n}(s,t)\} = \sup_{0\le s\le t}\{A_{Q,n}(s,t) - (c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t))\}.$$

In the next lemma we present an upper bound for $V_{q,n}(t)$ which follows immediately from the GPS discipline. The result is trivial for the workload at node 1, e.g., if the flow is backlogged it receives at least a service rate $\phi_{q,1}$. It will be used in deriving the upper and lower bound for the workload of flow i at node 2 in Sections 4 and 6. Since this lemma is a special case of Lemma 7.3, we omit the proof.

Lemma 3.2 (GPS upper bound workload 2-node tandem network) *For* $n \in \{1,2\}$,

$$V_{q,n}(t) \le W_q^{\tilde{\phi}_q}(t),$$

with $\tilde{\phi}_q = \phi_{q,1}$ *if* $n = 1$ *and* $\tilde{\phi}_q = \min\{\phi_{q,1}, \phi_{q,2}\}$ *if* $n = 2$.

4 Merging flows

We distinguish between the following two scenarios. In this section we assume the other flows which feed into the network to join the path of flow i, i.e., they are not allowed to leave this path (see Fig. 1). In Section 6 flows *are* allowed to leave the path of flow i. The latter model includes cross traffic as a special case.
In particular, we consider the following scenario in this section. We assume the GPS network to be fed by flow i and by two additional sets of flows. The set S_1 and flow i feed into node 1

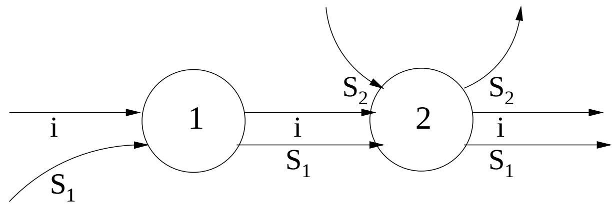

Figure 1: Two-node network with merging.

and are served both at nodes 1 and 2, while the set of flows S_2 feed into node 2 and receive only service at this node. We are interested in the distribution of the workload of flow i at node 2, $V_{i,2}$.
In this section we derive both a lower and an upper bound for $\mathbb{P}(V_{i,2} > x)$. The idea can be described as follows. If the flows other than i always showed exactly average behaviour, then $V_{i,2}$ would be equal in distribution to $W_i^{\tilde{c}_1,\tilde{c}_2}$. In reality, stochastic fluctuations in the activity of the other flows will cause $V_{i,2}$ to deviate somewhat from $W_i^{\tilde{c}_1,\tilde{c}_2}$. Accordingly, the bounds will relate $V_{i,2}$ to $W_i^{\tilde{c}_1,\tilde{c}_2}$ with some additional correction terms. In the subsequent section, we will then show that these terms can be neglected asymptotically, resulting in the exact workload asymptotics.
In both the upper and lower bound for $V_{i,2}(t)$ we need a manageable expression for the total workload at node 2. The following lemma provides such an expression.

Lemma 4.1 (alternative expression $V_2(t)$)

$$\begin{aligned} V_2(t) &= \sup_{0\le s\le t}\{A_i(s,t)+A_{S_1}(s,t)+A_{S_2}(s,t)+V_1(s)-c_2(t-s)\} \\ &\quad - \sup_{0\le s\le t}\{A_i(s,t)+A_{S_1}(s,t)-c_1(t-s)\}. \end{aligned}$$

Proof. Using (7) the total workload at node 2 is given by

$$V_2(t) = \sup_{0\le s\le t}\{A_{i,2}(s,t)+A_{S_1,2}(s,t)+A_{S_2}(s,t)-c_2(t-s)\}.$$

Using (6) to substitute for $A_{i,2}(s,t)+A_{S_1,2}(s,t)$ and then using (7) to substitute for $V_1(t)$ completes the proof. □

Before presenting the lower and upper bound, we introduce an additional variable. For $c < \rho_Q$, U_Q^c is defined to be a stochastic variable with the limiting distribution of $U_Q^c(t)$ for $t\to\infty$, with

$$U_Q^c(t) = \sup_{0\le s\le t}\{c(t-s)-A_Q(s,t)\}. \tag{8}$$

In words, $U_Q^c(t)$ is the workload at a node of a flow which feeds this node at constant rate c and receives an amount of service $A_Q(s,t)$ during a time interval $(s,t]$.

Throughout the analysis, we use the following properties of the sup operator,

$$\sup_t \{f(t) + g(t)\} \leq \sup_t \{f(t)\} + \sup_t \{g(t)\}, \tag{9}$$

which also implies

$$\sup_t \{f(t) + g(t)\} \geq \sup_t \{f(t)\} - \sup_t \{-g(t)\}. \tag{10}$$

The lower bound for $\mathbb{P}(V_{i,2} > x)$ is given in the following lemma.

Lemma 4.2 (lower bound $\mathbb{P}(V_{i,2} > x)$) *For any $\delta > 0$, $\epsilon > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,2} > x) \geq \mathbb{P}(W_i^{\tilde{c}_1-\epsilon,\tilde{c}_2+2\delta} > x + y)\mathbb{P}(Y^{\delta,\epsilon} \leq y), \tag{11}$$

with $Y^{\delta,\epsilon}$ a stochastic variable with the limiting distribution of $Y^{\delta,\epsilon}(t)$ for $t \to \infty$, where

$$Y^{\delta,\epsilon}(t) := U_{S_1}^{\rho_{S_1}-\delta}(t) + U_{S_2}^{\rho_{S_2}-\delta}(t) + W_{S_1}^{\rho_{S_1}+\epsilon}(t) + \sum_{q\in S_1} W_q^{\tilde{\phi}_q}(t) + \sum_{q\in S_2} W_q^{\tilde{\phi}_q}(t). \tag{12}$$

The stochastic variable $Y^{\delta,\epsilon}$ can be seen as the 'correction term' mentioned earlier, accounting for scenarios where $V_{i,2}(t)$ is smaller than $W_i^{\tilde{c}_1-\epsilon,\tilde{c}_2+2\delta}(t)$.

Proof. By definition,

$$V_{i,2}(t) = V_2(t) - V_{S_1,2}(t) - V_{S_2,2}(t).$$

According to Lemma 4.1,

$$\begin{aligned} V_2(t) &= \sup_{0\leq s\leq t} \{A_i(s,t) + A_{S_1}(s,t) + A_{S_2}(s,t) + V_1(s) - c_2(t-s)\} \\ &\quad - \sup_{0\leq s\leq t} \{A_i(s,t) + A_{S_1}(s,t) - c_1(t-s)\}. \end{aligned} \tag{13}$$

Using (10), the first supremum in (13) can be lower bounded by

$$\begin{aligned} &\sup_{0\leq s\leq t} \{A_i(s,t) - (\tilde{c}_2 + 2\delta)(t-s)\} - \sup_{0\leq s\leq t} \{(\rho_{S_1} - \delta)(t-s) - A_{S_1}(s,t)\} \\ &- \sup_{0\leq s\leq t} \{(\rho_{S_2} - \delta)(t-s) - A_{S_2}(s,t)\}. \end{aligned}$$

By definition, this is equal to

$$W_i^{\tilde{c}_2+2\delta}(t) - U_{S_1}^{\rho_{S_1}-\delta}(t) - U_{S_2}^{\rho_{S_2}-\delta}(t). \tag{14}$$

Using (9), the second supremum in (13) is upper bounded by

$$\begin{aligned} &\sup_{0\leq s\leq t} \{A_i(s,t) - (\tilde{c}_1 - \epsilon)(t-s)\} + \sup_{0\leq s\leq t} \{A_{S_1}(s,t) - (\rho_{S_1} + \epsilon)(t-s)\} \\ = \quad &W_i^{\tilde{c}_1-\epsilon}(t) + W_{S_1}^{\rho_{S_1}+\epsilon}(t). \end{aligned} \tag{15}$$

Finally we have to find an upper bound for $V_{S_1}(t) + V_{S_2}(t)$. Using Lemma 3.2,

$$V_{S_1,2}(t) + V_{S_2,2}(t) \leq \sum_{q\in S_1} W_q^{\tilde{\phi}_q}(t) + \sum_{q\in S_2} W_q^{\tilde{\phi}_q}(t). \tag{16}$$

Arranging the terms in (14), (15) and (16), we obtain, using (1) and (12),

$$V_{i,2}(t) \geq W_i^{\tilde{c}_1-\epsilon,\tilde{c}_2+2\delta}(t) - Y^{\delta,\epsilon}(t).$$

Hence, a lower bound is given by

$$\mathbb{P}(V_{i,2} > x) \geq \mathbb{P}(W_i^{\tilde{c}_1-\epsilon,\tilde{c}_2+2\delta} > x + y \text{ AND } Y^{\delta,\epsilon} \leq y),$$

for any y. Because $Y^{\delta,\epsilon}$ is independent of the traffic process of flow i, (11) follows. □

The next lemma provides an upper bound for $\mathbb{P}(V_{i,2} > x)$.

Lemma 4.3 (upper bound $\mathbb{P}(V_{i,2} > x)$) *For any $\eta > 0$, $\nu > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,2} > x) \leq \mathbb{P}(W_i^{\tilde{c}_1+\eta,\tilde{c}_2-2\nu} > x - y) + \mathbb{P}(W_i^{\tilde{\phi}_i} > x)\mathbb{P}(Z^{\eta,\nu} > y), \tag{17}$$

with $Z^{\eta,\nu}$ a stochastic variable with the limiting distribution of $Z^{\eta,\nu}(t)$ for $t \to \infty$, where

$$Z^{\eta,\nu}(t) := U_{S_1}^{\rho_{S_1}-\eta}(t) + W_{S_1}^{\rho_{S_1}+\nu}(t) + W_{S_2}^{\rho_{S_2}+\nu}(t). \tag{18}$$

Analogously to $Y^{\delta,\epsilon}$ in the lower bound, the stochastic variable $Z^{\eta,\nu}$ can be seen as the correction term, accounting for situations where $V_{i,2}(t)$ is larger than $W_i^{\tilde{c}_1+\eta,\tilde{c}_2-2\nu}(t)$.

Proof. By definition,

$$V_{i,2}(t) \leq V_2(t).$$

According to Lemma 4.1,

$$\begin{aligned} V_2(t) \quad = \quad & \sup_{0 \leq s \leq t} \{A_i(s,t) + A_{S_1}(s,t) + A_{S_2}(s,t) + V_1(s) - c_2(t-s)\} - \\ & \sup_{0 \leq s \leq t} \{A_i(s,t) + A_{S_1}(s,t) - c_1(t-s)\}. \end{aligned} \tag{19}$$

Using (7) to substitute for $V_1(s)$, we obtain for the first supremum in (19),

$$\sup_{0 \leq u \leq s \leq t} \{A_i(u,t) + A_{S_1}(u,t) + A_{S_2}(s,t) - c_1(s-u) - c_2(t-s)\},$$

which is upper bounded by, using (9),

$$\begin{aligned} & \sup_{0 \leq s \leq t} \{A_{S_2}(s,t) - (\rho_{S_2}+\nu)(t-s)\} + \sup_{0 \leq u \leq t} \{A_{S_1}(u,t) - (\rho_{S_1}+\nu)(t-u)\} + \\ & \sup_{0 \leq u \leq s \leq t} \{A_i(u,t) - (\tilde{c}_1-\nu)(s-u) - (\tilde{c}_2-2\nu)(t-s)\}. \end{aligned} \tag{20}$$

The first two suprema in (20) are equal to

$$W_{S_1}^{\rho_{S_1}+\nu}(t) + W_{S_2}^{\rho_{S_2}+\nu}(t). \tag{21}$$

Because $\tilde{c}_2 < \tilde{c}_1$, the third supremum in (20) is upper bounded by

$$\sup_{0 \leq s \leq t} \{A_i(s,t) - (\tilde{c}_2-2\nu)(t-s)\} = W_i^{\tilde{c}_2-2\nu}(t). \tag{22}$$

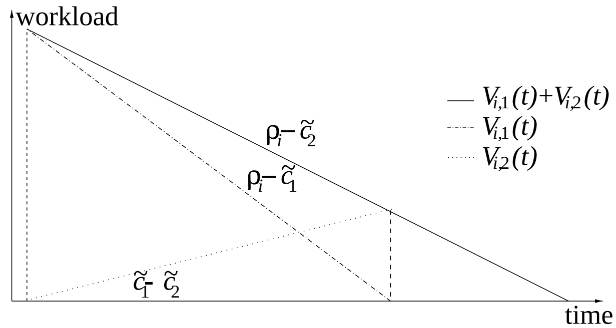

Figure 2: Overflow scenario for instantaneous traffic bursts.

Next we have to find a lower bound for the second supremum in (19). Using (10), we obtain as lower bound,

$$\begin{aligned} & \sup_{0\le s\le t}\{A_i(s,t)-(\tilde{c}_1+\eta)(t-s)\}-\sup_{0\le s\le t}\{(\rho_{S_1}-\eta)(t-s)-A_{S_1}(s,t)\} \\ = \; & W_i^{\tilde{c}_1+\eta}(t)-U_{S_1}^{\rho_{S_1}-\eta}(t). \end{aligned} \tag{23}$$

Arranging the terms in (21), (22) and (23), we obtain using (1) and (18),

$$V_{i,2}(t)\le W_i^{\tilde{c}_1+\eta,\tilde{c}_2-2\nu}(t)+Z^{\eta,\nu}(t).$$

Combining the above bound with the upper bound in Lemma 3.2,

$$V_{i,2}(t)\le \min\{W_i^{\tilde{\phi}_i}(t),W_i^{\tilde{c}_1+\eta,\tilde{c}_2-2\nu}(t)+Z^{\eta,\nu}(t)\}.$$

Hence, an upper bound is given by

$$\mathbb{P}(V_{i,2}>x)\le \mathbb{P}(W_i^{\tilde{\phi}_i}>x \text{ AND } (W_i^{\tilde{c}_1+\eta,\tilde{c}_2-2\nu}>x-y \text{ OR } Z^{\eta,\nu}>y)),$$

for any y, which leads to (17) because $Z^{\eta,\nu}$ is independent of the traffic process of flow i. □

5 Tail behaviour of the workload distribution

We now state our key theorem concerning the tail behaviour of the workload distribution.

Theorem 5.1 (asymptotic equivalence) *For the traffic scenarios described in Subsections 2.1 and 2.2, under Assumptions 3.1 and 3.2,*

$$\mathbb{P}(V_{i,2}>x)\sim \mathbb{P}(W_i^{\tilde{c}_1,\tilde{c}_2}>x),$$

where $\tilde{c}_1$ and $\tilde{c}_2$ represent the total service rate minus the aggregate average rate of all flows other than flow i at nodes 1 and 2 respectively, as defined in Section 3.

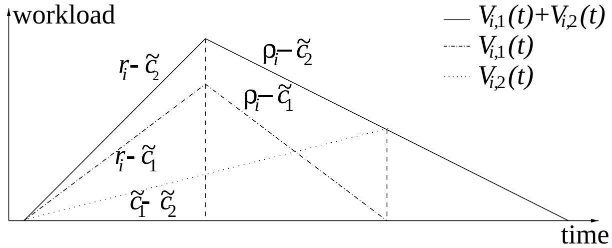

Figure 3: Overflow scenario for an on/off process.

According to this theorem, the workload distribution of flow i at node 2 is asymptotically equivalent to that in a tandem network where flow i is served in isolation at rates $\tilde{c}_1$ and $\tilde{c}_2$. Hence, the workload of flow i at node 2 is only affected by the characteristics of the other flows through their average rates, even when the other flows are 'heavier-tailed'. This suggests that an extremely large workload of flow i is most likely due to either a long on period or a large burst size of flow i itself. During the subsequent congestion period, the other flows continue to receive service at approximately their average rates. In the theorem this is represented by the constant rates $\tilde{c}_1$ and $\tilde{c}_2$. This result extends the result of [4] for the single-node case and shows that GPS is capable of isolating flows in networks as well.
The typical overflow scenario is schematically depicted in Fig. 2. At some point, flow i generates a large burst, causing $V_{i,1}(t)$ to reach some large value. After that, flow i returns to its average behaviour, producing traffic at rate ρ_i. Consequently, $V_{i,1}(t)$ will start to decrease roughly at rate $\rho_i - \tilde{c}_1$, and $V_{i,2}(t)$ will start to increase approximately at rate $\tilde{c}_1 - \tilde{c}_2$, until $V_{i,1}(t)$ reduces to zero at some point. From then on, $V_{i,1}(t)$ will remain relatively small, and $V_{i,2}(t)$ will also start to decrease, roughly at rate $\rho_i - \tilde{c}_2$, until $V_{i,2}(t)$ becomes zero as well. The corresponding behaviour for an on/off process is illustrated in Fig. 3.
A similar reduced-load equivalence result is obtained in [2] for a flow with subexponential on periods and a general light-tailed flow. Here, the other flows need *not* be light-tailed because of the GPS properties. Note however that Assumption 3.1 is crucial. If $\rho_q > \phi_{q,n}$ for some n then flow i may not receive service at a stable rate when other flows generate a large amount of traffic. Flows with an on period distribution or a burst size distribution which is heavier-tailed than that of flow i will then potentially affect the workload of flow i, see [5].

The above theorem follows from a general lemma which shows that the bounds of Lemmas 4.2 and 4.3 asymptotically coincide. Before giving this lemma, we first introduce some additional notation. Let R_i be some stochastic variable. For δ, ϵ, η and $\nu > 0$ let $C_{-i}^{\delta,\epsilon}$ and $D_{-i}^{\eta,\nu}$ also be stochastic variables.

Lemma 5.1 (general result) *If for δ, ϵ, η and $\nu > 0$ sufficiently small and any y,*

$$\mathbb{P}(R_i > x) \geq \mathbb{P}(W_i^{a_1-\epsilon, a_2+\delta} > x + y)\mathbb{P}(C_{-i}^{\delta,\epsilon} \leq y), \quad (24)$$

$$\mathbb{P}(R_i > x) \leq \mathbb{P}(W_i^{a_1+\eta, a_2-\nu} > x - y) + \mathbb{P}(W_i^{a} > x)\mathbb{P}(D_{-i}^{\eta,\nu} > y), \quad (25)$$

and $\mathbb{P}(W_i^a > x)$ *and* $\mathbb{P}(W_i^{a_1,a_2} > x)$ *satisfy Properties (2), (3) and (4), then*

$$\mathbb{P}(R_i > x) \sim \mathbb{P}(W_i^{a_1,a_2} > x). \tag{26}$$

Proof. The lower bound (24) implies, for any $\delta, \epsilon > 0$ sufficiently small and any y,

$$\frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \geq \frac{\mathbb{P}(W_i^{a_1-\epsilon,a_2+\delta} > x+y)}{\mathbb{P}(W_i^{a_1,a_2} > x+y)} \frac{\mathbb{P}(W_i^{a_1,a_2} > x+y)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \mathbb{P}(C_{-i}^{\delta,\epsilon} \leq y).$$

Using Properties (2) and (3), we obtain

$$\liminf_{x\to\infty} \frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \geq G_i(-\epsilon, \delta)\mathbb{P}(C_{-i}^{\delta,\epsilon} \leq y).$$

Letting $y \to \infty$ and then $\delta, \epsilon \downarrow 0$,

$$\liminf_{x\to\infty} \frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \geq 1. \tag{27}$$

Analogously, the upper bound (25) implies, for any η, $\nu > 0$ sufficiently small and any y,

$$\frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \leq \frac{\mathbb{P}(W_i^{a_1+\eta,a_2-\nu} > x-y)}{\mathbb{P}(W_i^{a_1,a_2} > x-y)} \frac{\mathbb{P}(W_i^{a_1,a_2} > x-y)}{\mathbb{P}(W_i^{a_1,a_2} > x)} + \frac{\mathbb{P}(W_i^{a} > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \mathbb{P}(D_{-i}^{\eta,\nu} > y).$$

Using Properties (2), (3) and (4), we have

$$\limsup_{x\to\infty} \frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \leq G_i(\eta, -\nu) + C\mathbb{P}(D_{-i}^{\eta,\nu} > y),$$

for some constant $C < \infty$. Letting $y \to \infty$ and $\eta, \nu \downarrow 0$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(R_i > x)}{\mathbb{P}(W_i^{a_1,a_2} > x)} \leq 1. \tag{28}$$

Combining Equations (27) and (28) gives the desired result. □

6 Splitting flows

Consider again a tandem network in which the following flows are served according to the GPS principle (see Fig. 4). As in Section 4, flow i and the set of flows S_1 feed into node 1 and are served both at nodes 1 and 2, and the set of flows S_2 feed into node 2. In addition, we consider in this section the set of flows S_3 which feed into node 1 but do not move on to node 2 after receiving service at node 1. We first derive a lower bound and an upper bound for the workload distribution of flow i at node 2, $\mathbb{P}(V_{i,2} > x)$. Then we use Lemma 5.1 to determine the tail behaviour of $\mathbb{P}(V_{i,2} > x)$.

In the following lemma we give an alternative expression for $V_2(t)$ which we need in the proof of the lower and upper bound for $\mathbb{P}(V_{i,2} > x)$.

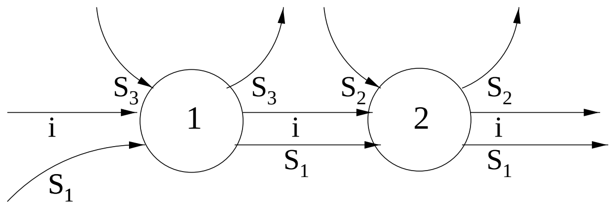

Figure 4: Two-node network with splitting.

Lemma 6.1 (alternative expression $V_2(t)$)

$$\begin{aligned} V_2(t) &= \sup_{0\le s\le t}\{A_i(s,t)+A_{S_1}(s,t)+A_{S_2}(s,t)+V_{i,1}(s)+V_{S_1,1}(s)-c_2(t-s)\} \\ &\quad -\sup_{0\le s\le t}\{A_i(s,t)+A_{S_1}(s,t)+A_{S_3}(s,t)-c_1(t-s)\}+V_{S_3,1}(t). \end{aligned}$$

Proof. Because of (7),

$$V_2(t)=\sup_{0\le s\le t}\{A_{i,2}(s,t)+A_{S_1,2}(s,t)+A_{S_2}(s,t)-c_2(t-s)\}.$$

Using (6) to substitute for $A_{i,2}(s,t)+A_{S_1,2}(s,t)$, we obtain

$$\begin{aligned} V_2(t) &= \sup_{0\le s\le t}\{A_i(s,t)+A_{S_1}(s,t)+A_{S_2}(s,t)+V_{i,1}(s)+V_{S_1,1}(s)-c_2(t-s)\} \\ &\quad -V_{i,1}(t)-V_{S_1,1}(t). \end{aligned}$$

As $V_1(t)=V_{i,1}(t)+V_{S_1,1}(t)+V_{S_3,1}(t)$, the proof is completed using (7) to substitute for $V_1(t)$. □

Analogously to Section 4, we introduce some additional variables. Due to the presence of the additional set of flows S_3, these variables are more complicated than in the previous section. For $\delta,\epsilon>0$, redefine $Y^{\delta,\epsilon}$ to be a stochastic variable with the limiting distribution of $Y^{\delta,\epsilon}(t)$ for $t\to\infty$, with

$$Y^{\delta,\epsilon}(t):=U_{S_1}^{\rho_{S_1}-\delta}(t)+U_{S_2}^{\rho_{S_2}-\delta}(t)+W_{S_1}^{\rho_{S_1}+\epsilon}(t)+W_{S_3}^{\rho_{S_3}+\epsilon}(t)+\sum_{q\in S_1}W_q^{\tilde{\phi}_q}(t)+\sum_{q\in S_2}W_q^{\tilde{\phi}_q}(t). \quad (29)$$

For $\eta,\nu>0$, redefine $Z^{\eta,\nu}$ to be a stochastic variable with the limiting distribution of $Z^{\eta,\nu}(t)$ for $t\to\infty$, with

$$Z^{\eta,\nu}(t):=U_{S_1}^{\rho_{S_1}-\eta}(t)+U_{S_3}^{\rho_{S_3}-\nu}(t)+U_{S_3}^{\rho_{S_3}-\eta}(t)+\sum_{j=1}^{3}W_{S_j}^{\rho_{S_j}+\nu}(t)+\sum_{q\in S_3}W_q^{\tilde{\phi}_q}(t). \quad (30)$$

Now we derive both an upper and a lower bound for $\mathbb{P}(V_{i,2}>x)$. These bounds are similar to the bounds in Lemmas 4.2 and 4.3, except for the structure of the correction terms $Y^{\delta,\epsilon}$ and $Z^{\eta,\nu}$. In the following lemma we give a lower bound for $\mathbb{P}(V_{i,2}>x)$.

Lemma 6.2 (lower bound $\mathbb{P}(V_{i,2}>x)$) *For any $\delta>0$, $\epsilon>0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,2}>x)\ge\mathbb{P}(W_i^{\tilde{c}_1-2\epsilon,\tilde{c}_2+2\delta}>x+y)\mathbb{P}(Y^{\delta,\epsilon}\le y). \quad (31)$$

Proof. By definition,

$$V_{i,2}(t) = V_2(t) - V_{S_1,2}(t) - V_{S_2,2}(t). \tag{32}$$

According to Lemma 6.1,

$$\begin{aligned} V_2(t) \quad \geq \quad & \sup_{0\leq s\leq t}\{A_i(s,t) + A_{S_1}(s,t) + A_{S_2}(s,t) - c_2(t-s)\} - \\ & \sup_{0\leq s\leq t}\{A_i(s,t) + A_{S_1}(s,t) + A_{S_3}(s,t) - c_1(t-s)\}. \end{aligned} \tag{33}$$

Using (10), the first supremum in (33) is lower bounded by

$$\begin{aligned} & \sup_{0\leq s\leq t}\{A_i(s,t) - (\tilde{c}_2 + 2\delta)(t-s)\} - \sup_{0\leq s\leq t}\{(\rho_{S_1} - \delta)(t-s) - A_{S_1}(s,t)\} - \\ & \sup_{0\leq s\leq t}\{(\rho_{S_2} - \delta)(t-s) - A_{S_2}(s,t)\}, \end{aligned}$$

which is equal to (using (8))

$$W_i^{\tilde{c}_2+2\delta}(t) - U_{S_1}^{\rho_{S_1}-\delta}(t) - U_{S_2}^{\rho_{S_2}-\delta}(t).$$

Next we need an upper bound for the second supremum in (33). Using (9) it is upper bounded by

$$W_i^{\tilde{c}_1-2\epsilon}(t) + W_{S_1}^{\rho_{S_1}+\epsilon}(t) + W_{S_3}^{\rho_{S_3}+\epsilon}(t).$$

Finally, using Lemma 3.2 we find a similar upper bound for $V_{S_1,2}(t)$ and $V_{S_2,2}(t)$ as in (16). Adding the three bounds and using (1) and (29),

$$V_{i,2}(t) \geq W_i^{\tilde{c}_1-2\epsilon,\tilde{c}_2+2\delta}(t) - Y^{\delta,\epsilon}(t).$$

Because $Y^{\delta,\epsilon}$ is independent of the traffic process of flow i, (31) follows. □

The following lemma provides an upper bound for $\mathbb{P}(V_{i,2} > x)$.

Lemma 6.3 (upper bound $\mathbb{P}(V_{i,2} > x)$) *For any $\eta > 0$, $\nu > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,2} > x) \leq \mathbb{P}(W_i^{\tilde{c}_1+2\eta,\tilde{c}_2-4\nu} > x - y) + \mathbb{P}(W_i^{\tilde{\phi}_i} > x)\mathbb{P}(Z^{\eta,\nu} > y). \tag{34}$$

Proof. By definition,

$$V_{i,2}(t) \leq V_2(t).$$

According to Lemma 6.1,

$$\begin{aligned} V_2(t) \quad = \quad & \sup_{0\leq s\leq t}\{A_i(s,t) + A_{S_1}(s,t) + A_{S_2}(s,t) + V_{i,1}(s) + V_{S_1,1}(s) - c_2(t-s)\} - \\ & \sup_{0\leq s\leq t}\{A_i(s,t) + A_{S_1}(s,t) + A_{S_3}(s,t) - c_1(t-s)\} + V_{S_3,1}(t). \end{aligned} \tag{35}$$

First observe that $V_{i,1}(s) + V_{S_1,1}(s) \leq V_1(s)$. Using (7) to substitute for $V_1(s)$, the first supremum in (35) is thus upper bounded by

$$\sup_{0\leq r\leq s\leq t}\{A_i(r,t) + A_{S_1}(r,t) + A_{S_3}(r,s) + A_{S_2}(s,t) - c_1(s-r) - c_2(t-s)\}. \tag{36}$$

Note that (36) can be written as

$$\sup_{0\le r\le s\le t}\{A_i(r,t)-(\tilde{c}_1-2\nu)(s-r)-(\tilde{c}_2-4\nu)(t-s)$$
$$+A_{S_1}(r,t)-(\rho_{S_1}+\nu)(t-r)+A_{S_2}(s,t)-(\rho_{S_2}+\nu)(t-s)$$
$$+A_{S_3}(r,t)-(\rho_{S_3}+\nu)(t-r)+(\rho_{S_3}-\nu)(t-s)-A_{S_3}(s,t)\}.$$

Using (9) and $\tilde{c}_1>\tilde{c}_2$, this is upper bounded by

$$\sup_{0\le r\le t}\{A_i(r,t)-(\tilde{c}_2-4\nu)(t-r)\}+\sup_{0\le r\le s\le t}\{-2\nu(s-r)\}+\sup_{0\le r\le t}\{A_{S_1}(r,t)-(\rho_{S_1}+\nu)(t-r)\}$$
$$+\sup_{0\le s\le t}\{A_{S_2}(s,t)-(\rho_{S_2}+\nu)(t-s)\}+\sup_{0\le r\le t}\{A_{S_3}(r,t)-(\rho_{S_3}+\nu)(t-r)\}$$
$$+\sup_{0\le s\le t}\{(\rho_{S_3}-\nu)(t-s)-A_{S_3}(s,t)\},$$

which by definition is equal to

$$W_i^{\tilde{c}_2-4\nu}(t)+W_{S_1}^{\rho_{S_1}+\nu}(t)+W_{S_2}^{\rho_{S_2}+\nu}(t)+W_{S_3}^{\rho_{S_3}+\nu}(t)+U_{S_3}^{\rho_{S_3}-\nu}(t).$$

Now we have to find a lower bound for the second supremum in (35). Using (10), this lower bound is given by

$$W_i^{\tilde{c}_1+2\eta}(t)-U_{S_1}^{\rho_{S_1}-\eta}(t)-U_{S_3}^{\rho_{S_3}-\eta}(t).$$

Finally, because of Lemma 3.2, we obtain for the third term in (35)

$$V_{S_3,1}(t)\le\sum_{q\in S_3}W_q^{\tilde{\phi}_q}(t).$$

Adding the three bounds and using (1) and (30),

$$V_{i,2}(t)\le W_i^{\tilde{c}_1+2\eta,\tilde{c}_2-4\nu}+Z^{\eta,\nu}(t).$$

Combining the above bound with the upper bound in Lemma 3.2, we obtain the following upper bound,

$$V_{i,2}(t)\le\min\{W_i^{\tilde{\phi}_i}(t),W_i^{\tilde{c}_1+2\eta,\tilde{c}_2-4\nu}(t)+Z^{\eta,\nu}(t)\}.$$

Because $Z^{\eta,\nu}$ is independent of the traffic process of flow i, (34) follows. □

Now we have all the ingredients to use Lemma 5.1, which gives the main result of this section.

Theorem 6.1 (asymptotic equivalence) *For the traffic scenarios described in Subsections 2.1 and 2.2, under Assumptions 3.1 and 3.2,*

$$\mathbb{P}(V_{i,2}>x)\sim\mathbb{P}(W_i^{\tilde{c}_1,\tilde{c}_2}>x),$$

where $\tilde{c}_1$ and $\tilde{c}_2$ represent the total service rate minus the aggregate average rate of all flows other than flow i at nodes 1 and 2 respectively, as defined in Section 3.

7 Preliminaries general networks

In the next two sections we extend the model of Section 6 and focus on the Nth node on the path of flow i. We assume this node to be the bottleneck node for flow i. Again we assume the flows to be served at each node according to the GPS mechanism. First we introduce some additional notation and present a number of lemmas which we use in the next sections. Then we analyse the behaviour of the workload of flow i at the bottleneck node on its path, if no other flows feed into any of the nodes on this path. Although this model is quite simple, it provides some useful intuition for the results in Sections 8 and 9.

We define S_j to be the set of flows that feed into node j and S_m^p to be the set of flows that feed into node m and leave the path of flow i at node p (so flows in S_m^p receive service at node p). For $q \in S_m^p$ we define $\tilde{\phi}_q := \min\{\phi_{q,m}, \ldots, \phi_{q,p}\}$, which is the minimum rate guaranteed to flow q on its path along node m up to and until p.

We now present some lemmas which we use in the next sections. The proofs can be found in Appendix B. The following lemma gives a lower bound for the amount of service flow q receives at node n during time interval $(s,t]$.

Lemma 7.1 (lower bound $B_{q,n}(s,t)$) *For $q \in S_m^p$, $1 \leq m \leq n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$B_{q,n}(s,t) \geq \gamma_q(t-s) - \sup_{s \leq s_m \leq t} \{\gamma_q(s_m - s) - A_q(s, s_m)\}. \tag{37}$$

Using this lemma, we can derive an upper bound for the total workload of flow $q \in S_m^p$ at nodes $m, \ldots, n$. This upper bound is presented in the next lemma.

Lemma 7.2 (upper bound total workload flow q) *For $q \in S_m^p$, $1 \leq m \leq n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$\sum_{j=m}^{n} V_{q,j}(t) \leq W_q^{\gamma_q}(t). \tag{38}$$

The above lemma immediately implies the following lemma, which includes Lemma 3.2 as a special case.

Lemma 7.3 (GPS upper bound workload) *For $q \in S_m^p$, $1 \leq m \leq n \leq p$,*

$$V_{q,n}(t) \leq W_q^{\tilde{\phi}_q}(t). \tag{39}$$

From Lemma 7.2 we can derive an upper bound for the amount of service that flow q receives during interval $(s,t]$ as well. This upper bound is given in the following lemma.

Lemma 7.4 (upper bound $B_{q,n}(s,t)$) *For $q \in S_m^p$, $1 \leq m \leq n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$B_{q,n}(s,t) \leq \gamma_q(t-s) + \sup_{0 \leq s_m \leq t} \{A_q(s_m, t) - \gamma_q(t - s_m)\}. \tag{40}$$

We now briefly discuss the workload behaviour at the Nth node of a network which is fed only by flow i. Take $m^* \in \arg\min_{n=1,\dots,N-1}\{\tilde{c}_n\}$. In Section 3 we assumed that $\tilde{c}_n > \tilde{c}_N$ (Assumption 3.2) for all $n = 1, \dots, N-1$, so that $\tilde{c}_{m^*} > \tilde{c}_N$. The workload distribution, $\mathbb{P}(V_{i,N} > x)$, is given in the following theorem.

Theorem 7.1 (workload node N) $\quad \mathbb{P}(V_{i,N} > x) = \mathbb{P}(W_i^{c_{m^*},c_N} > x).$

Proof. Observe that, because of the definition of m^*, the total workload at nodes $1, \dots, m^*$ is equivalent to that at a node with service rate c_{m^*} which is fed by the original traffic process of flow i (a formal proof can be found in Appendix B). Hence,

$$\sum_{j=1}^{m^*} V_{i,j}(t) = W_i^{c_{m^*}}(t). \tag{42}$$

Since $c_N < c_{m^*}$ (Assumption 3.2) we can apply the same reasoning to the total workload at nodes $1, \dots, N$ and we have

$$\sum_{j=1}^{N} V_{i,j}(t) = W_i^{c_N}(t).$$

In [9] the following observation is made. If $c_k > c_j$ for $k > j$ then the backlog at node k will always be zero in stationarity and this node can be removed from the network. Because the nodes succeeding node m^* (except N) have a service rate which is larger than c_{m^*}, the workload at these nodes is zero and we have, using (1),

$$V_{i,N}(t) = \sum_{j=1}^{N} V_{i,j}(t) - \sum_{j=1}^{m^*} V_{i,j}(t) = W_i^{c_{m^*},c_N}(t),$$

which completes the proof. □

The workload at node N in this network is equal to that at node 2 in a *two*-node tandem network serving flow i at rates c_{m^*} and c_N. Thus the distribution of the workload is entirely determined by the bottleneck nodes. Asymptotically, this is still true for the more general networks which we discuss in the next sections.

8 General network with merging

Analogously to Sections 4 and 6 we distinguish between two network scenarios. In this section we consider an extension of the network described in Section 4 and assume that each node on the path of flow i in the GPS network is fed by an additional set of flows (see Fig. 5 for the case where flow i traverses 4 nodes). These sets follow the path of flow i and do not leave before node N, the bottleneck node. In Section 9 we consider an extension of this network and the network described in Section 6 and allow the flows feeding into a node on the path of flow i to leave this path before the bottleneck node. In both sections we first derive an upper

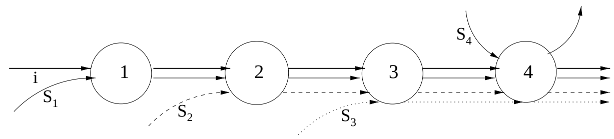

Figure 5: General network with merging.

and a lower bound for the workload distribution of flow i at the bottleneck node. Then we use Lemma 5.1 to determine the tail behaviour.
In this section we only give the proof of the lower and upper bound for $\mathbb{P}(V_{i,N} > x)$. The other proofs can be found in Appendix B.

Recall that in the two-node model the upper and lower bounds for $V_{i,2}(t)$ were derived from bounds for $V_1(t)$ and $V_2(t)$. Similarly, in the N-node case, the lower and upper bounds for $V_{i,N}(t)$ rely on bounds for the total workload at each node $n \in \{1, \ldots, N\}$. Define

$$X_n(t) := \sup_{0 \leq s_1 \leq \ldots \leq s_n \leq s_{n+1} = t} \left\{ A_i(s_1, t) + \sum_{j=1}^{n} \left[A_{S_j}(s_j, t) - c_j(s_{j+1} - s_j) \right] \right\}.$$

In the next lemma we give an expression for $V_n(t)$ in terms of $X_n(t)$. This expression will be used in deriving the upper and lower bounds for $V_{i,N}(t)$.

Lemma 8.1 (workload node n) *For $n \geq 2$,*

$$V_n(t) = X_n(t) - X_{n-1}(t). \tag{43}$$

In order to determine a lower and an upper bound for $V_n(t)$ we have to find a lower and an upper bound for $X_n(t)$. In the next lemma the lower bound for $X_n(t)$ is presented.

Lemma 8.2 (lower bound $X_n(t)$) *For any $\theta_1, \ldots, \theta_n$,*

$$X_n(t) \geq W_i^e(t) - \sum_{j=1}^{n} U_{S_j}^{\theta_j}(t),$$

with $e := \min_{m=1,\ldots,n}\{c_m - \sum_{j=1}^{m} \theta_j\}$.

The upper bound for $X_n(t)$ is given in the following lemma.

Lemma 8.3 (upper bound $X_n(t)$) *For any $\xi_1, \ldots, \xi_n$,*

$$X_n(t) \leq W_i^d(t) + \sum_{j=1}^{n} W_{S_j}^{\xi_j}(t),$$

with $d := \min_{m=1,\ldots,n}\{c_m - \sum_{j=1}^{m} \xi_j\}$.

We now introduce some additional notation similar to Section 4. For $\delta, \epsilon > 0$, define $Y^{\delta,\epsilon}$ as a stochastic variable with the limiting distribution of $Y^{\delta,\epsilon}(t)$ for $t \to \infty$, with

$$Y^{\delta,\epsilon}(t) := \sum_{j=1}^{N} U_{S_j}^{\rho_{S_j}-\delta}(t) + \sum_{j=1}^{N-1} W_{S_j}^{\rho_{S_j}+\epsilon}(t) + \sum_{j=1}^{N} \sum_{q \in S_j} W_q^{\tilde{\phi}_q}(t). \tag{44}$$

For $\eta, \nu > 0$, define $Z^{\eta,\nu}$ as a stochastic variable with the limiting distribution of $Z^{\eta,\nu}(t)$ for $t \to \infty$, with

$$Z^{\eta,\nu}(t) := \sum_{j=1}^{N} W_{S_j}^{\rho_{S_j}+\nu}(t) + \sum_{j=1}^{N-1} U_{S_j}^{\rho_{S_j}-\eta}(t). \tag{45}$$

We use the bounds for $X_n(t)$ to construct a lower and an upper bound for $\mathbb{P}(V_{i,N} > x)$. The lower bound is given in the following lemma.

Lemma 8.4 (lower bound $\mathbb{P}(V_{i,N} > x)$) *For any $\delta > 0$, $\epsilon > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,N} > x) \geq \mathbb{P}(W_i^{\tilde{c}_{m^*}-m^*\epsilon, \tilde{c}_N+N\delta} > x + y)\mathbb{P}(Y^{\delta,\epsilon} \leq y). \tag{46}$$

Proof. By definition,

$$V_{i,N}(t) = V_N(t) - \sum_{j=1}^{N} \sum_{q \in S_j} V_{q,N}(t).$$

Using Lemmas 7.3 and 8.1 this is lower bounded by

$$X_N(t) - X_{N-1}(t) - \sum_{j=1}^{N} \sum_{q \in S_j} W_q^{\tilde{\phi}_q}(t).$$

Now we can use the lower bound in Lemma 8.2 for $X_N(t)$ and the upper bound in Lemma 8.3 for $X_{N-1}(t)$. Taking $\theta_j = \rho_{S_j} - \delta$ in Lemma 8.2 and $\xi_j = \rho_{S_j} + \epsilon$ in Lemma 8.3 we obtain for $\delta > 0$, $\epsilon > 0$ sufficiently small,

$$V_{i,N}(t) \geq W_i^{\tilde{c}_N+N\delta}(t) - \sum_{j=1}^{N} U_{S_j}^{\rho_{S_j}-\delta}(t) - W_i^{\tilde{c}_{m^*}-m^*\epsilon}(t) - \sum_{j=1}^{N-1} W_{S_j}^{\rho_{S_j}+\epsilon}(t) - \sum_{j=1}^{N} \sum_{q \in S_j} W_q^{\tilde{\phi}_q}(t).$$

Using (1) and (44) yields,

$$V_{i,N}(t) \geq W_i^{\tilde{c}_{m^*}-m^*\epsilon, \tilde{c}_N+N\delta}(t) - Y^{\delta,\epsilon}(t).$$

Hence, the lower bound is given by

$$\mathbb{P}(V_{i,N} > x) \geq \mathbb{P}(W_i^{\tilde{c}_{m^*}-m^*\epsilon, \tilde{c}_N+N\delta} > x + y \text{ AND } Y^{\delta,\epsilon} \leq y).$$

Because $Y^{\delta,\epsilon}$ is independent of the traffic process of flow i, (46) follows immediately. $\square$

Note that the lower bound we found for $V_{i,2}(t)$ in Lemma 4.2 is indeed a special case of the lower bound for $V_{i,N}(t)$.

The upper bound for $\mathbb{P}(V_{i,N} > x)$ is given in the following lemma.

Lemma 8.5 (upper bound $\mathbb{P}(V_{i,N} > x)$) *For any $\eta > 0$, $\nu > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,N} > x) \leq \mathbb{P}(W_i^{\tilde{c}_{m^*}+m^*\eta,\tilde{c}_N-N\nu} > x-y) + \mathbb{P}(W_i^{\tilde{\phi}_i} > x)\mathbb{P}(Z^{\eta,\nu} > y) \tag{47}$$

Proof. By definition,

$$V_{i,N}(t) \leq V_N(t).$$

Thus, because of Lemma 8.1,

$$V_{i,N}(t) \leq X_N(t) - X_{N-1}(t).$$

Analogously to the proof of the lower bound we take the upper bound in Lemma 8.3 for $X_N(t)$ and the lower bound in Lemma 8.2 for $X_{N-1}(t)$. Taking $\xi_j = \rho_{S_j} + \nu$ in Lemma 8.3 and $\theta_j = \rho_{S_j} - \eta$ in Lemma 8.2, we obtain for $\eta > 0$, $\nu > 0$ sufficiently small,

$$V_{i,N}(t) \leq W_i^{\tilde{c}_N-N\nu}(t) + \sum_{j=1}^{N} W_{S_j}^{\rho_{S_j}+\nu}(t) - W_i^{\tilde{c}_{m^*}+m^*\eta}(t) + \sum_{j=1}^{N-1} U_{S_j}^{\rho_{S_j}-\eta}(t).$$

Using (1) and (45) yields,

$$V_{i,N}(t) \leq W_i^{\tilde{c}_{m^*}+m^*\eta,\tilde{c}_N-N\nu}(t) + Z^{\eta,\nu}(t).$$

Combining the above bound with the upper bound in Lemma 7.3, we obtain

$$\mathbb{P}(V_{i,N} > x) \leq \mathbb{P}(W_i^{\tilde{\phi}_i} > x \text{ AND } (W_i^{\tilde{c}_{m^*}+m^*\eta,\tilde{c}_N-N\nu} > x-y \text{ OR } Z^{\eta,\nu} > y)).$$

Because $Z^{\eta,\nu}$ is independent of the traffic process of flow i, (47) follows. □

Again note that the upper bound for $V_{i,2}(t)$ in Lemma 4.3 is a special case of the upper bound for $V_{i,N}(t)$.

We are now able to characterise the tail behaviour of $\mathbb{P}(V_{i,N} > x)$. It follows immediately from Lemma 5.1 and the lower and upper bound given in Lemmas 8.4 and 8.5.

Theorem 8.1 (asymptotic equivalence) *For the traffic scenarios described in Subsections 2.1 and 2.2, under Assumptions 3.1 and 3.2,*

$$\mathbb{P}(V_{i,N} > x) \sim \mathbb{P}(W_i^{\tilde{c}_{m^*},\tilde{c}_N} > x),$$

where $\tilde{c}_{m^*}$ and $\tilde{c}_N$ represent the total service rate minus the aggregate average rate of all flows other than flow i at nodes m^* and N, respectively, as defined in Section 3.

Remarkably, the workload distribution of flow i at the bottleneck node is asymptotically equivalent to that in a *two*-node tandem network where flow i is served in isolation at constant rates. In Sections 5 and 6 these rates are simply $\tilde{c}_1$ and $\tilde{c}_2$. For the N-node network we have to take the two smallest service rates for flow i when reduced by the aggregate average rates of the other flows, $\tilde{c}_{m^*}$ and $\tilde{c}_N$. Hence, for the network described in this section as well, the workload

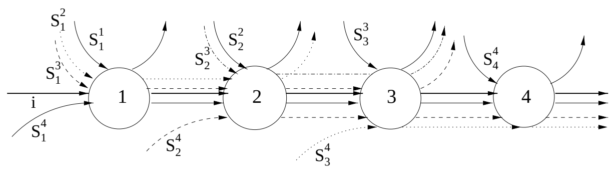

Figure 6: General network with splitting.

of flow i at the bottleneck node is only affected by the characteristics of the other flows through their average rates. This suggests that an extremely large workload of flow i at its bottleneck node is most likely due to either a long on period or a large burst of the flow itself and the other flows showing roughly their average behaviour. Consequently, we can consider flow i to be served in isolation at constant rates $\tilde{c}_1, \ldots, \tilde{c}_N$. Following the reasoning of [9] as in the proof of Theorem 7.1 we can then remove all nodes with capacity $\tilde{c}_n > \tilde{c}_{m^*}$, after which we are left with a two-node tandem network.

9 General network with splitting

In this section we extend the model of the previous section and assume that each node on the path of flow i is fed by an additional set of flows, which can leave this path before node N (see Fig. 6 for the case where flow i traverses 4 nodes).
As before we derive an upper and a lower bound for $\mathbb{P}(V_{i,N} > x)$ and we use Lemma 5.1 to determine the tail behaviour of this distribution. Analogously to the previous section we defer most of the proofs to Appendix B.

We first introduce some additional notation. Define $\hat{A}_k^p(s,t)$ to be the amount of work arriving at node k during the interval $(s,t]$ associated with flows entering the path of flow i at node k and passing through node $p \geq k$, i.e.,

$$\hat{A}_k^p(s,t) := \sum_{m=p}^{N} A_{S_k^m}(s,t).$$

Define $\hat{A}_{k,n}^p(s,t)$ to be the amount of work arriving at node n during the interval $(s,t]$ associated with flows entering the path of flow i before or at node k and passing through node $p \geq n \geq k$, i.e.,

$$\hat{A}_{k,n}^p(s,t) := \sum_{j=1}^{k} \sum_{m=p}^{N} A_{S_j^m,n}(s,t). \tag{48}$$

Similarly we define $V_k^p(t)$ to be the workload at node k at time t associated with flows passing through node $p \geq k$ (including flow i), i.e.,

$$V_k^p(t) := \sum_{j=1}^{k} \sum_{m=p}^{N} V_{S_j^m,k}(t) + V_{i,k}(t).$$

Finally we define $c_k^p(s,t)$ to be the amount of service available in node k during the interval $(s,t]$ for flows passing through node $p \geq k$, i.e.,

$$c_k^p(s,t) := c_k(t-s) - \sum_{j=1}^{k} \sum_{m=k}^{p-1} B_{S_j^m,k}(s,t). \tag{49}$$

The following lemma expresses the workload at node n at time t associated with the flows passing through node p, in terms of $X_n^p(t)$, with

$$X_n^p(t) := \sup_{0 \leq s_1 \leq \ldots \leq s_{n+1} = t} \left\{ A_i(s_1,t) + \sum_{k=1}^{n} \left[\hat{A}_k^p(s_k,t) - c_k^p(s_k,s_{k+1}) \right] \right\}.$$

Lemma 9.1 (workload node n) *For $2 \leq n \leq p$,*

$$V_n^p(t) = X_n^p(t) - X_{n-1}^p(t). \tag{50}$$

If we take p equal to N in (50) and $\sum_{j=1}^{k} \sum_{m=k}^{N-1} B_{S_j^m,k}(s,t) = 0$ so that $c_k^p(s_k,s_{k+1}) = c_k(s_{k+1}-s_k)$ for $k = 1,\ldots,N-1$, then we see that it reduces to the result in Lemma 8.1 where we assumed that flows cannot leave the path of flow i before node N.

Before presenting the upper and lower bound for $X_n^p(t)$ we first introduce some additional notation. Let R be the index set of the flows and γ, ζ and $\psi \in \mathbb{R}^R$. For any vector $x \in \mathbb{R}^R$, denote $x_{S_j^m} = \sum_{q \in S_j^m} x_q$.
Define

$$d_k^p := c_k - \sum_{j=1}^{k} \sum_{m=k}^{N} \gamma_{S_j^m} - \sum_{f=1}^{k-1} \sum_{j=1}^{f} \sum_{m=f}^{p-1} (\gamma_{S_j^m} - \psi_{S_j^m}),$$

and

$$e_k^p := c_k - \sum_{j=1}^{k} \sum_{m=k}^{N} \zeta_{S_j^m} - \sum_{f=1}^{k} \sum_{j=1}^{f} \sum_{m=f}^{p-1} (\zeta_{S_j^m} - \gamma_{S_j^m}).$$

In the next lemma we present the lower bound for $X_n^p(t)$.

Lemma 9.2 (lower bound $X_n^p(t)$) *For $n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$X_n^p(t) \geq W_i^{(e_n^p)^*}(t) - \sum_{k=1}^{n} \sum_{m=p}^{N} U_{S_k^m}^{\zeta_{S_k^m}}(t) - \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{m=k}^{p-1} \sum_{q \in S_j^m} \left\{ U_q^{\zeta_q}(t) + W_q^{\gamma_q}(t) \right\},$$

with $(e_n^p)^* := \min_{k=1,\ldots,n}\{e_k^p\}$.

In the following lemma an upper bound is given for $X_n^p(t)$.

Lemma 9.3 (upper bound $X_n^p(t)$) *For $n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$X_n^p(t) \leq W_i^{(d_n^p)^*}(t) + \sum_{k=1}^{n} \sum_{m=p}^{N} W_{S_k^m}^{\gamma_{S_k^m}}(t) + \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{m=k}^{p-1} \sum_{q \in S_j^m} \left\{ U_q^{\psi_q}(t) + W_q^{\gamma_q}(t) \right\},$$

with $(d_n^p)^* := \min_{k=1,\ldots,n}\{d_k^p\}$.

Note that if we take $p = N$ in Lemmas 9.2 and 9.3 and we omit all terms concerning flows that leave before node N, we obtain Lemmas 8.2 and 8.3, respectively. The additional terms reflect the fluctuations of the service capacities available for flow i. Before deriving a lower and upper bound for the workload of flow i in node N at time t, we first introduce some additional notation.

For $\delta, \epsilon > 0$, define $Y^{\delta,\epsilon}$ as a stochastic variable with the limiting distribution of $Y^{\delta,\epsilon}(t)$ for $t \to \infty$, with

$$\begin{aligned} Y^{\delta,\epsilon}(t) &= \sum_{k=1}^{N-1}\sum_{j=1}^{k}\sum_{m=k}^{N-1}\sum_{q\in S_j^m}\Big(U_q^{\rho_q-\delta}(t)+U_q^{\rho_q-\epsilon}(t)+W_q^{\rho_q+\delta}(t)+W_q^{\rho_q+\epsilon}(t)\Big) \\ &\quad +\sum_{k=1}^{N} U_{S_k^N}^{\rho_{S_k^N}-\delta|S_k^N|}(t)+\sum_{k=1}^{N-1} W_{S_k^N}^{\rho_{S_k^N}+\epsilon|S_k^N|}(t)+\sum_{j=1}^{N}\sum_{q\in S_j^N} W_q^{\tilde{\phi}_q}(t). \end{aligned} \tag{51}$$

For $\eta, \nu > 0$, define $Z^{\eta,\nu}$ as a stochastic variable with the limiting distribution of $Z^{\eta,\nu}(t)$ for $t \to \infty$, with

$$\begin{aligned} Z^{\eta,\nu}(t) &= \sum_{k=1}^{N-1}\sum_{j=1}^{k}\sum_{m=k}^{N-1}\sum_{q\in S_j^m}\Big(U_q^{\rho_q-\eta}(t)+U_q^{\rho_q-\nu}(t)+W_q^{\rho_q+\eta}(t)+W_q^{\rho_q+\nu}(t)\Big) \\ &\quad +\sum_{k=1}^{N} W_{S_k^N}^{\rho_{S_k^N}+\nu|S_k^N|}(t)+\sum_{k=1}^{N-1} U_{S_k^N}^{\rho_{S_k^N}-\eta|S_k^N|}(t). \end{aligned} \tag{52}$$

Also define

$$\sigma_k := \sum_{j=1}^{k}\sum_{m=k}^{N}|S_j^m| + 2\sum_{f=1}^{k-1}\sum_{j=1}^{f}\sum_{m=f}^{N-1}|S_j^m|. \tag{53}$$

In the next lemma the lower bound for $\mathbb{P}(V_{i,N} > x)$ is given.

Lemma 9.4 (lower bound $\mathbb{P}(V_{i,N} > x)$) *For any $\delta, \epsilon > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,N} > x) \geq \mathbb{P}(W_i^{\tilde{c}_{m^*}-\epsilon\sigma_{m^*},\tilde{c}_N+\delta\sigma_N} > x+y)\mathbb{P}(Y^{\delta,\epsilon} \leq y). \tag{54}$$

Proof. By definition,

$$V_{i,N}(t) = V_N(t) - \sum_{j=1}^{N}\sum_{q\in S_j^N} V_{q,N}(t) = V_N^N(t) - \sum_{j=1}^{N}\sum_{q\in S_j^N} V_{q,N}(t).$$

Using Lemmas 7.3 and 9.1 this is lower bounded by

$$X_N^N(t) - X_{N-1}^N(t) - \sum_{j=1}^{N}\sum_{q\in S_j^N} W_q^{\tilde{\phi}_q}(t).$$

Now we can use the lower bound for $X_N^N(t)$ as given in Lemma 9.2 and the upper bound for $X_{N-1}^N(t)$ as given in Lemma 9.3. We take in Lemma 9.2 $\zeta_q = \rho_q - \delta$ and $\gamma_q = \rho_q + \delta$, hence,

$$X_N^N(t) \geq W_i^{e_{N^*}^N}(t) - \sum_{k=1}^{N} U_{S_k^N}^{\rho_{S_k^N} - \delta|S_k^N|}(t) - \sum_{k=1}^{N}\sum_{j=1}^{k}\sum_{m=k}^{N-1}\sum_{q\in S_j^m}\left\{U_q^{\rho_q-\delta}(t) + W_q^{\rho_q+\delta}(t)\right\},$$

with for $k = 1, \ldots, N$,

$$\begin{aligned} e_k^N &= c_k - \sum_{j=1}^{k}\sum_{m=k}^{N}(\rho_{S_j^m} - |S_j^m|\delta) + 2\delta\sum_{f=1}^{k-1}\sum_{j=1}^{f}\sum_{m=f}^{N-1}|S_j^m| \\ &= \tilde{c}_k + \delta\sigma_k, \end{aligned}$$

and thus $e_{N^*}^N = \tilde{c}_N + \delta\sigma_N$ for $\delta > 0$ sufficiently small.
Analogously, we take in Lemma 9.3 $\gamma_q = \rho_q + \epsilon$ and $\psi_q = \rho_q - \epsilon$, hence,

$$X_{N-1}^N(t) \leq W_i^{d_{N-1^*}^N}(t) + \sum_{k=1}^{N-1} W_{S_k^N}^{\rho_{S_k^N} + \epsilon|S_k^N|}(t) + \sum_{k=1}^{N-1}\sum_{j=1}^{k}\sum_{m=k}^{N-1}\sum_{q\in S_j^m}\left\{U_q^{\rho_q-\epsilon}(t) + W_q^{\rho_q+\epsilon}(t)\right\},$$

with for $k = 1, \ldots, N-1$,

$$\begin{aligned} d_k^N &= c_k - \sum_{j=1}^{k}\sum_{m=k}^{N}(\rho_{S_j^m} + |S_j^m|\epsilon) - 2\epsilon\sum_{f=1}^{k-1}\sum_{j=1}^{f}\sum_{m=f}^{N-1}|S_j^m| \\ &= \tilde{c}_k - \epsilon\sigma_k, \end{aligned}$$

and thus $d_{N-1^*}^N = \tilde{c}_{m^*} - \epsilon\sigma_{m^*}$ for $\epsilon > 0$ sufficiently small.
Then using (1) and (51) we obtain,

$$V_{i,N}(t) \geq W_i^{\tilde{c}_{m^*} - \epsilon\sigma_{m^*}, \tilde{c}_N + \delta\sigma_N}(t) - Y^{\delta,\epsilon}(t).$$

Hence, the lower bound is given by,

$$\mathbb{P}(V_{i,N} > x) \geq \mathbb{P}(W_i^{\tilde{c}_{m^*} - \epsilon\sigma_{m^*}, \tilde{c}_N + \delta\sigma_N} > x + y \text{ AND } Y^{\delta,\epsilon} \leq y).$$

Because $Y^{\delta,\epsilon}$ is independent of the traffic process of flow i, (54) follows immediately. □

The upper bound for $\mathbb{P}(V_{i,N} > x)$ is given in the following lemma.

Lemma 9.5 (upper bound $\mathbb{P}(V_{i,N} > x)$) *For any $\eta, \nu > 0$ sufficiently small and any y,*

$$\mathbb{P}(V_{i,N} > x) \leq \mathbb{P}(W_i^{\tilde{c}_{m^*} + \eta\sigma_{m^*}, \tilde{c}_N - \nu\sigma_N} > x - y) + \mathbb{P}(W_i^{\tilde{\phi}_i} > x)\mathbb{P}(Z^{\eta,\nu} > y). \quad (55)$$

Proof. By definition,

$$V_{i,N}(t) \leq V_N(t) = V_N^N(t).$$

Using Lemma 9.1,

$$V_{i,N}(t) \leq X_N^N(t) - X_{N-1}^N(t).$$

Analogously to the proof of Lemma 9.4 we use the upper bound for $X_N^N(t)$ as given in Lemma 9.3 and the lower bound for $X_{N-1}^N(t)$ as given in Lemma 9.2. We take in Lemma 9.3 $\gamma_q = \rho_q + \nu$ and $\psi_q = \rho_q - \nu$. In Lemma 9.2 we take $\zeta_q = \rho_q - \eta$ and $\gamma_q = \rho_q + \eta$. Using (1) and (52) yields,

$$V_{i,N}(t) \leq W_i^{\tilde{c}_{m^*}+\eta\sigma_{m^*},\tilde{c}_N-\nu\sigma_N}(t) + Z^{\eta,\nu}(t).$$

Combining the above bound with the upper bound in Lemma 7.3, we obtain

$$\mathbb{P}(V_{i,N} > x) \leq \mathbb{P}(W_i^{\tilde{\phi}_i} > x \text{ AND } (W_i^{\tilde{c}_{m^*}+\eta\sigma_{m^*},\tilde{c}_N-\nu\sigma_N} > x - y \text{ OR } Z^{\eta,\nu} > y)).$$

Because $Z^{\eta,\nu}$ is independent of the traffic process of flow i, (55) follows. □

Note that the lower and upper bound for $V_{i,N}(t)$ in Lemmas 9.4 and 9.5 reduce to the lower and upper bound in Lemmas 8.4 and 8.5, in case we assume that no flows leave the path of flow i, i.e., $S_j^m = \emptyset$ for $m < N$.

We now have gathered all the elements to characterise the tail behaviour of the workload distribution in the most general class of networks that we consider.

Theorem 9.1 (asymptotic equivalence) *For the traffic scenarios described in Subsections 2.1 and 2.2, under Assumptions 3.1 and 3.2,*

$$\mathbb{P}(V_{i,N} > x) \sim \mathbb{P}(W_i^{\tilde{c}_{m^*},\tilde{c}_N} > x),$$

where $\tilde{c}_{m^*}$ and $\tilde{c}_N$ represent the total service rate minus the aggregate average rate of all flows other than i at nodes m^* and N, respectively, as defined in Section 3.

Again the workload distribution of flow i at the bottleneck node is asymptotically equivalent to that in a *two-node* tandem network where flow i is served in isolation at constant rates.

10 Concluding remarks

In this paper we analysed the workload behaviour under the GPS mechanism in networks fed by multiple flows. Specifically, we considered a particular flow i traversing the network and assumed it to have heavy-tailed traffic characteristics. We distinguished between two configurations of feed-forward networks, (i) other flows follow the path of flow i when they feed into any of the nodes on this path and (ii) other flows can leave the path of flow i. In addition, we considered two traffic scenarios for flow i, (i) flow i generates instantaneous traffic bursts and (ii) flow i generates traffic according to an on/off process. Under these conditions we showed that the tail behaviour of the workload distribution of flow i at its bottleneck node is equivalent to that in a *two*-node tandem network where flow i is served in isolation at *constant* rates. In case flow i traverses only two nodes and the second node is the bottleneck node, these rates are the service rates in the original network reduced by the average rates of the other flows. However, when flow i traverses more than two nodes, we have to take the rates from the nodes which are bottleneck when the service rate is reduced by the average rates of the

other flows. Hence, flow i is only affected by the characteristics of the other flows through their average rates. This suggests that the GPS mechanism is capable of isolating individual flows in networks, even when they have heavy-tailed traffic characteristics, while achieving significant multiplexing gains.
The results in this paper may be extended in several directions. We assumed for each flow the minimal rate guaranteed by the GPS mechanism to be larger than the average input rate. It may be possible to relax this assumption for a certain class of flows as in [6]. In this paper we only considered the workload distribution at nodes with the minimum average service rate for flow i on its path. The tail behaviour of the workload distribution of flow i at a node following the node with the minimal average service rate is an interesting topic for further research.

References

[1] V. Anantharam. Scheduling strategies and long-range dependence. *Queueing Systems* 33 (1-3), 73 – 89, 1999.

[2] R. Agrawal, A.M. Makowski and P. Nain. On a reduced load equivalence for fluid queues under subexponentiality. *Queueing Systems* 33 (1-3), 5 – 41, 1999.

[3] S. Aalto and W. Scheinhardt. Tandem fluid queues fed by homogeneous on-off sources. EURANDOM Report 99-022, 1999, to appear in *Operations Research Letters.*

[4] S.C. Borst, O.J. Boxma, and P.R. Jelenković. Generalized Processor Sharing with long-tailed traffic sources. In: *Teletraffic Engineering in a Competitive World, Proceedings of ITC-16*, Edinburgh, UK, eds. P. Key, D. Smith (North-Holland, Amsterdam), 345 – 354, 1999.

[5] S.C. Borst, O.J. Boxma, and P.R. Jelenković. Induced burstiness in Generalized Processor Sharing queues with long-tailed traffic sources. *Proceedings of the 37th Annual Allerton Conference on Communication, Control, and Computing*, Urbana-Champaign, Illinois, 1999.

[6] S.C. Borst, O.J. Boxma and P.R. Jelenković. Asymptotic behavior of Generalized Processor Sharing with long-tailed traffic flows. *Proceedings of Infocom 2000*, Tel Aviv, 912 – 922, 2000.

[7] O.J. Boxma and V. Dumas. The busy period in the fluid queue. *Performance Evaluation Review* 26, 100 – 110, 1998.

[8] P.R. Jelenković and A.A. Lazar. Asymptotic results for multiplexing subexponential on-off processes. *Advances in Applied Probability* 31, 394 – 421, 1999.

[9] O. Kella and W. Whitt. A tandem fluid network with Lévy input. *Queues and Related Models*, I. Basawa and U. Bhat (eds.), Oxford, Oxford University Press, 112 – 128, 1992.

[10] W.E. Leland, M.S. Taqqu, W. Willinger and D.V. Wilson. On the self-similar nature of Ethernet traffic (extended version). *IEEE/ACM Transactions on Networking* 2, 1 – 15, 1994.

[11] A.G. Pakes. On the tails of waiting-time distributions. *Journal of Applied Probability* 12, 555 – 564, 1975.

[12] A.K. Parekh and R.G. Gallager. A generalized processor sharing approach to flow control in integrated services networks: the single node case. *IEEE/ACM Transactions on Networking* 1 (3), 344 – 357, 1993.

[13] A.K. Parekh and R.G. Gallager. A generalized processor sharing approach to flow control in integrated services networks: the multiple node case. *IEEE/ACM Transactions on Networking* 2 (2), 137 – 150, 1994.

[14] V. Paxson and S. Floyd. Wide area traffic: the failure of Poisson modelling. *IEEE/ACM Transactions on Networking* 3 (3), 226 – 244, 1995.

[15] K. Ramanan and P. Dupuis. Large deviation properties of data streams that share a buffer. *Annals of Applied Probability* 8 (4), 1070 – 1129, 1998.

[16] D. Stiliadis and A. Varma. Efficient fair queueing algorithms for packet-switched networks. *IEEE/ACM Transactions on Networking* 6 (2), 175 – 185, 1998.

[17] W. Willinger, M.S. Taqqu, W.E. Leland and D.V. Wilson. Self-similarity in high-speed packet traffic: analysis and modeling of Ethernet traffic measurements. *Statistical Science* 10, 67 – 85, 1995.

[18] A.P. Zwart. Tail asymptotics for the busy period in the GI/G/1 queue. Technical report COSOR 99-12 Eindhoven University of Technology. Submitted for publication, 1999.

A Definitions

Definition A.1. A distribution function $F(\cdot)$ on $[0,\infty]$ is called *long-tailed* ($F(\cdot) \in \mathcal{L}$) if

$$\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = 1, \text{ for all real } y.$$

Definition A.2. A distribution function $F(\cdot)$ on $[0,\infty]$ is called *subexponential* ($F(\cdot) \in \mathcal{S}$) if

$$\lim_{x\to\infty} \frac{1-F^{2*}(x)}{1-F(x)} = 2,$$

where $F^{2*}(\cdot)$ is the 2-fold convolution of $F(\cdot)$ with itself, i.e., $F^{2*}(x) = \int_0^x F(x-y)F(\mathrm{d}y)$.

A relevant subclass of $\mathcal{S}$ is the class $\mathcal{R}$ of *regularly-varying* distributions (which contains the Pareto distribution).

Definition A.3. A distribution function $F(\cdot)$ on $[0,\infty]$ is called *regularly varying of index* $-\nu$ ($F(\cdot) \in \mathcal{R}_{-\nu}$) if

$$F(x) = 1 - \frac{l(x)}{x^{\nu}}, \;\; \nu \geq 0,$$

where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of slow variation, i.e., $\lim_{x\to\infty} \frac{l(\eta x)}{l(x)} = 1, \eta > 0$.

A technical extension of $\mathcal{R}$ is the class $\mathcal{IR}$ of *intermediately regularly varying* distributions.

Definition A.4. A distribution function $F(\cdot)$ on $[0,\infty]$ is called *intermediately regularly varying* ($F(\cdot) \in \mathcal{IR}$) if

$$\lim_{\eta\uparrow 1} \limsup_{x\to\infty} \frac{1 - F(\eta x)}{1 - F(x)} = 1.$$

Examples of subexponential distributions which do not belong to $\mathcal{IR}$ include the Weibull, lognormal and Benktander distributions.

B Proofs

Lemma 3.1 Assuming $V_{Q,n}(0) = 0$,

$$V_{Q,n}(t) = \sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - B_{Q,n}(s,t)\} = \sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - (c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t))\}.$$

Proof. We show
(i)

$$V_{Q,n}(t) \leq \sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - (c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t))\},$$

(ii)

$$\sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - (c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t))\} \leq \sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - B_{Q,n}(s,t)\}$$

and (ii)

$$\sup_{0\leq s\leq t} \{A_{Q,n}(s,t) - B_{Q,n}(s,t)\} \leq V_{Q,n}(t).$$

(i) Define

$$s^* := \max\{s | V_{Q,n}(s) = 0, 0 \leq s \leq t\},$$

i.e., s^* is the last time before t at which the workload of all the flows $q \in Q$ at node n was 0. Note that s^* is well-defined since $V_{Q,n}(0) = 0$. Because of the definition of s^*, $V_{Q,n}(s) > 0$ for all $s \in (s^*, t]$. Recall that the GPS mechanism is work-conserving, so that

$$B_{Q,n}(s^*,t) + B_{S^{(n)}\setminus Q,n}(s^*,t) = c_n(t - s^*),$$

and hence,

$$\begin{aligned} V_{Q,n}(t) &= A_{Q,n}(s^*,t) + V_{Q,n}(s^*) - B_{Q,n}(s^*,t) \\ &= A_{Q,n}(s^*,t) - (c_n(t-s^*) - B_{S^{(n)}\setminus Q,n}(s^*,t)) \\ &\le \sup_{0\le s\le t}\{A_{Q,n}(s,t) - (c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t))\}. \end{aligned}$$

(ii) By definition,

$$B_{Q,n}(s,t) \le c_n(t-s) - B_{S^{(n)}\setminus Q,n}(s,t),$$

for all $s \in [0,t]$.
(iii) From (5),

$$V_{Q,n}(t) \ge A_{Q,n}(s,t) - B_{Q,n}(s,t)$$

for all $s \in [0,t]$. Hence,

$$V_{Q,n}(t) \ge \sup_{0\le s\le t}\{A_{Q,n}(s,t) - B_{Q,n}(s,t)\},$$

for all $t \ge 0$. □

Lemma 7.1 *For $q \in S^p_m$, $1 \le m \le n \le p$ and $\gamma_q \le \tilde{\phi}_q$,*

$$B_{q,n}(s,t) \ge \gamma_q(t-s) - \sup_{s\le s_m\le t}\{\gamma_q(s_m - s) - A_q(s,s_m)\}.$$

Proof. We will prove by induction on r that for each $r \in \{0,\ldots,n-m\}$,

$$B_{q,n}(s,t) \ge \gamma_q(t-s) - \sup_{s\le s_{n-r}\le t}\{\gamma_q(s_{n-r} - s) - A_{q,n-r}(s,s_{n-r})\}, \tag{56}$$

which gives immediately the desired result for $r = n-m$.
For $r = 0$, (56) reduces to

$$B_{q,n}(s,t) \ge \gamma_q(t-s) - \sup_{s\le s_n\le t}\{\gamma_q(s_n - s) - A_{q,n}(s,s_n)\}, \tag{57}$$

which can be verified as follows. We distinguish between two cases.
(i) If $V_{q,n}(s_n) > 0$ for all $s_n \in [s,t]$, then flow q is continuously backlogged at node n during $[s,t]$, meaning that

$$B_{q,n}(s,t) \ge \phi_{q,n}(t-s) \ge \tilde{\phi}_q(t-s) \ge \gamma_q(t-s).$$

Obviously we then immediately obtain (57).
(ii) If the workload $V_{q,n}(s_n)$ is equal to 0 for some $s_n \in [s,t]$, then define $s^*_n := \max\{s_n | V_{q,n}(s_n) = 0, 0 \le s_n \le t\}$. We have,

$$\begin{aligned} B_{q,n}(s,t) &= B_{q,n}(s,s^*_n) + B_{q,n}(s^*_n,t) \\ &= V_{q,n}(s) + A_{q,n}(s,s^*_n) - V_{q,n}(s^*_n) + B_{q,n}(s^*_n,t). \end{aligned}$$

Since $V_{q,n}(s_n^*) = 0$ and flow q is continuously backlogged at node n during $(s_n^*, t]$, this is lower bounded by

$$\begin{aligned} & A_{q,n}(s, s_n^*) + \phi_{q,n}(t - s_n^*) \geq A_{q,n}(s, s_n^*) + \gamma_q(t - s_n^*) \\ = \; & \gamma_q(t-s) - (\gamma_q(s_n^* - s) - A_q(s, s_n^*)) \geq \gamma_q(t-s) - \sup_{s \leq s_n \leq t} \{\gamma_q(s_n - s) - A_{q,n}(s, s_n)\}. \end{aligned}$$

Now assume (56) to hold for $r-1$, i.e.,

$$B_{q,n}(s,t) \geq \gamma_q(t-s) - \sup_{s \leq s_{n-r+1} \leq t} \{\gamma_q(s_{n-r+1} - s) - A_{q,n-r+1}(s, s_{n-r+1})\}. \tag{58}$$

As in (57),

$$B_{q,n-r}(s, s_{n-r+1}) \geq \gamma_q(s_{n-r+1} - s) - \sup_{s \leq s_{n-r} \leq s_{n-r+1}} \{\gamma_q(s_{n-r} - s) - A_{q,n-r}(s, s_{n-r})\}.$$

Using (6) to substitute $B_{q,n-r}(s, s_{n-r+1})$ for $A_{q,n-r+1}(s_n, s_{n-r+1})$ in (58) yields (56). □

Lemma 7.2 *For $q \in S_m^p$, $1 \leq m \leq n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$\sum_{j=m}^{n} V_{q,j}(t) \leq W_q^{\gamma_q}(t).$$

Proof. By induction on r we prove that for each $r \in \{0, \ldots, n-m\}$,

$$V_{q,n}(t) = \sum_{j=n-r}^{n} V_{q,j}(s) + A_{q,n-r}(s,t) - B_{q,n}(s,t) - \sum_{j=n-r}^{n-1} V_{q,j}(t), \text{ for all } 0 \leq s \leq t. \tag{59}$$

For $r = 0$, (59) reduces to (5). Assume (59) to hold for $r-1$. Substituting (6) for $A_{q,n-r+1}(s,t)$ we immediately obtain (59).
Taking $r = n - m$ in (59) and choosing time s such that $\sum_{j=m}^{n} V_{q,j}(s) = 0$ (for example $s = 0$) yields,

$$\sum_{j=m}^{n} V_{q,j}(t) = A_q(s,t) - B_{q,n}(s,t).$$

Rewriting the lower bound for $B_{q,n}(s,t)$ in Lemma 7.1 to

$$- \sup_{s \leq s_m \leq t} \{-A_q(s, s_m) - \gamma_q(t - s_m)\},$$

we obtain,

$$\sum_{j=m}^{n} V_{q,j}(t) \leq A_q(s,t) + \sup_{s \leq s_m \leq t} \{-A_q(s, s_m) - \gamma_q(t - s_m)\} = \sup_{s \leq s_m \leq t} \{A_q(s_m, t) - \gamma_q(t - s_m)\}$$

and the proof is completed. □

Lemma 7.4 *For $q \in S_m^p$, $1 \leq m \leq n \leq p$ and $\gamma_q \leq \tilde{\phi}_q$,*

$$B_{q,n}(s,t) \leq \gamma_q(t-s) + \sup_{0 \leq s_m \leq t} \{A_q(s_m, t) - \gamma_q(t - s_m)\}.$$

Proof. We first prove by induction on r that for each $r \in \{0, \dots, n-m\}$,

$$B_{q,n}(s,t) \leq A_{q,n-r}(s,t) + \sum_{j=n-r}^{n} V_{q,j}(s). \tag{60}$$

For $r = 0$, (60) reduces to the upper bound which immediately follows from (5). Assume (60) to hold for $r-1$. Substituting (6) for $A_{q,n-r}(s,t)$ yields (60).
Taking $r = n-m$ in (60) and using Lemma 7.2 we obtain

$$B_{q,n}(s,t) \leq A_q(s,t) + W_q^{\gamma_q}(s) \leq A_q(s,t) + \sup_{0 \leq s_m \leq t}\{A_q(s_m, s) - \gamma_q(s - s_m)\}.$$

□

Equation (42)

$$\sum_{j=1}^{m^*} V_{i,j}(t) = \sup_{0 \leq s \leq t}\{A_i(s,t) - c_{m^*}(t-s)\}$$

Proof. As in the proof of Lemma 8.1, for any $n \geq 1$,

$$\sum_{j=1}^{n} V_{i,j}(t) = \sup_{0 \leq s_1 \leq \ldots \leq s_n \leq s_{n+1} = t}\{A_i(s_1, t) - \sum_{j=1}^{n} c_j(s_{j+1} - s_j)\} =: W_i^{(n)}(t).$$

We now show with a lower and upper bound that

$$W_i^{(n)}(t) = W_i^{c_{j^*}}(t), \tag{61}$$

with $c_{j^*} := \min_{j=1,\ldots,n}\{c_j\}$. We first show that the right-hand side is a lower bound for the left-hand side. Imposing a restriction on the optimising arguments, the supremum becomes smaller. Hence, choosing $s = s_1 = \ldots = s_{j^*}$ and $s_{j^*+1} = \ldots = t$,

$$\sup_{0 \leq s_1 \leq \ldots \leq s_{n+1} = t}\{A_i(s_1, t) - \sum_{j=1}^{n} c_j(s_{j+1} - s_j)\} \geq \sup_{0 \leq s \leq t}\{A_i(s,t) - c_{j^*}(t-s)\}.$$

Next we show that the right-hand side is in fact also an upper bound. Because $c_j \leq c_{j^*}$ for all $j = 1, \ldots, n$,

$$\sum_{j=1}^{n} c_j(s_{j+1} - s_j) \geq \sum_{j=1}^{n} c_{j^*}(s_{j+1} - s_j) = c_{j^*}(t - s_1),$$

and the proof is completed. □

Lemma 8.1 *For $n \geq 2$,*

$$V_n(t) = X_n(t) - X_{n-1}(t).$$

Proof. Note that $S^{(n-m)} = S^{(n-m-1)} \cup S_{n-m}$ and $S^{(n-m-1)} \cap S_{n-m} = \emptyset$. We prove by induction on m that for each $m \in \{0, \ldots, n-1\}$,

$$\begin{aligned} V_n(t) \quad = \quad & \sup_{0 \leq s_{n-m} \leq \ldots \leq s_n \leq s_{n+1} = t}\{A_{S^{(n-m-1)}, n-m}(s_{n-m}, t) \\ & + \sum_{j=n-m}^{n}(A_{S_j}(s_j, t) - c_j(s_{j+1} - s_j))\} - \sum_{j=n-m}^{n-1} V_j(t), \end{aligned} \tag{62}$$

with the notational convention that $S^{(0)} = \{i\}$. If $m = 0$, then (62) reduces to (7). Assume (62) to hold for $m-1$. Using (6) to substitute for $A_{S^{(n-m)},n-m+1}(s_{n-m+1}, t)$,

$$V_n(t) = \sup_{0 \le s_{n-m+1} \le \ldots \le s_n \le s_{n+1} = t} \{A_{S^{(n-m)},n-m}(s_{n-m+1}, t) + V_{n-m}(s_{n-m+1}) - V_{n-m}(t) + \sum_{j=n-m+1}^{n} (A_{S_j}(s_j, t) - c_j(s_{j+1} - s_j))\} - \sum_{j=n-m+1}^{n-1} V_j(t).$$

Substituting (7) for $V_{n-m}(s_{n-m+1})$, and arranging the terms yields (62).
Taking $m = n-1$ in (62), we obtain

$$V_n(t) = X_n(t) - \sum_{m=1}^{n-1} V_m(t),$$

so that $V_n(t) = X_n(t) - X_{n-1}(t)$. □

Lemma 8.2 *For any* $\theta_1, \ldots, \theta_n$,

$$X_n(t) \ge W_i^e(t) - \sum_{j=1}^{n} U_{S_j}^{\theta_j}(t),$$

with $e := \min_{m=1,\ldots,n}\{c_m - \sum_{j=1}^{m} \theta_j\}$.

Proof. Using the fact that $\sum_{j=1}^{n} \theta_j(t - s_j) = \sum_{j=1}^{n} \theta^j(s_{j+1} - s_j)$ and $\theta^j := \sum_{m=1}^{j} \theta_m$, we write

$$X_n(t) = \sup_{0 \le s_1 \le \ldots \le s_{n+1} = t} \{A_i(s_1, t) - \sum_{j=1}^{n}(c_j - \theta^j)(s_{j+1} - s_j) + \sum_{j=1}^{n} \Big(A_{S_j}(s_j, t) - \theta_j(t - s_j)\Big)\}.$$

Because of (10) this is lower bounded by

$$\sup_{0 \le s_1 \le \ldots \le s_{n+1} = t} \{A_i(s_1, t) - \sum_{j=1}^{n}(c_j - \theta^j)(s_{j+1} - s_j)\} - \sum_{j=1}^{n} \sup_{0 \le s_j \le t} \{\theta_j(t - s_j) - A_{S_j}(s_j, t)\}.$$

Using (8) and (61) for the first supremum, the proof is completed. □

Lemma 8.3 *For any* $\xi_1, \ldots, \xi_n$,

$$X_n(t) \le W_i^d(t) + \sum_{j=1}^{n} W_{S_j}^{\xi_j}(t),$$

with $d := \min_{m=1,\ldots,n}\{c_m - \sum_{j=1}^{m} \xi_j\}$.

Proof. The proof is similar to that of the lower bound. First adding $\sum_{j=1}^{n} \xi_j(t - s_j) = \sum_{j=1}^{n} \xi^j(s_{j+1} - s_j)$ with $\xi^j := \sum_{m=1}^{j} \xi_m$ to $X_n(t)$, then subtracting it again and using (61) yields

$$X_n(t) \le \sup_{0 \le s \le t} \{A_i(s, t) - (c_k - \xi^k)(t - s)\} + \sum_{j=1}^{n} \sup_{0 \le s_j \le t} \{A_{S_j}(s_j, t) - \xi_j(t - s_j)\},$$

and the proof is completed. $\square$

Lemma 9.1 *For $2 \le n \le p$,*

$$V_n^p(t) = X_n^p(t) - X_{n-1}^p(t).$$

Proof. First we prove, using induction on r, that for each $r \in \{0, \dots, n-1\}$,

$$\begin{aligned} V_n^p(t) &= \sup_{0 \le s_{n-r} \le \dots \le s_{n+1} = t} \{A_{i,n-r}(s_{n-r}, t) + \hat{A}^p_{n-r-1,n-r}(s_{n-r}, t) \\ &\quad + \sum_{k=n-r}^{n} [\hat{A}_k^p(s_k, t) - c_k^p(s_k, s_{k+1})]\} - \sum_{k=n-r}^{n-1} V_k^p(t). \end{aligned} \tag{63}$$

For $r = 0$, (63) reduces to

$$V_n^p(t) = \sup_{0 \le s_n \le t} \left\{ A_{i,n}(s_n, t) + \hat{A}^p_{n,n}(s_n, t) - c_n^p(t - s_n) \right\},$$

which is true by virtue of Lemma 3.1.
Assume (63) to hold for $r-1$. Substituting (6) for $A_{i,n-r+1}(s_{n-r+1}, t) + \hat{A}^p_{n-r,n-r+1}(s_{n-r+1}, t)$ yields,

$$\begin{aligned} V_n^p(t) &= \sup_{0 \le s_{n-r+1} \le \dots \le s_{n+1} = t} \{A_{i,n-r}(s_{n-r+1}, t) + \hat{A}^p_{n-r,n-r}(s_{n-r+1}, t) + V^p_{n-r}(s_{n-r+1}) \\ &\quad + \sum_{k=n-r+1}^{n} [\hat{A}_k^p(s_k, t) - c_k^p(s_k, s_{k+1})]\} - \sum_{k=n-r+1}^{n-1} V_k^p(t) - V_{i,n-r}(t) - \sum_{j=1}^{n-r} \sum_{m=p}^{N} V_{S_j^m, n-r}(t). \end{aligned}$$

Using Lemma 3.1 to substitute for $V^p_{n-r}(s_{n-r+1})$, i.e., $V^p_{n-r}(s_{n-r+1}) =$

$$\sup_{0 \le s_{n-r} \le s_{n-r+1}} \{\hat{A}^p_{n-r,n-r}(s_{n-r}, s_{n-r+1}) + A_{i,n-r}(s_{n-r}, s_{n-r+1}) - c^p_{n-r}(s_{n-r}, s_{n-r+1})\},$$

and rewriting the supremum, we obtain (63).
Taking $r = n-1$ in (63) yields $\sum_{k=1}^{n} V_k^p(t) = X_n^p(t)$ for all $n \le p$, and thus we obtain the desired result. $\square$

Lemma 9.2 *For $n \le p$ and $\gamma_q \le \tilde{\phi}_q$,*

$$X_n^p(t) \ge W_i^{(e_n^p)^*}(t) - \sum_{k=1}^{n} \sum_{m=p}^{N} U_{S_k^m}^{\zeta_{S_k^m}}(t) - \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{m=k}^{p-1} \sum_{q \in S_j^m} \{U_q^{\zeta_q}(t) + W_q^{\gamma_q}(t)\},$$

with $(e_n^p)^* := \min_{k=1,\dots,n}\{e_k^p\}$.

Proof. Using the lower bound for $B_{q,k}(s_k, s_{k+1})$ as given in Lemma 7.1 and using (48) and (49),

$$\begin{aligned} X_n^p(t) &\ge \sup_{0 \le s_1 \le \dots \le s_{n+1} = t} \Bigg\{ A_i(s_1, t) + \sum_{k=1}^{n} \Bigg[\sum_{m=p}^{N} A_{S_k^m}(s_k, t) - c_k(s_{k+1} - s_k) \\ &\quad + \sum_{j=1}^{k} \sum_{m=k}^{p-1} \gamma_{S_j^m}(s_{k+1} - s_k) - \sum_{j=1}^{k} \sum_{m=k}^{p-1} \sum_{q \in S_j^m} \sup_{s_k \le s_m \le s_{k+1}} \{\gamma_q(s_m - s_k) - A_q(s_k, s_m)\} \Bigg] \Bigg\}. \end{aligned}$$

Observe that

$$\zeta_{S_k^m}(t-s_k)=\sum_{j=k}^{n}\zeta_{S_k^m}(s_{j+1}-s_j),$$

which means that

$$\sum_{k=1}^{n}\sum_{m=p}^{N}\zeta_{S_k^m}(t-s_k)=\sum_{k=1}^{n}\sum_{j=k}^{n}\sum_{m=p}^{N}\zeta_{S_k^m}(s_{j+1}-s_j).$$

First changing the order of summation and then interchanging the indices j and k, the latter term can be written as

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{m=p}^{N}\zeta_{S_k^m}(s_{j+1}-s_j)=\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=p}^{N}\zeta_{S_j^m}(s_{k+1}-s_k). \tag{64}$$

Hence, adding and subtracting $\sum_{k=1}^{n}\sum_{m=p}^{N}\zeta_{S_k^m}(t-s_k)$ yields,

$$X_n^p(t)\geq\sup_{0\leq s_1\leq\ldots\leq s_{n+1}=t}\Bigg\{A_i(s_1,t)-\sum_{k=1}^{n}(c_k-\sum_{j=1}^{k}\sum_{m=k}^{p-1}\gamma_{S_j^m}-\sum_{j=1}^{k}\sum_{m=p}^{N}\zeta_{S_j^m})(s_{k+1}-s_k)+$$
$$\sum_{k=1}^{n}\sum_{m=p}^{N}(A_{S_k^m}(s_k,t)-\zeta_{S_k^m}(t-s_k))-\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\sup_{s_k\leq s_m\leq s_{k+1}}\{\gamma_q(s_m-s_k)-A_q(s_k,s_m)\}\Bigg\}.$$

The inner supremum is upper bounded by

$$\sup_{s_k\leq s_m\leq s_{k+1}}\{\zeta_q(t-s_k)-A_q(s_k,t)\}+\sup_{s_k\leq s_m\leq s_{k+1}}\{A_q(s_m,t)-\gamma_q(t-s_m)\}+(\gamma_q-\zeta_q)(t-s_k). \tag{65}$$

Because

$$(\gamma_{S_j^m}-\zeta_{S_j^m})(t-s_k)=\sum_{f=k}^{n}(\gamma_{S_j^m}-\zeta_{S_j^m})(s_{f+1}-s_f),$$

we can follow the derivation of (64) to obtain

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}(\gamma_{S_j^m}-\zeta_{S_j^m})(t-s_k)=\sum_{k=1}^{n}\sum_{f=1}^{k}\sum_{j=1}^{f}\sum_{m=f}^{p-1}(\gamma_{S_j^m}-\zeta_{S_j^m})(s_{k+1}-s_k). \tag{66}$$

Using (65) and (66), we obtain for the lower bound,

$$\sup_{0\leq s_1\leq\ldots\leq s_{n+1}=t}\{A_i(s_1,t)-\sum_{k=1}^{n}e_k(s_{k+1}-s_k)\}-\sum_{k=1}^{n}\sum_{m=p}^{N}\sup_{0\leq s_k\leq t}\{\zeta_{S_k^m}(t-s_k)-A_{S_k^m}(s_k,t)\}$$
$$-\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\left(\sup_{0\leq s_k\leq t}\{\zeta_q(t-s_k)-A_q(s_k,t)\}+\sup_{0\leq s_m\leq t}\{A_q(s_m,t)-\gamma_q(t-s_m)\}\right).$$

Finally using (8) and (61) for the first supremum the proof is completed. □

Lemma 9.3 *For $n \le p$ and $\gamma_q \le \tilde{\phi}_q$,*

$$X_n^p(t) \le W_i^{(d_n^p)^*}(t) + \sum_{k=1}^{n}\sum_{m=p}^{N} W_{S_k^m}^{\theta_{S_k^m}}(t) + \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\{W_q^{\theta_q}(t) + U_q^{\psi_q}(t)\},$$

with $(d_n^p)^* := \min_{k=1,\ldots,n}\{d_k^p\}$.

Proof. Using the upper bound for $B_{q,k}(s_k, s_{k+1})$ as given in Lemma 7.4 and using (48) and (49) yields,

$$X_n^p(t) \le \sup_{0\le s_1\le\ldots\le s_{n+1}=t}\Bigg\{A_i(s_1,t) + \sum_{k=1}^{n}\Bigg[\sum_{m=p}^{N} A_{S_k^m}(s_k,t) - c_k(s_{k+1}-s_k) +$$
$$\sum_{j=1}^{k}\sum_{m=k}^{p-1}\gamma_{S_j^m}(s_{k+1}-s_k) + \sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\sup_{0\le s_m\le s_{k+1}}\{A_q(s_m,s_{k+1}) - \gamma_q(s_{k+1}-s_m)\}\Bigg]\Bigg\}.$$

Analogously to the proof of Lemma 9.2, we obtain

$$\sum_{k=1}^{n}\sum_{m=p}^{N}\gamma_{S_k^m}(t-s_k) = \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=p}^{N}\gamma_{S_j^m}(s_{k+1}-s_k).$$

Hence, adding and subtracting $\sum_{k=1}^{n}\sum_{m=p}^{N}\gamma_{S_k^m}(t-s_k)$ yields,

$$X_n^p(t) \le \sup_{0\le s_1\le\ldots\le s_{n+1}=t}\Bigg\{A_i(s_1,t) - \sum_{k=1}^{n}\left(c_k - \sum_{j=1}^{k}\sum_{m=k}^{p-1}\gamma_{S_j^m} - \sum_{j=1}^{k}\sum_{m=p}^{N}\gamma_{S_j^m}\right)(s_{k+1}-s_k) +$$
$$\sum_{k=1}^{n}\sum_{m=p}^{N}(A_{S_k^m}(s_k,t) - \gamma_{S_k^m}(t-s_k)) + \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\sup_{0\le s_m\le s_{k+1}}\{A_q(s_m,s_{k+1}) - \gamma_q(s_{k+1}-s_m)\}\Bigg\}.$$

The inner supremum is upper bounded by

$$\sup_{0\le s_m\le t}\{A_q(s_m,t) - \gamma_q(t-s_m)\} + \sup_{0\le s_{k+1}\le t}\{\psi_q(t-s_{k+1}) - A_q(s_{k+1},t)\} + (\gamma_q - \psi_q)(t-s_{k+1}).$$

Because

$$(\gamma_{S_j^m} - \psi_{S_j^m})(t-s_{k+1}) = \sum_{f=k+1}^{n}(\theta_{S_j^m} - \psi_{S_j^m})(s_{f+1}-s_f),$$

and following the reasoning in the proof of Lemma 9.2, it is easily seen that

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}(\gamma_{S_j^m} - \psi_{S_j^m})(t-s_{k+1}) = \sum_{k=1}^{n}\sum_{f=1}^{k-1}\sum_{j=1}^{f}\sum_{m=f}^{p-1}(\gamma_{S_j^m} - \psi_{S_j^m})(s_{k+1}-s_k).$$

Using this in the upper bound yields,

$$\sup_{0\le s_1\le\ldots\le s_{n+1}=t}\{A_i(s_1,t) - \sum_{k=1}^{n} d_k(s_{k+1}-s_k)\} + \sum_{k=1}^{n}\sum_{m=p}^{N}\sup_{0\le s_k\le t}\{A_{S_k^m}(s_k,t) - \gamma_{S_k^m}(t-s_k)\}$$
$$+\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{m=k}^{p-1}\sum_{q\in S_j^m}\left(\sup_{0\le s_m\le t}\{A_q(s_m,t) - \gamma_q(t-s_m)\} + \sup_{0\le s_{k+1}\le t}\{\psi_q(t-s_{k+1}) - A_q(s_{k+1},t)\}\right).$$

Then using (8) and (61) for the first supremum the proof is completed. □