The structure of graphs with a vital linkage of order 2*

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Abstract

A *linkage of order* k of a graph G is a subgraph with k components, each of which is a path. A linkage is *vital* if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.

1 Introduction

Robertson and Seymour [4] defined a *linkage* in a graph G as a subgraph in which each component is a path. The *order* of a linkage is the number of components. A linkage L of order k is *unique* if no other collection of paths connects the same pairs of vertices, it is *spanning* if $V(L) = V(G)$, and it is *vital* if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

Theorem 1.1 (Robertson and Seymour [4, Theorem 1.1])**.** *There exists an integer* w*, depending only on* k*, such that every graph with a vital linkage of order* k *has tree width at most* w*.*

Note that Robertson and Seymour use the term p-linkage to denote a linkage with p terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan [2] gave a shorter proof that avoids using the structure theorem.

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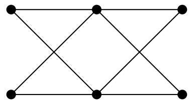

Figure 1: *The graph XX.*

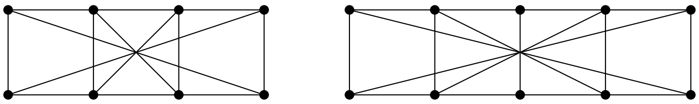

Figure 2: *The graphs $\ddot{U}_4$ and $\ddot{U}_5$.*

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls *almost regular*. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2. These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain $\Delta - Y$ operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

To state our main result we need a few more definitions. Fix a graph G and a spanning linkage L of order k. A *path edge* is a member of $E(L)$; edges in $E(G) - E(L)$ are called *rung edges*. A *linkage minor* of G with respect to L is a minor H of G such that all path edges in $E(G) - E(H)$ have been contracted, and all rung edges in $E(G) - E(H)$ have been deleted. If the linkage L is clear from the context we simply say that H is a linkage minor of G.

The graph *XX* is depicted in Figure 1. For each integer n, the graph $\ddot{U}_n$ is the graph with $V(\ddot{U}_n) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\}$, and

$$E(\ddot{U}_n) = \{v_i v_{i+1} \mid i = 1, \ldots, n-1\} \cup \{u_i u_{i+1} \mid i = 1, \ldots, n-1\} \cup \\ \{u_i v_i \mid i = 1, \ldots, n\} \cup \{u_i v_{n+1-i} \mid i = 1, \ldots, n\}. \tag{1}$$

We denote by L_n the linkage of $\ddot{U}_n$ consisting of all edges $v_i v_{i+1}$ and $u_i u_{i+1}$ for $i = 1, \ldots, n-1$. In Figure 2 the graphs $\ddot{U}_4$ and $\ddot{U}_5$ are depicted.

Finally, we say that G is a *Truemper graph* if G is a linkage minor of $\ddot{U}_n$ for some n. The main result of this paper is the following:

Theorem 1.2. *Let G be a graph. The following statements are equivalent:*

(i) *G has a vital linkage of order 2;*

(ii) *G has a spanning linkage of order 2 with no XX linkage minor;*

(iii) *G is a Truemper graph.*

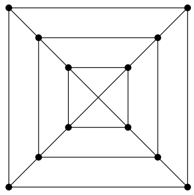

Figure 3: *The graph $\ddot{U}_6$. The linkage is formed by the two diagonally drawn paths.*

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with $k \leq 5$ terminal vertices have path width at most k. A weaker claim is the following:

Corollary 1.3. *Let G be a graph with a vital linkage of order* 2*. Then G has path width at most 4.*

Another consequence of our result is that graphs with vital linkage of order 2 are projective-planar:

Corollary 1.4. *Let G be a graph with a vital linkage of order 2. Then G can be embedded on a Möbius strip.*

Both corollaries can be seen to be true by considering an alternative depiction of $\ddot{U}_{2n}$, analogous to Figure 3.

2 Proof of Theorem 1.2

We start with a few more definitions. Suppose L is a linkage of order 2 with components P_1 and P_2, such that the terminal vertices of P_1 are s_1 and t_1, and those of P_2 are s_2 and t_2. We order the vertices on the paths in a natural way, as follows. If v and w are vertices of P_i, then we say that v is *(strictly) to the left* of w if the graph distance from s_i to v in the subgraph P_i is (strictly) smaller than the graph distance from s_i to w. The notion *to the right* is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.

Lemma 2.1. *Let G be a graph with spanning linkage L of order 2. Let P_1 and P_2 be the components of L, with terminal vertices respectively s_1, t_1 and s_2, t_2. Let H be a linkage minor of G. If v and w are on P_i, and v is to the left of w, then the vertex corresponding to v in H is to the left of the vertex corresponding to w in H.*

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.

Lemma 2.2. *Let G be a graph with a spanning linkage L of order 2. Then L is vital if and only if G has no XX linkage minor with respect to L.*

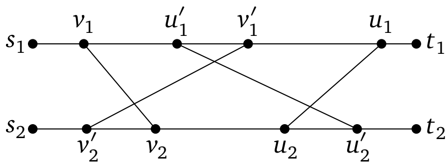

Figure 4: *Detail of the proof of Lemma 2.2.*

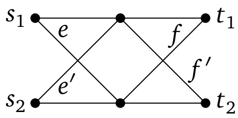

Figure 5: *Detail of the proof of Lemma 2.2.*

Proof. First we suppose that there exists a graph G with a non-vital spanning linkage L of order 2 such that G has no XX linkage minor. Let P_1, P_2 be the paths of L, where P_1 runs from s_1 to t_1, and P_2 runs from s_2 to t_2. Let P_1', P_2' be different paths connecting the same pairs of vertices. Let $e = v_1v_2$ be an edge of P_1' such that the subpath $s_1 - v_1$ of P_1' is also a subpath of P_1, but e is not an edge of P_1. Let $f = u_2u_1$ be an edge of P_1' such that the subpath $u_1 - t_1$ of P_1' is also a subpath of P_2, but f is not an edge of P_2. Similarly, let $e' = v_2'v_1'$ be an edge of P_2' such that the subpath $s_2 - v_2'$ of P_2' is also a subpath of P_2, but e' is not an edge of P_2. Let $f' = u_1'u_2'$ be an edge of P_2' such that the subpath $u_2' - t_2$ of P_2' is also a subpath of P_2, but f' is not on P_2. See Figure 4.

Since P_1' and P_2' are vertex-disjoint, v_2' must be strictly to the left of v_2 and u_2. For the same reason, v_1' must be strictly between v_1 and u_1. Likewise, u_2' must be strictly to the right of v_2 and u_2, and u_1' must be strictly between v_1 and u_1. Now construct a linkage minor H of G, as follows. Contract all edges on the subpaths $s_1 - v_1$, $v_1' - u_1'$, and $u_1 - t_1$ of P_1, contract all edges on the subpaths $s_2 - v_2'$, $v_2 - u_2$, and $u_2' - t_2$ of P_2, delete all rung edges but $\{e, f, e', f'\}$, and contract all but one of the edges of each series class in the resulting graph. Clearly H is isomorphic to XX, a contradiction.

Conversely, suppose that G has an XX linkage minor H, but that L is unique. Let $e = v_1v_2$, $f = u_2u_1$, $e' = v_2'v_1'$, and $f' = u_1'u_2'$ be the rung edges of G used by H, so that the edges in H are as in Figure 5. By Lemma 2.1, v_1 is strictly to the left of v_1' and u_1', and u_1 is strictly to the right of v_1' and u_1'. Likewise, v_2' is strictly to the left of v_2 and u_2, and u_2' is strictly to the right of v_2 and u_2. Now let P_1' be the union of the paths $s_1 - v_1$, $v_1 - v_2$, $v_2 - u_2$, $u_2 - u_1$, and $u_1 - t_1$. Let P_2' be the union of the paths $s_2 - v_2'$, $v_2' - v_1'$, $v_1' - u_1'$, $u_1' - u_2'$, and $u_2' - t_2$. Then $L := P_1' \cup P_2'$ is a linkage distinct from L connecting the same pairs of vertices, a contradiction. □

Next we show that the third statement of Theorem 1.2 implies the second.

Lemma 2.3. *For all n, $\ddot{U}_n$ has no XX linkage minor with respect to L_n.*

Proof. The result holds for $n \leq 2$, because then $|V(\ddot{U}_n)| < |V(XX)|$. Suppose the lemma fails for some $n \geq 3$, but is valid for all smaller n. Every edge of XX is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any XX linkage minor. But after deleting those edges from $\ddot{U}_n$ the end vertices have degree one, and hence the edges incident with them will not be in any XX linkage minor. Contracting these four edges produces $\ddot{U}_{n-2}$, a contradiction. □

For the final implication we need some more definitions. Let $e = v_1v_2$ and $f = u_1u_2$ be two rung edges, with v_1, u_1 on P_1 and v_2, u_2 on P_2. We say that e and f *cross* if v_1 is strictly to the left of u_1 and v_2 is strictly to the right of u_2, or vice versa. We say that e and f *meet* if $v_1 = u_1$ or $v_2 = u_2$. A partition (A, B) of the rung edges is *valid* if the edges in A pairwise cross or meet, and no pair of edges in B is crossing.

Reversing a path P_i means exchanging the labels of vertices s_i and t_i, thereby reversing the order on the vertices of the path. The following observation is straightforward; we omit the proof.

Lemma 2.4. *Let G be a graph, and L a spanning linkage of G of order 2, consisting of paths P_1, running from s_1 to t_1, and P_2, running from s_2 to t_2. Let G' be obtained from G by reversing P_2. If (A, B) is a valid partition of the rung edges of G, then (B, A) is a valid partition of the rung edges of G'.*

Lemma 2.5. *Let G be a graph, and L a spanning linkage of order 2 of G consisting of paths P_1, running from s_1 to t_1, and P_2, running from s_2 to t_2. If G has no XX linkage minor, then there is a valid partition of the rung edges.*

Proof. Suppose the lemma fails for G. Choose G with as few edges as possible. The result holds if either P_1 or P_2 consists of a single vertex, so this is not the case. Let $e = s_1v$ be the leftmost edge of P_1. In G/e we can partition the rung edges into sets A and B with the desired properties. A *conflict* is a pair of edges e_1, e_2 such that either $e_1, e_2 \in A$ and e_1, e_2 do not cross and do not meet in G, or $e_1, e_2 \in B$ and e_1, e_2 cross in G.

Since the rung edges of G cannot be partitioned, there must be a conflict. We assume that the valid partition (A, B) of G/e has been chosen so that the number of conflicts in G is minimal. By reversing P_2 and exchanging A and B as necessary, we may assume that there are edges $e_1, e_2 \in B$ that cross in G. One of these, say e_1, must be incident with s_1, and then e_2 must be incident with v.

We will now refine our partition, making sure we don't increase the number of conflicts in G. Suppose there is an edge $f \in A$ that does not meet or cross e_1. Suppose, next, that there is an edge $f' \in B$ that crosses f. Since edges in B do not cross in G/e, f' must meet P_2 to the right of e_1. But then G has an XX minor, a contradiction. Hence f does not cross any edge of B. It follows that the partition $(A - \{f\}, B \cup \{f\})$ is a valid partition in G/e, and is moreover a partition with no more conflicts in G than (A, B) had.

It follows that we may assume without loss of generality that every edge $f \in A$ meets or crosses e_1. But then the partition $(A \cup \{e_1\}, B - \{e_1\})$

is a valid partition of G/e that has strictly fewer conflicts in G than (A,B) had, a contradiction. □

Lemma 2.6. *Let G be a graph, and L a spanning linkage of order 2 of G consisting of paths P_1, running from s_1 to t_1, and P_2, running from s_2 to t_2. Let (A,B) be a valid partition of the rung edges. If G has no XX linkage minor, then G is a linkage minor of $\ddot{U}_n$ with respect to L_n for some integer n.*

The lemma can be strengthened to require that $\ddot{U}_n$ has a valid partition (A',B') with $A \subseteq A'$ and $B \subseteq B'$. We omit the proof of this fact.

Proof. Suppose the statement is false. Let G be a counterexample with as few edges as possible. If some end vertex of a path, say s_1, has degree one (with $e = s_1 v$ the only edge), then we can embed G/e in $\ddot{U}_n$ for some n. Let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1, s_1's_2', s_1't_2', s_2'u_1, s_2't_1', v_nt_1', u_nt_2', t_1't_2'$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor.

Hence we may assume that each end vertex of P_1 and P_2 has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that G has an *XX* minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge $e = s_1s_2$.

Since e crosses no edge, we can change our partition so that $e \in B$. By our assumption $G \backslash e$ can be embedded in $\ddot{U}_n$ for some n. Again, let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1, s_1's_2', s_1't_2', s_2'u_1, s_2't_1', v_nt_1', u_nt_2', t_1't_2'$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor, a contradiction. □

Now we have all ingredients of our main result.

Proof of Theorem 1.2. From Lemma 2.2 we learn that (*i*)⟺(*ii*). From Lemma 2.3 we learn that (*iii*)⇒(*ii*), and from Lemmas 2.5 and 2.6 we conclude that (*ii*)⇒(*iii*). □

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