Robust Price of Anarchy for Atomic Games with Altruistic Players

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Abstract

We study the inefficiency of equilibria for various classes of games when players are (partially) altruistic. We model altruistic behavior by assuming that player i's perceived cost is a convex combination of $1-\beta_i$ times his direct cost and β_i times the social cost. Tuning the parameters β_i allows smooth interpolation between purely selfish and purely altruistic behavior. Within this framework, we study altruistic extensions of linear congestion games, fair cost-sharing games and valid utility games.

We derive (tight) bounds on the price of anarchy of these games for several solution concepts. Thereto, we suitably adapt the *smoothness* notion introduced by Roughgarden and show that it captures the essential properties to determine the *robust price of anarchy* of these games. Our bounds reveal that for congestion games and cost-sharing games, the worst-case robust price of anarchy increases with increasing altruism, while for valid utility games, it remains constant and is not affected by altruism. However, the increase in the price of anarchy is not a universal phenomenon: for symmetric singleton linear congestion games, we derive a bound on the price of anarchy for pure Nash equilibria that decreases as the level of altruism increases. Since the bound is also strictly lower than the robust price of anarchy, it exhibits a natural example in which Nash equilibria are more efficient than more permissive notions of equilibrium.

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1 Introduction

Many large-scale decentralized systems, such as infrastructure investments or traffic on roads or computer networks, bring together large numbers of individuals with different and oftentimes competing objectives. When these individuals choose actions to benefit themselves, the result is frequently suboptimal for society as a whole. This basic insight has led to a study of such systems from the viewpoint of game theory, focusing on the inefficiency of stable outcomes. Traditionally, "stable outcomes" have been associated with pure Nash equilibria of the corresponding game. The notions of *price of anarchy* [22] and *price of stability* [2] provide natural measures of the system degradation, by capturing the degradation of the worst and best Nash equilibria, respectively, compared to the socially optimal outcome.

However, the predictive power of such bounds has been questioned on (at least) two grounds:

1. The adoption of Nash equilibria as a prescriptive solution concept implicitly assumes that players are able to reach such equilibria. In particular in light of several known hardness results for finding Nash equilibria, this assumption is very suspect for computationally bounded players. In response, recent work has begun analyzing the outcomes of natural response dynamics [7, 8, 34], as well as more permissive solution concepts such as correlated or coarse correlated equilibria [3, 18, 35]. This general direction of inquiry has become known as "robust price of anarchy".
2. The assumption that players seek only to maximize their own utility is at odds with altruistic behavior routinely observed in the real world. While modeling human incentives and behavior accurately is a formidable task, several papers have proposed natural models of altruism [23, 24] and analyzed its impact on the outcomes of games [11, 12, 13, 15, 20].

The goal of this paper is to begin a thorough investigation of the effects of relaxing both of the standard assumptions simultaneously, i.e., considering the combination of weaker solution concepts and notions of partially altruistic behavior by players. In Section 2, we formally define the *altruistic extension* with parameters (β_i) of an n-player game in the spirit of past work on altruism (see [23, p. 154] and [11, 12]): informally, player i's cost (or payoff) is a convex combination of $(1-\beta_i)$ times his direct cost (or payoff) and β_i times the social cost (or social welfare). By tuning the parameters β_i, this model allows smooth interpolation between pure selfishness ($\beta_i = 0$) and pure altruism ($\beta_i = 1$).

In order to analyze the degradation of system performance in light of partially altruistic behavior, we extend the notion of *robust price of anarchy* [34] to altruistic extensions, and show that a suitably adapted notion of *smoothness* [34] captures the properties of a system that determine its robust price of anarchy. We use these insights to analyze three classes of games:

1. In Sections 3 and 4, we analyze *linear congestion games* [33], in which players choose subsets of resources whose costs increase (linearly) with the number of players using them.
2. In Section 5, we study *fair cost-sharing games* [2], in which players choose subsets of "infrastructure" to build, and all players choosing the same item share the item's cost evenly.
3. In Section 6, we apply our framework to *valid utility games* [36], in which players again choose "infrastructure" to build, deriving (submodular) utility of the chosen set. The total welfare is determined by a submodular function of the union of all chosen sets.

We derive (mostly tight) bounds on the robust price of anarchy for these games under general altruism distributions. In fact, the same bounds are tight when the altruism level is uniform, i.e., $\beta_i = \beta$ for all i. Our bounds reveal a somewhat counterintuitive trend: for congestion games, the worst-case robust price of anarchy actually *increases* (as $(5+4\beta)/(2+\beta)$) with increasing altruism β, and it does so unboundedly (as $n/(1-\beta)$) for cost-sharing games. On the other hand, for valid utility games, the worst-case robust price of anarchy remains at 2, unaffected by altruism.

The intuition behind the increase in the price of anarchy is the following: there are instances in which all players get stuck choosing the wrong resources. A deviation by one player affects not only him, but also others: for congestion games, the player may increase the cost on the resources he switches to, while for cost-sharing games, there will be fewer remaining players to share the cost of the player's current item. Thus, partially altruistic players have even stronger disincentive to deviate from the suboptimal strategy, meaning that even worse system states are stable.

The above explanation intuitively corresponds to altruistic players "accepting" more states as "stable". This suggests that the best stable solution can also be chosen from a larger set, and the price of stability should thus decrease. Our results lend partial support to this intuition: for congestion games, we derive an upper bound on the price of stability which decreases as $2/(1+\beta)$; similarly, for cost-sharing games, we establish an upper bound which decreases as $(1-\beta)H_n+\beta$.

It should be noted that the increase in price of anarchy is not a universal phenomenon. Indeed, for linear symmetric singleton congestion games (in which all players have the same strategy set, consisting of all sets of exactly one resource), we establish a bound of $4/(3+\beta)$ for the price of anarchy with respect to pure Nash equilibria. This bound is noteworthy not only because it shows improvements resulting from the presence of altruism; it also establishes that pure Nash equilibria can result in strictly lower price of anarchy than weaker solution concepts, as Lücking et al. [25] gave an example of a class of linear symmetric singleton congestion games whose price of anarchy under mixed Nash equilibria can be arbitrarily close to 2. In particular, this establishes a natural example of a class of games whose price of anarchy cannot be established using the smoothness framework.

The paper is rounded out by a more in-depth analysis of the effect of combining players with different altruism levels in singleton congestion games (Section 4), and a mathematical investigation of the set of smoothness parameters that can occur in games (Section 7).

Related Work. Much of our analysis is based on extensions of the notion of *smoothness* as proposed by Roughgarden [34] (see Section 2.2). The basic idea is to bound the sum of cost increases of individual players switching strategies by a combination of the costs of two states. Because these types of bounds capture local improvement dynamics, they bound the price of anarchy not only for Nash equilibria, but also more general solution concepts, including coarse correlated equilibria. The smoothness notion was recently refined in the *local smoothness* framework by Roughgarden and Schoppmann [35]. They require the types of bounds described above only for nearby states, thus obtaining tighter bounds, albeit only for more restrictive solution concepts and convex strategy sets. Using the local smoothness framework, they obtained optimal upper bounds for atomic splittable congestion games. Nadav and Roughgarden [28] showed that smoothness bounds apply all the way to solution concepts called "average coarse correlated equilibrium", but not beyond.

A comparison between the costs in worst-case outcomes under solution concepts of different generality was recently undertaken by Bradonjic et al. [9] under the name "price of mediation": specifically for the case of symmetric singleton congestion games with convex latency functions, they showed that the ratio between the most expensive correlated equilibrium and the most expensive Nash equilibrium can grow exponentially in the number of players.

Hayrapetyan et al. [19] studied the impact of "collusion" in network congestion games, where players form coalitions to minimize their collective cost. These coalitions are assumed to be formed exogeneously, i.e., conceptually, each coalition is replaced by a "super-player" that acts on behalf of its members. The authors show that collusion in network congestion games can lead to Nash equilibria that are inferior to the ones of the collusion-free game (in terms of social cost). They also derive bounds on the the price of anarchy caused by collusion. Note that the cooperation within each coalition can be interpreted as a kind of "locally" altruistic behavior, i.e., each player only cares about the cost of the members of his coalition. In a sense, the setting considered in [19] can therefore be regarded as being orthogonal to the viewpoint that we adopt in this paper: in their setting, players are assumed to be entirely altruistic but locally attached to their coalitions. In contrast, in our setting, players may have different levels of altruism but locality does not play a role.

Several recent studies investigate "irrational" player behavior in games; examples include studies on malicious (or spiteful) behavior [5, 10, 13, 21] and unpredictable (or Byzantine) behavior [8, 27, 31]. The work that is most related to our work in this context is the one by Blum et al. [8]. The authors consider repeated games in which every player is assumed to minimize his own regret. They derive bounds on the inefficiency, called *total price of anarchy*, of the resulting outcomes for certain classes of games, including congestion games and valid utility games. The exhibited bounds exactly match the respective price of anarchy and even continue to hold if only some of the players minimize their regret while the others are Byzantine. The latter result is surprising: in the context of valid utility games because it means that the price of total anarchy remains at 2, even if additional players are added to the game that behave arbitrarily. Our findings allow us to draw an even more dramatic conclusion. Our bounds on the robust price of anarchy also extend to the total price of anarchy of the respective repeated games (see Section 2.3). As a consequence, our result for valid utility games implies that the price of total anarchy would remain at 2, even if the "Byzantine" players were to act altruistically. That is, while the result in [8] suggests that arbitrary behavior does not harm the inefficiency of the final outcome, our result shows that altruistic behavior does not help.

If players' altruism levels are not uniform, then even the existence of pure Nash equilibria is not obvious. Hoefer and Skopalik established it for several subclasses of atomic congestion games [20]; for the generalization of arbitrary player-specific cost functions, Milchtaich [26] showed existence for singleton congestion games, and Ackermann et al. [1] for matroid congestion games, in which the strategy space of each player is the basis of a matroid on the set of resources.

Models of Altruism. Models of altruism either identical or very similar to the one in this paper have been studied in several papers. Perhaps the first published suggestion of a similar model is due to Ledyard [23], but since then, different variations of it have been studied more extensively, e.g., [11, 12, 13, 15]. The main difference is that in some of these models, linear combinations (rather than convex combinations) are considered, e.g., with the selfish term having a factor of 1. For most of these variations, a straightforward scaling of the coefficients shows equivalence with the model we consider here. The altruism model can be naturally extended to include $\beta_i < 0$, modeling spiteful behavior (see, e.g., [13]). While the modeling extension is natural, several results in this and other papers do not continue to hold directly for negative β_i. Our model is strictly more general than some of the previous work in that the social cost function need not be the sum of all players' costs, but rather only needs to be bounded by the sum.

Besides models based on linear combinations of individual players' costs (as well as social welfare), several other approaches have been studied. Generally, altruism or other "other-regarding" social behavior has received some attention in the behavioral economics literature (e.g., [17]). Alternative models of altruism and spite have been proposed by Levine [24], Rabin [30] and Geneakoplos et al. [16]. These models are designed more with the goal of modeling the psychological processes underlying spite or altruism (and reciprocity): they involve players forming beliefs about other players. As a result, they are well-suited for experimental work, but perhaps not as directly suited for the type of analysis in this paper.

2 Preliminaries

Let $G = (N, \{\Sigma_i\}_{i\in N}, \{C_i\}_{i\in N})$ be a finite strategic game, where $N = \{1,\dots,n\}$ is the set of players, Σ_i the strategy space of player i, and $C_i : \Sigma \to \mathbb{R}$ the cost function of player i, mapping every joint strategy $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ to the player's direct cost. Unless stated otherwise, we assume that every player $i \in N$ wants to minimize his individual cost function C_i. We also call such games *cost-minimization games*. A *social cost* function $C : \Sigma \to \mathbb{R}$ maps strategies to social costs. We require that C be *sum-bounded*, i.e., $C(s) \le \sum_{i=1}^n C_i(s)$ for all $s \in \Sigma$.

In this paper, we study the *altruistic extension* of strategic games equipped with sum-bounded social cost functions, defined as follows:

Definition 1. Let $G = (N, \{\Sigma_i\}_{i\in N}, \{C_i\}_{i\in N})$ be a cost-minimization game with a sum-bounded social cost function C. Let β be a vector in $[0,1]^n$. The β*-altruistic extension* of G (or simply β*-altruistic game*) is defined as the strategic game $G^\beta = (N, \{\Sigma_i\}_{i\in N}, \{C_i^\beta\}_{i\in N})$, where for every $i \in N$ and $s \in \Sigma$,

$$C_i^\beta(s) \quad = \quad (1-\beta_i)C_i(s) + \beta_i C(s).$$

Thus, the perceived cost that a player experiences is a convex combination of his direct (selfish) cost and the social cost; we call such a player β_i-*altruistic*.[1] When $\beta_i = 0$, player i is entirely selfish; thus, $\beta = \mathbf{0}$ recovers the original game. Similarly, a player with $\beta_i = 1$ is entirely altruistic. Given an altruism vector $\beta \in [0,1]^n$, we let $\hat{\beta} = \max_{i \in N} \beta_i$ and $\check{\beta} = \min_{i \in N} \beta_i$ denote the maximum and minimum altruism level of a player, respectively. An important special case occurs when all players have the same altruism level $\beta_i = \beta$. We call such games *uniformly* β*-altruistic games*, with β being a scalar (instead of a vector) that characterizes the common altruism level.

The altruistic extension of a *payoff-maximization game*, in which players seek to maximize their payoff functions $\{\Pi_i\}_{i \in N}$, with a social welfare function Π is defined analogously to Definition 1.

2.1 Equilibrium Concepts

We study the inefficiency of equilibria in altruistic extensions of various games. The most general equilibrium concept that we will deal with is the following one.

Definition 2. A *coarse equilibrium* (or *coarse correlated equilibrium*) of a game G is a probability distribution σ over $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ with the following property: if s is a random variable with distribution σ, then for each player i, and all $s_i^* \in \Sigma_i$: $\mathbf{E}_{s \sim \sigma}[C_i(s)] \leq \mathbf{E}_{s_{-i} \sim \sigma_{-i}}[C_i(s_i^*, s_{-i})]$, where σ_{-i} is the projection of σ on $\Sigma_{-i} = \Sigma_1 \times \cdots \times \Sigma_{i-1} \times \Sigma_{i+1} \times \cdots \times \Sigma_n$.

The set of all coarse equilibria is also known as the *Hannan Set* (see, e.g., [37]). It includes several other solution concepts, such as correlated equilibria, mixed Nash equilibria and pure Nash equilibria. (A brief review of the definitions of these notions is given in Appendix A.)

The *price of anarchy (PoA)* [22] and *price of stability (PoS)* [2] are natural ways of quantifying the inefficiency of equilibria for classes of games:

Definition 3. Let $S \subseteq \Sigma$ be a set of strategy profiles for a cost-minimization game G with social cost function C, and let s^* be a strategy profile that minimizes C. We define

$$\mathrm{PoA}(S,G) = \sup_{s \in S} \frac{C(s)}{C(s^*)} \quad \text{and} \quad \mathrm{PoS}(S,G) = \inf_{s \in S} \frac{C(s)}{C(s^*)}.$$

The *coarse* (or *correlated, mixed, pure*) *price of anarchy* (or *price of stability*) of a class of games $\mathscr{G}$ is the supremum over all games in $\mathscr{G}$ and all strategy profiles in the respective set of equilibrium outcomes.

Notice that the PoA and PoS are defined with respect to the *original* social cost function C, not accounting for the altruistic components. This reflects our desire to understand the overall performance of the system (or strategic game), which is not affected by different *perceptions* of costs by individuals. Note, however, that if all players have a uniform altruism level $\beta_i = \beta \in [0,1]$ and the social cost function C is equal to the sum of all players' individual costs, then for every strategy profile $s \in \Sigma$, $C^{\beta}(s) = (1 - \beta + \beta n)C(s)$, where $C^{\beta}(s) = \sum_{i \in N} C_i^{\beta}(s)$ denotes the sum of all players' perceived costs. In particular, bounding the PoA (or PoS) with respect to C is equivalent to bounding the PoA (or PoS) with respect to total perceived cost C^{β} in this case.

We extend Definition 3 in the obvious way to payoff-maximization games G with social welfare function Π by considering the ratio $\Pi(s^*)/\Pi(s)$, where s^* refers to a strategy profile maximizing Π.

2.2 Smoothness

Many proofs bounding the price of anarchy for specific games (e.g., [33, 36]) use the fact that deviating from an equilibrium to the strategy at optimum is not beneficial for any player. The addition of these inequalities,

[1] We note that the altruistic part of an individual's perceived cost does not recursively take other players' *perceived* cost into account. Such recursive definitions of altruistic utility have been studied, e.g., by Bergstrom [6], and can be reduced to our definition under suitable technical conditions.

combined with suitable properties of the social cost function, then gives a bound on the equilibrium's cost. Roughgarden [34] recently captured the essence of this type of argument with his definition of (λ,μ)-*smoothness* of a game, thus providing a generic template for proving bounds on the price of anarchy. Indeed, because such arguments only reason about local moves by players, they immediately imply bounds not only for Nash equilibria, but all classes of equilibria defined in Section 2.1, as well as the outcomes of no-regret sequences of play [8, 7]. Recent work has explored both the limits of this concept [28] and a refinement requiring smoothness only in local neighborhoods [35]. The latter permits more fine-grained analysis of games, but applies only to correlated equilibria and their subclasses.

We extend the concept of (λ,μ)-smoothness to altruistic extensions of strategic games. This allows us to quantify the price of anarchy of these games with respect to the very broad class of coarse correlated equilibria. For notational convenience, we define $C_{-i}(s) = C(s) - C_i(s) \leq \sum_{j\neq i} C_j(s)$.

Definition 4. Let G^{β} be a β-altruistic extension of a game with sum-bounded social cost function C. G^{β} is (λ,μ,β)-*smooth* iff for any two strategy profiles $s, s^* \in \Sigma$,

$$\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) + \beta_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s)) \leq \lambda C(s^*) + \mu C(s). \tag{1}$$

For $\beta = \mathbf{0}$, this definition coincides with Roughgarden's notion of (λ,μ)-smoothness. To gain some intuition, consider two strategy profiles $s, s^* \in \Sigma$, and a player $i \in N$ who switches from his strategy s_i under s to s_i^*, while the strategies of the other players remain fixed at s_{-i}. The contribution of player i to the left-hand side of (1) then accounts for the individual cost that player i perceives after the switch plus β_i times the difference in social cost caused by this switch. The sum of these contributions needs to be bounded by $\lambda C(s^*) + \mu C(s)$.

2.3 Preliminary Results

We first show that many of the results in [34] following from (λ,μ)-smoothness carry over to our altruistic setting using the extended (λ,μ,β)-smoothness notion (Definition 4). Even though some care has to be taken in extending these results, most of the proofs of the propositions in this section follow along similar lines as their analogues in [34]; we therefore defer these proofs to Appendix B.

Proposition 1. *Let G^{β} be a β-altruistic game. If G^{β} is (λ,μ,β)-smooth with $\mu < 1$, then the coarse price of anarchy of G^{β} is at most $\frac{\lambda}{1-\mu}$.*

As coarse equilibria include correlated equilibria, mixed Nash equilibria and pure Nash equilibria, the above theorem also holds for these solution concepts.

As we show later, for many important classes of games, the bounds obtained by (λ,μ,β)-smoothness arguments are actually tight, even for pure Nash equilibria. Therefore, as in [34], we define the *robust price of anarchy* as the best possible bound on the coarse price of anarchy obtainable by a (λ,μ,β)-smoothness argument.

Definition 5. The *robust price of anarchy* of a β-altruistic game G^{β} is defined as

$$\mathrm{RPoA}_G(\beta) = \inf\left\{\tfrac{\lambda}{1-\mu} : G^{\beta} \text{ is } (\lambda,\mu,\beta)\text{-smooth}, \mu < 1\right\}.$$

For a class $\mathscr{G}$ of games, we define $\mathrm{RPoA}_{\mathscr{G}}(\beta) = \sup\{\mathrm{RPoA}_G(\beta) : G \in \mathscr{G}\}$. We omit the subscript when the game (or class of games) is clear from the context.

The smoothness condition also proves useful in the context of no-regret sequences and the *price of total anarchy*, introduced by Blum et al. [8].

Proposition 2. *Let s^* be a strategy profile minimizing the social cost function C of a β-altruistic game G^β, and $s^1, \dots, s^T$ a sequence of strategy profiles in which every player $i \in N$ experiences vanishing average external regret, i.e.,*

$$\sum_{t=1}^{T} C_i^\beta(s^t) \le \left(\min_{s_i' \in \Sigma_i} \sum_{t=1}^{T} C_i^\beta(s_i', s_{-i}^t) \right) + o(T).$$

The average cost of this sequence of T strategy profiles then satisfies $\frac{1}{T}\sum_{t=1}^{T} C(s^t) \le RPoA(\beta) \cdot C(s^*)$ as $T \to \infty$.

Roughgarden [34, Proposition 2.6] shows that for games that have an *underestimating exact potential function*, best response dynamics[2] converge rapidly to a strategy profile of social cost close to the robust price of anarchy times the optimum social cost of the game; see [34] for a precise statement of this result and the accompanying definitions. Proposition 2.6 in [34] and its proof straightforwardly carry over to (λ, μ, β)-smooth games that have such an underestimating exact potential function.

The results in this section continue to hold for altruistic extensions of payoff-maximization games if we adapt Definition 4 as follows. Let G^β be a β-altruistic extension of a payoff-maximization game with social welfare function Π. Define $\Pi_{-i}(s) = \Pi(s) - \Pi_i(s)$. G^β is (λ, μ, β)-smooth iff for every two strategy profiles $s, s^* \in \Sigma$,

$$\sum_{i=1}^{n} (\Pi_i(s_i^*, s_{-i}) + \beta_i(\Pi_{-i}(s_i^*, s_{-i}) - \Pi_{-i}(s))) \ge \lambda \Pi(s^*) - \mu \Pi(s). \tag{2}$$

Given this smoothness definition, all the results above hold when we replace $\frac{\lambda}{1-\mu}$ by $\frac{1+\mu}{\lambda}$ and $\mu < 1$ by $\mu > -1$ in Definition 5.

3 Congestion Games

In an atomic congestion game $G = (N, E, \{d_e\}_{e \in E}, \{\Sigma_i\}_{i \in N})$, we are given a set of players $N = \{1, \dots, n\}$, a set of *facilities* E with *delay functions* $d_e : \mathbb{N} \to \mathbb{R}$ for every facility $e \in E$, and a strategy set $\Sigma_i \subseteq 2^E$ for every player $i \in N$. For a joint strategy $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, define $x_e(s) = |\{i : e \in s_i\}|$ as the number of players using facility $e \in E$. The social objective function is $C(s) = \sum_{i=1}^{n} C_i(s)$, where $C_i(s) = \sum_{e \in s_i} d_e(x_e(s))$ is the cost of player $i \in N$. In a *linear* congestion game, the delay function of every facility $e \in E$ is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{Q}^+$ are non-negative rational numbers. By using standard transformation arguments, we can assume without loss of generality that the delay functions are of the form $d_e(x) = x$; the details are in Appendix C.2.

Linear congestion games have the advantage that pure Nash equilibria of their altruistic extensions always exist [20], which may not be the case for arbitrary congestion games. The price of anarchy of linear congestion games in the purely selfish setting ($\beta = 0$) is known to be $\frac{5}{2}$ [4, 14]. We extend this result to altruistic congestion games and the robust price of anarchy.

Theorem 1. *The robust price of anarchy of β-altruistic linear congestion games is at most $\frac{5+2\hat{\beta}+2\check{\beta}}{2-\hat{\beta}+2\check{\beta}}$.*

If all players have uniform altruism level $\beta \in [0,1]$, the above bound reduces to $\frac{5+4\beta}{2+\beta}$. This is tight, as shown in Example 1 below.

In order to prove Theorem 1, we need the following technical lemma, whose proof is in Appendix C.2.

Lemma 1. *For every two integers $x, y \in \mathbb{N}_0$ and $\hat{\beta}, \check{\beta} \in [0,1]$ with $\hat{\beta} \ge \check{\beta}$,*

$$((1+\hat{\beta})x + 1)y + \check{\beta}(1-x)x \le \tfrac{5+2\hat{\beta}+2\check{\beta}}{3} y^2 + \tfrac{1+\hat{\beta}-2\check{\beta}}{3} x^2.$$

[2] Best response dynamics are a natural way of searching for a pure Nash equilibrium: if the current strategy profile is not a Nash equilibrium, then pick a player who can improve his cost and change his strategy to one that minimizes his cost.

Proof of Theorem 1. We show that the β-altruistic extension G^β of a linear congestion game is $(\frac{1}{3}(5+2\hat{\beta}+2\check{\beta}), \frac{1}{3}(1+\hat{\beta}-2\check{\beta}), \beta)$-smooth.

Let s and s^* be two strategy profiles, and write $x_e = x_e(s), x_e^* = x_e(s^*)$. The left-hand side of the smoothness condition (1) is equivalent to

$$\sum_{i=1}^{n} ((1-\beta_i)C_i(s_i^*, s_{-i}) + \beta_i(C(s_i^*, s_{-i}) - C(s)) + \beta_i C_i(s))$$

$$= \sum_{i=1}^{n} \left((1-\beta_i) \left(\sum_{e \in s_i^* \setminus s_i} (x_e+1) + \sum_{e \in s_i \cap s_i^*} x_e \right) + \beta_i \left(\sum_{e \in s_i^* \setminus s_i} (2x_e+1) + \sum_{e \in s_i \setminus s_i^*} (1-2x_e) \right) + \beta_i C_i(s) \right)$$

$$\leq \sum_{i=1}^{n} \left(\sum_{e \in s_i^*} ((1+\beta_i)x_e + 1) + \beta_i \sum_{e \in s_i} (1-x_e) \right) \leq \sum_{e \in E} \left(((1+\hat{\beta})x_e + 1)x_e^* + \check{\beta}(1-x_e)x_e \right).$$

In the above derivation, the first inequality follows from the fact that $(1-\beta_i)x_e \leq (1+\beta_i)x_e + 1 + \beta_i(1-2x_e)$ for every $e \in s_i \cap s_i^*$. The second inequality holds because for every $i \in N$ and $e \in s_i$, $1 - x_e \leq 0$ and by the definition of $\hat{\beta}$ and $\check{\beta}$. The bound on the robust price of anarchy now follows from Lemma 1. $\square$

The following example shows that the upper bound on the robust price of anarchy given above is tight for uniformly β-altruistic games, even for pure Nash equilibria.

Example 1. Consider a game with three β-altruistic players and six resources $E = E_1 \cup E_2$, $E_1 = \{h_0, h_1, h_2\}$, $E_2 = \{g_0, g_1, g_2\}$. The delay functions are given by $d_e(x) = (1+\beta)x$ for $e \in E_1$, and $d_e(x) = x$ for $e \in E_2$. Each player i has two pure strategies: $\{h_{i-1}, g_{i-1}\}$ and $\{h_{(i-2) \pmod 3}, h_{i \pmod 3}, g_{i \pmod 3}\}$. The strategy profile in which every player selects his first strategy is a social optimum of cost $(1+\beta)\cdot 3 + 3 = (2+\beta)\cdot 3$. Consider the strategy profile s in which every player chooses his second strategy. We argue that s is a Nash equilibrium: Each player's perceived individual cost is $c_1 = (1-\beta)(4(1+\beta)+1) + \beta(5+4\beta)\cdot 3$, whereas if a player unilaterally deviates to his first strategy, the new social cost would become $(5+4\beta)\cdot 3 + 1 - \beta$. Thus, the player's new perceived individual cost is $c_2 = (1-\beta)(3(1+\beta)+2) + \beta((5+4\beta)\cdot 3 + 1 - \beta)$. Because $c_1 = c_2$, s is a Nash equilibrium, of cost $4(1+\beta)\cdot 3 + 3 = (5+4\beta)\cdot 3$. We conclude that PoA $\geq \frac{5+4\beta}{2+\beta}$ for $\beta \in [0,1]$.

We turn to the pure price of stability of β-altruistic congestion games. Clearly, an upper bound on the pure price of stability extends to the mixed, correlated and coarse price of stability. The proof of the following proposition exploits a standard technique to bound the pure PoS of exact potential games (see, e.g., [29]) and is deferred to Appendix C.2.

Proposition 3. *The pure price of stability of uniformly β-altruistic linear congestion games is at most $\frac{2}{1+\beta}$.*

4 Singleton Congestion Games

While the price of anarchy for general congestion games *increases* with β, the situation is markedly different for the PoA of *symmetric singleton* congestion games. In a symmetric singleton congestion game $G = (N, E, \{\Sigma_i\}_{i \in N}, \{d_e\}_{e \in E})$, every player chooses one facility (also called *edge*) from $E = \{1, \ldots, m\}$, and all strategy sets are identical, i.e., $\Sigma_i = E$ for every i. We assume that the delay functions are of the form $d_e(x) = a_e x + b_e$. Note that the transformation of Appendix C.2 does not work in this setting.

In this section, we analyze perhaps the two most fundamental cases with respect to altruistic singleton congestion games: the uniform case, and the case when all altruism levels are in $\{0,1\}$, i.e., each player is either completely altruistic or completely selfish. For both settings, we establish bounds on the pure PoA which *improve* with the total altruism level in the system, i.e., decrease in β or the fraction of selfish players. This stands in marked contrast to the bounds in the previous section.

Theorem 2. *The pure price of anarchy of uniformly β-altruistic extensions of symmetric singleton linear congestion games is $\frac{4}{3+\beta}$.*

While in the previous section, we were able to derive tight bounds via smoothness arguments, this is not possible for altruistic extensions of symmetric singleton congestion games. For example, by the above theorem, the price of anarchy in the purely selfish setting is $4/3$, whereas Lücking et al. [25, Theorem 5.4] showed that the mixed price of anarchy for symmetric singleton congestion games with delay functions $d_e(x) = x$ is $1+\min\{\frac{m-1}{n}, \frac{n-1}{m}\}$. That is, for $n = m$, the mixed price of anarchy approaches 2 as n increases. The bound given in Theorem 2 can therefore not be derived via a smoothness argument.

Theorem 2 implies that the pure PoA is 1 if all players are completely altruistic. We remark that this continues to hold true for the more general class of semi-convex[3] delay functions (see Corollary 1 below).

Proof of Theorem 2. Let s be a pure Nash equilibrium of G^β and s^* an optimal strategy profile. We write $x_e = x_e(s)$ and $x_e^* = x_e(s^*)$. For every edge $e \in E$, define $\Delta_e = x_e - x_e^*$. Let E^+ and E^- be the set of edges with $\Delta_e > 0$ and $\Delta_e < 0$, respectively. Define $\Delta = \sum_{e\in E^+} \Delta_e > 0$. Because s and s^* assign the same number of players to edges, $\Delta = \sum_{e\in E^+} \Delta_e = -\sum_{e\in E^-} \Delta_e$. If $\Delta = 0$, then the PoA is 1. Hence, we assume that $\Delta > 0$, in which case both E^+ and E^- are non-empty.

By definition, $x_e > x_e^* \geq 0$ for every edge $e \in E^+$. Because s is a Nash equilibrium of G^β, we have for every edge $e \in E^+$ and $\bar{e} \in E$ that

$$\begin{aligned}
& (1-\beta)(a_e x_e + b_e) + \beta((a_e x_e^2 + b_e x_e) + (a_{\bar{e}} x_{\bar{e}}^2 + b_{\bar{e}} x_{\bar{e}})) \\
\leq\ & (1-\beta)(a_{\bar{e}}(x_{\bar{e}}+1) + b_{\bar{e}}) + \beta\left((a_e(x_e-1)^2 + b_e(x_e-1)) + (a_{\bar{e}}(x_{\bar{e}}+1)^2 + b_{\bar{e}}(x_{\bar{e}}+1))\right),
\end{aligned}$$

which is equivalent to

$$(1+\beta)a_e x_e + b_e - \beta a_e \leq (1+\beta)a_{\bar{e}} x_{\bar{e}} + b_{\bar{e}} + a_{\bar{e}}. \tag{3}$$

We can use this relation in order to show that

$$\begin{aligned}
& \sum_{e\in E^+} \Delta_e((1+\beta)a_e x_e^* + b_e + a_e\Delta_e) + \sum_{e\in E^-} \Delta_e((1+\beta)a_e x_e^* + b_e + \beta a_e \Delta_e) \\
& \leq \Delta\left(\max_{e\in E^+}\{(1+\beta)a_e x_e + b_e - \beta a_e\} - \min_{e\in E^-}\{(1+\beta)a_e x_e + b_e + a_e\}\right) \leq 0.
\end{aligned} \tag{4}$$

The first inequality follows from the definition of Δ_e and because $\Delta_e \geq 1$ for every $e \in E^+$ and $\Delta_e \leq -1$ for every $e \in E^-$; the last inequality follows from (3). Thus,

$$\begin{aligned}
C(s) &= \sum_{e\in E}(x_e^* + \Delta_e)(a_e(x_e^* + \Delta_e) + b_e) \\
&= \sum_{e\in E}(a_e x_e^{*2} + b_e x_e^*) + \sum_{e\in E^+} \Delta_e(2a_e x_e^* + b_e + a_e\Delta_e) + \sum_{e\in E^-} \Delta_e(2a_e x_e^* + b_e + a_e\Delta_e) \\
&\leq C(s^*) + (1-\beta)\left(\sum_{e\in E^+} \Delta_e a_e x_e^* + \sum_{e\in E^-} \Delta_e a_e x_e\right) \leq C(s^*) + \tfrac{1}{4}(1-\beta)\sum_{e\in E^+} a_e(x_e^* + \Delta_e)^2 \\
&\leq C(s^*) + \tfrac{1}{4}(1-\beta)C(s).
\end{aligned}$$

The first inequality holds because of (4). The second inequality uses that $xy \leq \frac{1}{4}(x+y)^2$ for arbitrary real numbers x, y and that $\Delta_e a_e x_e \leq 0$ for every $e \in E^-$. Hence, the pure price of anarchy is at most $4/(3+\beta)$.

To see that this bound is tight, consider the β-altruistic extension of a congestion game with two players and two edges $E = \{1, 2\}$ with delay functions $d_1(x) = x$ and $d_2(x) = 2+\beta$. If the players use different edges, we obtain an optimal strategy profile of cost $3+\beta$. If both players use edge 1, we obtain a Nash equilibrium of cost 4. □

[3] The delay functions $(d_e)_{e\in E}$ are *semi-convex* if for every edge $e \in E$, $x \cdot d_e(x)$ is convex.

Next, we focus on a second very natural special case: when all altruism levels are either 0 or 1. This kind of scenario, in which each player is either completely selfish or completely altruistic, has some natural relationship with *Stackelberg routing games* [32], and constitutes another class of examples where system performance *improves* with the total amount of altruism present.

Theorem 3. *The pure price of anarchy of β-altruistic extensions of symmetric singleton linear congestion games with $\beta \in \{0,1\}^n$ is at most $1+\frac{n_0}{2n+n_0}$, where n_0 is the number of selfish players.*

Let s be a pure Nash equilibrium of G^β and s^* an optimal strategy profile. Again, let $x_e = x_e(s)$ and $x_e^* = x_e(s^*)$. Based on the strategy profile s, we partition the edges in E into sets E_0, E_1.

$$E_1 = \{e \in E : \exists i \in N \text{ with } \beta_i = 1 \text{ and } s_i = \{e\}\},$$

is the set of edges having at least one altruistic player, while $E_0 = E \setminus E_1$ is the set of edges that are used exclusively by selfish players or not used at all. Let N_1 and N_0 refer to the respective player sets that are assigned to E_1 and E_0. N_1 may contain both altruistic and selfish players, while N_0 consists of selfish players only. Let $k_1 = \sum_{e\in E_1} x_e$ and $k_0 = n - k_1$ denote the number of players in N_1 and N_0, respectively.

The high-level approach of our proof is as follows: We split the total cost $C(s)$ of the pure Nash equilibrium into $C(s) = \gamma C(s) + (1-\gamma)C(s)$ for some $\gamma \in [0,1]$ such that $\gamma C(s) = \sum_{e\in E_0} x_e d_e(x_e)$ and $(1-\gamma)C(s) = \sum_{e\in E_1} x_e d_e(x_e)$. We bound these two contributions separately to show that

$$\tfrac{3}{4}\gamma C(s) + (1-\gamma)C(s) \leq C(s^*). \tag{5}$$

The pure price of anarchy is therefore at most $(\frac{3}{4}\gamma + (1-\gamma))^{-1} = \frac{4}{4-\gamma}$. The bound of $1+\frac{n_0}{2n+n_0}$ then follows by deriving an upper bound on γ in Lemma 4.

Lemma 2. *Assume that the delay functions $(d_e)_{e\in E}$ are semi-convex. Then there is an optimal strategy profile s^* such that $x_e \leq x_e^*$ for every edge $e \in E_1$.*

The proof of this lemma is given in Appendix D. Note that Lemma 2 implies that at least for singleton congestion games, entirely altruistic players will ensure that Nash equilibria are optimal.

Corollary 1. *The pure price of anarchy of* 1*-altruistic extensions of symmetric singleton congestion games with semi-convex delay functions is* 1.

Henceforth, we assume that s^* is an optimal strategy profile that satisfies the statement of Lemma 2.

Lemma 3. *Define y^* as $y_e^* = x_e^* - x_e \geq 0$ for every $e \in E_1$, and $y_e^* = x_e^*$ for all edges $e \in E_0$. Then, $\sum_{e\in E_0} x_e d_e(x_e) \leq \frac{4}{3}\sum_{e\in E} y_e^* d_e(x_e^*)$.*

Proof. Consider the game $\bar{G}$ induced by G^β if all k_1 players in N_1 are fixed on the edges in E_1 according to s. Note that all remaining $k_0 = n - k_1$ players in N_0 are selfish. That is, $\bar{G}$ is a symmetric singleton congestion game with player set N_0, edge set E and delay functions $(\bar{d}_e)_{e\in E}$, where $\bar{d}_e(z) = d_e(x_e + z)$ if $e \in E_1$ and $\bar{d}_e(z) = d_e(z)$ for $e \in E_0$. Let $\bar{s}$ be the restriction of s to the players in N_0, and define $\bar{x}$ as $\bar{x}_e = 0$ for $e \in E_1$ and $\bar{x}_e = x_e$ for $e \in E_0$. It is not hard to verify that $\bar{s}$ is a pure Nash equilibrium of the game $\bar{G}$. Let $\bar{s}^*$ be a socially optimum profile for $\bar{G}$, and for each edge e, let $\bar{x}_e^*$ be the total number of players on e under $\bar{S}^*$. Then,

$$\sum_{e\in E_0} x_e d_e(x_e) = \sum_{e\in E} \bar{x}_e \bar{d}_e(\bar{x}_e) \leq \tfrac{4}{3}\sum_{e\in E} \bar{x}_e^* \bar{d}_e(\bar{x}_e^*) \leq \tfrac{4}{3}\sum_{e\in E} y_e^* \bar{d}_e(y_e^*) = \tfrac{4}{3}\sum_{e\in E} y_e^* d_e(x_e^*),$$

where the first inequality follows from Theorem 2 and the second inequality follows from the optimality of $\bar{x}^*$. □

Lemma 4. *It holds that $\gamma \leq \frac{2n_0}{n+n_0}$.*

The proof is given in Appendix 4.

Proof of Theorem 3. Using the above lemmas, we can show that the relation in (5) holds:

$$\tfrac{3}{4}\gamma C(s)+(1-\gamma)C(s)=\tfrac{3}{4}\sum_{e\in E_0}x_e d_e(x_e)+\sum_{e\in E_1}x_e d_e(x_e)\le\sum_{e\in E}x_e^* d_e(x_e^*)+\sum_{e\in E_1}(x_e d_e(x_e)-x_e d_e(x_e^*))\le C(s^*),$$

where the first inequality follows from Lemma 3 and the last inequality follows from Lemma 2 and because delay functions are monotone increasing. We conclude that the pure price of anarchy is at most $(\frac{3}{4}\gamma+(1-\gamma))^{-1}=\frac{4}{4-\gamma}$. The stated bound now follows from Lemma 4. □

5 Fair Cost-sharing Games

In a *fair cost-sharing game*, players choose facilities, and the cost of each selected facility is shared evenly among the players that have selected it. Formally, a cost-sharing game is a congestion game $G=(N,E,\{d_e\}_{e\in E},\{\Sigma_i\}_{i\in N})$ with decreasing delay functions of the form $d_e(x)=c_e/x$ for every $e\in E$, where $c_e\in\mathbb{Q}^+$ is the non-negative cost of facility e. Let $E(s)=\bigcup_{i\in N}s_i$ be the union of all facilities used under s. The social objective function is $C(s)=\sum_{i=1}^n C_i(s)=\sum_{e\in E(s)}c_e$.

It is well-known that the pure price of anarchy of fair cost-sharing games is n [29]. We show that it can get significantly worse in the presence of altruistic players: the following theorem gives a much worse upper bound, which we subsequently show to be tight.

Theorem 4. *The robust price of anarchy of β-altruistic cost-sharing games is at most $\frac{n}{1-\hat\beta}$ (with $n/0=\infty$).*

Proof. The claim is true for $\hat\beta=1$ because $\mathrm{RPoA}(\beta)\le\infty$ holds trivially. We show that G^β is $(n,\hat\beta,\beta)$-smooth for $\hat\beta\in[0,1)$. Let s and s^* be two strategy profiles. Fix an arbitrary player $i\in N$. We have

$$C(s_i^*,s_{-i})-C(s)=\sum_{e\in E(s_i^*,s_{-i})}c_e-\sum_{e\in E(s)}c_e\le\sum_{e\in s_i^*\setminus E(s)}c_e.$$

We use this inequality obtain the following bound:

$$\begin{aligned}(1-\beta_i)C_i(s_i^*,s_{-i})+\beta_i(C(s_i^*,s_{-i})-C(s)) &\le (1-\beta_i)\sum_{e\in s_i^*}\frac{c_e}{x_e(s_i^*,s_{-i})}+\beta_i\sum_{e\in s_i^*\setminus E(s)}\frac{c_e}{x_e(s_i^*,s_{-i})}\\ &\le \sum_{e\in s_i^*}\frac{c_e}{x_e(s_i^*,s_{-i})}\le\sum_{e\in s_i^*}\frac{n\cdot c_e}{x_e(s^*)}.\end{aligned}$$

The first inequality holds because $x_e(s_i^*,s_{-i})=1$ for every $e\in s_i^*\setminus E(s)$, and the last inequality follows from $x_e(s_i^*,s_{-i})\ge x_e(s^*)/n$ for every $e\in s_i^*$. Using the above,

$$\sum_{i=1}^n((1-\beta_i)C_i(s_i^*,s_{-i})+\beta_i(C(s_i^*,s_{-i})-C(s))+\beta_i C_i(s))\le\sum_{i=1}^n\left(\sum_{e\in s_i^*}\frac{n\cdot c_e}{x_e(s^*)}\right)+\hat\beta C(s)=nC(s^*)+\hat\beta C(s).$$

We conclude that the robust price of anarchy is at most $\frac{n}{1-\hat\beta}$. Example 2 shows that this bound is tight, even for pure Nash equilibria. □

Example 2. Consider the cost-sharing game in which n players can choose between two different facilities e_1 and e_2 of cost 1 and $n/(1-\beta)$, respectively. Let $s^*=(e_1,\dots,e_1)$ and $s=(e_2,\dots,e_2)$ refer to the strategy profiles in which every player chooses e_1 and e_2, respectively. Then $C(s^*)=1$ and $C(s)=n/(1-\beta)$. Note that s is a pure Nash equilibrium of the β-altruistic extension of this game because for every player i we have

$$(1-\beta)C_i(s)+\beta C(s)=1+\beta\frac{n}{1-\beta}=C_i^\beta(\{e_1\},s_{-i}).$$

The pure price of anarchy is therefore at least $n/(1-\beta)$.

As with congestion games, the PoS does improve with increased altruism. The proof of the following proposition is given in Appendix E.

Proposition 4. *The pure price of stability of uniformly β-altruistic cost-sharing games is at most $(1-\beta)H_n+\beta$.*

6 Valid Utility Games

A valid utility game [36] is a payoff maximization game given by $G=(N,E,\{\Sigma_i\}_{i\in N},\{\Pi_i\}_{i\in N},V)$, where E is a ground set, the strategy sets Σ_i are subsets of 2^E, Π_i is the payoff function of player i, and V is a submodular[4] and non-negative function on E. Every player strives to maximize his individual payoff function Π_i.

For a joint strategy $s\in\Sigma$, let $U(s)\subseteq E$ be the union of all players' strategies under s. The social welfare function $\Pi:\Sigma\to\mathbb{R}$ to be maximized is $\Pi(s)=V(U(s))$, and thus depends only on the union of the players' chosen strategies, evaluated by V. The individual payoff functions of all players $i\in N$ are assumed to satisfy[5] $\Pi_i(s)\geq\Pi(s)-\Pi(\emptyset,s_{-i})$ for every strategy profile $s\in\Sigma$. Intuitively, this means that the individual payoff of a player is at least his contribution to the social welfare. Moreover, it is assumed that $\Pi(s)\geq\sum_{i=1}^n\Pi_i(s)$ for every $s\in\Sigma$. See [36] for a detailed description and justification of these assumptions.

Examples of games falling into this framework include natural game-theoretic variants of the facility location, k-median and network routing problems [36]. Vetta [36] proved a bound of 2 on the pure price of anarchy for valid utility games with non-decreasing V, and Roughgarden showed in [34] how this bound is achieved via a (λ,μ)-smoothness argument. We extend this result to altruistic extensions of these games.

Theorem 5. *The robust price of anarchy of β-altruistic valid utility games is 2.*

Proof. We show that the β-altruistic extension G^β of a valid utility game is $(1,1,\beta)$-smooth.

Fix two strategy profiles $s,s^*\in\Sigma$ and consider an arbitrary player $i\in N$. By assumption, $\Pi_i(s)\geq\Pi(s)-\Pi(\emptyset,s_{-i})$. For each player $i\in N$, we therefore have

$$\begin{aligned}\Pi(s_i^*,s_{-i})-\Pi(s)+\Pi_i(s) &= (\Pi(s_i^*,s_{-i})-\Pi(\emptyset,s_{-i}))-(\Pi(s)-\Pi(\emptyset,s_{-i}))+\Pi_i(s)\\ &\geq \Pi(s_i^*,s_{-i})-\Pi(\emptyset,s_{-i}).\end{aligned}\tag{6}$$

Now let $U_i=\bigcup_{j=1}^n s_j\cup\bigcup_{j=1}^i s_j^*$. Summing over all $i\in N$,

$$\begin{aligned}&\sum_{i=1}^n((1-\beta_i)\Pi_i(s_i^*,s_{-i})+\beta_i(\Pi(s_i^*,s_{-i})-\Pi(s)+\Pi_i(s)))\;\geq\;\sum_{i=1}^n(\Pi(s_i^*,s_{-i})-\Pi(\emptyset,s_{-i}))\\ &=\sum_{i=1}^n(V(U(s_i^*,s_{-i}))-V(U(\emptyset,s_{-i})))\;\geq\;\sum_{i=1}^n(V(U_i)-V(U_{i-1}))\;\geq\;\Pi(s^*)-\Pi(s).\end{aligned}$$

Here, the first inequality follows from (6) and because $\Pi_i(s)\geq\Pi(s)-\Pi(\emptyset,s_{-i})$ for every i, the second inequality holds because V is submodular, and the final inequality follows from V being non-decreasing. We conclude that G^β is $(1,1,\beta)$-smooth, which proves an upper bound of 2 on the robust price of anarchy. This bound is tight, as shown by Example 3. □

Example 3. Consider a valid utility game G with a set $N=\{1,2\}$ of two players, a ground set $E=\{1,2\}$ of two elements and strategy sets $\Sigma_1=\{\{1\},\{2\}\}$, $\Sigma_2=\{\emptyset,\{1\}\}$. Define $V(S)=|S|$ for every subset $S\subseteq E$. Note that V is non-negative, non-decreasing and submodular.

For a given strategy profile $s\in\Sigma$, the individual profits $\Pi_1(s)$ and $\Pi_2(s)$ of player 1 and player 2, respectively, are defined as follows: $\Pi_1(s)=1$ for all strategy profiles s. $\Pi_2(s)=1$ if $s=(\{2\},\{1\})$ and $\Pi_2(s)=0$ otherwise. It is not hard to verify that for every player i and every strategy profile $s\in\Sigma$ we have

[4] For a finite set E, a function $f:2^E\to\mathbb{R}$ is called *submodular* iff $f(A\cup\{x\})-f(A)\geq f(B\cup\{x\})-f(B)$ for any $A\subseteq B\subseteq E, x\in E$.

[5] We abuse notation and write $\Pi(\emptyset,s_{-i})$ to denote $V(U(s)\backslash s_i)$.

$\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$. Moreover, $\Pi(s) \geq \Pi_1(s) + \Pi_2(s)$ for every $s \in \Sigma$. We conclude that G is a valid utility game.

Let $\beta \in [0,1]^2$, and consider the β-altruistic extension G^β of G. We claim that $s = (\{1\}, \emptyset)$ is a pure Nash equilibrium of G^β: the profit of player 1 under s is $(1-\beta_1)+\beta_1 = 1$. His profit remains 1 if he switches to strategy $\{2\}$. The profit of player 2 under s is β_2. If he switches to strategy $\{1\}$, then his profit is β_2 as well. Thus, s is a pure Nash equilibrium. Since $\Pi(s) = 1$ and $\Pi((\{2\},\{1\})) = 2$, the pure price of anarchy of G is 2.

7 General Properties of Smoothness

For the game classes that we analyzed (with the exception of symmetric singleton congestion games), we used (λ, μ, β)-smoothness as our main tool to derive bounds on the price of anarchy. In this section, we provide some general results about (λ, μ, β)-smoothness. The proofs can be found in Appendix F.

Proposition 5. *Suppose that $\mathscr{G}$ is a class of cost-minimization games equipped with sum-bounded social cost functions. The set $S_{\mathscr{G}} = \{(\lambda, \mu, \beta) : \forall G \in \mathscr{G}, G^\beta$ is (λ, μ, β)-smooth$\}$ is convex.*

A natural question to ask is whether the robust price of anarchy is also a convex function of β. This turns out not to be the case. For instance, the robust price of anarchy for uniformly β-altruistic congestion games is $\frac{5+4\beta}{2+\beta}$ (see Section 3), which is a non-convex function. However, we can prove a somewhat weaker statement: For a subset $S \subseteq \mathbb{R}^n$, we call a function $f : S \to \mathbb{R}$ *quasi-convex* iff $f(\gamma x + (1-\gamma)y) \leq \max\{f(x), f(y)\}$ for all $\gamma \in [0,1]$.

Theorem 6. *Let $\mathscr{G}$ be a class of games equipped with sum-bounded social cost functions. Then $RPoA_{\mathscr{G}}(\beta)$ is a quasi-convex function of β.*

The quasi-convexity of $\text{RPoA}_{\mathscr{G}}$ implies:

Corollary 2. *The points β that minimize $RPoA_{\mathscr{G}}(\beta)$ on the domain $[0,1]^n$ form a convex set. The set of points β that maximize $RPoA_{\mathscr{G}}(\beta)$ on the domain $[0,1]^n$ includes at least one point that is a 0-1 vector.*

8 Conclusions and Future Work

Intuitively, one would expect the worst-case price of anarchy of a game to improve when the altruism level β gets closer to $\mathbf{1}$, but we have seen that this is not the case. Indeed, there are important classes of games for which the robust price of anarchy turns out to be tight, and actually gets worse as the altruism level of the players increases. The fact that the price of anarchy does not *necessarily* get worse in all cases is exemplified by our analysis of symmetric singleton congestion games.

The most immediate future directions include analyzing singleton congestion games with more general delay functions than linear ones. While the PoA of such functions increases (e.g., the PoA for polynomials increases exponentially in the degree [4, 14]), this also creates room for potentially larger reductions due to altruism. Similarly, the characterization of the robust price of anarchy of altruistic congestion games with more general delay functions (e.g., polynomials) is left for future work.

For games where the smoothness argument cannot give tight bounds, would a refined smoothness argument like local smoothness in [33] work? For symmetric singleton congestion games, this seems unlikely, as the PoA bounds are already different between pure and mixed Nash equilibria.

It is also worth trying to apply the smoothness argument or its refinements to analyze the PoA for other dynamics in other classes of altruistic games, for example, (altruistic) network vaccination games [12], which are known to not always possess pure Nash equilibria. Furthermore, while the existence of pure Nash equilibria has been shown for singleton and matroid congestion games with player-specific latency functions [1, 26], the PoA (for pure Nash equilibria or more general equilibrium concepts) has not yet been addressed. Studying the PoA in such a general setting (in which our setting with altruism can be embedded) by either smoothness-based techniques or other methods is undoubtedly intriguing.

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A Equilibrium Notions

Some of the natural notions of equilibrium studied in the literature include coarse correlated equilibria (as defined in Section 2.1), correlated equilibria, mixed Nash equilibria, and pure Nash equilibria.

Informally, the difference between a coarse equilibrium and a *correlated equilibrium* is the following: In a coarse equilibrium, it is required that a player "adheres" to s when he is informed of the distribution σ from which s is drawn. In a correlated equilibrium, a player is only required to adhere to s when he is informed of the distribution σ as well as the strategy that has been drawn for him, i.e., that he will play under s. More formally, this means that in a correlated equilibrium, for all $s_i^* \in \Sigma_i$,

$$\mathbf{E}_{s\sim\sigma}\left[C_i(s)\right] \quad \leq \quad \mathbf{E}_{s\sim\sigma}\left[C_i(s_i^*, s_{-i})\right]. \tag{7}$$

(By (s_i^*, s_{-i}), we mean the strategy profile obtained from s when we replace s_i with s_i^*.)

A *mixed Nash equilibrium* is a coarse equilibrium whose distribution σ is the product of *independent* distributions $\sigma_1, \ldots, \sigma_n$ for the players. Thus, any mixed Nash equilibrium is also a correlated equilibrium. A *pure Nash equilibrium* is a strategy profile s such that for each player i, $C_i(s) \leq C_i(s_i^*, s_{-i})$ for all $s_i^* \in \Sigma_i$. A pure Nash equilibrium is a special case of a mixed Nash equilibrium where the support of σ_i has cardinality 1 for all i.

B Missing Proofs of Section 2

Proposition 1. *Let G^β be a β-altruistic game. If G^β is (λ, μ, β)-smooth with $\mu < 1$, then the coarse price of anarchy of G^β is at most $\frac{\lambda}{1-\mu}$.*

Proof. Let σ be a coarse equilibrium of G^β, s a random variable with distribution σ, and $s^* \in \Sigma$ an arbitrary strategy profile. The coarse equilibrium condition implies that for every player $i \in N$:

$$\mathbf{E}\left[(1-\beta_i)C_i(s) + \beta_i C(s)\right] \leq \mathbf{E}\left[(1-\beta_i)C_i(s_i^*, s_{-i}) + \beta_i C(s_i^*, s_{-i})\right].$$

By linearity of expectation, for every player $i \in N$:

$$\mathbf{E}\left[C_i(s)\right] \leq \mathbf{E}[C_i(s_i^*, s_{-i}) + \beta_i(C(s_i^*, s_{-i}) - C_i(s_i^*, s_{-i})) - \beta_i(C(s) - C_i(s))].$$

By summing over all players and using linearity of expectation, we obtain

$$\mathbf{E}\left[C(s)\right] \leq \mathbf{E}\left[\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) + \beta_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s))\right].$$

Now we use the smoothness property (1) to conclude

$$\mathbf{E}\left[C(s)\right] \leq \mathbf{E}\left[\lambda C(s^*) + \mu C(s)\right] = \lambda C(s^*) + \mu \mathbf{E}\left[C(s)\right].$$

Hence, the coarse price of anarchy is at most $\frac{\lambda}{1-\mu}$. □

Proposition 2. *Let s^* be a strategy profile minimizing the social cost function C of a β-altruistic game G^β, and $s^1, \ldots, s^T$ a sequence of strategy profiles in which every player $i \in N$ experiences vanishing average external regret, i.e.,*

$$\sum_{t=1}^{T} C_i^\beta(s^t) \leq \left(\min_{s_i' \in \Sigma_i} \sum_{t=1}^{T} C_i^\beta(s_i', s_{-i}^t)\right) + o(T).$$

The average cost of this sequence of T strategy profiles then satisfies $\frac{1}{T}\sum_{t=1}^{T} C(s^t) \leq RPoA(\beta) \cdot C(s^*)$ as $T \to \infty$.

Proof. Consider a sequence $s^1, \ldots, s^T$ of strategy profiles of a β-altruistic game G^β that is (λ, μ, β)-smooth with $\mu < 1$. For every $i \in N$ and $t \in \{1, \ldots, T\}$, define

$$\delta_i^\beta(s^t) = C_i^\beta(s^t) - C_i^\beta(s_i^*, s_{-i}^t).$$

Let $\Delta(s^t) = \sum_{i=1}^n \delta_i^\beta(s^t)$. We have

$$\begin{aligned}
\Delta(s^t) &= \sum_{i=1}^n C_i^\beta(s^t) - C_i^\beta(s_i^*, s_{-i}^t) \\
&= \sum_{i=1}^n \big((1-\beta_i)C_i(s^t) + \beta_i C(s^t) - \big((1-\beta_i)C_i(s_i^*, s_{-i}^t) + \beta_i C(s_i^*, s_{-i}^t)\big)\big) \\
&= C(s^t) - \sum_{i=1}^n \left(C_i(s_i^*, s_{-i}^t) + \beta_i(C_{-i}(s_i^*, s_{-i}^t) - C_{-i}(s^t))\right).
\end{aligned}$$

Exploiting the (λ, μ, β)-smoothness property, we obtain

$$C(s^t) \quad \leq \quad \tfrac{\lambda}{1-\mu} C(s^*) + \tfrac{1}{1-\mu}\Delta(s^t). \tag{8}$$

Suppose that $s^1, \ldots, s^T$ is a sequence of strategy profiles in which every player experiences vanishing average external regret, i.e.,

$$\sum_{t=1}^T C_i^\beta(s^t) \leq \left(\min_{s_i' \in \Sigma_i} \sum_{t=1}^T C_i^\beta(s_i', s_{-i}^t)\right) + o(T).$$

We obtain that for every player $i \in N$:

$$\frac{1}{T}\sum_{t=1}^T \delta_i(t) \leq \frac{1}{T}\left(\sum_{t=1}^T C_i^\beta(s^t) - \min_{s_i' \in \Sigma_i} \sum_{t=1}^T C_i^\beta(s_i', s_{-i}^t)\right) = o(1).$$

By summing over all players, we obtain that the average cost of the sequence of T strategy profiles is

$$\frac{1}{T}\sum_{t=1}^T C(s^t) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{1-\mu}\sum_{i=1}^n \left(\frac{1}{T}\sum_{t=1}^T \delta_i(t)\right) \xrightarrow{T\to\infty} \frac{\lambda}{1-\mu} C(s^*).$$

□

C Missing Proofs of Section 3

C.1 Reducing to Identity Cost Functions

We show that for general linear congestion games, we can assume without loss of generality that all delay functions are $d_e(x) = x$. We may assume that for every delay function d_e, the a_e and b_e coefficients are integers. If this is not the case, then we can multiply all coefficients among all facilities by their least common multiple in order to obtain a game in which the coefficients are integers. In this new game, the price of anarchy is the same, as is the set of all equilibria.

Next, we can assume that $b_e = 0$ for all $e \in E$. For we can simply replace any facility $e \in E$ with delay function $d(x) = a_e x + b_e$ by $n+1$ facilities facilities $e_0, \ldots, e_n$ with delay functions $d_{e_0}(x) = a_e x$ and $d_{e_i}(x) = b_e x$ for $1 \leq i \leq n$. We then adapt the strategy space Σ_i of each player i as follows: we replace every strategy $s_i \in \Sigma_i$ in which e occurs by the strategy $s_i \setminus \{e\} \cup \{e_0, e_i\}$. There is an obvious bijection between the joint strategies in the original game and those in the new game, preserving the values of individual cost functions and the social objective function. (Notice that this construction exploits the fact that players are unit size, and would not carry over to weighted congestion games.)

Finally, for the same reason we can also assume that $a_e = 1$ for all $e \in E$. We replace e with facilities $e_1, \ldots, e_{a_e}$, each having delay function $d_{e_i}(x) = x$, and adapt the strategy space Σ_i of each player i by replacing each strategy s_i in which e occurs by $s_i \setminus \{e\} \cup \{e_1, \ldots, a_{a_e}\}$. Now, all delay functions are $d_e(x) = x$.

C.2 Proofs of Lemmas

We begin with the following lemma:

Lemma 5. *For all $x, y \in \mathbb{N}_0$, $\beta \in [0,1]$ and $\alpha \in [0,1]$, it holds that*

$$((1+\beta)x+1)y+\alpha\beta(1-x)x \leq (2+\beta-\gamma)y^2+\gamma x^2$$

for all $\gamma \in [\frac{1}{3}(1+\beta-2\alpha\beta), 1+\beta]$.

Proof. The inequality is equivalent to

$$((1+\beta)x+1)y+\alpha\beta(1-x)x-(2+\beta)y^2 \leq \gamma(x^2-y^2).$$

Assume that $x = y$. The inequality is then trivially satisfied because $x \leq x^2$ for all $x \in \mathbb{N}_0$. Next suppose that $x > y$. Then

$$\gamma \geq \frac{((1+\beta)x+1)y+\alpha\beta(1-x)x-(2+\beta)y^2}{x^2-y^2}.$$

We show that the maximum of the expression on the right-hand side is attained by $x = 2$ and $y = 1$. First, we fill in these values and conclude that for these values, $\gamma \geq \frac{1}{3}(1+\beta-2\alpha\beta) \geq 0$. We now write x as $y+a, a \geq 1$, and rewrite the right-hand side as

$$f(y,a) = \frac{(1+\beta)y+\alpha\beta}{2y+a} + \frac{(1+\alpha\beta)(y-y^2)}{a(2y+a)} - \alpha\beta. \tag{9}$$

Because we know that there are choices of x and a for which $f(y,a)$ is positive (e.g., when $y = 1$ and $a = 1$), and because a only occurs in the denominators, we know that (9) reaches its maximum when $a = 1$. So we assume $a = 1$. When we then fill in $y = 0$, we see that $f(0,1) = 0$, so $f(1,1) \geq f(0,1)$. When $y > 1$ we can write y as $w+2$, where $w \geq 0$, and we can now further rewrite $f(y,a)$ as

$$f(w+2,1) = \frac{2\beta-6\alpha\beta}{2w+5} - \frac{(2-\beta+5\alpha\beta)w+(1+\alpha\beta)w^2}{2w+5} \leq \frac{2\beta-6\alpha\beta}{2w+5}.$$

When $2\beta-6\alpha\beta$ is negative, this term is certainly less than $f(1,1)$. When $2\beta-6\alpha\beta$ is positive, we have

$$f(w+2,1) \leq \frac{2\beta-6\alpha\beta}{2w+5} \leq \frac{2\beta-6\alpha\beta}{5} \leq \frac{1}{3}(2\beta-6\alpha\beta) \leq \frac{1}{3}(1+\beta-2\alpha\beta) = f(1,1).$$

This shows that $\gamma \geq f(1,1) = \frac{1}{3}(1+\beta-2\alpha\beta)$.

The final case is when $x < y$. Then,

$$\gamma \leq \frac{(2+\beta)y^2-((1+\beta)x+1)y-\alpha\beta(1-x)x}{y^2-x^2}.$$

We show that the minimum of the expression on the right-hand side is attained by $x = 0$ and $y = 1$. First, we fill in these values and conclude that for these values, $\gamma \leq 1+\beta$. We now write y as $x+a$, $a \geq 1$, and rewrite the right-hand side as

$$g(x,a) = \frac{(1+\alpha\beta)x^2-(1+a+(a+\alpha)\beta)x-a}{a(2x+a)} + 2+\beta.$$

Suppose first that $x = 0$ and that $a \geq 2$. Then we can write a as $1+b$, $b > 0$, and therefore $f(0,1+b) = 2+\beta-\frac{1}{1+b} \geq \frac{3}{2}+\beta \geq 1+\beta = f(0,1)$. When $x \geq 1$, we can write x as $1+b$, $b \geq 0$. We then have

$$f(1+b,a) = 2+\beta-\frac{2+\beta+(1-\beta)b}{2b+2+a} + \frac{(1+\alpha\beta)(b^2+b)}{a(2b+2+a)}.$$

The last of these terms is positive, hence

$$\begin{aligned} f(1+b,a) &\geq 2+\beta - \frac{2+\beta+(1-\beta)b}{2b+2+a} \geq 2+\beta - \frac{2+1+b}{2b+2+a} \\ &\geq 2+\beta-1 = 1+\beta = f(0,1). \end{aligned}$$

This shows that $\gamma \leq f(0,1) = 1+\beta$. □

We use the above lemma to complete the proof of Lemma 1.
Lemma 1. *For every two integers $x,y \in \mathbb{N}_0$ and $\hat{\beta}, \check{\beta} \in [0,1]$ with $\hat{\beta} \geq \check{\beta}$*

$$((1+\hat{\beta})x+1)y + \check{\beta}(1-x)x \leq \tfrac{5+2\hat{\beta}+2\check{\beta}}{3}y^2 + \tfrac{1+\hat{\beta}-2\check{\beta}}{3}x^2.$$

Proof. Choose $\alpha \in [0,1]$ such that $\check{\beta} = \alpha\hat{\beta}$. Using Lemma 5 above, we obtain

$$((1+\hat{\beta})x+1)y + \check{\beta}(1-x)x = ((1+\hat{\beta})x+1)y + \alpha\hat{\beta}(1-x)x \leq (2+\hat{\beta}-\gamma)y^2 + \gamma x^2,$$

where $\gamma \in [\frac{1}{3}(1+\hat{\beta}-2\alpha\hat{\beta}), 1+\hat{\beta}]$. By choosing $\gamma = \frac{1}{3}(1+\hat{\beta}-2\alpha\hat{\beta})$, we obtain

$$((1+\hat{\beta})x+1)y + \check{\beta}(1-x)x \leq \frac{5+2\hat{\beta}+2\alpha\hat{\beta}}{3}y^2 + \frac{1+\hat{\beta}-2\alpha\hat{\beta}}{3}x^2.$$

Substituting $\alpha\hat{\beta} = \check{\beta}$ yields the claim. □

We remark that the choice of γ in the proof above has been made in order to minimize the expression $\lambda/(1-\mu)$ (which is an increasing function in γ).

Proposition 3. *The pure price of stability of uniformly β-altruistic linear congestion games is at most $\frac{2}{1+\beta}$.*

Proof. Let G^β be a uniformly β-altruistic extension of a linear congestion game. It is not hard to verify that G^β is an exact potential game with potential function $\Phi^\beta(s) = (1-\beta)\Phi(s) + \beta C(s)$, where $\Phi(s) = \sum_{e\in E}\sum_{i=1}^{x_e(s)} i$ is Rosenthal's potential function. We have $\frac{1}{2}(1+\beta)C(s) \leq \Phi^\beta(s) \leq C(s)$. Let s be a strategy profile that minimizes Φ^β, and let s^* be an optimal strategy profile that minimizes the social cost function C. Note that s is a pure Nash equilibrium of G^β. We have

$$C(s) \leq \frac{2}{1+\beta}\Phi^\beta(s) \leq \frac{2}{1+\beta}\Phi^\beta(s^*) \leq \frac{2}{1+\beta}C(s^*),$$

which proves the claim. □

D Missing Proofs of Section 4

Lemma 2. *Assume that the delay functions $(d_e)_{e\in E}$ are semi-convex. Then there is an optimal strategy profile s^* such that $x_e \leq x_e^*$ for every edge $e \in E_1$.*

Proof. Let s^* be an optimal strategy profile, and assume that $x_e^* < x_e$ for some $e \in E_1$. Then there is some edge $\bar{e} \in E$ with $x_{\bar{e}}^* > x_{\bar{e}}$. Consider an altruistic player $i \in N_1$ with $s_i = \{e\}$. (Note that i must exist by the definition of E_1.) Because s is a pure Nash equilibrium, player i has no incentive to deviate from e to $\bar{e}$, i.e., $C(\{\bar{e}\}, s_{-i}) \geq C(s)$, or, equivalently,

$$(x_{\bar{e}}+1)d_{\bar{e}}(x_{\bar{e}}+1) - x_{\bar{e}}d_{\bar{e}}(x_{\bar{e}}) \geq x_e d_e(x_e) - (x_e-1)d_e(x_e-1). \tag{10}$$

Since $x_e^* < x_e$ and $x_{\bar{e}} < x_{\bar{e}}^*$, the semi-convexity of the delay functions implies

$$(x_e^*+1)d_e(x_e^*+1) - x_e^*d_e(x_e^*) \leq x_e d_e(x_e) - (x_e-1)d_e(x_e-1), \tag{11}$$

$$(x_{\bar{e}}+1)d_{\bar{e}}(x_{\bar{e}}+1)-x_{\bar{e}}d_{\bar{e}}(x_{\bar{e}}) \quad \leq \quad x^*_{\bar{e}}d_{\bar{e}}(x^*_{\bar{e}})-(x^*_{\bar{e}}-1)d_{\bar{e}}(x^*_{\bar{e}}-1). \tag{12}$$

By combining (10), (11) and (12) and re-arranging terms, we obtain

$$(x^*_e+1)d_e(x^*_e+1)+(x^*_{\bar{e}}-1)d_{\bar{e}}(x^*_{\bar{e}}-1) \leq x^*_e d_e(x^*_e)+x^*_{\bar{e}}d_{\bar{e}}(x^*_{\bar{e}}).$$

The above inequality implies that by moving a player j with $s^*_j=\{\bar{e}\}$ from $\bar{e}$ to e, we obtain a new strategy profile $s'=(\{e\},s^*_{-j})$ of cost $C(s')\leq C(s^*)$. (Note that j must exist because $x^*_{\bar{e}}>x_{\bar{e}}\geq 0$.) Moreover, the number of players on e under the new strategy profile s' increased by one. We can therefore repeat the above argument (with s' in place of s^*) until we obtain an optimal strategy profile that satisfies the claim. □

Lemma 4. *We have* $\gamma \leq \frac{2n_0}{n+n_0}$.

Proof. The claim follows directly from Theorem 2 if $N_1=\emptyset$. Assume that $N_1\neq\emptyset$, and let $j\in N_1$ with $s_j=\{\bar{e}\}$. Let $\bar{C}(s)=\sum_{i\in N_0}C_i(s)/k_0$ be the average cost experienced by players in N_0. We first show $C_j(s)\geq \frac{1}{2}\bar{C}(s)$. If $N_0=\emptyset$, then $C_j(s)\geq\frac{1}{2}\bar{C}(s)$ trivially holds. Suppose that $N_0\neq\emptyset$, and let $i\in N_0$ with $s_i=\{e\}$. Recall that i is selfish. Because s is a Nash equilibrium, we have $C_i(s)=a_ex_e+b_e\leq a_{\bar{e}}(x_{\bar{e}}+1)+b_{\bar{e}}\leq 2(a_{\bar{e}}x_{\bar{e}}+b_{\bar{e}})=2C_j(s)$. By summing over all k_0 selfish players in N_0, we obtain $C_j(s)\geq\frac{1}{2}\bar{C}(s)$ and thus $\sum_{j\in N_1}C_j(s)\geq\frac{1}{2}k_1\bar{C}(S)$. We have

$$\gamma \leq \frac{k_0\bar{C}(S)}{k_0\bar{C}(S)+\frac{1}{2}k_1\bar{C}(S)}=\frac{2k_0}{n+k_0}\leq\frac{2n_0}{n+n_0},$$

where the last inequality follows because $k_0\leq n_0$. □

E Missing Proofs of Section 5

Proposition 4. *The pure price of stability of uniformly β-altruistic cost-sharing games is at most* $(1-\beta)H_n+\beta$.

Proof. Let G^β be a uniformly β-altruistic cost-sharing game. G^β is an exact potential game with potential function $\Phi^\beta(s)=(1-\beta)\Phi(s)+\beta C(s)$, where $\Phi(s)=\sum_{e\in E}\sum_{i=1}^{x_e(s)}c_e/i$. It is not hard to verify that $C(s)\leq\Phi^\beta(s)\leq(1-\beta)H_nC(s)+\beta C(s)$. The claim now follows by using similar arguments as in the proof of Proposition 3. □

F Missing Proofs of Section 7

Proposition 5. *Suppose that $\mathcal{G}$ is a class of cost-minimization games equipped with sum-bounded social cost functions. The set $S_{\mathcal{G}}=\{(\lambda,\mu,\beta):\forall G\in\mathcal{G}$, G^β is (λ,μ,β)-smooth$\}$ is convex.*

Proof. Pick an arbitrary game $G\in\mathcal{G}$. It suffices to show that $S_G=\{(\lambda,\mu,\beta):G^\beta$ is (λ,μ,β)-smooth$\}$ is convex, because the intersection of any collection of convex sets is always convex.

Let $(\lambda_1,\mu_1,\beta^1),(\lambda_2,\mu_2,\beta^2)\in S_G$ be two elements in S_G, and pick an arbitrary $\gamma\in[0,1]$. For all pairs (s,s^*) of strategy profiles of G,

$$\begin{aligned} &\gamma\sum_{i=1}^{n}(C_i(s^*_i,s_{-i})+\beta^1_i(C_{-i}(s^*_i,s_{-i})-C_{-i}(s))) \\ &+(1-\gamma)\sum_{i=1}^{n}(C_i(s^*_i,s_{-i})+\beta^2_i(C_{-i}(s^*_i,s_{-i})-C_{-i}(s))) \\ \leq\; &\gamma(\lambda_1C(s^*)+\mu_1C(s))+(1-\gamma)(\lambda_2C(s^*)+\mu_2C(s)). \end{aligned}$$

By rewriting both sides of the above inequality, we obtain

$$\begin{aligned} & \sum_{i=1}^{n}(C_i(s_i^*,s_{-i})+(\gamma\beta_i^1+(1-\gamma)\beta_i^2)(C_{-i}(s_i^*,s_{-i})-C_{-i}(s))) \\ \leq\quad & (\gamma\lambda_1+(1-\gamma)\lambda_2)C(s^*)+(\gamma\mu_1+(1-\gamma)\mu_2)C(s). \end{aligned}$$

We conclude that G is $(\gamma(\lambda_1,\mu_1,\beta^1)+(1-\gamma)(\lambda_2,\mu_2,\beta^2))$-smooth. Therefore, S_G is convex. □

Theorem 6. *Let $\mathscr{G}$ be a class of games equipped with sum-bounded social cost functions. Then $RPoA_{\mathscr{G}}(\beta)$ is a quasi-convex function of β.*

Proof. Let $G\in\mathscr{G}$. We show that for any $\beta^1,\beta^2\in\mathbb{R}^n$ and $\gamma\in[0,1]$,

$$\text{RPoA}(\gamma\beta^1+(1-\gamma)\beta^2)\leq\max\{\text{RPoA}(\beta^1),\text{RPoA}(\beta^2)\}.$$

Let $(\varepsilon_1,\varepsilon_2,\dots)$ be a decreasing sequence of positive real numbers that tends to 0. Moreover, let

$$((\lambda_{1,1},\mu_{1,1},\beta^1),(\lambda_{1,2},\mu_{1,2},\beta^1),\dots)\quad\text{and}\quad((\lambda_{2,1},\mu_{2,1},\beta^2),(\lambda_{2,2},\mu_{2,2},\beta^2),\dots)$$

be sequences of elements in S_G (where S_G is as defined in the proof of Proposition 5) such that

$$\text{RPoA}(\beta^1)+\varepsilon_j=\tfrac{\lambda_{1,j}}{1-\mu_{1,j}}\quad\forall j\quad\text{and}\quad\text{RPoA}(\beta^2)+\varepsilon_j=\tfrac{\lambda_{2,j}}{1-\mu_{2,j}}\quad\forall j.$$

By Proposition 5, we know that for all j,

$$\begin{aligned} & \sum_{i=1}^{n}(C_i(s_i^*,s_{-i})+(\gamma\beta_i^1+(1-\gamma)\beta_i^2)(C_{-i}(s_i^*,s_{-i})-C_{-i}(s))) \\ \leq\quad & \gamma(\lambda_{1,j}C(s^*)+\mu_{1,j}C(s))+(1-\gamma)(\lambda_{2,j}C(s^*)+\mu_{2,j}C(s)) \\ \leq\quad & \max\{\lambda_{1,j}C(s^*)+\mu_{1,j}C(s),\lambda_{2,j}C(s^*)+\mu_{2,j}C(s)\}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{RPoA}(\gamma\beta^1+(1-\gamma)\beta^2) \quad & \leq \quad \max\left\{\frac{\lambda_{1,j}}{1-\mu_{1,j}},\frac{\lambda_{2,j}}{1-\mu_{2,j}}\right\} \\ & \leq \quad \max\{\text{RPoA}(\beta^1),\text{RPoA}(\beta^2)\}+\varepsilon_j, \end{aligned}$$

for all j. By taking the limit of j going to infinity, we conclude RPoA $\leq\max\{\text{RPoA}(\beta^1),\text{RPoA}(\beta^2)\}$, which proves the claim. □