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Uniqueness and Mixing Properties of Gibbs Measures

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A short introduction to the theory of (finite-range) Gibbs measures and phase transitions is given, starting with the famous Ising model. Some recent results on uniqueness and mixing, based on percolation-type arguments, are summarized and compared with more classical results.

1. The Ising model

The basic Ising model is a simple but very useful model of a *ferromagnet*. In the 2-dimensional case it is as follows. Consider a finite part Λ (for instance an $n \times n$ box centered at O) of the square lattice (this is the graph whose vertices are the elements of $\mathbb{Z}^2$ and where two vertices share an edge iff their distance is 1). Assign to each vertex i a variable σ_i, which can take the value $+1$ or -1, and which represents the spin of the 'elementary magnet' at i. It is assumed that only neighbours (vertices at distance 1) interact, namely in the following way. Each pair i, j contributes an amount J to the total *energy* if $\sigma_i \neq \sigma_j$, and an amount $-J$ otherwise. Here $J > 0$ is the interaction parameter of the model. The total energy H_Λ is the sum of all such contributions. Further, based on statistical/physical arguments, it is assumed that each configuration $\sigma \in \{-1, +1\}^\Lambda$ occurs with a probability proportional to $\exp^{-H_\Lambda(\sigma)}$. So we have

$$\mu_\Lambda(\sigma) := \frac{\exp^{-H_\Lambda(\sigma)}}{Z}, \tag{1}$$

where Z is a normalizing constant. This probability distribution is called a *finite-volume Gibbs measure*.

It is clear that this probability distribution 'has a preference' for neighbour spins to be the same. It is also clear that if Λ is fixed and we make J very large, then, with very high probability, there is either a vast majority of $+1$-spins or a vast majority of -1-spins. More surprising (and certainly not trivial) is that if we first fix J at a sufficiently high value, then this *symmetry breaking* persists for arbitrary large Λ. More precisely, there is a critical value J_c such that if $J < J_c$, the following, related, properties hold

– If we let the size of Λ go to ∞ then the frequency of $+$ spins tends (in a strong probabilistic sense) to $1/2$. Moreover, the $+1$-spins and -1-spins are typically well mixed so that no macroscopic magnetic field is created.

– Suppose we have a so-called boundary condition τ. (This means that each vertex i on the outer boundary of Λ has a fixed value τ_i which influences the energy, and hence the distribution, by its interaction with its neighbour inside Λ). Then, no matter which boundary condition we choose, the probability that the spin at the origin is $+1$ tends to $1/2$ as the size of Λ tends to ∞.

– There is a unique *infinite-volume Gibbs measure.* This means that all probability distributions on $\{-1,+1\}^{\mathbb{Z}^2}$ which arise as (weak) limits of finite-volume Gibbs measures, are the same.

On the other hand, if $J > J_c$, these properties no longer hold. In particular, there is a $p > 1/2$ (depending on J), such that if Λ is very large, then with high probability either the fraction of $+$-spins is close to p or the fraction of $-$-spins is close to p. (The behaviour in the 3-dimensional case can be more complicated). This symmetry breaking causes a macroscopic magnetic field (spontaneous magnetization). This behaviour for very large boxes corresponds with the *non-uniqueness* of infinite-volume Gibbs measures.

Summarizing, we say that this model has a *phase transition.* Since these phenomena correspond reasonably well to what is observed for real feromagnets, this model, and several variations, have become very popular in theoretical physics.

2. Finite-range Gibbs measures; Markov property

The Ising model is an example of a finite-range Gibbs model. In general we have a *single-site state space* S, and for each $A \subset \mathbb{Z}^d$ with diameter $\leq r$ (the *range* of interaction) we have an interaction function which assigns to each element of S^A a real number, the contribution to the total energy H_Λ. Then, as in the Ising model, we consider the finite-volume Gibbs measure on S^Λ defined analogously to (1).

Again we can introduce a boundary condition, and one of the central questions is whether the influence of the boundary condition vanishes (as the size of Λ grows to ∞) and the infinite-volume Gibbs measure is unique.

Gibbs measures originate from Statistical Physics. More recently they became important in the context of statistical image analysis, randomized optimization algorithms and certain large communication networks (Kelly [11]).

In the last context the notion of phase transition is important because it reflects a certain instability: an event very far away (for instance the breakdown of a node) may have a non-negligible impact throughout the network.

In fact, finite-range Gibbs measures are, from a mathematical point of view, quite natural objects for the following reason. It is not difficult to check that they satisfy the following *Markov property.* For each vertex x the conditional distribution of σ_x, given all other spins, depends only on the spins of the vertices at distance $\leq r$ from x. Moreover, the reverse als holds (which is not trivial): if the above Markov property holds, then the distribution is a Gibbs distribution with respect to an interaction with range r.

3. Mixing

As we stated before, an important question is whether the influence of a boundary condition vanishes as the size of a box goes to ∞. More generally, we are interested in convenient bounds for the influence of two sets A and B of vertices on each other. First we have to quantify the notion 'influence'. This is often done as follows. We assume that we deal with a finite box Λ. Take two possible spin configurations τ and τ' on A, and consider the corresponding conditional Gibbs ditributions on $\Lambda \setminus A$, restricted to B. Take their variational distance and maximise this over all pairs τ, τ'. We call the result the influence of A on B (in Λ), denoted by $f_\Lambda(A, B)$. If the vertices x and y are neighbours (notation: $x \sim y$), then, somewhat confusing, we define the *local influence* of x on y as follows: let τ and τ' be two configurations on the set of neighbours of y which differ only at x. Again take the variational distance of the corresponding distributons of σ_y, and maximise over all such pairs τ, τ'. The result is denoted by $g(x, y)$. (It is important to note that each $g(x, y)$ can be explicitly calculated when the interaction functions are given.) It can be shown (by a clever argument) that f_λ is dominated by a function f'_Λ which satisfies, for arbitrary $x, y \in \Lambda$,

$$f'_\Lambda(x, y) \leq \sum_{i \sim y} f'_\Lambda(x, i) g(i, y). \tag{2}$$

Related to this, it can be shown that if the supremum over $j \in \mathbb{Z}^d$ of $\sum_{i \sim j} g(i, j)$ is smaller than 1, then there exist C and $\gamma > 0$ (independent of Λ) such that for all finite boxes $\Lambda \subset \mathbb{Z}^d$, and all $A, B \subset \Lambda$,

$$f_\Lambda(A, B) \leq C|A| \exp^{-\mathrm{dist}(A,B)},$$

where $|A|$ is the size of A and $\mathrm{dist}(A, B)$ the distance between A and B. This is an example of a *mixing* property. In particular, the boundary influence then vanishes and there is a unique infinite-volume Gibbs measure. This is a classical result of Dobrushin [6].

A few years ago Van den Berg and Maes have obtained an alternative result (see also Van den Berg [2] and Van den Berg & Steif [3]). The idea was to compare the 'spread of influence' with a *percolation* process. In percolation models each vertex x is, independent of the others, open with probability p_x and closed with probability $1 - p_x$. One is interested in the probability that there is an open path between vertices at large distance of each other, and, in particular, the existence of infinite open paths. If all p_x are smaller than some critical p_c (which depends on the graph), then, with probability 1, only finite open paths exist. Van den Berg and Maes have shown the following. Let, for each vertex x,

$$p_x = f_{N_x}(\tilde{N}_x, x), \tag{3}$$

with f as above, N_x the set of vertices at distance $\leq r$ from x, and $\tilde{N}_x = N_x \setminus \{x\}$. Like $g(x,y)$, p_x can be explicitly calculated when the interaction functions are given. Then

$$f_\Lambda(A, B) \leq P_{\{p_x\}}(A \to B), \tag{4}$$

where the right hand side is the probability of an open path from A to B in the percolation model with parameters $p_x, x \in \mathbb{Z}^d$ (and 'path' refers to the graph whose vertices are the elements of $\mathbb{Z}^d$ and where two different vertices share an edge iff their distance is at most r). As a consequence, using results from percolation theory, if $\sup_x p_x < p_c$, then there exist C' and $\gamma' > 0$ such that $f_\Lambda(A,B) \leq C'|A| \exp^{-\gamma' \mathrm{dist}(A,B)}$, and the infinite-volume Gibbs measure is unique.

It appears that for some models, in particular low-range 2-dimensional hard-core models and antiferromagnetic Ising models (where the interaction parameter is negative, and hence there is a tendency for neighbours to disagree), this alternative condition gives a better result than Dobrushin's condition. Moreover, this percolation-like approach appears to be more robust with respect to non-homogeneity: If a non-zero fraction of the vertices j of $\mathbb{Z}^d$ has $\sum_{i \sim j} g(i,j) > 1$, then the inequality (2) is quite useless. However, if a non-zero fraction of the vertices have large p_x, then the right hand side of (4) may still be exponentially small in $\mathrm{dist}(A,B)$ when the other vertices have sufficiently small p_x.

4. Random interactions

The above mentioned robustness was exploited by Gielis & Maes [9] for the case of *random interactions*, i.e. when the interaction functions are not fixed but first chosen according to some random mechanism, after which the Gibbs measures for that particular choice are studied. Random interactions are used to model disordered materials and networks (for instance, a dilute ferromagnet where not every vertex but a fraction of the vertices contains an 'elementary magnet'; or a communication network which does not correspond with a regular grid, but which does have much regularity in a statistical sense). Usually, to make these models mathematically tractable, a lot of spatial indepence is assumed, as well as translation invariance of the interaction process.

Related to what we said at the end of the previous section, if the vertices have a positive probability (no matter how small) to have bad local interactions, then (2) is not useful. Bassalygo & Dobrushin [7] give an elementary but complicated method which does give useful results in many of these situations. The approach of Gielis and Maes, which uses (4), seems to be more intuitively appealing and elegant. In fact, the flexibility (towards randomness of the interactions) of the percolation approach is related to the following: a percolation model with *random*, independent parameters $p_x, x \in \mathbb{Z}^d$ is, essentially, equivalent to one with fixed parameters equal to the *expectations* of the p_x's. Although Gibbs models with random interactions lead, via (4), to percolation models with random but *locally dependent* parameters, this local dependence creates only minor difficulties. As a result, Gielis and Maes obtain a theorem of the form that if for each x the *expectation* of the right hand side of (3) is sufficiently small, then (for almost all realizations of the interactions) there is a unique Gibbs measure, and some kind of mixing holds.

5. RESCALING

DOBRUSHIN & SHLOSMAN [7] present a so-called *constructive* version of Dobrushin's uniqueness result. Instead of local influences on single sites it involves local influences on boxes (e.g. cubes of size l), which we will not define here precisely. They obtain a result of the form: if, for some l, each cube of size l satisfies a local influence condition, then the system is, in some sense, mixing, and, in particular, there is a unique Gibbs measure.

This naturally leads to the question if the percolation approach also has a 'constructive extension'. Recently VAN DEN BERG [5] has, for the 2-dimensional case, obtained a rescaled version of the Van den Berg-Maes result, and combined this with the above mentioned idea of Gielis and Maes. The result for random interactions is of the following form: if, for some n, the *expected* 'influence' of the boundary of a $3n \times 3n$ square on the $n \times n$ square in its middle, as well as the expected 'influence' of the horizontal sides of an $n \times n$ square on the horizontal strip of width $r - 1$ in its middle (and the analog for vertical instead of horizontal) are sufficiently small, then (almost surely) there is a unique infinite-volume Gibbs measure, and some mixing property holds. (We write 'influence' because it is not exactly the same notion of influence defined before). Moreover, an extension (to random interactions) is obtained of the interesting result (of MARTINELLI, OLIVIERI & SCHONMANN [12]) that, for 2-dimensional spin systems two mixing properties, of which one is seemingly stronger than the other, are actually equivalent.

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