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Asymptotic Density in a Coalescing Random Walk Model

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Abstract. We consider a system of particles, each of which performs a continuous time random walk on $\mathbb{Z}^d$. The particles interact only at times when a particle jumps to a site at which there are a number of other particles present. If there are j particles present, then the particle which just jumped is removed from the system with probability p_j. We show that if p_j is increasing in j and if the dimension d is at least 6 and if we start with one particle at each site of $\mathbb{Z}^d$, then $p(t) := P\{\text{there is at least one particle at the origin at time } t\} \sim C(d)/t$. The constant $C(d)$ is explicitly identified. We think the result holds for every dimension $d \geq 3$ and we briefly discuss which steps in our proof need to be sharpened to weaken our assumption $d \geq 6$.

The proof is based on a justification of a certain mean field approximation for $dp(t)/dt$. The method seems applicable to many more models of coalescing and annihilating particles.

1991 Mathematics Subject Classification: 60K35, 82C22

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1. Introduction.

Annihilating and coalescing random walks were studied as simple interacting particle systems by Bramson and Griffeath (1980), and Arratia (1981). They considered the following systems. Particles move according to a continuous time random walk on $\mathbb{Z}^d$. The particles only interact when a particle at some site x jumps to a site y which already contains a particle. At this time, the two particles annihilate each other and disappear from the system, or they coalesce to only one particle at y, which continues with its random walk until it again coincides with another particle. The former system is called *annihilating random walk* and the latter system is called *coalescing random walk*. In this paper we shall call the above models the *basic models*. These systems first arose as duals to the "anti-voter model" and the "voter model" and were used as tools to analyze the voter model (see Holley and Liggett (1975), Harris (1976) and Liggett (1985), Section V.1 and Examples III.4.16, 17). Further motivation comes from models for chemical reactions. For chemical reactions one often considers particles of two types and allows only particles of different types to annihilate each other (or to form an inert compound). Such systems have received considerable attention in the literature (see Bramson and Lebowitz (1991a, b) and Lee and Cardy (1995), (1997)). Here we shall restrict ourselves to systems with particles of one type only.

Usually one starts at time 0 with one particle at each site of $\mathbb{Z}^d$, although some results are valid for more general translation invariant initial states. It is further common to let the particles move according to continuous time simple random walk. That is, the particle jumps at the times of a rate 1 Poisson process, and when it jumps from position x, then it jumps to any one of the $2d$ neighbors of x with probability $1/(2d)$. For this version of the model, Bramson and Griffeath and Arratia found the asymptotic behavior of

$$p(t) := P\{\mathbf{0} \text{ is occupied at time } t\}.$$

For the coalescing random walk in dimension $d \geq 3$ one has (Bramson and Griffeath (1980))

$$p(t) \sim \frac{1}{\gamma_d t}, \tag{1.1}$$

where

$$\gamma_d = P\{\text{simple random walk in } \mathbb{Z}^d \text{ never returns to the origin after first leaving it}\}.$$

For annihilating random walk in $d \geq 3$ Arratia (1981) shows

$$p(t) \sim \frac{1}{2\gamma_d t}. \tag{1.2}$$

These articles also find the asymptotic behavior of $p(t)$ for $d = 1$ or 2, but we shall only be concerned with $d \geq 3$ here. In fact, the proof of our principal result requires $d \geq 6$. Bramson and Griffeath and Arratia base their proof on an ingenious derivation by Sawyer (1979) of the limit distribution of the number of particles in the voter model at time t which have taken their opinion from the same individual as the origin (the so-called 'patch size'). Bramson and Griffeath use the so-called duality between the basic coalescing random walk and the voter model to deduce (1.1) from Sawyer's result. It is not clear how robust Sawyer's derivation is. If one wants to consider small variations in the interaction rules for the particles, then proving an analogue of (1.1) and (1.2) via Sawyer's method seems very difficult (see also Remark (iv) after the Theorem). On the other hand, there is an intuitively appealing, heuristic derivation of (1.1) and (1.2), which will be shown below. The main purpose of this paper is to turn those heuristic arguments into a rigorous and quite robust proof. We first give this heuristic explanation.

It is not hard to see that the forward equation for $p(t)$ is

$$\frac{d}{dt}p(t) = -P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

for the coalescing random walk, and

$$\frac{d}{dt}p(t) = -2P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

for the annihilating random walk; here e_1 denotes the site $(1, 0, \dots, 0)$. For brevity we only discuss the coalescing random walk. Now if $\mathbf{0}$ and e_1 are occupied at time t, then the particles at these two sites must have been at some sites x and y, respectively, at the earlier time $t-\Delta$, and the paths of the particles from x to $\mathbf{0}$ and from y to e_1 must not have coincided during $[t-\Delta, t]$. One can expect that if Δ becomes large with t, then only the contributions from pairs x, y far apart will play a role. When x and y are far apart, particles which are at x and y at time $t - \Delta$ should not have 'felt each other' before time $t - \Delta$. It therefore seems reasonable to believe that in this case the events

$$\{x \text{ is occupied at time } t - \Delta\} \text{ and } \{y \text{ is occupied at time } t - \Delta\}$$

are nearly independent, so that for Δ chosen properly as a function of t, the dependence between

$$\{\mathbf{0} \text{ is occupied at time } t\} \text{ and } \{e_1 \text{ is occupied at time } t\} \tag{1.3}$$

is almost entirely due to the requirement that the paths from x to $\mathbf{0}$ and from y to e_1 do not coincide during $[t - \Delta, t]$. Let $\{S_s\}_{s\geq 0}, \{S'_s\}_{s\geq 0}, \{S''_s\}_{s\geq 0}$ be independent copies of a continuous time simple random walk starting at $\mathbf{0}$. Then one is led to approximate

$$P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

by

$$\begin{aligned}
&\sum_{x,y} P\{x \text{ is occupied at } t-\Delta\} P\{y \text{ is occupied at } t-\Delta\} \\
&\qquad \times P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, x + S'_s \neq y + S''_s \text{ for } 0 \le s \le \Delta\} \\
&= p^2(t-\Delta) \sum_{x,y} P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, x + S'_s \neq y + S''_s \text{ for } 0 \le s \le \Delta\}.
\end{aligned}$$

Let $\{\widetilde{S}'_s\}_{s\ge 0}$ and $\{\widetilde{S}''_s\}_{s\ge 0}$ be independent copies of the time-reversed random walk. For simple random walk these are again simple random walks, but in general $\widetilde{S}'$ satisfies for $0 = s_0 < s_1 < \cdots < s_\ell = \Delta$, and Borel sets B_i,

$$P\{\widetilde{S}'_{s_i} - \widetilde{S}'_{s_{i-1}} \in B_i, 1 \le i \le \ell\} = P\{S_{\Delta - s_{i-1}} - S_{\Delta - s_i} \in -B_i, 1 \le i \le \ell\}. \quad (1.4)$$

The same relation holds when $\widetilde{S}'$ is replaced by $\widetilde{S}''$. By time reversal one then has

$$\begin{aligned}
&P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, x + S'_s \neq y + S''_s \text{ for } 0 \le s \le \Delta\} \\
&= P\{\widetilde{S}'_\Delta = x, e_1 + \widetilde{S}''_\Delta = y, \widetilde{S}'_s \neq e_1 + \widetilde{S}''_s \text{ for } 0 \le s \le \Delta\}.
\end{aligned}$$

It is an exercise in random walk to show that the right hand side here is well approximated by

$$P\{\widetilde{S}'_\Delta = x\} P\{e_1 + \widetilde{S}''_\Delta = y\} P\{\widetilde{S}'_s \neq e_1 + \widetilde{S}''_s \text{ for } 0 \le s \le \Delta\},$$

and of course, for large Δ and simple random walk,

$$P\{\widetilde{S}'_s \neq e_1 + \widetilde{S}''_s \text{ for } 0 \le s \le \Delta\} \sim P\{\widetilde{S}'_s \neq e_1 + \widetilde{S}''_s \text{ for } s \ge 0\} = \gamma_d.$$

We will explicitly estimate the errors in Lemmas 11–14, but for now we shall just ignore them. This leads to

$$\begin{aligned}
&P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\} \\
&\approx \gamma_d \sum_x P\{\widetilde{S}'_\Delta = x\} p(t-\Delta) \sum_y P\{e_1 + \widetilde{S}''_\Delta = y\} p(t-\Delta) \\
&= \gamma_d \sum_x P\{S'_\Delta = -x \text{ and } x \text{ is occupied at } t-\Delta\} \\
&\qquad\qquad \times \sum_y P\{S''_\Delta = e_1 - y \text{ and } y \text{ is occupied at } t-\Delta\} \\
&\approx \gamma_d P\{\mathbf{0} \text{ is occupied at } t\} P\{e_1 \text{ is occupied at } t\} = \gamma_d p^2(t),
\end{aligned}$$

where $A \approx B$ means that $A-B$ is negligeable for our purposes. From these relations we can expect $p(t)$ to behave asymptotically like the solution of the equation

$$\frac{d}{dt}y(t) = -\gamma_d y^2(t)$$

which vanishes at ∞, namely,

$$y(t) = \frac{1}{\gamma_d t}. \tag{1.5}$$

This is the heuristic reason for (1.1). The idea of switching off the interaction during an interval $[t-\Delta, t]$ we took from Arratia (1981).

It is precisely these approximations which our paper makes rigorous. To show the power of our method we treat the model in which the particles perform a continuous time random walk, but in which particles only coalesce with a probability which may be < 1. As far as we know this model has not been analyzed before. Specifically, let $\{S_t\}_{t\geq 0}$ be a continuous time random walk starting at $\mathbf{0}$. We denote by $q(y)$ the probability that $S_{\cdot}$ has a jump of size y when it jumps (thus, $q(\mathbf{0}) = 0$). Assume that the motion of a particle starting at x is distributed like $\{x + S_t\}$, independent of the motion of all other particles. However, if a particle jumps to a site which already contains j particles, then it colaesces with one of these j particles with a certain probability p_j. For our purposes it is simpler to say that the particle which jumps is removed from the system, and (with the exception of the proofs of Lemmas 9 and 14) we shall follow this convention. (Of course there are other problems for which one wants to keep track of the mass of particles. One then assumes that when two particles of masses m_1 and m_2 coalesce, they form a particle of mass $m_1 + m_2$. However, we shall not do this and only consider the number of particles at a site.)

Our principal result is the following theorem.

Theorem. *Assume that*

$$p_0 = 0, \quad p_1 > 0 \tag{1.6}$$

and that

$$p_j \text{ is increasing in } j. \tag{1.7}$$

Assume further that the particles perform continuous time random walks which are distributed as translates of $\{S_t\}$ *and that*

$$ES_t = t\sum_{y\in\mathbb{Z}^d} yq(y) = \mathbf{0} \text{ and } \sum_{y\in\mathbb{Z}^d} \|y\|^2 q(y) < \infty. \tag{1.8}$$

Finally, assume $d \geq 6$. *Then in the above coalescing model there exists a* $\zeta = \zeta(d) > 0$ *such that*

$$p(t) - \frac{1}{C(d)t} = O\Big(\frac{1}{t^{1+\zeta}}\Big), \quad t \to \infty, \tag{1.9}$$

with

$$C(d) = p_1 \sum_{m=0}^{\infty} (1-p_1)^m P\{S^\sigma_\cdot \text{ returns exactly } m \text{ times to } \mathbf{0} \text{ after first leaving it}\}$$
$$= \frac{p_1 \gamma}{1-(1-p_1)(1-\gamma)}, \tag{1.10}$$

where $S^\sigma_\cdot$ *is the difference of two independent copies of* $S_\cdot$, *and* γ *is the probability that* $S^\sigma_\cdot$ *never returns to the origin after first leaving it. Also*

$$E\{\text{number of particles at } \mathbf{0} \text{ at time } t\} - \frac{1}{C(d)t} = O\big(\frac{1}{t^{1+\zeta}}\big) \tag{1.11}$$

and

$$P\{\text{there are } \geq 2 \text{ particles at } \mathbf{0} \text{ at time } t\} = O\big(\frac{1}{t^2}\big), \quad t \to \infty. \tag{1.12}$$

Remarks (i) It is crucial for our Theorem that (1.6) holds. If $p_0 > 0$, then $p(t)$ will usually decrease exponentially in t. If $p_0 = p_1 = 0$, then $p(t)$ will usually decrease like $t^{-\rho}$ for some $\rho < 1$. Models with $p_0 = p_1 = 0$ are presently being investigated by D. S. Stephenson.

(ii) Although we think that the global structure of our proof is 'what it should be', certain steps are not optimal and therefore our proof works only when $d \geq 6$. We believe that the conclusion of our theorem is valid for $d \geq 3$. This is known for the basic coalescing model with $p_0 = 0, p_j = 1, j \geq 1$ (see Bramson and Griffeath (1980)). For the basic coalescing model our proof too can be improved (and even shortened) to work for all $d \geq 3$. If $p_0 = 0 < p_1 \leq p_2 \leq \cdots \leq p_M = p_{M+j} = 1$ for some finite M and all $j \geq 1$, then (with a lot of extra work) (1.9) can still be proven for $d \geq 4$. We hope to return to these improvements in a separate paper; see also Remark (v) in Section 3.

(iii) The heuristics above form a basic outline of our proof. The principal technical tool to estimate the correlation between events such as in (1.3) is a bound on the variance of

$$\sum_x \beta(x)\xi_t(x)$$

for suitable $\beta(\cdot)$. This variance estimate is derived in Section 3 by what is sometimes called the 'method of bounded differences'.

(iv) We point out that we only consider the expected number of particles at the origin at time t, or the probability that there is at least one such particle. We do not keep track of how many particles have coalesced to form the particles at $\mathbf{0}$ at time t. More specifically, one can define the *mass* of a surviving particle by taking the mass of each particle at time 0 equal to 1, and by taking the mass of a particle which arises when two particles of masses m_1 and m_2 coalesce equal to $m_1 + m_2$. If $M(t)$ denotes the total mass of the particles at $\mathbf{0}$ at time t, then the result of Sawyer (1979) for the basic coalescing model is equivalent to an exponential limit law for $p(t)M(t)$, conditioned on $\{M(t) > 0\} = \{\mathbf{0}$ is occupied at time $t\}$ (when $d \geq 2$). For our more general models we do not know how to prove such a conditional limit law for $p(t)M(t)$, even though we believe that such a conditional limit theorem still holds. However, even if we could prove such a limit law, we do not see how to use the method of Bramson and Griffeath (1980) to deduce the asymptotic behavior of $E(t)$ and $p(t)$ from this. This is so because Bramson and Griffeath use the Markov property for the dual model of the coalescing random walk (see their Lemma 2). We do not know how to construct a useful dual to our more general model. We therefore have not pursued limit laws for $M(t)$, even though this is an interesting problem in its own right.

Another related interesting problem is the spatial structure of the collection of particles at time 0 which–through coalescence– end at the origin at time t. For the basic model this is investigated by Bramson, Cox and LeGall (1998).

2. Description and construction of the Markov process. Since in our system of random walks there can be arbitrarily many particles at a given site, the standard existence theorems do not seem to cover the present set-up. We therefore prove in this section that there exists a Markov process which corresponds to the intuitive description given just before the Theorem in Section 1.

Throughout the p_j are fixed. For the mere construction of the Markov process the monotonicity condition (1.7) is not needed. However, we do use (1.7) to establish some desirable properties of our Markov process. On the initial state and the random walks by which the particles move we only put the weak restriction that $\xi_0 \in \Xi$ (see (2.10)) plus the irreducibility condition (2.2).

The state space of our Markov process will be a subset of

$$\Xi_0 := \{0, 1, \dots\}^{\mathbb{Z}^d}.$$

A generic point of Ξ_0 is denoted by $\xi = \{\xi(x) : x \in \mathbb{Z}^d\}$. ξ_t denotes the state of our system at time t. Its x-coordinate is denoted by $\xi_t(x)$ or sometimes as $\xi(x,t)$; it represents the number of particles at position x at time t. The most useful construction of the process for our purposes is essentialy one based on a graphical representation, as discussed in Griffeath (1979). Let $\tau_1(x,k) < \tau_2(x,k) < \dots$ be the jumptimes of a Poisson process $\{\mathcal{N}_t(x,k)\}_{t \geq 0}$ (with $\mathcal{N}_0(x,k) = 0$). Set $\tau_0(x,k)$ $= 0$. We assume that

$$\text{all processes } \mathcal{N}(x,k), x \in \mathbb{Z}^d, k \geq 1, \text{ are independent.} \tag{2.1}$$

Without the interaction each particle would perform a continuous time random walk which jumps at the times of a rate 1 Poisson process, and when it jumps from position x, then it jumps to y with probability $q(y-x) \geq 0$ ($q(\mathbf{0}) = 0$, $\sum_z q(z) = 1$). We denote a random walk with these jump probabilities and which starts at the origin by $\{S_t\}_{t \geq 0}$. Throughout we assume that

$$S_\cdot \text{ is irreducible,} \tag{2.2}$$

that is, for all x, $P\{S_t = x\} > 0$ for some (and hence for all) $t > 0$. We now attach to each jump time $\tau_n(x,k)$ of the Poisson process $\mathcal{N}(x,k)$ a position $y = y_n(x,k)$ and a collection of random variables $X(n,x,k,j), j \geq 0$. The y here will specify the position to which a particle will jump from x (if any particle will jump from x at time $\tau_n(x,k)$). $X(n,x,k,j)$ takes the value 0 or 1, and specifies whether a particle which jumps from x at time $\tau_n(x,k)$ is removed from the system or not. If there are j particles present at $y_n(x,k)$ at time τ_n (i.e., $\xi(y, \tau_n-) = j$), then the particle which jumps from x to y at τ_n is removed from the system if and only if $X(n,x,k,j) = 0$. We take our sample paths right continuous, so if a particle is removed at τ, then it is not counted in ξ_τ. We assume that

$$\begin{aligned}&\text{all } y_n(x,k) \text{ and } X(n,x,k,\cdot) \text{ for different } (n,x,k)\\ &\text{are independent of each other and of all Poisson processes.}\end{aligned} \tag{2.3}$$

Further, for fixed (n,x,k),

$$y_n(x,k) \text{ and } X(n,x,k,\cdot) \text{ are independent,} \tag{2.4}$$

but the $X(n,x,k,j)$ for different j are coupled. We let $U(n,x,k), x \in \mathbb{Z}^d, n, k \geq 1$, be a family of uniform random variables on $[0,1]$ which are independent of all y's and of all Poisson processes $\mathcal{N}$. We then define the joint distribution of $y_n(x,k)$ and $U(n,x,k)$ by

$$P\{y_n(x,k) = y, U(n,x,k) \leq \lambda\} = q(y-x)\lambda, \quad 0 \leq \lambda \leq 1. \tag{2.5}$$

Further

$$X(n,x,k,j) = 0 \text{ if and only if } U(n,x,k) \leq p_j. \tag{2.6}$$

In particular,

$$P\{X(n,x,k,j) = 0\} = p_j. \tag{2.7}$$

To make the description of our Markov process complete we have to tell when particles jump. The intuitive description is that if there are ℓ particles at x at a certain time t, then the next jump from x occurs at the first jump of one of the processes $\mathcal{N}(x,k)$ with $1 \leq k \leq \ell$. If that jump is at time $\tau_n(x,k)$, then the particle jumps to $y = y_n(x,k)$ and is removed if and only if $X(n,x,k,j) = 0$, where $j = \xi_{\tau_n(x,k)-}(y)$ is the number of particles at y at time $\tau_n(x,k)-$.

If our initial state is a finite state, that is, a state with only finitely many particles present, then there is no difficulty in formalizing the above description. Indeed if we start with n_0 particles, then at all times there are at most n_0 particles present, and therefore with probability 1 the times at which any of the existing particles jumps have no finite accumulation point. On the null set on which there is an accumulation point we can give any value to ξ_t; for instance we can take $\xi_t(x) = 0$ for all x and $t \geq$ first accumulation point of the jump times for the existing particles. We do not give any further details but take it for granted that for any finite initial state the Markov process $\{\xi_t\}$ is completely specified by the description in the preceding paragraph. In fact, this gives us a definition of ξ_t as a function of the initial state ξ_0, all the $\tau_n(x,k), y_n(x,k)$ and the $X(n,x,k,j)$, $x \in \mathbb{Z}^d, n,k \geq 1, j \geq 0$. ξ_t is with probability 1 defined simultaneously for all finite initial states (note that there are only countably many finite states). It will be necessary on occasion to consider ξ_t for various initial states. If we have to indicate the initial state explicitly we shall write $\xi_t(\eta)$ for the process with initial state η. Of course $\xi_t(\eta)$ is also a function of the $\mathcal{N}, y_n$ and the X's, but we do not indicate this in the notation. In accordance with this notation $Ef(\xi_t(\eta))$ is the expectation of $f(\xi_t)$ over all the $\mathcal{N}, y_n, X(n,x,k,j)$ when the initial state $\xi_0 = \eta$. For the time being this is only meaningful for a finite state η.

Extra work is needed to define the ξ-process when we allow infinitely many particles in the system. Our construction more or less follows Liggett (1985), Section IX.1. To describe the state space when we allow infinitely many particles we introduce the norms

$$N_t(\xi) := \sum_{x \in \mathbb{Z}^d} \xi(x)\alpha_t(x), \quad t > 0, \tag{2.8}$$

where

$$\alpha_t(x) = P\{S_t = -x\}. \tag{2.9}$$

We take as state space for our process the space

$$\Xi := \{\xi \in \Xi_0 : N_t(\xi) < \infty \text{ for all } t > 0\}. \tag{2.10}$$

For any $\eta \in \Xi$ we let $\eta^{(N)}$ be the *finite* state given by

$$\eta^{(N)}(x) = \eta(x) I[|x| \leq N]. \tag{2.11}$$

For $\xi_0 \in \Xi$ we can then form the process $\xi_t(\xi_0^{(N)})$ (that is, we first truncate ξ_0 to a finite state and then construct the Markov process with the truncated state as its initial state). We are going to show that the process with the initial state ξ_0 can be defined as $\xi_t = \lim_{N\to\infty} \xi_t(\xi_0^{(N)})$. The principal estimate used to show that this makes sense is based on a comparison lemma of chains with different finite initial states. Let ξ_0', ξ_0'' and $\xi_0^{\#}$ be finite initial states which satisfy

$$\xi_0'(x) \le \xi_0^{\#}(x) \le \xi_0'(x) + \xi_0''(x) \text{ for all } x. \tag{2.12}$$

We now take $\{\xi_t'\}$ and $\{\xi_t^{\#}\}$ to be the processes $\{\xi_t(\xi_0')\}$ and $\{\xi_t(\xi_0^{\#})\}$, respectively. We also introduce a process $\{\xi_t''\}$. This will *not* be the process $\{\xi_t(\xi_0'')\}$, but an equivalent process which is coupled with the ξ'-process and the $\xi^{\#}$-process in such a way that

$$\text{the } \xi'\text{-process and the } \xi''\text{-process are independent.} \tag{2.13}$$

The three processes are coupled in that they use the same $\mathcal{N}, y_n$ and $U(n,\cdot,\cdot)$, as we now specify. In order to describe the three processes together we keep track of the system to which a particle belongs, so that we distinguish $\#$-particles, $'$-particles and $''$-particles. However, we do not distinguish the particles in a single system, so we really only keep track of the number of particles of each type at each site. These numbers at x at time t are $\xi_t'(x), \xi_t''(x)$ and $\xi_t^{\#}(x)$, respectively. If $\xi_t'(x) = \ell', \xi_t''(x) = \ell''$ and $\xi_t^{\#}(x) = \ell^{\#}$, then a $'$-particle jumps from x at the next jump of any of $\mathcal{N}(x,k), 1 \le k \le \ell'$, and a $''$-particle jumps at the next jump of any of $\mathcal{N}(x,k), \ell' < k \le \ell' + \ell''$. Also a $\#$-particle jumps at the next jump of any of $\mathcal{N}(x,k), 1 \le k \le \ell^{\#}$. If a particle jumps at time $\tau_n(x,k)$, then it jumps to $y_n(x,k)$. If it is a $'$-particle, then it is removed if and only if $X(n,x,k,\xi_{\tau_n-}'(y_n)) = 0$. The corresponding rules with $''$ and $\#$ instead of $'$ hold for $''$-particles and $\#$-particles. Note that a $'$-particle and a $\#$-particle may jump at the same time. However, with probability 1 there are no times at which both a $'$-particle and a $''$-particle jump. Thus the $'$-process and the $''$-process never use the same $y_n(x,k)$ or $U(n,x,k)$ and therefore are independent as claimed in (2.13).

Lemma 1. *Assume (1.7). If ξ_0', ξ_0'' and $\xi_0^{\#}$ are finite states which satisfy (2.12), then, under the above coupling, it holds with probability 1 that for all $t \ge 0$*

$$\xi_t'(x) \le \xi_t^{\#}(x) \le \xi_t'(x) + \xi_t''(x) \textit{ for all } x \in \mathbb{Z}^d. \tag{2.14}$$

The left hand inequality remains valid even without (1.7).

Proof. We shall assume (1.7) and leave it to the reader to verify that this is only needed when proving the right hand inequality in (2.14).

Let $s_0 = 0$ and define $s_i, i \ge 1$, recursively as follows. First let $x_1^{(i)}, x_2^{(i)}, \dots, x_{p(i)}^{(i)}$ be the finitely many sites with

$$\xi_{s_i}'(x) + \xi_{s_i}''(x) + \xi_{s_i}^{\#}(x) > 0.$$

Then define

$$\begin{aligned} s_{i+1} = {} & \text{first jump time } > s_i \text{ of any } \mathcal{N}(x,k) \\ & \text{with } x \in \{x_1^{(i)}, x_2^{(i)}, \dots, x_{p(i)}^{(i)}\}, \quad k \le \xi_{s_i}'(x) + \xi_{s_i}''(x). \end{aligned}$$

Now assume that the coupling is such that (2.14) holds for all $t \le s_i$ for some i. We shall prove that (2.14) also holds for $t \le s_{i+1}$. By our construction, $\xi'_t(x), \xi''_t(x)$ and $\xi^{\#}_t(x)$ are all constant for all x and $s_i \le t < s_{i+1}$. (Note that

$$\xi^{\#}_t(x^{(i)}_r) \le \xi'_t(x^{(i)}_r) + \xi''_t(x^{(i)}_r)$$

for $t = s_i$, so $\xi^{\#}_t(x^{(i)}_r)$ indeed does not jump for $s_i < t < s_{i+1}$.) If

$$s_{i+1} = \tau_{n(i+1)}(x^{(i)}_r, k(i+1)),$$

then some particle jumps at time s_{i+1} from $x^{(i)}_r$ to $y_{n(i+1)}(x^{(i)}_r, k(i+1))$, but for $x \ne x^{(i)}_r, y_{n(i+1)}(x^{(i)}_r, k(i+1))$, none of $\xi'_t(x), \xi''_t(x), \xi^{\#}_t(x)$ change at $t = s_{i+1}$. In order to prove (2.14) for $t \le s_{i+1}$, we therefore only have to, check that (2.14) again holds right after the jump at $t = s_{i+1}$ for $x = x^{(i)}_r$ and for $x = y_{n(i+1)}(x^{(i)}_r, k(i+1))$. We distinguish three cases:

$$\begin{aligned} &(a) \qquad 1 \le k(i+1) \le \xi'_{s_i}(x^{(i)}_r); \\ &(b) \qquad \xi'_{s_i}(x^{(i)}_r) < k(i+1) \le \xi^{\#}_{s_i}(x^{(i)}_r); \\ &(c) \qquad \xi^{\#}_{s_i}(x^{(i)}_r) < k(i+1) \le \xi'_{s_i}(x^{(i)}_r) + \xi''_{s_i}(x^{(i)}_r). \end{aligned}$$

By (2.14) for $t = s_i$, these are the only possibilities.

Case (a): In this case a $'$-particle and a $\#$-particle jump simultaneously from $x^{(i)}_r$ to $y_{n(i+1)} = y_{n(i+1)}(x^{(i)}_r, k(i+1))$ (because we also have $k(i+1) \le \xi^{\#}_{s_i}(x^{(i)}_r)$, by virtue of (2.14)). However, no $''$-particle jumps. The particle which jumps is removed from the system in the $'$-system if and only if

$$X' := X(n(i+1), x^{(i)}_r, k(i+1), \xi'_{s_i}(y_{n(i+1)})) = 0 \tag{2.15}$$

and similarly in the $\#$-system. Therefore,

$$\begin{aligned} \xi'_{s_{i+1}}(x^{(i)}_r) &= \xi'_{s_i}(x^{(i)}_r) - 1, \\ \xi''_{s_{i+1}}(x^{(i)}_r) &= \xi''_{s_i}(x^{(i)}_r), \\ \xi^{\#}_{s_{i+1}}(x^{(i)}_r) &= \xi^{\#}_{s_i}(x^{(i)}_r) - 1. \end{aligned}$$

Also

$$\begin{aligned} \xi'_{s_{i+1}}(y_{n(i+1)}) &= \xi'_{s_i}(y_{n(i+1)}) + X', \\ \xi''_{s_{i+1}}(y_{n(i+1)}) &= \xi''_{s_i}(y_{n(i+1)}), \\ \xi^{\#}_{s_{i+1}}(y_{n(i+1)}) &= \xi^{\#}_{s_i}(y_{n(i+1)}) + X^{\#}. \end{aligned}$$

It is clear from the first set of these relations that (2.14) still holds at $t = s_{i+1}, x = x_r^{(i)}$. From the second set of relations we see immediately that the left hand inequality in (2.14) also holds at $t = s_{i+1}, x = y_{n(i+1)}$ if $\xi^{\#}_{s_i}(y_{n(i+1)}) > \xi'_{s_i}(y_{n(i+1)})$. And if $\xi^{\#}_{s_i}(y_{n(i+1)}) = \xi'_{s_i}(y_{n(i+1)}) = \xi'_{s_i}$ say, for short, then

$$X' = X(n(i+1), x_r^{(i)}, k(i+1), \xi'_{s_i}) = X(n(i+1), x_r^{(i)}, k(i+1), \xi^{\#}_{s_i}) = X^{\#}, \quad (2.16)$$

so that even in this case the left hand inequality of (2.14) holds at $t = s_{i+1}, x = y_{n(i+1)}$.

The right hand inequality in (2.14) follows by noticing that under (1.7),

$$X(n, x, k, j) \text{ is decreasing in } j \quad (2.17)$$

(see (2.6)). Thus, (2.14) at $t = s_i$ and the definition of $X', X^{\#}$ (compare (2.15)) show that $X' \geq X^{\#}$. Hence (2.14) holds for $t \leq s_{i+1}$ in case (a).

Case (b): Now no $'$-particle jumps, but a #-particle and a $''$ -particle jump from $x_r^{(i)}$ to $y_{n(i+1)} = y_{n(i+1)}(x_r^{(i)}, k(i+1))$. The #-particle will be removed from the system if $X^{\#} = 0$ and similarly for the $''$-particle. This time we therefore have

$$\begin{aligned}
\xi'_{s_{i+1}}(x_r^{(i)}) &= \xi'_{s_i}(x_r^{(i)}),\\
\xi''_{s_{i+1}}(x_r^{(i)}) &= \xi''_{s_i}(x_r^{(i)}) - 1,\\
\xi^{\#}_{s_{i+1}}(x_r^{(i)}) &= \xi^{\#}_{s_i}(x_r^{(i)}) - 1.
\end{aligned}$$

Also

$$\begin{aligned}
\xi'_{s_{i+1}}(y_{n(i+1)}) &= \xi'_{s_i}(y_{n(i+1)}),\\
\xi''_{s_{i+1}}(y_{n(i+1)}) &= \xi''_{s_i}(y_{n(i+1)}) + X'',\\
\xi^{\#}_{s_{i+1}}(y_{n(i+1)}) &= \xi^{\#}_{s_i}(y_{n(i+1)}) + X^{\#}.
\end{aligned}$$

The right hand inequality in (2.14) at $t = s_{i+1}, x = x_r^{(i)}$ is clear from the former set of equations. The left hand inequality can only go wrong if $\xi'_{s_i}(x_r^{(i)}) = \xi^{\#}_{s_i}(x_r^{(i)})$, but this is impossible in case (b). The left hand inequality in (2.14) at $t = s_{i+1}, x = y_{n(i+1)}$ is immediate from the last set of equations. Finally, the right hand inequality in (2.14) at $t = s_{i+1}, x = y_{n(i+1)}$ is again obvious if $\xi'_{s_i}(y_{n(i+1)}) + \xi''_{s_i}(y_{n(i+1)}) > \xi^{\#}_{s_i}(y_{n(i+1)})$. If we have equality here, then $\xi''_{s_i}(y_{n(i+1)}) \leq \xi^{\#}_{s_i}(y_{n(i+1)})$ and therefore $X'' \geq X^{\#}$ by (2.17). Thus (2.14) at $t = s_{i+1}, x = y_{n(i+1)}$ again holds in this case.

Case (c): Now only a $''$-particle jumps from $x_r^{(i)}$ to $y_{n(i+1)}$. We leave the simple verification of (2.14) at $t = s_{i+1}$ in this case to the reader.

We now have proven that (2.14) holds for $t \le s_{i+1}$ in all cases and therefore (2.14) holds by induction for all $t \le \lim_{i\to\infty} s_i$. However, let

$$\begin{aligned}\mathcal{F}_s = &\sigma\text{-field generated by all } \mathcal{N}_u(x,k) \text{ for } u \le s \\ &\text{and all } y_n(x,k) \text{ and } U(n,x,k) \text{ attached to some } \tau_n(x,k) \le s.\end{aligned} \tag{2.18}$$

Then the conditional distribution of $s_{i+1} - s_i$ given $\mathcal{F}_{s_i}$ is exponential with mean

$$\frac{1}{\sum_{x\in\mathbb{Z}^d}[\xi'_{s_i}(x) + \xi''_{s_i}(x)]} \ge \frac{1}{\sum_{x\in\mathbb{Z}^d}[\xi'_0(x) + \xi''_0(x)]}.$$

Consequently, with probability 1, $s_i \to \infty$ and (2.14) holds for all $t \ge 0$. ■

The same argument as for the right hand inequality of (2.14) shows that if (1.7) holds and if we have finite initial states $\xi_0(\cdot), \xi_0(\cdot;1), \dots, \xi_0(\cdot;r)$ such that

$$\xi_0(x) \le \sum_{i=1}^r \xi_0(x;i) \text{ for all } x, \tag{2.19}$$

then there exist independent processes $\xi_t(\cdot), \xi_t(\cdot\,;1), \dots, \xi_t(\cdot\,;r)$ so that $\{\xi_t(\cdot)\}_{t\ge0}$, $\{\xi_t(\cdot\,;i)\}_{t\ge0}$ have the same distribution as $\{\xi_t(\xi_0(\cdot))\}_{t\ge0}$ and $\{\xi_t(\xi_0(\cdot\,;i))\}_{t\ge0}$, respectively, and so that

$$\xi_t(x) \le \sum_{i=1}^r \xi_t(x;i). \tag{2.20}$$

In particular, (2.19) implies

$$E\xi_t(\sum_1^r \xi_0(\cdot\,;i))(x) \le \sum_1^r E\xi_t(\xi_0(\cdot\,;i))(x). \tag{2.21}$$

The next lemma compares processes with the same initial states, but with different collections of p_j. We shall not need the full strength of (1.7) but instead that

$$p_0 = 0. \tag{2.22}$$

The largest and smallest p_j which satisfy this side condition are

$$p_j^* := \begin{cases} 0 & \text{if } j = 0 \\ 1 & \text{if } j > 0, \end{cases} \tag{2.23}$$

and

$$\overline{p}_j := 0 \text{ for all } j,$$

respectively. Correspondingly we take

$$X^*(n,x,k,j) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}$$

and

$$\overline{X}(n,x,k,j) = 1, \quad j \geq 0.$$

Based on these X^* and $\overline{X}$ we can now define processes $\{\xi_t^*(\xi_0)\}_{t\geq 0}$ and $\{\overline{\xi}_t(\xi_0)\}_{t\geq 0}$ for any finite initial state ξ_0. These will use the same $\mathcal{N}$ and y_n as the process $\{\xi_t(\xi_0)\}_{t\geq 0}$ which we already defined (and which uses $X(n,x,k,j)$ in its construction). The following lemma compares the coupled processes $\xi^*, \overline{\xi}$ and ξ.

Lemma 2. *Assume that (2.22) holds. Then with probability 1 for any finite initial state ξ_0 and any $x \in \mathbb{Z}^d$, $t \geq 0$,*

$$\xi_t^*(x) \leq \xi_t(x) \leq \overline{\xi}_t(x).$$

The right hand inequality remains valid even without (2.22).

The intuitive content of this lemma is fairly clear. In the ξ^*-process we always remove a particle which jumps to a site which is already occupied. In this process there can be at most one particle at a site and we remove particles at a maximal rate. This yields the smallest process. In the $\overline{\xi}$-process we remove as few particles as possible, that is, we never remove a particle and this process is simply a process of non-interacting random walks. It is the largest process of the type considered here. We shall not prove Lemma 2. The general outline of its proof is the same as for Lemma 1 and in fact various cases are easier in this lemma.

We can now show that $\lim_{N\to\infty} \xi_t(\xi_0^{(N)})$ exists with probability 1. Note that this does neither depend on assumption (1.7), nor on (1.6).

Lemma 3. *Let $\xi_0 \in \Xi$. With probability 1 it holds that for all $x \in \mathbb{Z}^d, t \geq 0$,*

$$\xi_t(\xi_0^{(N)})(x) \text{ increases to a finite limit}, \xi_t(x) \text{ say.} \tag{2.24}$$

Since $\xi_t(\xi_0^{(N)})(x)$ is integer valued this actually means that with probability 1, for fixed x and t, $\xi_t(\xi_0^{(N)})(x)$ is eventually constant in N.

Proof. By Lemma 1 we have for $N < M$ with probability 1 that

$$\xi_t(\xi_0^{(M)})(x) \geq \xi_t(\xi_0^{(N)})(x) \text{ for all } x, t,$$

because this inequality holds for $t = 0$. Thus $\xi_t(\xi_0^{(N)})(x)$ is increasing in N and we only have to prove that its limit $\xi_t(x)$ is with probability 1 finite for all t simultaneously.

To prove the finiteness of the limit we use (2.20). This shows that

$$\xi_t(\xi_0^{(N)})(x) \le \overline{\xi}_t(x;N), \tag{2.25}$$

where $\overline{\xi}_t(\cdot\,;N)$ is a system which starts with

$$\xi_0^{(N)}(y) = \xi_0(y)I[|y| \le N] \le \xi_0(y)$$

particles at y for each $y \in \mathbb{Z}^d$, and whose particles merely perform independent random walks without interaction. It follows from this and the monotonicity of $\xi_t(\xi_0^{(N)})$ in N that for fixed x

$$\begin{aligned} &P\{\sup_N \xi_t(\xi_0^{(N)})(x) = \infty \text{ for some } t \le r\} \\ &= \lim_{A\to\infty} P\{\xi_t(\xi_0^{(N)})(x) > A \text{ for some } t \le r \text{ and for some } N\} \\ &= \lim_{A\to\infty} \lim_{N\to\infty} P\{\xi_t(\xi_0^{(N)})(x) > A \text{ for some } t \le r\} \\ &\le \lim_{A\to\infty} \limsup_{N\to\infty} P\{\overline{\xi}_t(x;N) > A \text{ for some } t \le r\}. \end{aligned} \tag{2.26}$$

But, if $\overline{\xi}_t(x;N) > A$ at some stopping time $t \le r$, then (for large A) with probability $\ge 1/2$ there will still be at least $(1/2)e^{-r}A$ particles at x at time r, because each particle present at time t will stay till time r with (conditional) probability $e^{-(r-t)}$. Thus

$$P\{\overline{\xi}_t(x;N) > A \text{ for some } t \le r\} \le 2P\{\overline{\xi}_r(x;N) \ge \frac{1}{2}e^{-r}A\}.$$

Consequently

$$\begin{aligned} &P\{\sup_N \xi_t(\xi_0^{(N)})(x) = \infty \text{ for some } t \le r\} \\ &\le 2 \lim_{A\to\infty} \limsup_{N\to\infty} P\{\overline{\xi}_r(x;N) \ge \frac{1}{2}e^{-r}A\}. \end{aligned} \tag{2.27}$$

Next we note that (recall (2.9))

$$\begin{aligned} \alpha_{r+1}(y) &= P\{y + S_{r+1} = \mathbf{0}\} \\ &\ge P\{y + S_r = x, S_{r+1} - S_r = -x\} \\ &= P\{S_1 = -x\}P\{y + S_r = x\}. \end{aligned} \tag{2.28}$$

This implies that, uniformly in N,

$$\begin{aligned} E\,\overline{\xi}_r(x;N) &\le \sum_y \xi_0(y)P\{y + S_r = x\} \\ &\le \frac{1}{P\{S_1 = -x\}} N_{r+1}(\xi_0) < \infty \text{ (see (2.8))}. \end{aligned} \tag{2.29}$$

Thus the right hand side of (2.27) is zero, for each r. Then, the limit in (2.24) exists and is finite for a fixed x with probability 1 for all t simultaneously. This implies the lemma because there are only countably many x. ∎

We now define

$$\xi_t := \lim_{N\to\infty} \xi_t(\xi_0^{(N)}) \tag{2.30}$$

(with $\xi_t(y) = 1$ for all y on the exceptional set where (2.24) fails for some x, t). We wish to show that this process can reasonably be regarded as the Markov process whose intuitive description was given before the Theorem in Section 1. Before we tackle any details of the distribution of ξ_t, let us show that it lives on Ξ.

Lemma 4. *Let* $\xi_0 \in \Xi$. *Then*

$$E\{\xi_t(x)\} \le \sum_y \xi_0(y) P\{y + S_t = x\} < \infty, \quad t \ge 0, \tag{2.31}$$

and

$$P\{\xi_t(\xi_0) \in \Xi \textit{ for all } t \ge 0\} = 1. \tag{2.32}$$

Proof. This is a small elaboration of the preceding proof. We saw (cf. (2.25)) that

$$E\xi_t(\xi_0)(x) = \lim_{N\to\infty} E\xi_t(\xi_0^{(N)})(x) \le \limsup_{N\to\infty} E\overline{\xi}_t(x;N) = \sum_y \xi_0(y)P\{y + S_t = x\}.$$

This, together with (2.29), proves (2.31). To obtain (2.32) we first note that for $s \le u$

$$\alpha_u(x) \ge \alpha_s(x) P\{S_{u-s} = \mathbf{0}\} \ge \alpha_s(x) e^{-(u-s)}, \tag{2.33}$$

so that for any ξ

$$N_s(\xi) \le e^u N_u(\xi), \quad s \le u. \tag{2.34}$$

Therefore (compare (2.26))

$$\begin{aligned} &P\{N_s(\xi_t) = \infty \text{ for some } t \le r,\ s \le u\} \\ &= P\{N_u(\xi_t) = \infty \text{ for some } t \le r\} \\ &\le \lim_{A\to\infty} P\{\sup_N \sup_{t\le r} N_u(\xi_t(\xi_0^{(N)})) > A\} \\ &\le \lim_{A\to\infty} \limsup_{N\to\infty} P\{\sup_{t\le r} N_u(\overline{\xi}_t(\cdot\,;N)) > A\}. \end{aligned} \tag{2.35}$$

Furthermore, for $t \le s$,

$$\begin{aligned} &E\{N_u(\overline{\xi}_s(\cdot\,;N)|\mathcal{F}_t\} \\ &= \sum_{x\in\mathbb{Z}^d} \alpha_u(x) E\{\overline{\xi}_s(x;N)|\mathcal{F}_t\} \\ &= \sum_{x\in\mathbb{Z}^d} \alpha_u(x) \sum_{y\in\mathbb{Z}^d} \overline{\xi}_t(y;N)\alpha_{s-t}(y-x) \\ &= \sum_{y\in\mathbb{Z}^d} \overline{\xi}_t(y;N)\alpha_{s-t+u}(y) \\ &= N_{s-t+u}(\overline{\xi}_t(\cdot\,;N)) \ge e^{-(s-t)} N_u(\overline{\xi}_t(\cdot\,;N)). \end{aligned} \tag{2.36}$$

Thus $t \mapsto -e^t N_u((\overline{\xi}_t(\cdot\,;N))$ is a negative supermartingale. For each fixed N this supermartingale is right continuous, so that by a well known inequality (see for instance Meyer (1966), Theorem VI.1)

$$\begin{aligned} &P\{\sup_{t\le r} N_u(\overline{\xi}_t(\cdot\,;N)) > A\} \\ &\le P\{\inf_{t\le r} -e^t N_u(\overline{\xi}_t(\cdot\,;N)) < -A\} \\ &\le \frac{1}{A} E\{e^r N_u(\overline{\xi}_r(\cdot\,;N))\} \\ &= \frac{1}{A} e^r N_{u+r}(\overline{\xi}_0(\cdot\,;N)) \text{ (by (2.36) with } t=0). \end{aligned}$$

Since $\limsup_{N\to\infty} N_{u+r}(\overline{\xi}_0(\cdot\,;N)) = N_{u+r}(\xi_0) < \infty$ for $\xi_0 \in \Xi$, we now obtain from (2.35)

$$P\{N_s(\xi_t) = \infty \text{ for some } t \le r,\ s \le u\} = 0,$$

which is just (2.32). ∎

Now define

$$\mathcal{G}_n = \sigma\text{-field of subsets of } \Xi \text{ generated by the coordinate functions } \xi(x) \text{ with } |x| \le n,$$

and

$$\mathcal{G} = \bigvee \mathcal{G}_n.$$

Then for $\eta \in \Xi, B \in \mathcal{G}$, define

$$K_t(\eta, B) = P\{\xi_t(\eta) \in B\}.$$

For fixed η, t, $K_t(\eta, \cdot)$ is a probability measure on $\mathcal{G}$. If $B \in \mathcal{G}_n$, then B is of the form $B = \{\xi \in \Xi : \{\xi(x)\}_{|x| \le n} \in C\}$ with C a subset of $\mathbb{Z}^M$ with $M =$ the number of x with $|x| \le n$. Then

$$K_t(\eta, B) = P\{\{\xi_t(\eta)(x)\}_{|x| \le n} \in C\} = \lim_{N \to \infty} P\{\xi_t(\eta^{(N)}) \in B\} \text{ (by Lemma 3).}$$

Since $\eta^{(N)}$ can take on only countably many values, $P\{\xi_t(\eta^{(N)}) \in B\}$ is clearly a $\mathcal{G}$-measurable function of η. Therefore, for any fixed $B \in \mathcal{G}_n$, $\eta \mapsto K_t(\eta, B)$ is $\mathcal{G}$-measurable. Standard momotone class arguments show then that this remains valid for all $B \in \mathcal{G}$. The main fact now is that K_t defines a semigroup, as shown in the next lemma.

Lemma 5. *If $\eta \in \Xi$ and $B \in \mathcal{G}$, then*

$$K_{t+s}(\eta, B) = \int_\Xi K_s(\eta, d\xi) K_t(\xi, B). \tag{2.37}$$

Proof. Both sides of (2.37) are probability measures as functions of B. Therefore, by the $\pi - \lambda$ theorem (see Billingsley (1986), Theorem 3.2) it suffices to show that (2.37) holds for $B \in \cup_n \mathcal{G}_n$, that is, for B of the form $\{\xi : \{\xi(x)\}_{|x| \le n} \in C\}$. It even suffices to take B of the form

$$\{\xi : \xi(x) \ge a(x) \text{ for } |x| \le n\} \tag{2.38}$$

with $a(x) \in \mathbb{R}$. For such a B the left hand side of (2.37) equals

$$\lim_{N \to \infty} P\{\xi_{t+s}(\eta^{(N)}) \in B\}.$$

By the Markov property for the process $\{\xi_t(\eta^{(N)})\}_{t \ge 0}$ we have

$$P\{\xi_{t+s}(\eta^{(N)}) \in B\} = \int_\Xi P\{\xi_s(\eta^{(N)}) \in d\lambda\} P\{\xi_t(\lambda) \in B\}. \tag{2.39}$$

Now the right hand side here can be written as $Ef(\xi_s(\eta^{(N)}))$, where $f(\lambda) = P\{\xi_t(\lambda) \in B\}$, and the expectation is over the system of Poisson processes $\mathcal{N}(x,k)$ and the attached $y_n(x,k), U(n,x,k)$. When these random elements are fixed (outside an exceptional set of probability zero), then for each x, $\xi_s(\eta^{(N)})(x) = \xi_s(\eta)(x)$, for all large N (by Lemma 3). In particular, for each fixed M, and large N

$$\big(\xi_s(\eta)\big)^{(M)}(y) = \xi_s(\eta)(y) I[|y| \le M] \le \xi_s(\eta^{(N)})(y) \le \xi_s(\eta)(y) \text{ for all } y. \tag{2.40}$$

But since B depends on finitely many coordinates only, we have from Lemma 3 that

$$f(\lambda^{(M)}) \to f(\lambda) \text{ as } M \to \infty \text{ for } \lambda \in \Xi. \tag{2.41}$$

Moreover for B of the special form (2.38), $f(\lambda)$ is increasing in λ. Thus (2.40) and (2.41) imply that

$$Ef(\xi_s(\eta)) = \lim_{M\to\infty} Ef((\xi_s(\eta))^{(M)}) \le \lim_{N\to\infty} Ef(\xi_s(\eta^{(N)})) \le Ef(\xi_s(\eta)).$$

In other words, the limit as $N \to \infty$ of the right hand side of (2.39) equals

$$\begin{aligned} Ef(\xi_s(\eta)) &= EP\{\xi_t(\xi_s(\eta)) \in B\} \\ &= \int_\Xi P\{\xi_s(\eta) \in d\lambda\} P\{\xi_t(\lambda) \in B\} \\ &= \int K_s(\eta, d\lambda) K_t(\lambda, B). \end{aligned}$$

■

The preceding lemma shows that $\{\xi_t\}_{t\ge 0}$ has the Markov property. In order to show that this process corresponds to the description given before the Theorem in Section 1 we also show that its semigroup has the 'correct' generator, at least when applied to functions with sufficient continuity. Formally the description corresponds to the generator

$$\begin{aligned} \Omega f(\eta) = \sum_x \eta(x) \sum_y q(y-x) \big\{ & p_{\eta(y)}[f(\eta - e_x) - f(\eta)] \\ & + (1 - p_{\eta(y)})[f(\eta + e_y - e_x) - f(\eta)] \big\}, \end{aligned} \tag{2.42}$$

where e_x is the vector with $e_x(y) = 1$ if $y = x$ and 0 otherwise (here we interpret Ξ_0 as a vectorspace in the obvious way). We shall define $\Omega f(\eta)$ by (2.42) whenever

$$\sum_x \eta(x) \sum_y q(y-x) \big\{ p_{\eta(y)} \big| f(\eta - e_x) - f(\eta) \big| + (1 - p_{\eta(y)}) \big| f(\eta + e_y - e_x) - f(\eta) \big| \big\}$$

converges. We now indicate how to prove a proposition which is an analogue of Theorem IX.1.14 in Liggett (1985). For a bounded $\mathcal{G}$-measurable function $f : \Xi \to \mathbb{R}$ we define

$$K_t f(\eta) = \int_\Xi K_t(\eta, d\xi) f(\xi), \quad \eta \in \Xi.$$

It is convenient to formulate the proposition using a slightly larger class than the cylinder functions, because K_t does not preserve cylinder functions, that is $K_t f$

does not have to be a cylinder function even if f is. The class $\mathcal{L}$ with which we shall work is more or less a class of Lipschitz functions. To define this class we set for $f : \Xi \to \mathbb{R}$,

$$L_t(f) := \sup_x \sup_{\eta \in \Xi} \frac{|f(\eta + e_x) - f(\eta)|}{\alpha_t(x)}.$$

Then we define

$$\mathcal{L} = \{f : f \text{ is bounded, } \mathcal{G}\text{-measurable and } L_t(f) < \infty \text{ for some } t > 0\}.$$

It is not hard to check that $\mathcal{L}$ contains all bounded cylinder functions. The following simple lemma shows that K_t does preserve $\mathcal{L}$.

Lemma 6. *Let $s_0 > 0$ and $f \in \mathcal{L}$ such that $L_{s_0}(f) < \infty$ and let $t \geq 0$. Then the following hold:*
(a)

$$K_t f(\eta) = \lim_{N \to \infty} E f(\xi_t(\eta^{(N)})), \quad \eta \in \Xi; \tag{2.43}$$

(b) If (1.7) holds, then

$$|K_t f(\eta + e_z) - K_t f(\eta)| \leq L_{s_0}(f) \alpha_{t+s_0}(z), \quad z \in \mathbb{Z}^d, \eta \in \Xi; \tag{2.44}$$

(c) If (1.7) holds and $u \geq t + s_0$, then

$$L_u(K_t f) \leq L_{s_0}(f) e^{u-t-s_0}. \tag{2.45}$$

Proof. (a) By definition of $L_{s_0}(f)$,

$$\begin{aligned} &|K_t f(\eta) - E f(\xi_t(\eta^{(N)})| \\ &= \big|E[f(\xi_t(\eta)) - f(\xi_t(\eta^{(N)}))]\big| \\ &\leq L_{s_0}(f) \sum_x E|\xi_t(\eta)(x) - \xi_t(\eta^{(N)})(x)| \alpha_{s_0}(x). \end{aligned} \tag{2.46}$$

Also by Lemma 2, $|\xi_t(\eta)(x) - \xi_t(\eta^{(N)})(x)| \to 0$ as $N \to \infty$. Moreover, by the monotonicity in (2.24) this difference is bounded by $\xi_t(\eta)(x)$. (2.43) now follows from the dominated convergence theorem applied to the right hand side of (2.46).

(b) Again as in (2.46), the left hand side of (2.44) is bounded by

$$L_{s_0}(f) \sum_x E|\xi_t(\eta + e_z)(x) - \xi_t(\eta)(x)| \alpha_{s_0}(x). \tag{2.47}$$

Now, if (1.7) holds we can use the right hand inequality of (2.14) to deduce that

$$|\xi_t(\eta + e_z)(x) - \xi_t(\eta)(x)| \leq \overline{\xi}_t(x),$$

where $\overline{\xi}$ is a process which starts with a single particle at z and which does not interact with any particle. Consequently,

$$E|\xi_t(\eta + e_z)(x) - \xi_t(\eta)(x)| \le E\,\overline{\xi}_t(x) = P\{z + S_t = x\},$$

and (2.47) is at most

$$L_{s_0}(f) \sum_x P\{S_{s_0} = -x\} P\{z + S_t = x\} = L_{s_0}(f)\alpha_{t+s_0}(z).$$

(c) The estimate (2.45) now follows from (2.44) plus

$$\alpha_u(x) \ge e^{(-u+t+s_0)}\alpha_{t+s}(x) \text{ (see (2.33)).} \qquad \blacksquare$$

Proposition 7. *Assume that (1.7) holds and that $f : \Xi \mapsto \mathbb{R}$ is $\mathcal{G}$-measurable and bounded on Ξ. Then*

(a) $$K_{t+s}f = K_t(K_s f).$$

For (b) - (f) assume in addition that $f \in \mathcal{L}$, say $L_{s_0}(f) < \infty$, and that $\eta \in \Xi$. Then

(b) $\Omega(K_t f)(\eta)$ is well defined and

$$K_t f(\eta) = f(\eta) + \int_0^t \Omega(K_s f)(\eta) ds;$$

(c) $$|K_t f(\eta) - f(\eta)| \le te^t L_{s_0}(f)[N_{t+s_0}(\eta) + 2eN_{t+s_0+1}(\eta)];$$

(d) $$\lim_{s\downarrow 0} \Omega(K_s f)(\eta) = \Omega f(\eta);$$

(e) $$\lim_{t\downarrow 0} \frac{K_t f(\eta) - f(\eta)}{t} = \Omega f(\eta)$$

and the fraction in the left hand side is bounded for $t \le 1, N_{t+s_0+1}(\eta) \le A$ for any fixed $A < \infty$;

(f) $$\Omega(K_t f)(\eta) = K_t(\Omega f)(\eta) = E(\Omega f)(\xi_t(\eta)).$$

Proof. (a) is immediate from Lemma 5. For (b) we use the Markov property for the ξ-process starting in a finite state (and which consequently has a bounded number of particles at all times). This gives (compare Dynkin (1965), Vol I, equations I.2.1.4 and I.2.1.5)

$$Ef(\xi_t(\eta^{(N)})) = f(\eta^{(N)}) + \int_0^t \Omega(Ef(\xi_s(\eta^{(N)})))ds, \tag{2.48}$$

where $\Omega(Ef(\xi_s(\eta^{(N)})))$ stands for $\Omega(Ef(\xi_s(\cdot)))$ evaluated at $\eta^{(N)}$. Note that

$$\begin{aligned}
&\Omega(Ef(\xi_s(\eta^{(N)}))) \\
&= \sum_{|x|\le N} \eta(x) \sum_y q(y-x)\big\{p_{\eta^{(N)}(y)}[Ef(\xi_s(\eta^{(N)}-e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\qquad\qquad + (1-p_{\eta^{(N)}(y)})[Ef(\xi_s(\eta^{(N)}+e_y-e_x)) - Ef(\xi_s(\eta^{(N)}))]\big\}
\end{aligned} \tag{2.49}$$

is only a finite sum over x. This sum converges without any smoothness assumptions on f. We now want to take the limit $N \to \infty$ in (2.48). If we use that $L_{s_0}(f) < \infty$ then by Lemma 6 the left hand side of (2.48) converges to $Ef(\xi_t(\eta))$. Similarly, for each fixed x, y

$$\begin{aligned}
&p_{\eta^{(N)}(y)}[Ef(\xi_s(\eta^{(N)}-e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\qquad\qquad + (1-p_{\eta^{(N)}(y)})[Ef(\xi_s(\eta^{(N)}+e_y-e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\to p_{\eta(y)}[Ef(\xi_s(\eta-e_x)) - Ef(\xi_s(\eta))] + (1-p_{\eta(y)})[Ef(\xi_s(\eta+e_y-e_x)) - Ef(\xi_s(\eta))].
\end{aligned}$$

From (2.44) and (2.33) we further have the following bound for (2.49) (when $s \le t$):

$$\begin{aligned}
&\sum_x \eta(x) \sum_y q(y-x) L_{s_0}(f)\big\{\alpha_{s+s_0}(x) + \alpha_{s+s_0}(y)\big\} \\
&\le L_{s_0}(f) e^t \big\{N_{t+s_0}(\eta) + \sum_x \eta(x) \sum_y q(y-x)\alpha_{t+s_0}(y)\big\}.
\end{aligned} \tag{2.50}$$

Furthermore,

$$\begin{aligned}
&P\{x + S_u = \mathbf{0} \text{ for some } u \in [t+s_0, t+s_0+1]\} \\
&\ge \sum_y P\{S_{t+s_0} = -y\} P\{S_u = y - x \text{ for some } 0 \le u \le 1\} \\
&\ge \sum_y \alpha_{t+s_0}(y) P\{\text{first jump of } S_\cdot \text{ occurs during } [0,1] \text{ and is from } \mathbf{0} \text{ to } y-x\} \\
&= \sum_y \alpha_{t+s_0}(y)(1-e^{-1}) q(y-x) \ge \frac{1}{2} \sum_y \alpha_{t+s_0}(y) q(y-x).
\end{aligned} \tag{2.51}$$

Consequently

$$\begin{aligned}
&\sum_y q(y-x)\alpha_{t+s_0}(y) \\
&\le 2P\{x + S_u = \mathbf{0} \text{ for some } u \in [t+s_0, t+s_0+1]\} \\
&\le 2e\, P\{x + S_{t+s_0+1} = \mathbf{0}\} \le 2e\alpha_{t+s_0+1}(x).
\end{aligned} \tag{2.52}$$

Substituting this estimate into (2.50) we find that (2.49) is at most

$$L_{s_0}(f)e^t\big\{N_{t+s_0}(\eta) + 2eN_{t+s_0+1}(\eta)\big\}.$$

Essentially the same estimates as used to bound (2.49) show that the formal series for $\Omega(K_s f)$ converges and that

$$|\Omega(K_s f)(\eta)| \le L_{s_0}(f)e^t\big\{N_{t+s_0}(\eta) + 2eN_{t+s_0+1}(\eta)\big\}. \tag{2.53}$$

With these bounds and the dominated convergence theorem it is easy to justify that

$$\lim_{N\to\infty}\int_0^t \Omega(Ef(\xi_s(\eta^{(N)})))ds = \int_0^t \Omega(Ef(\xi_s(\eta)))ds.$$

This proves (b).

(c) follows from (b) and (2.53).

We obtain (d) by taking the limit $s \downarrow 0$ in the explicit expression for $\Omega(K_s f)(\eta)$, which is given by (2.49) with $\eta^{(N)}$ replaced by η and the sum over x extended over all x. The estimates (2.44) and (2.50)-(2.52) and the dominated convergence theorem justify taking the limit $s \downarrow 0$ inside the double sum over x, y.

(e) is immediate from (b), (d) and (2.53).

Finally, (f) is proven in essentially the same way as part (g) in Theorem IX.1.14 in Liggett (1985). ■

3. A variance estimate. Throughout this section we take the initial state to be $\xi_0 = \mathbb{1}$, that is

$$\xi_0(x) = 1, \quad x \in \mathbb{Z}^d,$$

although the argument works for any ξ_0 with $\xi_0(x)$ bounded. We also use for the first time the hypothesis

$$\sum_{x\in\mathbb{Z}^d} \|x\|^2 q(x) < \infty. \tag{3.1}$$

To simplify notation somewhat, we write just ξ_t for $\xi_t(\mathbb{1})$ and $\xi_{N,t}$ for $\xi_t(\mathbb{1}^{(N)})$. The following estimate is the basic result of this section.

Proposition 8. *Assume (1.7) and (3.1). Then there exists a constant C_0, which is independent of β, K, t and the p_j, such that for $\beta(x) \in \mathbb{R}$ and $K < \infty$ it holds that*

$$\mathrm{Var}\Big\{\sum_{|x|\le K} \beta(x)\xi_t(x)\Big\} \le C_0 \log(t+2) \sum_{x\in\mathbb{Z}^d} \beta^2(x). \tag{3.2}$$

If

$$\sum_{x\in\mathbb{Z}^d} |\beta(x)| E\xi_t(x) < \infty, \tag{3.3}$$

then also

$$\text{Var}\Big\{\sum_{x\in\mathbb{Z}^d}\beta(x)\xi_t(x)\Big\} \le C_0\log(t+2)\sum_{x\in\mathbb{Z}^d}\beta^2(x). \tag{3.4}$$

Remark (v) The estimate (3.4) can, by quite a lot of extra work, be improved to

$$\text{Var}\Big\{\sum_{x\in\mathbb{Z}^d}\beta(x)\xi_t(x)\Big\} \le C_0t^{-1/4}\log(t+2)\sum_{x\in\mathbb{Z}^d}\beta^2(x). \tag{3.5}$$

If this improved estimate is used throughout Section 4, then one obtains that (1.9) remains valid even in $d = 5$. This improvement is obtained by directly comparing the ξ' and the ξ''-processes, rather than comparing each one separately with the $\widetilde{\xi}$-process (these processes are introduced a little before (3.17) below). As we already stated in Remark (ii) one can even prove (1.9) for d = 4 if one assumes that $p_M = 1$ for some M. To deal with the special case where $p_M = 1$, one needs not only (3.5), but also an improved version of Lemma 12 which shows that if $p_M = 1$ for some M, then

$$E\Lambda_t(u_1,\dots,u_p) \le C_6(p)t^{-p}. \tag{3.6}$$

(Λ_t is defined in (4.18)). In turn, (3.6) is obtained by comparing the process with $p_M = 1$ with a process which has p_j replaced by $p'_j = (j/M') \wedge 1$ for some large M' so that $p'_j \le p_j$ for all j. It can be shown that the process with parameters p'_j satisfies the analogue of Lemma 1 of Arratia (1981), to wit

$$P\{\xi_t(x_i)\ge m_i, 1\le i\le r\} \le \prod_{i=1}^r P\{\xi_t(x_i)\ge m_i\}, \quad x_i\in\mathbb{Z}^d, m_i\ge 1, r\ge 1. \tag{3.7}$$

For such processes our proof even works for $d \ge 3$. (Note that the model with $M' = 1$ is the basic model mentioned in the beginning of this paper.)

We hope to discuss the somewhat lengthy proofs of these improvements elsewhere.

Before we can start on the proof proper of this Proposition we need an apriori estimate for

$$E(t) := E\xi_t(x) \tag{3.8}$$

(this is independent of x).

Lemma 9. *Assume (1.6) and (3.1). Then, for* $d \ge 3$, *there exist constants* $0 < C_1 \le C_2 < \infty$ *such that*

$$\frac{C_1}{t} \le E(t) \le \frac{C_2}{t}, \quad t \ge 1. \tag{3.9}$$

Proof. These estimates basically come from Arratia (1983) and Bramson and Griffeath (1980). By Lemma 2

$$E(t) \geq E^*(t) := E\,\xi_t^*(\mathbf{0}),$$

where ξ_t^* is the process with removal probabilities p_j^*, given by (2.23) (and initial state $\mathbb{1}$). This ξ^*-process is the basic coalescing random walk model, except that $S_.$ does not have to be a simple random walk. We can therefore not simply use (1.1). However, by Lemma 1 of Arratia (1983) one has for $S_.$ an arbitrary random walk,

$$E^*(t) \geq \frac{C_1}{t}. \tag{3.10}$$

Thus the left hand inequality of (3.9) holds.

The right hand inequality of (3.9) is proven in exactly the same way as the case $d \geq 3$ of Theorem 1 of Bramson and Griffeath (1980), but we nevertheless need three comments about this. The first, rather trivial comment is that for the inequality three lines below (2.5) in Bramson and Griffeath we need the right hand inequality of (2.14), or better yet, (2.20). The second comment concerns Lemma 3 of Bramson and Griffeath. Their proof is based on the fact that in the basic model, when $p_j = p_j^*$ (see (2.23)) one has for any finite initial state ξ_0 that

$$\sum_{x\in\mathbb{Z}^d} \xi_0(x) - E\Big\{\sum_{x\in\mathbb{Z}^d} \xi_s(\xi_0)(x)\Big\} \geq \Big[\sum_{x\in\mathbb{Z}^d} \xi_0(x) - 1\Big] \min_{\xi_0(u),\xi_0(v)>0} H_s(u-v), \tag{3.11}$$

where

$$H_s(z) = P\{S_t^\sigma = z \text{ for some } t \leq s\}$$

and $\{S_t^\sigma\}$ is as in the Theorem of Section 1. The min of H_s is taken over all u, v with $\xi_0(u) > 0, \xi_0(v) > 0$. We need the analogue of (3.11) (with a factor p_1 in the right hand side) for general p_j satisfying (1.6) and (1.7), not just for $p_j = p_j^*$.

In order to show that (3.11) remains valid for such p_j we have to use another construction for ξ_t than the one used in Section 2. In this construction we distinguish the different particles and keep track of the position of the individual particles, not merely of the number of particles at each site. For the present purposes it is also better to let a particle coalesce with another particle after a jump, rather than removing it. At time 0 we label the particles at any given site x as (x, k) with $1 \leq k \leq \xi_0(x)$ (in some arbitrary ordering of the particles at x). We further pick for each such particle a random walk path $\{S_t^{(x,k)}\}_{t\geq 0}$. The $\{S_t^{(x,k)}\}_{t\geq 0}$ are i.i.d., each with the distribution of $\{S_t\}_{t\geq 0}$. We further attach to each particle (x, k) further random variables $\{U_n^{(x,k)}, V_{n,j}^{(x,k)}, j \geq 1, n \geq 1\}$. Random variables with different values of (x, k) or n are independent. Also, for fixed (x, k), all $U_n^{(x,k)}$ are independent of all $V_{m,j}^{(x,k)}$. All the $U_n^{(x,k)}$ are uniform on $[0,1]$ and each $V_{n,j}^{(x,k)}$ takes values in $\{1,\dots,j\}$ with

$$P\{V_{n,j}^{(x,k)} = \ell\} = \frac{1}{j}, \quad 1 \le \ell \le j.$$

Now the particle labeled (x,k) moves along the path $t \mapsto x + S_t^{(x,k)}$ until it first jumps to a site, y say, which already contains another particle. At such a jump the (x,k)-particle may coalesce with one of the particles present at the site y. Whether the (x,k)-particle does coalesce, and with which particle, is a function of the $\{U_n^{(x,k)}, V_{n,j}^{(x,k)}\}$. Suppose that the (x,k)-particle did not coalesce with another particle at one of the first $n-1$ jumps of $S_\cdot^{(x,k)}$ and that at its n-th jump this particle jumps to y. Suppose at that time there are j particles at y. Number these particles in some order, say in the order of their arrival times at y. Then the (x,k)-particle coalesces with one of the j particles at y if and only if $U_{n,j}^{(x,k)} \le p_j$. If this is the case, then it coalesces with the particle with the number $V_{n,j}^{(x,k)}$. After this coalescing event the (x,k)-particle no longer follows the path $t \mapsto S_t^{(x,k)}$, but follows the path of the particle with which it coalesced. Note that it is always the variables associated with the particle which just jumped which determine whether coalescence takes place. It is also the particle which just jumped which 'gives up' its own trajectory and starts following the trajectory of the particle with which it coalesced.

If we start with finitely many particles, then the construction of the preceding paragraph assigns with probability 1 a unique trajectory to each particle. If the (x,k)-particle and the (y,ℓ)-particle have coalesced, then they both move according to one of the trajectories $t \mapsto z + S_t^{(z,m)}$; (z,m) may be (x,k) or (y,ℓ) or yet another particle with which both the (x,k)-particle and the (y,ℓ)-particle have coalesced. This allows us to define $\xi_t(x)$ again as the number of particles present at x at time t.

We shall not prove that the preceding construction is equivalent to the one of Section 2, in the sense that the joint distribution of the $\{\xi_t(x)\}_{t\ge 0}, x \in \mathbb{Z}^d$, is the same under both constructions (we need this only for finite initial states). It is further left to the reader to verify that the proof of Bramson and Griffeath's lemma 3 for (3.11) (with an extra factor p_1 in the right hand side) goes through for the newly constructed $\{\xi_t\}$. But if (3.11) holds for one of the constructions of $\{\xi_t\}$, then it holds for all constructions, since (3.11) depends only on the joint distribution of the $\{\xi_t(x)\}_{t\ge 0}, x \in \mathbb{Z}^d$.

Our final comment concerns the lower bound for $\inf_{\|z\|\le r} H_{r^2}(z)$ which Bramson and Griffeath (1980) derive in their Lemma 5 when $S_\cdot$ is simple random walk. This lemma remains valid under condition (3.1) only, because as Bramson and Griffeath argue, one merely needs a lower bound (of size $C_1 r^{2-d}$) on

$$\inf_{\|z\|\le r}\int_0^{r^2} P\{S_s^\sigma = z\}ds$$
$$\ge \int_{r^2/2}^{r^2} ds \sum_{k=r^2/4}^{2r^2} P\{S_\cdot^\sigma \text{ has } k \text{ jumps during } [0,s]\} \inf_{\|z\|\le r} q_\sigma^{*k}(z),$$

where $q_\sigma(z) = [q(z)+q(-z)]/2 = P\{S_\cdot^\sigma$ jumps from $\mathbf{0}$ to z at its first jump$\}$. The required lower bound follows directly from the local central limit theorem (see Spitzer (1976), Proposition 7.9).

In all other respects the proof of the right hand inequality in (3.9) follows Bramson and Griffeath (1980). ■

Proof of Proposition 8. First choose a $K < \infty$ and let

$$Z = \sum_{|x|\le K} \beta(x)\xi_t(x),$$
$$Z_N = \sum_{|x|\le K} \beta(x)\xi_{N,t}(x).$$

(Recall that $\xi_{N,t}$ is the state at time t if we start with $\xi_0(y)I[|y|\le N] = I[|y|\le N]$ particles at y.) Now

$$EZ_N = \sum_{|x|\le K} \beta(x)E\xi_{N,t}(x)$$

and, as $N \to \infty$,

$$E\xi_{N,t}(x) \uparrow E\xi_t(x) \le \sum_y P\{y + S_t = x\} = 1$$

by Lemma 3, the monotone convergence theorem and (2.31). Hence

$$EZ_N \to EZ \quad (N\to\infty). \tag{3.12}$$

By Fatou's lemma we then get

$$\mathrm{Var}(Z) = EZ^2 - (EZ)^2 \le \liminf_{N\to\infty} \mathrm{Var}(Z_N). \tag{3.13}$$

It therefore suffices for (3.2) to prove

$$\mathrm{Var}\big(\sum_{|x|\le K} \beta(x)\xi_{N,t}(x)\big) \le C_0 \log(t+2) \sum_x \beta^2(x). \tag{3.14}$$

Now let $\mathcal{F}_s$ be as in (2.18) and define

$$\Delta_\ell = \Delta_\ell(p) = \Delta_\ell(p,N,t) = E\{Z_N|\mathcal{F}_{\ell t/p}\} - E\{Z_N|\mathcal{F}_{(\ell-1)t/p}\}.$$

Then for each integer $p \geq 1$

$$Z_N - EZ_N = \sum_1^p \Delta_\ell$$

and

$$\mathrm{Var}(Z_N) = \sum_1^p E\Delta_\ell^2(p) = \liminf_{p\to\infty} \sum_1^p E\Delta_\ell^2(p) = \liminf_{p\to\infty} \sum_1^p E\big\{E\{\Delta_\ell^2(p)|\mathcal{F}_{(\ell-1)t/p}\}\big\}.$$

We fix N and write $W_\ell = W_\ell(p,N)$ for the random elements which summarize all the information which becomes available between time $(\ell-1)t/p$ and $\ell t/p$. More precisely, W_ℓ stands for all the increments $\mathcal{N}_u(x,k) - \mathcal{N}_{(\ell-1)t/p}(x,k)$ of the Poisson processes with $(\ell-1)t/p < u \leq \ell t/p$, and the $y_n(x,k), U(n,x,k)$ associated to jump times during $((\ell-1)t/p, \ell t/p]$ of any of these processes. We skip the tedious explicit construction of a probability space on which these random variables are defined. Whatever this probability space for the W is, we shall have

$$\mathcal{F}_{\ell t/p} = \sigma\{W_1,\dots,W_\ell\}$$

and the W_ℓ for different ℓ are independent. Also, W_ℓ has a distribution which we denote by μ_ℓ (i.e., $\mu_\ell(dw) = P\{W_\ell \in dw\}$). $Z_N = f(W_1,W_2,\dots,W_p)$ for a suitably measurable function $f = f_N$ and therefore

$$\begin{aligned} &E\{Z_N|\mathcal{F}_{\ell t/p}\} \\ &= \int \prod_{i=\ell+1}^p \mu_i(dw_i) f(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p) \\ &= \int \prod_{i=\ell}^p \mu_i(dw_i) f(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p). \end{aligned}$$

Note that the last member also includes an integration with respect to $\mu_\ell(dw_\ell)$; this integration can be added because the integrand does not depend on w_ℓ. Therefore

$$\begin{aligned} \Delta_\ell = \int \prod_{i=\ell}^p \mu_i(dw_i)\big[&f(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p) \\ &- f(W_1,\dots,W_{\ell-1},w_\ell,w_{\ell+1},\dots,w_p)\big]. \end{aligned} \tag{3.15}$$

Note that Δ_ℓ is a function of $W_1, \dots, W_\ell$, and that therefore

$$E\{\Delta_\ell^2 | \mathcal{F}_{(\ell-1)t/p}\} = \int \mu_\ell(dW_\ell)\Delta_\ell^2$$

and

$$E\Delta_\ell^2 = \int \prod_{j\le\ell} \mu_j(dW_j)\Delta_\ell^2.$$

By Schwarz' inequality applied to (3.15) we find

$$\Delta_\ell^2 \le \int \prod_{i=\ell}^{p} \mu_i(dw_i)\big[f(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p) - f(W_1,\dots,W_{\ell-1},w_\ell,\dots,w_p)\big]^2,$$

and we now turn to an estimate for

$$\big[f(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p) - f(W_1,\dots,W_{\ell-1},w_\ell,\dots,w_p)\big]^2. \tag{3.16}$$

The expression in square brackets here is the change in Z_N due to the change from w_ℓ to W_ℓ in the time interval $((\ell-1)t/p, \ell t/p]$, while keeping all other random elements in $[0,(\ell-1)t/p]$ fixed at $W_1,\dots,W_{\ell-1}$ and the random elements in $(\ell t/p, t]$ fixed at $w_{\ell+1},\dots,w_p$. We shall use that at all times the number of particles present in the $\xi(\mathbb{1}^{(N)})$-process is at most

$$\sum_x \xi_{N,0}(x) = \sum_{|x|\le N} 1 = (2N+1)^d.$$

The location of these particles at time $(\ell-1)t/p$ is determined by $W_1,\dots,W_{\ell-1}$ and is therefore $\mathcal{F}_{(\ell-1)t/p}$-measurable. We shall write $I_\ell[\ge k \text{ jumps}]$ for the indicator function of the event that the particles present at time $(\ell-1)t/p$ have at least k jumps during $((\ell-1)t/p, \ell t/p]$. (Repeated jumps by the same particle are counted as different jumps; we anyway only keep track of the ξ's so do not know which particle jumps.) $I_\ell[1 \text{ jump}]$ and $I_\ell[\text{no jump}]$ have similar self-evident definitions. If $I_\ell[\ge 2 \text{ jumps}](W_1,\dots,W_{\ell-1},w_\ell,\dots,w_p) = 1$ or if $I_\ell[\ge 2 \text{ jumps}](W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p) = 1$, then we simply estimate (3.16) by

$$\begin{aligned}&[2\sup|Z_N|]^2\\ &\le \Big[2\sup_{|x|\le N}|\beta(x)|\sum_x \xi_0(x)\Big]^2\\ &= 4(2N+1)^{2d}\Big[\sup_{|x|\le N}|\beta(x)|\Big]^2.\end{aligned} \tag{3.17}$$

The same bound applies when there is at least one jump in *both* the configurations $W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p$ and $W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p$. We shall soon see that the contributions to $\sum E\Delta_\ell^2$ in all these configurations go to 0 as $p \to \infty$. When in both configurations $W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p$ and $W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p$ no particle at all jumps during $((\ell-1)t/p, \ell t/p]$, then $W_\ell = w_\ell$ and (3.16) equals 0. Therefore (3.16) is at most equal to the sum of the following three terms:

$$\begin{aligned} &12(2N+1)^{2d}\Big[\sup_{|x|\le N}|\beta(x)|\Big]^2 \\ &\times \Big[I_\ell[\ge 2 \text{ jumps}](W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p) \\ &\qquad\qquad + I_\ell[\ge 2 \text{ jumps}](W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p) \\ &\qquad + I_\ell[\ge 1 \text{ jump}](W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p) \\ &\qquad\qquad \times I_\ell[\ge 1 \text{ jump}](W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p)\Big]^2; \end{aligned} \tag{3.18}$$

$$\begin{aligned} &3\big[f(W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p) - f(W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p)\big]^2 \\ &\quad \times I[1 \text{ jump}](W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p) \cdot I[\text{no jump}](W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p) \end{aligned} \tag{3.19}$$

and (3.19) with W_ℓ and w_ℓ interchanged.

We first show that the contribution of (3.18) to $\sum E\Delta_\ell^2$ becomes negligeable as $p \to \infty$. The square of the sum of the indicator functions between square brackets in (3.18) is at most

$$\begin{aligned} &3I_\ell[\ge 2 \text{ jumps}](W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p) \\ &+ 3I_\ell[\ge 2 \text{ jumps}](W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p) \\ &+ 3I_\ell[\ge 1 \text{ jump}](W_1, \ldots, W_{\ell-1}, w_\ell, \ldots, w_p) \\ &\qquad\qquad \times I_\ell[\ge 1 \text{ jump}](W_1, \ldots, W_\ell, w_{\ell+1}, \ldots, w_p). \end{aligned}$$

We only estimate the contribution of the last term here. The other terms can be estimated in the same way (but are actually easier to treat). Note that $I_\ell[\ge 1 \text{ jump}](W_1, \ldots W_{\ell-1}, w_\ell, \ldots, w_p)$ depends on $W_1, \ldots, W_{\ell-1}$ and w_ℓ only, while $I_\ell[\ge$

1 jump]$(W_1,\dots W_\ell,w_{\ell+1},\dots,w_p)$ depends on $W_1,\dots,W_\ell$ only. Therefore

$$\begin{aligned}
&\int\mu_\ell(dW_\ell)\int\mu_\ell(dw_\ell)\int\prod_{i=\ell+1}^{p}\mu_i(dw_i)I_\ell[\geq 1\text{ jump}](W_1,\dots W_{\ell-1},w_\ell,\dots,w_p)\\
&\qquad\qquad\qquad\times I_\ell[\geq 1\text{ jump}](W_1,\dots W_\ell,w_{\ell+1},\dots,w_p)\\
&=\int\mu_\ell(dW_\ell)I_\ell[\geq 1\text{ jump}](W_1,\dots W_\ell,w_{\ell+1},\dots,w_p)\\
&\qquad\qquad\qquad\times\int\mu_\ell(dw_\ell)I_\ell[\geq 1\text{ jump}](W_1,\dots W_{\ell-1},w_\ell,\dots,w_p)\\
&=\Big[P\{\text{at least one jump occurs during }((\ell-1)t/p,\ell t/p]\big|W_1,\dots,W_{\ell-1}\}\Big]^2\\
&\leq\Big[\sum_x\xi_{(\ell-1)t/p}(\mathbb{1}^{(N)})(x)\frac{t}{p}\Big]^2\\
&\leq\frac{t^2}{p^2}(2N+1)^{2d}.
\end{aligned}$$

If we finally integrate the left hand side also with respect to $\prod_1^{\ell-1}\mu(dW_j)$ and then sum over ℓ from 1 to p we find a contribution to $\sum_1^p E\Delta_\ell^2$ of at most

$$\frac{t^2}{p}(2N+1)^{2d},$$

and this tends to 0 as $p\to\infty$. Thus the contribution of (3.18) can be ignored.

Because of the symmetry between W_ℓ and w_ℓ in our estimates, (3.19) and the term with W_ℓ and w_ℓ interchanged gives the same contribution. We therefore only have to estimate (3.19). To this end let us write ξ_t' for the $\xi(\mathbb{1}^{(N)})$-process in the configuration $W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p$ and ξ_t'' for the $\xi(\mathbb{1}^{(N)})$-process in the configuration $W_1,\dots,W_{\ell-1},w_\ell,\dots,w_p$. Through time $(\ell-1)t/p$ these two processes agree, so that

$$\xi'_{(\ell-1)t/p}=\xi''_{(\ell-1)t/p}.$$

Suppose there is exactly one jump during $((\ell-1)t/p,\ell t/p]$ in the configuration $W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p$, that is, in the ξ'-process. Let this jump be from x' to y'. Assume further that there is no jump in the configuration $W_1,\dots,W_{\ell-1},w_\ell,\dots,w_p$. Then

$$\begin{aligned}
\xi''_{\ell t/p}(x)&=\xi''_{(\ell-1)t/p}(x)=\xi'_{(\ell-1)t/p}(x)\text{ for all }x,\\
\xi''_{\ell t/p}(x)&=\xi'_{\ell t/p}(x)\text{ if }x\neq x',y',\\
\xi'_{\ell t/p}(x')&=\xi'_{(\ell-1)t/p}(x')-1,\\
\xi'_{\ell t/p}(y')&=\xi'_{(\ell-1)t/p}(y')\text{ or }\xi'_{(\ell-1)t/p}(y')+1.
\end{aligned}$$

In any case, $\xi'_{\ell t/p}$ and $\xi''_{\ell t/p}$ differ at most on the two sites x', y' and there they differ by at most 1. Rather than compare ξ'_t directly with ξ''_t, we compare each of them with a third process $\widetilde{\xi}_t$ which we define as the process which behaves like ξ' except that the particle which jumps from x' to y' during $((\ell-1)t/p, \ell t/p]$ is removed immediately after the jump in the $\widetilde{\xi}$-process. After time $\ell t/p$ it develops by the prescribed rules in the configurations $w_{\ell+1}, \dots, w_p$. Of course it may be that $\widetilde{\xi} \equiv \xi'$, namely if the particle which jumps from x' to y' is also removed in the ξ'-process. If this particle is not removed in the ξ'-process, then the ξ'-process has one particle more than the $\widetilde{\xi}$-process at time $\ell t/p$, and this extra particle is located at y'. Therefore, by Lemma 1

$$\widetilde{\xi}_t(x) \le \xi'_t(x) \le \widetilde{\xi}_t(x) + \overline{\xi}_t(y')(x), \tag{3.20}$$

where $\overline{\xi}(y')$ is a process which starts with a single particle at y' at time $\ell t/p$ which moves according to the random walk but does not interact with anything. This process is not defined for times $< \ell t/p$. However, $\widetilde{\xi}_\cdot$ and $\overline{\xi}_\cdot(y')$ are coupled and are defined as functions of y' and the Poisson processes $\mathcal{N}_s(x,k), x \in \mathbb{Z}^d, k \ge 1, s \ge \ell t/p$, as well as the $y_n(x,k), U(n,x,k)$ corresponding to jumps at or after time $\ell t/p$, as described for the ξ' and ξ''-processes just before Lemma 1. (Note that the present ξ', ξ'' do *not* have the same meaning as in Lemma 1.) Thus

$$\begin{aligned}\overline{\xi}_t(y')(x) = I[&\text{extra particle in } \xi' \text{ which is at } y' \text{ at } \ell t/p \\ &\text{moves to } x \text{ at time } t].\end{aligned} \tag{3.21}$$

Similarly,

$$\widetilde{\xi}_t(x) \le \xi''_t(x) \le \widetilde{\xi}_t(x) + \overline{\xi}_t(x')(x),$$

where $\overline{\xi}(x')$ is a process which starts with a single particle at x' at time $\ell t/p$ and which does not interact with anything. Therefore, if there is exactly one jump in the ξ'-process and no jump in the ξ''-process, then

$$\big|f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) - f(W_1, \dots, W_{\ell-1}, w_\ell, \dots, w_p)\big|$$

is at most

$$\begin{aligned}&\sum_{|x| \le K} |\beta(x)||\xi'_t(x) - \xi''_t(x)| \\ &\le \sum_{|x|\le K} |\beta(x)| \sum_{x',y'} I_\ell[\text{a single jump from } x' \text{ to } y' \text{ occurs during} \\ &\qquad ((\ell-1)t/p, \ell t/p](W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p)\big[\overline{\xi}_t(y')(x) + \overline{\xi}_t(x')(x)\big].\end{aligned} \tag{3.22}$$

Let us estimate the contribution of the term involving $\overline{\xi}_t(y')$. Note that

$$\Big[\sum_{|x|\le K}|\beta(x)|\sum_{x',y'} I_\ell[\text{a single jump from } x' \text{ to } y' \text{ occurs during}$$

$$((\ell-1)t/p,\ell t/p](W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p)\overline{\xi}_t(y')(x)\Big]^2$$

$$=\sum_{x',y'} I_\ell[\text{a single jump from } x' \text{ to } y' \text{ occurs during}$$

$$((\ell-1)t/p,\ell t/p](W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p)\Big[\sum_{|x|\le K}|\beta(x)|\overline{\xi}_t(y')(x)\Big]^2,$$

because only for one pair x',y' do we have

$$I_\ell[\text{a single jump from } x' \text{ to } y' \text{ occurs during}$$
$$((\ell-1)t/p,\ell t/p](W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p)\ne 0.$$

This yields the following contribution to $E\Delta_\ell^2$:

$$\int\prod_{j\le \ell-1}\mu_j(dW_j)\int\mu_\ell(dW_\ell)\int\mu_\ell(dw_\ell)\int\prod_{i=\ell+1}^{p}\mu_i(dw_i)\sum_{x',y'}$$
$$I_\ell[\text{a jump from } x' \text{ to } y' \text{ occurs during } ((\ell-1)t/p,\ell t/p](W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p)$$
$$\Big[\sum_{|x|\le K}|\beta(x)|\overline{\xi}_t(y')(x)\Big]^2. \tag{3.23}$$

(Note that integrating over w_i, $\ell+1\le i\le p$, in (3.23) includes taking the expectation over $\overline{\xi}_t(y')$, since $\overline{\xi}_t$ is a function of the processes $\mathcal{N}_s(x,k), s\ge \ell t/p$, as described after (3.20).) The same method will work for the term involving $\overline{\xi}_t(x')(x)$ in (3.22). We can handle (3.23) by noting that $\overline{\xi}_t(y')(x)\ne 0$ for exactly one x. Let us denote this position by z_t. Then $\overline{\xi}_t(y')(z_t)=1$ and

$$\Big[\sum_{|x|\le K}|\beta(x)|\overline{\xi}_t(y')(x)\Big]^2=|\beta(z_t)|^2 I[|z_t|\le K].$$

Moreover, conditionally on $\mathcal{F}_{\ell t/p}$, z_t is just the position of a random walk at time t which starts at y' at time $\ell t/p$. Thus

$$\int\prod_{i=\ell+1}^{p}\mu_i(dw_i)\Big[\sum_{|x|\le K}|\beta(x)|\overline{\xi}_t(y')(x)\Big]^2$$
$$\le\sum_z|\beta(z)|^2P\{y'+S_{t-\ell t/p}=z\}.$$

Therefore (by (2.52)) (3.23) is at most

$$\int \prod_{j\le \ell-1} \mu_j(dW_j) \sum_{x',y'} \xi_{(\ell-1)t/p}(x') \frac{t}{p} q(y'-x') \sum_z |\beta(z)|^2 P\{y' + S_{t-\ell t/p} = z\}$$
$$\le \int \prod_{j\le \ell-1} \mu_j(dW_j) \sum_{x'} \xi_{(\ell-1)t/p}(x') \frac{t}{p} \sum_z |\beta(z)|^2 \, 2e\, P\{S_{t-\ell t/p+1} = z - x'\}. \tag{3.24}$$

But, if $(\ell-1)t/p \ge 1$, then by Lemma 9

$$\int \prod_{j\le \ell-1} \mu_j(dW_j) \xi_{(\ell-1)t/p}(x') = E\xi_{(\ell-1)t/p}(x') \le C_2 \frac{p}{(\ell-1)t}.$$

Also, by (2.31), for any $(\ell-1)t/p$, $E\xi_{(\ell-1)t/p}(x') \le \sum_y P\{S_{(\ell-1)t/p} = x' - y\} = 1$. Substituting these estimates into (3.24) shows that (3.23) is at most

$$C_3 \frac{t}{p} \min\{\frac{p}{(\ell-1)t}, 1\} \sum_z |\beta(z)|^2.$$

With a similar estimate for the other term in (3.22) we finally obtain after summing over ℓ the estimate

$$\begin{aligned}
&\liminf_{p\to\infty} \sum_1^p E\Delta_\ell^2 \\
&\le C_3 \sum_z |\beta(z)|^2 t \liminf_{p\to\infty} \frac{1}{p} \Big[\sum_{1\le \ell < p/t+1} 1 + \sum_{p/t+1 \le \ell \le p} \frac{p}{(\ell-1)t} \Big] \\
&\le C_0 \sum_z |\beta(z)|^2 \log(t+2)
\end{aligned}$$

for some constant C_0, which is the desired inequality (3.2).

Once we have (3.2) we can obtain (3.4) under (3.3) exactly as in (3.12),(3.13). Indeed we have

$$E \sum_{|x|\le K} \beta(x)\xi_t(x) \to E \sum_{x\in\mathbb{Z}^d} \beta(x)\xi_t(x) \quad (K\to\infty)$$

and

$$\mathrm{Var}\Big\{ \sum_{x\in\mathbb{Z}^d} \beta(x)\xi_t(x) \Big\} \le \liminf_{K\to\infty} \mathrm{Var}\Big\{ \sum_{|x|\le K} \beta(x)\xi_t(x) \Big\}.$$

■

4. An approximate differential equation for the expected number of particles per site.

Again we start with one particle at each site ($\xi_0 = \mathbb{1}$) and we write ξ_t instead of $\xi_t(\mathbb{1})$. Also $\xi_{N,t}$ stands for $\xi_t(\mathbb{1}^{(N)})$. We define

$$\gamma_t(k) = P\{\xi_t(x) = k\}.$$

γ_t is independent of x. Note that

$$p(t) = \sum_{k=1}^{\infty} \gamma_t(k) = P\{\xi_t(x) > 0\} \tag{4.1}$$

and

$$E(t) = \sum_{k=1}^{\infty} k\gamma_t(k). \tag{4.2}$$

We first derive a differential equation for $E(t)$.

Lemma 10. *$E(t)$ is differentiable and*

$$\frac{d}{dt}E(t) = -\sum_{x \in \mathbb{Z}^d} E\big\{\xi_t(\mathbf{0})q(x)p_{\xi_t(x)}\big\}. \tag{4.3}$$

Proof. For $0 < \Delta \le 1$

$$\begin{aligned}
&\xi_{N,t+\Delta}(\mathbf{0}) - \xi_{N,t}(\mathbf{0}) \\
&\quad = (\text{number of particles in } \mathbf{0} \text{ at } t+\Delta) - (\text{number of particles in } \mathbf{0} \text{ at } t) \\
&\quad = (\text{number of particles moving from } \mathbb{Z}^d \setminus \mathbf{0} \text{ to } \mathbf{0} \text{ during } (t, t+\Delta]) \\
&\qquad\qquad - (\text{number of particles moving from } \mathbf{0} \text{ to } \mathbb{Z}^d \setminus \mathbf{0} \text{ during } (t, t+\Delta]) \\
&\qquad\qquad - (\text{number of particles removed from } \mathbf{0} \text{ during } (t, t+\Delta]).
\end{aligned} \tag{4.4}$$

Here 'number of particles' refers to the number of particles in the $\xi_{N,\cdot}$-process. Also we include in the first and second term in the right hand side particles which are removed from the system at or after their move to $\mathbf{0}$ or $\mathbb{Z}^d \setminus \mathbf{0}$, respectively. For this proof we use $\overline{\xi}_t(x)$ to denote the number of particles at x at time t in the system of noninteracting particles which starts in the state $\xi_0(\cdot)$. (This is $\overline{\xi}_t(x;\infty)$ in the notation of (2.25).) We now take expectations of each of the numbers in the right hand side of (4.4).

$$\begin{aligned}
&E\{\text{number of particles moving from } \mathbb{Z}^d \setminus \mathbf{0} \text{ to } \mathbf{0} \text{ during } (t, t+\Delta]\} \\
&= \sum_{x \ne \mathbf{0}} E\xi_{N,t}(x) P\{x + S_\Delta = \mathbf{0}\}.
\end{aligned} \tag{4.5}$$

Moreover, if we write $q^{*k}(\cdot)$ for the k-th convolution power of $q(\cdot)$, then, for $x \neq \mathbf{0}$,

$$P\{x + S_\Delta = \mathbf{0}\} = \sum_{k=1}^{\infty} e^{-\Delta} \frac{\Delta^k}{k!} q^{*k}(-x).$$

Since $E\xi_{N,t}(x) \leq E\overline{\xi}_t(\mathbf{0}) < \infty$, it follows that

$$\begin{aligned} &\big|E\{\text{number of particles moving from } \mathbb{Z}^d \setminus \mathbf{0} \text{ to } \mathbf{0} \text{ during } (t, t+\Delta]\} \\ &\qquad\qquad - \sum_{x \neq \mathbf{0}} E\xi_{N,t}(x) q(-x)\Delta\big| \\ &\leq C_1 \Delta^2 \end{aligned} \tag{4.6}$$

for some constant C_1 (which is independent of N, t and Δ). Essentially the same argument shows that

$$\begin{aligned} &\big|E\{\text{number of particles moving from } \mathbf{0} \text{ to } \mathbb{Z}^d \setminus \mathbf{0} \text{ during } (t, t+\Delta]\} \\ &\qquad\qquad - \sum_{x \neq \mathbf{0}} E\xi_{N,t}(\mathbf{0}) q(x)\Delta\big| \\ &\leq C_1 \Delta^2. \end{aligned} \tag{4.7}$$

Next,

$$\begin{aligned} &\Big|E\{\text{number of particles removed from } \mathbf{0} \text{ during } (t, t+\Delta]\} \\ &\qquad\qquad - E\big\{\sum_{x \neq \mathbf{0}} \xi_{N,t}(x) \int_t^{t+\Delta} q(-x) p_{\xi_{N,s}}(\mathbf{0}) ds\big\}\Big| \\ &\leq E\{\text{number of particles which visit } \mathbf{0} \text{ during } (t, t+\Delta] \\ &\qquad\qquad \text{and which make } \geq 2 \text{ jumps during} (t, t+\Delta]\} \\ &\leq \sum_{x \neq \mathbf{0}} \sum_{k \geq 2} \sum_{\ell \geq k} e^{-\Delta} \frac{\Delta^\ell}{\ell!} q^{*k}(-x) \\ &\leq C_2 \Delta^2 \end{aligned} \tag{4.8}$$

for a suitable constant C_2 (again independent of N, t and Δ). Finally

$$\begin{aligned} &\Big|E\big\{\sum_{x \neq \mathbf{0}} \xi_{N,t}(x) \int_t^{t+\Delta} q(-x) p_{\xi_{N,s}}(\mathbf{0}) ds - E \sum_{x \neq \mathbf{0}} \xi_{N,t}(x) \int_t^{t+\Delta} q(-x) p_{\xi_{N,t}}(\mathbf{0}) ds\big\}\Big| \\ &\leq \sum_{x \neq \mathbf{0}} q(-x) E\big\{\overline{\xi}_t(x) \int_t^{t+\Delta} \big|p_{\xi_{N,s}}(\mathbf{0}) - p_{\xi_{N,t}}(\mathbf{0})\big| ds\big\} \\ &\leq \Delta \sum_{x \neq \mathbf{0}} q(-x) E\big\{\overline{\xi}_t(x) I[\xi_{N,s}(\mathbf{0}) \text{ is not constant for } s \in [t, t+\Delta]]\big\}. \end{aligned} \tag{4.9}$$

But the distribution of $\overline{\xi}_t(x)$ is independent of x and

$$\begin{aligned}
&P\{\xi_{N,s}(\mathbf{0}) \text{ is not constant for } s \in [t, t+\Delta]\} \\
&\le P\{\text{a particle leaves } \mathbf{0} \text{ during } (t, t+\Delta]\} \\
&\qquad\qquad + P\{\text{a particle jumps to } \mathbf{0} \text{ during } (t, t+\Delta]\} \\
&\le E\xi_{N,t}(\mathbf{0})\Delta + \sum_{x \neq \mathbf{0}} \sum_{k \ge 1} \sum_{\ell \ge k} E\xi_{N,t}(x) e^{-\Delta} \frac{\Delta^\ell}{\ell!} q^{*k}(-x) \\
&\le C_3 \Delta,
\end{aligned}$$

for a constant C_3 independent of N, t. These observations show that the left hand side of (4.9) is $o(\Delta)$ as $\Delta \downarrow 0$, uniformly in N.

Combining (4.4)-(4.9) we find that for $\Delta > 0$

$$\begin{aligned}
&E\xi_{N,t+\Delta}(\mathbf{0}) - E\xi_{N,t}(\mathbf{0}) \\
&= \Delta \sum_{x \neq \mathbf{0}} E\xi_{N,t}(x) q(-x) - \Delta \sum_{x \neq \mathbf{0}} E\xi_{N,t}(\mathbf{0}) q(x) \\
&\qquad - \Delta \sum_{x \neq \mathbf{0}} E\big\{\xi_{N,t}(x) q(-x) p_{\xi_{N,t}(\mathbf{0})}\big\} + o(\Delta),
\end{aligned} \tag{4.10}$$

where $o(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$, uniformly in N.

We can now take the limit $N \to \infty$. Taking into account that the distribution of $\xi_t(x)$ is independent of x and that we took $q(\mathbf{0}) = 0$, we obtain

$$\begin{aligned}
E\xi_{t+\Delta}(\mathbf{0}) - E\xi_t(\mathbf{0}) &= -\Delta \sum_{x \neq \mathbf{0}} E\{\xi_t(x) q(-x) p_{\xi_t(\mathbf{0})}\} + o(\Delta) \\
&= -\Delta \sum_{x \in \mathbb{Z}^d} E\{\xi_t(\mathbf{0}) q(x) p_{\xi_t(x)}\} + o(\Delta).
\end{aligned}$$

By taking the limit $\Delta \downarrow 0$ this gives us that the right hand derivative of $E(t)$ exists and is given by (4.3). However, for $\Delta < 0$ and $t > 0$ we can interchange the role of t and $t+\Delta$ in the above derivation to get estimates for $E(t) - E(t+\Delta)$. This shows that for $t > 0$ also the left derivative of $E(t)$ exists and is given by (4.3). ■

The remainder of this section is devoted to showing that (4.3) can be replaced by

$$\frac{d}{dt} E(t) = -C(d)(1 + o(1)) E^2(t), \tag{4.11}$$

where $o(1) \to 0$ as $t \to \infty$. *Throughout we assume (1.6), (1.7) and $d \ge 6$.* (For most lemmas $d \ge 5$ is enough.) To this end we follow the heuristic outline of the introduction to approximate $E\{\xi_t(\mathbf{0}) p_{\xi_t(x)}\}$ for $x \neq \mathbf{0}$. We want the estimates to be uniform in $x \neq 0$. $C_i, i \ge 1$, will be used for various strictly positive, finite constants whose precise value is of no importance to us. The same C_i may take different values in different formulae.

Lemma 11. *Assume (1.6) and (1.7). Then for $d \geq 5$,*

$$0 \leq E(t) - p(t) \leq E(t) - P\{\xi_t(\mathbf{0}) = 1\} \leq \frac{C_3}{t^2}. \tag{4.12}$$

Proof.

$$E(t) - p(t) = \sum_{k \geq 2}^{\infty} (k-1) P\{\xi_t(\mathbf{0}) = k\} \geq 0$$

(see (4.1) and (4.2)). On the other hand,

$$\begin{aligned} E(t) - p(t) &\leq E(t) - P\{\xi_t(\mathbf{0}) = 1\} = E\{\xi_t(\mathbf{0}); \xi_t(\mathbf{0}) \geq 2\} \\ &= E\Big\{E\{\xi_t(\mathbf{0}); \xi_t(\mathbf{0}) \geq 2 | \mathcal{F}_{t/2}\}\Big\} \\ &= \lim_{N \to \infty} E\Big\{E\{\xi_t(\mathbb{1}^{(N)})(\mathbf{0}); \xi_t(\mathbb{1}^{(N)})(\mathbf{0}) \geq 2 | \mathcal{F}_{t/2}\}\Big\} \end{aligned} \tag{4.13}$$

(by the monotone convergence theorem). Now write, as before, $\xi_{N,t/2}$ for $\xi_{t/2}(\mathbb{1}^{(N)})$ and let $z_1, \dots, z_r$ be the positions at time $t/2$ of the particles present at time $t/2$ in $\xi_{N,t/2}$. Here each position occurs with the proper multiplicity; if $\xi_{N,t/2}(x) = k$, for some x, then k of the z_i equal x. Hence $r = \sum_x \xi_{N,t/2}(x)$.

To estimate the conditional expectation in the right hand side of (4.13) we apply (2.20) to the process $\{\xi_{N,t/2+s}\}_{s \geq 0}$, conditioned on $\mathcal{F}_{t/2}$. According to (2.20) there exist independent processes $\{\overline{\xi}_s(z_i)(\cdot)\}_{s \geq 0}$, $1 \leq i \leq r$, such that $\overline{\xi}_0(z_i)(x) = 1$ for $x = z_i$ and $= 0$ otherwise and such that $\{\overline{\xi}_s(z_i)(x)\}_{x \in \mathbb{Z}^d}$ has the distribution of $\{I[z_i + S_s = x]\}_{x \in \mathbb{Z}^d}$. Moreover these processes are coupled with $\xi_{N,t/2+s}$ so that

$$\xi_{N,t/2+s}(x) \leq \sum_{i=1}^{r} \overline{\xi}_s(z_i)(x).$$

In particular

$$\xi_{N,t}(x) \leq \sum_{i=1}^{r} \overline{\xi}_{t/2}(z_i)(x). \tag{4.14}$$

Consequently

$$
\begin{aligned}
&E\big\{\xi_{N,t}(\mathbf{0});\xi_{N,t}(\mathbf{0})\ge 2|\mathcal{F}_{t/2}\big\}\\
&\le E\big\{\sum_{i=1}^{r}\overline{\xi}_{t/2}(z_i)(\mathbf{0});\sum_{i=1}^{r}\overline{\xi}_{t/2}(z_i)(\mathbf{0})\ge 2\big\}\\
&= E\big\{\sum_{i=1}^{r}\overline{\xi}_{t/2}(z_i)(\mathbf{0})\big\}-P\{\sum_{i=1}^{r}\overline{\xi}_{t/2}(z_i)(\mathbf{0})=1\}\\
&= \sum_x \alpha_{t/2}(x)\xi_{N,t/2}(x)-\sum_x \xi_{N,t/2}(x)\frac{\alpha_{t/2}(x)}{1-\alpha_{t/2}(x)}\prod_{y\in\mathbb{Z}^d}\big[1-\alpha_{t/2}(y)\big]^{\xi_{N,t/2}(y)}\\
&\le \sum_x \alpha_{t/2}(x)\xi_{N,t/2}(x)\Big[1-\prod_{y\in\mathbb{Z}^d}\big[1-\alpha_{t/2}(y)\big]^{\xi_{N,t/2}(y)}\Big]\\
&\le \big[\sum_x \alpha_{t/2}(x)\xi_{N,t/2}(x)\big]^2, \qquad (4.15)
\end{aligned}
$$

because for any numbers $0\le\gamma_i\le 1$, $1-\prod_i(1-\gamma_i)\le\sum_i\gamma_i$. Finally, by virtue of (3.9) and Proposition 8

$$
\begin{aligned}
&E(t)-P\{\xi_t(\mathbf{0})=1\}\le \lim_{N\to\infty}E\big\{\big[\sum_x \alpha_{t/2}(x)\xi_{N,t/2}(x)\big]^2\big\}\\
&\le \lim_{N\to\infty}\big[\sum_x \alpha_{t/2}(x)E\xi_{N,t/2}(x)\big]^2+\limsup_{N\to\infty}\mathrm{Var}\Big(\sum_x \alpha_{t/2}(x)\xi_{N,t/2}(x)\Big)\\
&\le \big[\frac{2C_2}{t}\big]^2+C_0\log(t/2+2)\sum_x \alpha^2_{t/2}(x)\\
&\le \frac{4C_2^2}{t^2}+C_0\log(t+2)\sup_x \alpha_{t/2}(x)\\
&\le C_4[t^{-2}+t^{-d/2}\log(t+2)], \qquad (4.16)
\end{aligned}
$$

where in the last inequality we used the estimate

$$
\sup_y \alpha_s(y)=\sup_y P\{y+S_s=\mathbf{0}\}\le\frac{C_5}{(s+1)^{d/2}}, \qquad (4.17)
$$

which, in turn, follows from the local central limit theorem (see Spitzer (1976), Proposition 7.9 and the Remark following it). This gives (4.12) when $d\ge 5$. ■

Let $u_1,\ldots,u_p\in\mathbb{Z}^d$ (not necessarily distinct). Define

$$
\sum_{i=1}^{p}{}^{*}\xi_t(u_i) \qquad (4.18)
$$

to be the sum of the $\xi_t(u_i)$ only over the distinct u_i in $\{u_1, \dots, u_p\}$. Thus if a given u appears several times among the u_i, there is still only one summand $\xi_t(u)$ in (4.18). Define further

$$\Lambda_t(u_1, u_2, \dots, u_p) \\ = \Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i)\Big)\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - 1\Big)\dots\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - p + 1\Big). \tag{4.19}$$

$\Lambda_t(u_1, \dots, u_p)$ represents the number of ordered p-tuples of distinct particles which we can select from the $\sum{}^{*}\xi_t(u_i)$ particles present at the sites $u_1, \dots, u_p$ at time t.

Lemma 12. *Assume (1.6), (1.7) and $d \geq 5$. Then for any $u, v \in \mathbb{Z}^d$*

$$E\Lambda_t(u, v) \leq C_5 t^{-2}. \tag{4.20}$$

Also, for any $p \geq 3, u_1, \dots, u_p \in \mathbb{Z}^d$ and $0 < \varepsilon < 1/2$,

$$E\Lambda_t(u_1, \dots, u_p) \leq C_6(\varepsilon, p)[t^{-p} \vee t^{-d(1-\varepsilon)/2}]. \tag{4.21}$$

Proof. Without loss of generality we may take $u \neq v$ in (4.20) because $\Lambda_t(u, u) \leq \Lambda_t(u, v)$ for any v. Similarly we may take the u_i distinct in (4.21). We note further that it suffices to prove (4.20) and (4.21) when $\xi_t(u_i)$ is replaced by $\xi_{N,t}(u_i)$ (with constants C_5, C_6 which are independent of N). We shall write $\Lambda_{N,t}$ instead of Λ_t when this replacement is made. We take z_i and $\xi_s(\cdot\,; i)$ as in the preceding lemma and repeatedly use (4.14).

Now to prove (4.20), we have from(4.14) that

$$\begin{aligned}\Lambda_{N,t}(u, v) &\leq \Big(\sum_{i=1}^{r}[\overline{\xi}_{t/2}(z_i)(u) + \overline{\xi}_{t/2}(z_i)(v)]\Big) \\ &\qquad \times \Big(\sum_{i=1}^{r}[\overline{\xi}_{t/2}(z_i)(u) + \overline{\xi}_{t/2}(z_i)(v)] - 1\Big) \\ &= \sum_{i=1}^{r}\sum_{\substack{j=1 \\ j\neq i}}^{r}[\overline{\xi}_{t/2}(z_i)(u) + \overline{\xi}_{t/2}(z_i)(v)] \\ &\qquad \times [\overline{\xi}_{t/2}(z_j)(u) + \overline{\xi}_{t/2}(z_j)(v)].\end{aligned}$$

The right hand side equals the number of ordered pairs of distinct particles starting at some z_i at time $t/2$ and ending at u or v at time t. These particles are the ones counted by the $\overline{\xi}_{t/2}(z_i)$ and they just move according to random walks without interaction. At time $t/2$ we have $\xi_{N,t/2}(z)$ particles at z to choose from (for any $z \in \mathbb{Z}^d$). The number of choices for starting pairs, one from z and one from z' (with $z = z'$ allowed), is $\Lambda_{N,t/2}(z,z') \le \xi_{N,t/2}(z)\xi_{N,t/2}(z')$. The probability that the two different particles of the pair end at u or v at time t is

$$\big(\alpha_{t/2}(z-u) + \alpha_{t/2}(z-v)\big)\big(\alpha_{t/2}(z'-u) + \alpha_{t/2}(z'-v)\big).$$

We now sum over all possible z, z' and take the conditional expectation given $\mathcal{F}_{t/2}$ to find

$$\begin{aligned} &E\{\Lambda_{N,t}(u,v)|\mathcal{F}_{t/2}\} \\ &\le \sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r} \big(\alpha_{t/2}(z_i-u) + \alpha_{t/2}(z_i-v)\big) \\ &\qquad\qquad\qquad \times \big(\alpha_{t/2}(z_j-u) + \alpha_{t/2}(z_j-v)\big) \\ &\le \Big[\sum_{z\in\mathbb{Z}^d} \xi_{N,t/2}(z)\big(\alpha_{t/2}(z-u) + \alpha_{t/2}(z-v)\big)\Big]^2. \end{aligned} \tag{4.22}$$

Finally, as in (4.16), for $d \ge 5$, and uniformly in u,

$$E\Big\{\Big[\sum_z \alpha_{t/2}(z-u)\xi_{N,t/2}(z)\Big]^2\Big\} \le C_7 t^{-2} \tag{4.23}$$

and the same estimate holds when u is replaced by v. (4.20) is immediate from (4.22) and (4.23), if we take expectation once again.

The argument for (4.21) begins in the same way as for (4.20). By an application of (4.14) we can bound $\Lambda_{N,t}(u_1,\dots,u_p)$ by the number of p-tuples of distinct particles which start at some z_i at time $t/2$ and end at one the u_j at time t. Therefore, we get as in (4.22) that

$$E\Lambda_{N,t}(u_1,\dots,u_p) \le E\Big\{\Big[\sum_z \xi_{N,t/2}(z)\sum_{j=1}^{p}\alpha_{t/2}(z-u_j)\Big]^p\Big\}.$$

It therefore suffices for (4.21) to show that, uniformly in $u \in \mathbb{Z}^d$,

$$E\Big\{\Big[\sum_z \xi_{N,t/2}(z)\alpha_{t/2}(z-u)\Big]^p\Big\} \le C_8(\varepsilon,p)[t^{-p} \vee t^{-d(1-\varepsilon)/2}], \quad p \ge 3. \tag{4.24}$$

To prove (4.24) let us use the abbreviation

$$U = \sum_z \alpha_{t/2}(z-u)\xi_{N,t/2}(z). \tag{4.25}$$

Note that $U \geq 0$. We further know from Lemma 9 that

$$EU \leq C_2 t^{-1}, \tag{4.26}$$

and from Proposition 8 and (4.16) that

$$\mathrm{Var}(U) = E\{(U - EU)^2\} \leq C_9 t^{-d/2} \log(t+2), \quad E\{U^2\} \leq C_9 t^{-2}. \tag{4.27}$$

Now use

$$\begin{aligned} U^p &\leq C_{10}(p)\big[|U - EU|^p + (EU)^p\big] \\ &\leq C_{10}(p)|U - EU|^{2-\varepsilon}|U - EU|^{p-2+\varepsilon} + C_{11}(p)t^{-p}. \end{aligned}$$

Combined with Hölder's inequality this shows that

$$E\{U^p\} \leq C_{10}\Big[E\{(U - EU)^2\}\Big]^{(1-\varepsilon/2)}\Big[E\{|U - EU|^{2(p-2+\varepsilon)/\varepsilon}\}\Big]^{\varepsilon/2} + C_{11}(p)t^{-p}. \tag{4.28}$$

Assume for the moment that we have proven for any integer $q \geq 1$

$$E\{U^q\} \leq C_{12}(q), \tag{4.29}$$

(with C_{12} independent of N). Then by Jensen's inequality this holds for any $q \geq 1$ and also

$$E\{|U - EU|^q\} \leq C_{13}(q) \tag{4.30}$$

follows. Together with (4.28) this shows

$$E\{U^p\} \leq C_{14}(\varepsilon, p)[\mathrm{Var}(U)]^{(1-\varepsilon/2)} + C_{11}t^{-p}.$$

Together with (4.27) this implies (4.24) and hence (4.21).

The proof of (4.21) has therefore been reduced to (4.29). We now turn to its proof. We note that $\sum_z \alpha_{t/2}(z-u) = 1$, so that by Jensen's inequality

$$U^q \leq \sum_z \alpha_{t/2}(z-u)\xi^q_{N,t/2}(z)$$

and hence

$$E\{U^q\} \leq \sup_x E\xi^q_{N,t/2}(x).$$

Next we again use (4.14) (compare also with (2.21)). Then

$$\begin{aligned} E\{U^q\} &\leq \sup_x E\xi^q_{N,t/2}(x) = E\xi^q_{N,t/2}(\mathbf{0}) \\ \text{(by translation invariance)} &\leq E\Big[\sum_z \overline{\xi}_{t/2}(z)(\mathbf{0})\Big]^q \\ &\leq C_{15}(q)\sum_{k=1}^{q}\sum_{n_1,\dots,n_k}\sum_{\substack{z_1,\dots,z_k\\ \text{distinct}}} E\{\overline{\xi}^{n_1}_{t/2}(z_1)(\mathbf{0})\}\cdots E\{\overline{\xi}^{n_k}_{t/2}(z_k)(\mathbf{0})\}, \end{aligned}$$

where $n_1, \ldots, n_k$ runs over the partitions of q into k nonzero integers. Since $\overline{\xi}_{t/2}(z)(\mathbf{0})$ can take on only the values 0 or 1 and $P\{\overline{\xi}_{t/2}(z)(\mathbf{0}) = 1\} = P\{z + S_{t/2} = \mathbf{0}\} = \alpha_{t/2}(z)$, we find that

$$E\{U^q\} \leq C_{15}(q) \sum_{k=1}^{q} \sum_{n_1,\ldots,n_k} \sum_{z_1,\ldots,z_k} \prod_{i=1}^{k} P\{z_i + S_{t/2} = \mathbf{0}\} \leq C_{16}(q),$$

as desired. ■

The next lemma is an estimate for noninteracting random walks. If $s \mapsto u + S_s^{(u)}$ and $s \mapsto v + S_s^{(v)}$ are two random walk paths, then we shall say that *they meet at least m times during a time interval J* if there exist m times $s_1 < s_2 < \ldots s_m$ in J such that each s_i is a jumptime of one of these paths for which $u + S_{s_i}^{(u)} = v + S_{s_i}^{(v)}$. We say that the paths *meet exactly m times during J* if they meet at least m times during J but not at least $(m+1)$ times.

Lemma 13. *Let $d \geq 3$ and let $\{S_t^{(u)}\}_{t\geq 0}$, $u \in \mathbb{Z}^d$, be independent copies of $\{S_t\}_{t\geq 0}$. Also let $\Delta \geq 1$. Define for $u, v, y \in \mathbb{Z}^d$*

$$\begin{aligned}\mathcal{E}(u,v,m) =& \mathcal{E}(u,v,m,\Delta,y) = \{u + S_\Delta^{(u)} = \mathbf{0}, v + S_\Delta^{(v)} = y \textit{ and the paths} \\ & s \mapsto u + S_s^{(u)}, s \mapsto v + S_s^{(v)} \textit{ meet exactly } m \textit{ times during } (0,\Delta]\}.\end{aligned}$$

Then, there exists a $\delta = \delta(d)$ with $0 < \delta(d) \leq 1$ such that uniformly in y and m,

$$\begin{aligned}&\sum_{u,v\in\mathbb{Z}^d} \Big| P\{\mathcal{E}(u,v,m,\Delta,y)\} \\ &\qquad - P\{s \mapsto S_s^{(\mathbf{0})} \textit{ and } s \mapsto -y + S_s^{(-y)} \textit{ meet exactly } m \textit{ times during } (0,\infty)\} \\ &\qquad\qquad \times \alpha_\Delta(u)\alpha_\Delta(v-y)\Big| \\ &\leq C_{17}\Delta^{-\delta}. \qquad (4.31)\end{aligned}$$

Remark (vi) We can take

$$\delta(d) = \frac{d-2}{3d^2 - 3d - 4}. \qquad (4.32)$$

Proof. Let $\{S'_s\}_{s\geq 0}$ and $\{S''_s\}_{s\geq 0}$ be two independent copies of $\{S_s\}_{s\geq 0}$. Also let $\{\widetilde{S}'_s\}_{s\geq 0}$ and $\{\widetilde{S}''_s\}_{s\geq 0}$ be two independent copies of the corresponding time reversed random walk which satisfies (1.4). We first use time reversal to write $P\{\mathcal{E}(u,v,m)\}$ as

$$P\{\widetilde{S}'_\Delta = u, y + \widetilde{S}''_\Delta = v \text{ and the paths } s \mapsto \widetilde{S}'_s, s \mapsto y + \widetilde{S}''_s \\ \text{meet exactly } m \text{ times during } (0,\Delta]\},$$

If we put

$$\widetilde{\alpha}_s(u) = P\{\widetilde{S}_s = -u\} = P\{S_s = u\} = \alpha_s(-u),$$

then

$$\alpha_\Delta(u)\alpha_\Delta(v-y) = \widetilde{\alpha}_\Delta(-u)\widetilde{\alpha}_\Delta(y-v).$$

Moreover,

$$\begin{aligned} &P\{s \mapsto S_s^{(\mathbf{0})} \text{ and } s \mapsto -y + S_s^{(-y)} \text{ meet exactly } m \text{ times during } (0,\infty)\} \\ &= P\big\{\{S_s^{(\mathbf{0})} - S_s^{(-y)}\}_{s\ge 0} = -y \text{ for exactly } m \text{ jump times of } \{S_s^{(\mathbf{0})} - S_s^{(-y)}\}_{s\ge 0}\big\} \\ &= P\big\{\{-S_s^{(\mathbf{0})} + S_s^{(y)}\}_{s\ge 0} = y \text{ for exactly } m \text{ jump times of } \{-S_s^{(\mathbf{0})} + S_s^{(y)}\}_{s\ge 0}\big\} \\ &= P\{s \mapsto \widetilde{S}'_s \text{ and } s \mapsto y + \widetilde{S}''_s \text{ meet exactly } m \text{ times during } (0,\infty)\}. \end{aligned}$$

Therefore, the left hand side of (4.31) equals

$$\begin{aligned} \Big| &P\{\widetilde{S}'_\Delta = u, y + \widetilde{S}''_\Delta = v \text{ and the paths } s \mapsto \widetilde{S}'_s, s \mapsto y + \widetilde{S}''_s \\ &\qquad \text{meet exactly } m \text{ times during } (0,\Delta]\} \\ &- P\{s \mapsto \widetilde{S}'_s \text{ and } s \mapsto y + \widetilde{S}''_s \\ &\qquad \text{meet exactly } m \text{ times during } (0,\infty)\}\widetilde{\alpha}_\Delta(-u)\widetilde{\alpha}_\Delta(y-v)\Big|. \end{aligned}$$

To simplify notation we drop the tildes and introduce

$$\nu(J) := \text{ number of times } s \mapsto S'_s \text{ and } s \mapsto y + S''_s \text{ meet during } J.$$

We shall prove, merely from assumption (1.8), that

$$\begin{aligned} \sum_{u,v} \Big| &P\{S'_\Delta = u, y + S''_\Delta = v, \nu((0,\Delta]) = m\} \\ &\qquad - P\{\nu((0,\infty)) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v)\Big| \\ &\le C_{17}\Delta^{-\delta}. \end{aligned} \tag{4.33}$$

If we apply this to the random walk $\{\widetilde{S}_s\}$ (which also satisfies (1.8)) and reverse time we obtain (4.31).

The estimate (4.33) will be obtained by estimating various pieces. For the time being we fix an arbitrary $\delta > 0$. First we drop the sum over the terms with $\|u\| > \Delta^{(1+\delta)/2}$ or $\|v-y\| > \Delta^{(1+\delta)/2}$. Since

$$E\|S_\Delta\|^2 = \sum_{k=0}^{\infty} e^{-\Delta}\frac{\Delta^k}{k!}k\sum_x \|x\|^2 q(x) \le C_{18}\Delta$$

for some constant $C_{18} < \infty$, we see from Chebyshev's inequality that the terms with such u, v add up to at most

$$P\{\|S'_\Delta\| > \Delta^{(1+\delta)/2}\} + P\{\|S''_\Delta\| > \Delta^{(1+\delta)/2}\} \le 2C_{18}\Delta^{-\delta}. \tag{4.34}$$

Next we fix $1 \le \Gamma \le \Delta/2$. For the time being Γ is otherwise arbitrary. We next replace

$$P\{S'_\Delta = u, y + S''_\Delta = v, \nu((0,\Delta]) = m\}$$

by

$$P\{S'_\Delta = u, y + S''_\Delta = v, \nu((0,\Gamma]) = m\}$$

and

$$P\{\nu((0,\infty)) = m\}$$

by

$$P\{\nu((0,\Gamma]) = m\}.$$

This changes the left hand side of (4.33) by at most

$$\begin{aligned}
&2P\{s \mapsto S'_s \text{ and } s \mapsto y + S''_s \text{ meet at least once during } (\Gamma,\infty)\} \\
&\le 4E\{\text{amount of time in } (\Gamma,\infty) \text{ that } S'_s = y + S''_s\} \\
&\le \int_\Gamma^\infty P\{S'_s - S''_s = y\}ds \le C_{19}\int_\Gamma^\infty \frac{ds}{s^{d/2}} \le C_{20}\Gamma^{1-d/2}.
\end{aligned} \tag{4.35}$$

Combining (4.34) and (4.35) we see that the left hand side of (4.33) is at most

$$\begin{aligned}
&2C_{18}\Delta^{-\delta} + C_{20}\Gamma^{1-d/2} \\
&+ \sum_{\|u\|\vee\|v-y\|\le\Delta^{(1+\delta)/2}} \Big| P\{S'_\Delta = u, y + S''_\Delta = v, \nu((0,\Gamma]) = m\} \\
&\qquad - P\{\nu((0,\Gamma]) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v)\Big|.
\end{aligned} \tag{4.36}$$

Next we fix a $\gamma > 0$ and write

$$
\begin{aligned}
&P\{S'_\Delta = u, y + S''_\Delta = v, \nu((0,\Gamma]) = m\} \\
&= \sum_{a,b\in\mathbb{Z}^d} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\}\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) \\
&= \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\} \\
&\qquad\qquad \times \alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) + E_1,
\end{aligned}
$$

where the error $E_1 = E_1(u,v)$ satisfies

$$
\begin{aligned}
0 \le E_1 &= \sum_{\|a\|\vee\|b-y\|>\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\} \\
&\qquad\qquad \times \alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) \\
&\le P\{\|S'_\Gamma\| > \Gamma^{(1+\gamma)/2} \text{ or } \|S''_\Gamma\| > \Gamma^{(1+\gamma)/2}\} \sup_{z_1,z_2} \alpha_{\Delta-\Gamma}(z_1)\alpha_{\Delta-\Gamma}(z_2) \\
&\le C_{21}\Gamma^{-\gamma}\Delta^{-d} \text{ (by Chebyshev, (4.17) and } \Gamma \le \Delta/2). \qquad (4.37)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&P\{\nu((0,\Gamma]) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v) \\
&= \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\} \\
&\qquad\qquad \times \alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) + E_2,
\end{aligned}
$$

for an error $E_2 = E_2(u,v)$ with

$$
0 \le E_2 \le C_{21}\Gamma^{-\gamma}\Delta^{-d}. \qquad (4.38)
$$

Finally we note that

$$
\begin{aligned}
&\Big| \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\} \\
&\qquad - \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\Big| \\
&\le \Big| \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\} - P\{\nu((0,\Gamma]) = m\}\Big| \\
&\qquad + \Big| \sum_{\|a\|,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y) - P\{\nu((0,\Gamma]) = m\}\Big| \\
&\le 2P\{\|S'_\Gamma\| > \Gamma^{(1+\gamma)/2} \text{ or } \|S''_\Gamma\| > \Gamma^{(1+\gamma)/2}\} \le 4C_{18}\Gamma^{-\gamma}.
\end{aligned}
$$

Now for any positive measures μ_1, μ_2 of total mass $\le A$ on some space Ω and a function $f : \Omega \mapsto \mathbb{R}$ one has the general and simple inequality

$$\begin{aligned}&\Big|\int \mu_1(d\omega)f(\omega) - \int \mu_2(d\omega)f(\omega)\Big| \\ &\le \big|\mu_1(\Omega) - \mu_2(\Omega)\big| \sup_\omega |f(\omega)| + A \sup_{\omega_1,\omega_2} |f(\omega_1) - f(\omega_2)|.\end{aligned}$$

We apply this with

$$\begin{aligned}\mu_1(a,b) &= P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\}, \\ \mu_2(a,b) &= P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y).\end{aligned}$$

We then obtain

$$\begin{aligned}&\Big|\sum_{a,b\in\mathbb{Z}^d} P\{S'_\Gamma = a, y + S''_\Gamma = b, \nu((0,\Gamma]) = m\}\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) \\ &- \sum_{a,b\in\mathbb{Z}^d} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v)\Big| \\ &\le E_1 + E_2 + C_{22}\Gamma^{-\gamma}\Delta^{-d} \\ &+ \sup_{\substack{\|a_1-a_2\|\le 2\Gamma^{(1+\gamma)/2} \\ \|b_1-b_2\|\le 2\Gamma^{(1+\gamma)/2}}} \Big|\alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Big|.\end{aligned} \tag{4.39}$$

We now sum (4.39) over $\|u\|, \|v-y\| \le \Delta^{(1+\delta)/2}$. Since there are at most $C_{23}\Delta^{d(1+\delta)}$ points u, v satisfying these restrictions we find by means of (4.36)-(4.38) that the left hand side of (4.33) is at most

$$\begin{aligned}&2C_{18}\Delta^{-\delta} + C_{20}\Gamma^{1-d/2} + C_{24}\Delta^{d\delta}\Gamma^{-\gamma} \\ &+ C_{23}\Delta^{d(1+\delta)} \sup_{\substack{\|a_1-a_2\|\le 2\Gamma^{(1+\gamma)/2} \\ \|b_1-b_2\|\le 2\Gamma^{(1+\gamma)/2}}} \Big|\alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) \\ &\qquad - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Big|.\end{aligned} \tag{4.40}$$

Finally, denote by $\Phi_t(\theta) = E\{\exp(i\theta \cdot S_t)\}$, $\theta \in \mathbb{R}^d$, the characteristic function of S_t. Then standard arguments (compare Spitzer (1976), Propositions 7.7, 7.8) show that there exists some $C_{25}, C_{26} > 0, \eta > 0$ such that

$$|\Phi_t(\theta)| \le e^{-C_{25}t\|\theta\|^2} \text{ for } \|\theta\| \le \eta,$$

and

$$|\Phi_t(\theta)| \le e^{-C_{26}t} \text{ for } \eta < \|\theta\|, \theta \in [-\pi,\pi]^d.$$

Consequently,

$$\begin{aligned}
&\sup_{\substack{\|a_1-a_2\|\le 2\Gamma^{(1+\gamma)/2}\\ \|b_1-b_2\|\le 2\Gamma^{(1+\gamma)/2}}} \Big|\alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Big| \\
&\le 2 \sup_{\|c_1-c_2\|\le 2\Gamma^{(1+\gamma)/2}} \Big|\alpha_{\Delta-\Gamma}(c_1) - \alpha_{\Delta-\Gamma}(c_2)\Big| \sup_v \alpha_{\Delta-\Gamma}(v) \\
&\le C_{27}\Delta^{-d/2} \sup_{\|c_1-c_2\|\le 2\Gamma^{(1+\gamma)/2}} \int_{\theta\in[-\pi,\pi]^d} \Big|e^{-i\theta\cdot c_1} - e^{-i\theta\cdot c_2}\Big| |\Phi_{\Delta-\Gamma}(\theta)| d\theta \\
&\le C_{28}\Delta^{-d/2}\Gamma^{(1+\gamma)/2}\Delta^{-(d+1)/2}. \qquad (4.41)
\end{aligned}$$

Substituting this estimate into (4.40) yields the upper bound

$$C_{29}\Big[\Delta^{-\delta} + \Gamma^{1-d/2} + \Delta^{d\delta}\Gamma^{-\gamma} + \Delta^{d\delta-1/2}\Gamma^{(1+\gamma)/2}\Big]$$

for the left hand side of (4.33). It remains to choose

$$\begin{aligned}
\Gamma &= \Delta^{1/(1+3\gamma)}, \\
\gamma &= \frac{(d+1)\delta}{1-3(d+1)\delta} = \frac{(d+1)(d-2)}{2}, \\
\delta &= \frac{d-2}{3d^2-3d-4},
\end{aligned}$$

to find that (4.33) holds for the given δ. ■

We define

$$\rho(m,y) = P\{s \mapsto S_s^{(\mathbf{0})} \text{ and } s \mapsto -y + S_s^{(-y)} \text{ meet exactly } m \text{ times during } [0,\infty)\} \qquad (4.42)$$

and

$$D(y) = p_1 \sum_{m=0}^{\infty} (1-p_1)^m \rho(m,y). \qquad (4.43)$$

We also define $\Lambda_t^*(u,v)$ as the number of ordered pairs of distinct particles, the first particle being present at u at time t, and the second particle at v at time t. Comparison with (4.19) shows immediately that $\Lambda_t^*(u,v) \le \Lambda_t(u,v)$.

Lemma 14. *Let $1 \le \Delta < t/2$. Then for $y \ne \mathbf{0}, d \ge 5, 0 < \varepsilon < 1/2$,*

$$\begin{aligned}
&\big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\} - D(y) \sum_{u,v\in\mathbb{Z}^d} \alpha_\Delta(u)\alpha_\Delta(v-y)E\{\Lambda^*_{t-\Delta}(u,v)\}\big| \\
&\le C_{30}\Delta[t^{-3} \vee t^{-d(1-\varepsilon)/2}] + C_{30}\Delta^{-\delta(d)}t^{-2}. \qquad (4.44)
\end{aligned}$$

Proof. First we show that

$$\big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\} - p_1 P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\}\big| \le C_{31}(\varepsilon)[t^{-3} \vee t^{-d(1-\varepsilon)/2}]. \qquad (4.45)$$

To see this we take $\xi_{N,t}, z_1, \dots, z_r$ and $\overline{\xi}(z_i)(\cdot)$ as in Lemma 11. Then

$$\begin{aligned}
&\big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\} - p_1 E\{\xi_t(\mathbf{0})I[\xi_t(y) = 1]\}\big| \\
&\le E\{\xi_t(\mathbf{0})\xi_t(y)I[\xi_t(y) \ge 2]\} \\
&\le E\Big\{E\{\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(\mathbf{0}) \sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y) I[\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y) \ge 2]\big|\mathcal{F}_{t/2}\}\Big\}. \qquad (4.46)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\big|E\{\xi_t(\mathbf{0})I[\xi_t(y) = 1]\} - P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\}\big| \\
&\le E\{\xi_t(\mathbf{0})I[\xi_t(\mathbf{0}) \ge 2]\xi_t(y)\} \\
&= E\{\xi_t(-y)I[\xi_t(-y) \ge 2]\xi_t(\mathbf{0})\}.
\end{aligned}$$

Since this has the same form as (4.46), it suffices for (4.45) to estimate (4.46).

It is easy to see that

$$\begin{aligned}
&E\{\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(\mathbf{0}) \sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y)\, I[\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y) \ge 2]\big|\mathcal{F}_{t/2}\} \\
&\le E\{\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(\mathbf{0})\big|\mathcal{F}_{t/2}\}\, E\{\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y)\, I[\sum_{i=1}^r \overline{\xi}_{t/2}(z_i)(y) \ge 2]\big|\mathcal{F}_{t/2}\}, \qquad (4.47)
\end{aligned}$$

because $\overline{\xi}_{t/2}(z_i)(\mathbf{0})$ can only take the values 0 or 1, and if it equals 1, then $\overline{\xi}_{t/2}(z_i)(y) = 0$. The second factor in the right hand side of (4.47) was estimated in (4.15), except for a trivial translation. This factor is at most

$$\big[\sum_{z\in\mathbb{Z}^d} \alpha_{t/2}(z-y)\xi_{N,t/2}(z)\big]^2.$$

The first factor in the right hand side of (4.47) trivially equals $\sum_{x\in\mathbb{Z}^d}\alpha_{t/2}(x)\xi_{N,t/2}(x)$. We therefore find that (4.46) is bounded by

$$E\Big\{\sum_{x\in\mathbb{Z}^d}\alpha_{t/2}(x)\xi_{N,t/2}(x)\big[\sum_{z\in\mathbb{Z}^d}\alpha_{t/2}(z-y)\xi_{N,t/2}(z)\big]^2\Big\}$$
$$\le\Big[E\Big\{\big[\sum_{x\in\mathbb{Z}^d}\alpha_{t/2}(x)\xi_{N,t/2}(x)\big]^3\Big\}\Big]^{1/3}\Big[E\Big\{\big[\sum_{z\in\mathbb{Z}^d}\alpha_{t/2}(z-y)\xi_{N,t/2}(z)\big]^3\Big\}\Big]^{2/3}. \tag{4.48}$$

(4.45) now follows from (4.27)-(4.29).

Next we approximate

$$P\{\xi_t(\mathbf{0})=\xi_t(y)=1\}. \tag{4.49}$$

It is easiest to carry out this part of the proof if we construct ξ_t as in Lemma 9, so that we can speak of the trajectory of a particle. We only know how to carry out such a construction for a system with a finite initial state. Formally the remaining estimates in this lemma must therefore be carried out for the process ξ_N, and then the limit $N\to\infty$ must be taken in the final estimates (4.52) and (4.54) below. For simplicity we have written the proof as if it applies directly to the full process ξ. Here we want to condition on $\mathcal{F}_{t-\Delta}$, so that we think of $t-\Delta$ as the origin of the time axis. Thus, the label (x,k) refers to the k-th particle at position x at time $t-\Delta$. $s\mapsto\{S_s^{(x,k)}\}_{s\ge0}$ describes the motion of this particle until it coalesces, that is, its position at time $t-\Delta+s$ is $x+S_s^{(x,k)}$, if it did not coalesce during $(t-\Delta,t-\Delta+s]$. Of course we take the $\{S_s^{(x,k)}\}_{s\ge0}$ to be independent copies of $\{S_s\}_{s\ge0}$. If $\xi_t(\mathbf{0})=\xi_t(y)=1$, then there must be two different particles, π' and π'' say, in the system at time $t-\Delta$ which move to $\mathbf{0}$ and y, respectively, at time t, without coalescing with another particle during $(t-\Delta,t]$. Let the positions of these particles at time $t-\Delta$ be u and v, respectively. Then there must exist $1\le k\le\xi_{t-\Delta}(u), 1\le\ell\le\xi_{t-\Delta}(v)$ and random walk paths $s\mapsto S_s^{(u,k)}, s\mapsto S_s^{(v,\ell)}$ with $u+S_\Delta^{(u,k)}=\mathbf{0}, v+S_\Delta^{(v,\ell)}=y$.

As a first step in approximating (4.49) we bound the probability of the event $\mathcal{G}$ that there exist two different particles π',π'' with labels (u,k) and (v,ℓ), which move along the trajectories $s\mapsto u+S_s^{(u,k)}, s\mapsto v+S_s^{(v,\ell)}$ for $0\le s\le\Delta$, satisfying $u+S_\Delta^{(u,k)}=\mathbf{0}, v+S_\Delta^{(v,\ell)}=y$, and that there exists another particle π such that π coincides with π' or π'' at some time $s\in(0,\Delta]$. If π coincides with π', then the probability that they stay together for one unit of time after their paths first coincide is at least e^{-2}. We can therefore estimate $P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\}$ by e^2 times the conditional expectation given $\mathcal{F}_{t-\Delta}$ of the amount of time in $(t-\Delta,t+1]$ at which a particle π coincides with π' or π''. A decomposition with respect to the position z at which π coincides with π' or π'', as well as with respect to the starting positions

u, v, w of π', π'' and π, shows that this expectation is bounded by

$$\sum_{u,v,w} \Lambda_{t-\Delta}(u,v,w) \int_0^{\Delta+1} \Big[\sum_z \alpha_s(u-z)\alpha_s(w-z)\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) + \alpha_\Delta(u) \sum_z \alpha_s(v-z)\alpha_s(w-z)\alpha_{\Delta-s}(z-y) \Big] ds.$$

Therefore, taking expectation,

$$\begin{aligned} P\{\mathcal{G}\} &\le e^2 \sum_{u,v,w} E\{\Lambda_{t-\Delta}(u,v,w)\} \\ &\quad \times \int_0^{\Delta+1} \Big[\sum_z \alpha_s(u-z)\alpha_s(w-z)\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) \\ &\qquad + \alpha_\Delta(u) \sum_z \alpha_s(v-z)\alpha_s(w-z)\alpha_{\Delta-s}(z-y) \Big] ds \\ &\le e^2 C_6 [t^{-3} \vee t^{-d(1-\varepsilon)/2}] \sum_{u,v,w} \int_0^{\Delta+1} \Big[\sum_z \alpha_s(u-z)\alpha_s(w-z)\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) \\ &\qquad + \alpha_\Delta(u) \sum_z \alpha_s(v-z)\alpha_s(w-z)\alpha_{\Delta-s}(z-y) \Big] ds \text{ (by (4.21))} \\ &= e^2 C_6 [t^{-3} \vee t^{-d(1-\varepsilon)/2}] \int_0^{\Delta+1} 2ds = 2e^2 C_6 (\Delta+1)[t^{-3} \vee t^{-d(1-\varepsilon)/2}]. \end{aligned} \tag{4.50}$$

Now on the complement of $\mathcal{G}$, $\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\}$ occurs if and only if the following two events occur:
(i) there exist $u, v \in \mathbb{Z}^d$ and a pair of particles π', π'' located at u, v, respectively at time $t - \Delta$, which move to $\mathbf{0}$ and y, respectively, at time t;
(ii) at each of the jumptimes of π' or π'' at which these two particles meet during $(t-\Delta, t]$, the corresponding $U_{n,1}^{\pi'}$ or $U_{n,1}^{\pi''}$ exceeds p_1.
In explanation of (ii) we point out that we do not want π' and π'' to coalesce. However, on $\mathcal{G}^c$, neither π' nor π'' coincide with a third particle π during $[t-\Delta, t]$. Thus, when π' jumps to the position of π'', then it jumps to a site which contains exactly one particle. If this is the n-th jump of π', then no coalescence takes place if and only if $U_{n,1}^{\pi'} > p_1$. A similar statement holds for π''.

Conditionally on $\mathcal{F}_{t-\Delta}$, the probability of (i) and (ii) is

$$\sum_{m=0}^\infty (1-p_1)^m P\Big\{ \bigcup_{u,v \in \mathbb{Z}^d} \bigcup_{\substack{1 \le k \le \xi_{t-\Delta}(u) \\ 1 \le \ell \le \xi_{t-\Delta}(v) \\ (u,k) \ne (v,\ell)}} \{u + S_\Delta^{(u,k)} = \mathbf{0}, v + S_\Delta^{(v,\ell)} = y \text{ and } s \mapsto u + S_s^{(u,k)} \text{ and } s \mapsto v + S_s^{(v,\ell)} \text{ meet exactly } m \text{ times during } (0, \Delta]\} \Big\}. \tag{4.51}$$

We shall write $\mathcal{E}_1(u,k,v,\ell,m) = \mathcal{E}_1(u,k,v,\ell,m,\Delta,y)$ for the event

$$\begin{aligned}\mathcal{E}_1(u,k,v,\ell,m) =&\{u + S_\Delta^{(u,k)} = \mathbf{0}, v + S_\Delta^{(v,\ell)} = y \text{ and } s \mapsto u + S_s^{(u,k)} \\ &\text{and } s \mapsto v + S_s^{(v,\ell)} \text{ meet exactly } m \text{ times during } (0,\Delta]\}.\end{aligned}$$

Then (4.51) shows that (with $\mathcal{E}$ as in Lemma 13)

$$\begin{aligned}&P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1 | \mathcal{F}_{t-\Delta}\} \\ &\le P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\} + \sum_{m=0}^\infty (1-p_1)^m \sum_{u,k,v,\ell} P\{\mathcal{E}_1(u,k,v,\ell,m)\} \\ &= P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\} + \sum_{m=0}^\infty (1-p_1)^m \sum_{u,v} \Lambda^*_{t-\Delta}(u,v) \\ &\qquad\qquad \times \big[P\{\mathcal{E}(u,v,m,\Delta,y)\} - \rho(m,y)\alpha_\Delta(u)\alpha_\Delta(v-y)\big] \\ &\qquad + \sum_{m=0}^\infty (1-p_1)^m \rho(m,y) \sum_{u,v} \Lambda^*_{t-\Delta}(u,v)\alpha_\Delta(u)\alpha_\Delta(v-y).\end{aligned} \tag{4.52}$$

Taking expectation once more and using (4.50) and Lemmas 12 and 13 we find

$$\begin{aligned}&p_1 P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\} \\ &\le 4p_1 e^2 C_6 \Delta\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big] \\ &\qquad + p_1 \sum_{m=0}^\infty (1-p_1)^m \frac{C_5}{t^2} \sum_{u,v} \big|P\{\mathcal{E}(u,v,m,\Delta,y)\} - \rho(m,y)\alpha_\Delta(u)\alpha_\Delta(v-y)\big| \\ &\qquad + p_1 \sum_{m=0}^\infty (1-p_1)^m \rho(m,y) \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y) \\ &\le 4p_1 e^2 C_6 \Delta\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big] + C_5 C_{17} \Delta^{-\delta(d)} t^{-2} \\ &\qquad + D(y) \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y).\end{aligned} \tag{4.53}$$

In the other direction, we have from the inclusion-exclusion principle that

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1 | \mathcal{F}_{t-\Delta}\} \\
&\geq -P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\} + \sum_{m=0}^{\infty} (1-p_1)^m P\Big\{ \bigcup_{u,v\in\mathbb{Z}^d} \bigcup_{\substack{1\leq k\leq \xi_{t-\Delta}(u) \\ 1\leq \ell \leq \xi_{t-\Delta}(v) \\ (u,k)\neq(v,\ell)}} \{u + S_\Delta^{(u,k)} = \mathbf{0}, v + S_\Delta^{(v,\ell)} = y \\
&\qquad \text{and } s\mapsto u + S_s^{(u,k)} \text{ and } s \mapsto v + S_s^{(v,\ell)} \text{ meet exactly } m \text{ times during } [0,\Delta]\}\Big\} \\
&\geq -P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\} + \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u,k,v,\ell} P\{\mathcal{E}_1(u,k,v,\ell,m)\} \\
&\qquad - \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u_i,k_i} P\{\mathcal{E}_1(u_1,k_1,u_2,k_2,m) \cap \mathcal{E}_1(u_3,k_3,u_4,k_4,m)\},
\end{aligned} \tag{4.54}
$$

where the last sum is over all 4-tuples $(u_1,k_1),\dots,(u_4,k_4)$ with $u_i \in \mathbb{Z}^d$, $1 \leq k_i \leq \xi_{t-\Delta}(u_i)$ and $(u_1,k_1) \neq (u_2,k_2)$, $(u_3,k_3) \neq (u_4,k_4)$, $\big((u_1,k_1),(u_2,k_2)\big) \neq \big((u_3,k_3),(u_4,k_4)\big)$. Let us first estimate the contribution to this sum from the 4-tuples with all four (u_i,k_i) distinct. Then for given $u_1,\dots,u_4$ we get a contribution

$$
\begin{aligned}
&\sum_{\substack{k_1,\dots,k_4 \\ \text{with all } (u_i,k_i) \text{ distinct}}} P\{\mathcal{E}_1(u_1,k_1,u_2,k_2,m) \cap \mathcal{E}_1(u_3,k_3,u_4,k_4,m)\} \\
&\leq \Lambda_{t-\Delta}(u_1,u_2,u_3,u_4)\alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y).
\end{aligned}
$$

After taking the expectation and multiplying by $(1-p_1)^m$ and summing over u_i, m these terms contribute at most

$$
\begin{aligned}
&\frac{1}{p_1} \sum_{u_1,\dots,u_4} E\{\Lambda_{t-\Delta}(u_1,u_2,u_3,u_4)\}\alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y) \\
&\leq \frac{1}{p_1} C_6(\varepsilon,4)\big[(t/2)^{-4} \vee (t/2)^{-d(1-\varepsilon)/2}\big] \sum_{u_1,\dots,u_4} \alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y) \\
&\leq C_{32}\big[t^{-4} \vee t^{-d(1-\varepsilon)/2}\big].
\end{aligned}
$$

Similarly the sum of the $P\{\mathcal{E}_1(u_1,k_1,u_2,k_2,m) \cap \mathcal{E}_1(u_3,k_3,u_4,k_4,m)\}$ over the (u_i,k_i) with only three distinct pairs contributes at most $C_{32}\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big]$. Combining these estimates we obtain

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1 | \mathcal{F}_{t-\Delta}\} \\
&\geq -P\{\mathcal{G}|\mathcal{F}_{t-\Delta}\} + \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u,k,v,\ell} P\{\mathcal{E}_1(u,k,v,\ell,m)\} - 2C_{32}\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big].
\end{aligned}
$$

Continuing as in (4.52) and (4.53) this yields

$$\begin{aligned} & p_1 P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\} \\ & \geq D(y) \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y) \\ & \qquad\qquad - C_{33}\Delta\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big] - C_{34}\Delta^{-\delta(d)}t^{-2}. \end{aligned} \tag{4.55}$$

Together with (4.45) and (4.53) this gives (4.44). ■

Proof of Theorem. Let $d \geq 6$. Then choose $\Delta = t^{1-\eta}$ with $0 < \eta < 1$ so small that for large t

$$\log(t+2)\Delta^{-d/2} \leq t^{-5/2}. \tag{4.56}$$

After that choose $\varepsilon \in (0, 1/2)$ so small that, again for large t,

$$\Delta t^{-d(1-\varepsilon)/2} \leq t^{-2-\eta/2}. \tag{4.57}$$

Lemmas 10 and 14 then show that there exists some $\zeta = \zeta(d) \in (0, \eta]$ and some constant $C_{35} < \infty$ such that

$$\Big|\frac{d}{dt}E(t) + \sum_y q(y)D(y)\sum_{u,v}\alpha_\Delta(u)\alpha_\Delta(v-y)E\{\Lambda^*_{t-\Delta}(u,v)\}\Big| \leq C_{35}t^{-2-\zeta}. \tag{4.58}$$

In addition, by the definition of $\Lambda^*_{t-\Delta}(u,v)$,

$$\begin{aligned} & \sum_{u,v}\alpha_\Delta(u)\alpha_\Delta(v-y)\Lambda^*_{t-\Delta}(u,v) \\ & = \sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v) - \sum_u \alpha_\Delta(u)\alpha_\Delta(u-y)\xi_{t-\Delta}(u). \end{aligned}$$

Therefore, by (3.9), (3.4) and (4.17), there exists a constant C_{36}, independent of y such that

$$\begin{aligned} & \Big|\sum_{u,v}\alpha_\Delta(u)\alpha_\Delta(v-y)E\{\Lambda^*_{t-\Delta}(u,v)\} \\ & \qquad - E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\}E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big| \\ & \leq \sigma\Big(\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big)\sigma\Big(\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big) + \frac{C_2}{t}\sum_u \alpha_\Delta(u)\alpha_\Delta(u-y) \\ & \leq C_0\log(t+2)\sum_u \alpha^2_\Delta(u) + \frac{C_2}{t}\sup_u \alpha_\Delta(u) \\ & \leq C_{36}\frac{\log(t+2)}{\Delta^{d/2}}. \end{aligned} \tag{4.59}$$

Substitution of this estimate into (4.58) and use of (4.56) yields

$$\Big|\frac{d}{dt}E(t)+\sum_y q(y)D(y)E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\}E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big|$$
$$\leq C_{35}t^{-2-\zeta}+C_{36}\frac{\log(t+2)}{\Delta^{d/2}}\leq 2C_{35}t^{-2-\zeta}. \tag{4.60}$$

Moreover, with $C(d)$ as in (1.10),

$$\sum_y q(y)D(y)=C(d). \tag{4.61}$$

Now for $\xi_t(y)\neq 0$ to occur, there must be at least one particle in the system at time $t-\Delta$ which moves to y during $[t-\Delta,t]$ without coalescing. The same kind of arguments as in Lemma 14 (but easier) now show that

$$E\{\xi_t(y)|\mathcal{F}_{t-\Delta}\}\geq \sum_v\sum_{\ell\leq\xi_{t-\Delta}(v)} P\{v+S^{(v,\ell)}=y,\ \text{and the path } s\mapsto v+S_s^{(v,\ell)}$$
$$\text{does not coincide with another path}$$
$$s\mapsto w+S_s^{(w,k)} \text{ for any } s\leq\Delta,\ k\leq\xi_{t-\Delta}(w)\}$$
$$\geq\sum_v\sum_{\ell\leq\xi_{t-\Delta}(v)}\alpha_\Delta(v-y)$$
$$-\sum_{v,w}e^2\int_0^{\Delta+1}\Lambda_{t-\Delta}(v,w)\}\sum_z\alpha_s(v-z)\alpha_s(w-z)\alpha_{\Delta-s}(z-y)ds$$

Therefore

$$E\xi_t(y)\geq E\Big\{\sum_v\alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}-C_{37}\Delta t^{-2}.$$

Of course we also have

$$E\xi_t(y)\leq E\Big\{\sum_v\alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}$$

so that

$$\Big|E\xi_t(y)-E\Big\{\sum_v\alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big|$$
$$\leq C_{37}\Delta t^{-2}=C_{37}t^{-1-\eta}\leq C_{37}t^{-1-\zeta}. \tag{4.62}$$

Combined with (4.60), (4.61) and (3.9) this yields

$$\Big|\frac{d}{dt}E(t)+C(d)E^2(t)\Big|\leq C_{39}t^{-2-\zeta}\leq C_{40}t^{-\zeta}E^2(t),\quad t\geq 1.$$

Integration now gives

$$\frac{1}{E(t)} - \frac{1}{E(0)} = -\int_0^t E^{-2}(s)\frac{dE(s)}{ds}ds = C(d)t + O(t^{1-\zeta}),$$

from which (1.11) follows. (1.9) and (1.12) then follow from Lemma 11.

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