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A Correlation Inequality for Connection Events in Percolation

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Abstract

It is well-known in percolation theory (and intuitively plausible) that two events of the form “there is an open path from s to a” are positively correlated. We prove the (not intuitively obvious) fact that this is still true if we condition on an event of the form “there is no open path from s to t”.

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1 Introduction and statement of results

We consider the usual bond percolation models on a (finite or countably infinite) graph $G = (V, E)$: each $e \in E$ is "open" (has value 1) with probability $p(e)$ and "closed" (has value 0) with probability $1 - p(e)$, independently of all other edges. We write P for the corresponding probability distribution on $\Omega := \{0,1\}^E$. For general background see [3].

For $s, a \in V$ we write $s \longleftrightarrow a$ for the event that there is an open path from s to a, and $s \not\longleftrightarrow a$ for the complementary event.

Positive (i.e. nonnegative) correlation of any two events $s \longleftrightarrow a$ and $s \longleftrightarrow b$ follows from Harris' inequality [5] (Theorem 2.1 below). The correlation inequality of the title says that this phenomenon persists if we condition on any event $s \not\longleftrightarrow t$.

Theorem 1.1 *For any $s, a, b, t \in V$*

$$P(s \longleftrightarrow a,\ s \longleftrightarrow b \,|\, s \not\longleftrightarrow t) \geq P(s \longleftrightarrow a \,|\, s \not\longleftrightarrow t)P(s \longleftrightarrow b \,|\, s \not\longleftrightarrow t).$$

The intuition for this is not very clear. In particular it is *not* true if we condition on $s \longleftrightarrow t$ rather than $s \not\longleftrightarrow t$. (Consider the graph with vertices s, a, b, t and each of s, t joined to each of a, b.)

From now on we fix $s \in V$, and set, for $X \subseteq V$, $Q_X = \{s \longleftrightarrow x\ \forall x \in X\}$ and $R_X = \{s \not\longleftrightarrow x\ \forall x \in X\}$.

Theorem 1.2 *For any $A, B, X, Y \subseteq V$,*

$$P(Q_A R_X)P(Q_B R_Y) \leq P(Q_{A\cup B} R_{X\cap Y})P(R_{X\cup Y}). \tag{1}$$

Remarks

1. Of course we recover Theorem 1.1 from Theorem 1.2 by taking $A = \{a\}$, $B = \{b\}$ and $X = Y = \{t\}$. This is not generalization for its own sake: the more general form is needed for the proof.

2. The perhaps intuitively more natural statement obtained by replacing $R_{X\cup Y}$ by $Q_{A\cap B} R_{X\cup Y}$ in Theorem 1.2 is *not* true: take $V(G) = \{s, x, y, a\}$, $E(G) = \{sx, xa, ay, ys\}$ and $X = \{x\}$, $Y = \{y\}$, $A = B = \{a\}$.

3. Note that if we replace A by $A \setminus B$ in Theorem 1.2, the r.h.s. of (1) remains the same and the l.h.s. does not decrease. So Theorem 1.2 as stated above is not more general than the case $A \cap B = \emptyset$.

4. The original motivation for Theorem 1.1 was a conjecture we learned from the late P.W. Kasteleyn (personal communication, circa 1985), a slightly informal description of which is as follows. Let $G = (V, E)$ be a finite graph, W some subset of V, and $\tilde{G} = (\tilde{V}, \tilde{E})$ a copy of G. For each $e \in E$ and $v \in V$, let $\tilde{e}$ and $\tilde{v}$ be the corresponding edge and vertex in $\tilde{G}$ respectively. Now we 'glue' G and $\tilde{G}$ together by identifying w with $\tilde{w}$ for $w \in W$, and on this new graph consider any percolation model with $p(\tilde{e}) = p(e)$ for all $e \in E$. The conjecture is then that, for every $a, b \in V$, $P(a \longleftrightarrow b) \geq P(a \longleftrightarrow \tilde{b})$. There is in fact a slight concrete connection with Theorem 1.1, in that a special case of the latter says that when $|W| = 2$, say $W = \{v, w\}$, one has $P(a \longleftrightarrow b | v \not\longleftrightarrow w) \geq P(a \longleftrightarrow \tilde{b} | v \not\longleftrightarrow w)$. But we feel that Theorem 1.1 is more interesting for its own sake, and believe it has potential applications in percolation theory in general.

2 Background

We just recall the two correlation inequalities we will need in Section 3. For more extensive discussions see [2].

An event $\mathcal{A}$ (i.e. a subset of Ω) is called *increasing* if $\mathcal{A} \ni \omega \leq \omega'$ implies $\omega' \in A$. (Here $\omega \leq \omega'$ means $\omega_e \leq \omega'_e$ for all $e \in E$). The following correlation inequality is due to Harris [5].

Theorem 2.1 *For any increasing* $\mathcal{A}, \mathcal{B} \subset \Omega$,

$$P(\mathcal{A}\mathcal{B}) \geq P(\mathcal{A})P(\mathcal{B}).$$

Of course this is equivalent to saying that for any increasing $\mathcal{A}$ and *decreasing* $\mathcal{B}$ $P(\mathcal{A}\mathcal{B}) \leq P(\mathcal{A})P(\mathcal{B})$.

There are a number of significant extensions of Harris' inequality, notably that of Fortuin, Kasteleyn and Ginibre [4]. Our main tool is the considerably more general Ahlswede-Daykin (or "Four Functions") Theorem [1], *viz.*

Theorem 2.2 *Let* N *be a finite set and let* $\mathcal{P}(N)$ *denote the set of all subsets of* N *Suppose* $\alpha, \beta, \gamma, \delta : \mathcal{P}(N) \to \mathbf{R}^+$ *satisfy*

$$\alpha(S)\beta(T) \leq \gamma(S \cap T)\delta(S \cup T) \quad \forall S, T \subseteq N. \tag{2}$$

Then $\sum \alpha(S) \sum \beta(S) \leq \sum \gamma(S) \sum \delta(S)$ *(where the sums are over all* $S \subseteq N$*).*

3 Proof of Theorem 1.2

We assume G is finite. (If G is countably infinite, the result follows from the finite case by obvious limit arguments). The proof is by induction on the number of vertices $|V|$. If $|V| = 1$, the result is trivial. Suppose it always holds if $|V| \leq n$ and consider a graph G with $n+1$ vertices.

Set $X \cap Y = Z$. If $Z = \emptyset$ then (1) follows from Harris' inequality:

$$\begin{aligned} P(Q_A R_X)P(Q_B R_Y) &\leq P(Q_A)P(R_X)P(Q_B)P(R_Y) \\ &\leq P(Q_A Q_B)P(R_X R_Y) \\ &= P(Q_{A \cup B} R_{X \cap Y})P(R_{X \cup Y}). \end{aligned}$$

If $Z \neq \emptyset$ we proceed as follows: Set $N = \{y \notin Z : y \sim Z\}$ (where $y \sim Z$ means y is adjacent to at least one vertex of Z). Define the (random) set

$$\mathbf{S} = \{y \in N : \text{there is an open edge from } y \text{ to } Z\}.$$

We use S, T for possible values of $\mathbf{S}$ and write $P(S)$ for $P(\mathbf{S} = S)$ and $P(\cdot|S)$ for the conditional distribution given $\mathbf{S} = S$. We may expand

$$P(Q_A R_X) = \sum_S P(S)P(Q_A R_X|S)$$

(where the sum is over all subsets of N), and similarly for the other terms in (1). Thus if we define

$$\begin{aligned} \alpha(S) &= P(S)P(Q_A R_X|S), \\ \beta(S) &= P(S)P(Q_B R_Y|S), \\ \gamma(S) &= P(S)P(Q_{A \cup B} R_{X \cap Y}|S), \\ \delta(S) &= P(S)P(R_{X \cup Y}|S), \end{aligned}$$

then (1) becomes

$$\sum \alpha(S) \sum \beta(S) \leq \sum \gamma(S) \sum \delta(S),$$

where S runs over the subsets of N. Theorem 2.2 says that to verify this we just need to establish (2), which, since (as one can easily check) $P(S)P(T) = P(S \cup T)P(S \cap T)$, is the same as

$$P(Q_A R_X|S)P(Q_B R_Y|T) \leq P(Q_{A\cup B} R_{X\cap Y}|S \cap T)P(R_{X\cup Y}|S \cup T). \tag{3}$$

Let P' refer to the percolation model for the graph G', obtained from G by removing Z, with edge probabilities as in our original percolation model on G. Then it is easy to see that for any $C, W \subseteq V \setminus Z$ and $S \subseteq N$,

$$P(Q_C R_{W\cup Z}|S) = P'(Q_C R_{W\cup S}). \tag{4}$$

Now we obtain (3) as follows: Let $X' = X \setminus Z$ and $Y' = Y \setminus Z$. We have

$$\begin{aligned} P(Q_A R_X|S)P(Q_B R_Y|T) &= P'(Q_A R_{X'\cup S})P'(Q_B R_{Y'\cup T}) \\ &\leq P'(Q_{A\cup B} R_{(X'\cup S)\cap(Y'\cup T)})P'(R_{(X'\cup S)\cup(Y'\cup T)}) \\ &\leq P'(Q_{A\cup B} R_{(S\cap T)})P'(R_{(X'\cup Y')\cup(S\cup T)}) \\ &= P(Q_{A\cup B} R_{X\cap Y}|S \cap T)P(R_{X\cup Y}|S \cup T), \end{aligned}$$

where the first equality follows from applying (4) twice (with $W = X'$ and $W = Y'$ respectively), the first inequality from the induction hypothesis (which says that (1) holds for G'), the second inequality from $(S \cap T) \subseteq (X' \cup S) \cap (Y' \cup T)$, and the second equality from again applying (4) twice (with $W = \emptyset$ and $W = X' \cup Y'$ respectively). $\square$

References

[1] R. Ahlswede and D.E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, *Z. Wahrscheinl. Geb.* **43** (1978), 183-185.

[2] B. Bollobás, *Combinatorics*, Cambridge Univ. Pr., Cambridge, 1986.

[3] G. Grimmett, *Percolation*, Springer, New York, 1989.

[4] C.M. Fortuin, P.W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets, *Comm. Math. Phys.* **22** (1971), 89-103.

[5] T.E. Harris, A lower bound for the critical probability in a certain percolation process, *Proc. Cam. Phil. Soc.* **56** (1960), 13-20.