# Numerical modelling of strongly anisotropic dissipative effects in MHD

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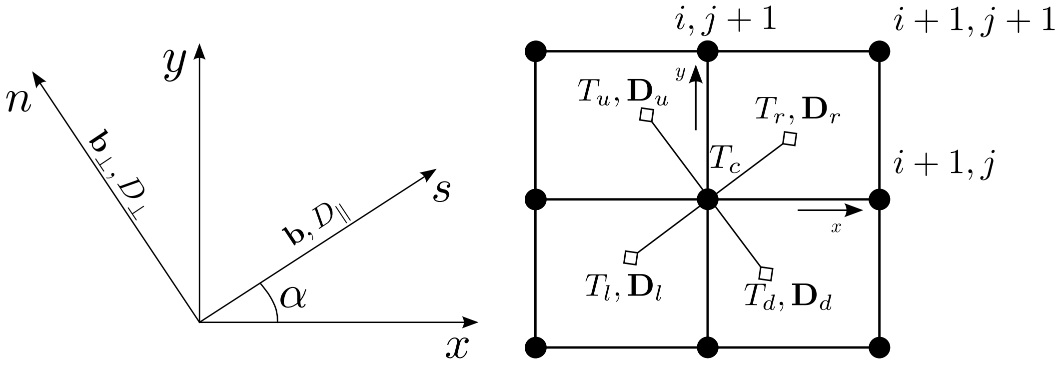
## Introduction

Anisotropic diffusion is a common physical phenomenom describing processes where the diffusion of some scalar quantity is directionally dependent. Anisotropic diffusive processes include Darcy's flow for porous media, large scale turbulence where turbulence scales are anisotropic in size and heat conduction and momentum dissipation in fusion plasmas. Given the high level of anisotropy in tokamak plasmas, a numerical approximation may introduce large perpendicular errors if the magnetic field direction is strongly misaligned with the grid. Here, misaligned means that the directions of diffusion are not aligned with the grid points. Problems that may arise with highly anisotropic diffusion problems on non-aligned meshes are in general; significant numerical diffusion perpendicular to the magnetic field lines due to grid misalignment (see e.g. Umansky et al[1]), non-positivity near high gradients (see e.g. Sharma et al[2]), mesh locking, stagnation of convergence dependent on anisotropy (see e.g. Babuška and Suri[3]), convergence loss in case of variable diffusion tensor (see e.g. Günter et al[4]). To confidently perform simulations of phenomena that rely heavily on the resolution of the perpendicular temperature gradient we must apply a scheme that is robust in terms of accuracy in the case of varying anisotropy and misalignment.

## Methods

As a novel approach we suggest to use a finite difference scheme that is approximately aligned with the field lines. First, we write the diffusion equation in terms of locally aligned coordinates $(s,n)$ where $s$ is aligned with the field line and $n$ is perpendicular to the field line, see figure 1a. The aligned equation is discretised with central differencing with fixed stepsizes $(\Delta s, \Delta n)$. The stepsizes $\Delta s, \Delta n$ are free but bounded parameters, as long as the aligned stencil points stay within the interpolation region. The stencil to solve the discretised scheme is found by simply taking two straight lines through the point $(i, j)$ in the directions $\mathbf{b}$ and $\mathbf{b}_\perp$ and picking two points on each line, see figure 1b. This gives us an aligned five-point stencil. The values for $\mathbf{b}, D_\parallel, D_\perp$, ($\mathbf{D}$) and $T$ on this aligned stencil, i.e. at points $r, l, u, d, c$, are found by interpolation of the surrounding nodes which are placed on a colocated mesh. For the interpolation we apply two sets of coefficients, the Vandermonde coefficients and equivalent (and partially equal) symmetric coefficients. The Vandermonde coefficients follow from $\mathbf{c} = \mathbf{V}^{-1}\mathbf{u}$ where $\mathbf{V}$ is the well-known Vandermonde matrix and $\mathbf{u}$ is a vector containing the values for the interpolated quantities at the surrounding nodes. Realizing that the coefficients approximate differential terms we can rewrite some of the coefficients to obtain a more symmetric formulation. The results for the aligned method with Vandermonde coefficients and symmetric coefficients are denoted as *aligned Vandermonde* and *aligned symmetric* respectively. Another approach where the interpolation function is applied directly to the diffusion equation, is called *interp. Vandermonde* or *interp. symmetric* depending on the coefficients used.



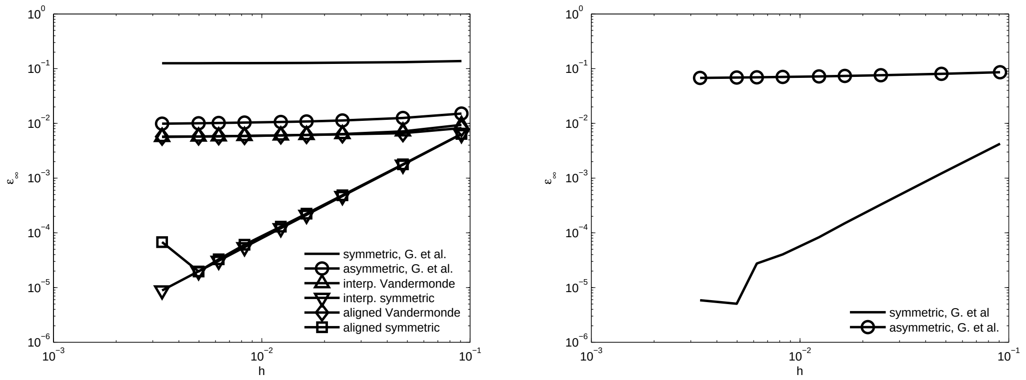

(a) Explanation of symbols

(b) locally transformed grid, 5-point stencil

For comparison we apply a symmetric finite difference scheme (which is mimetic) and an asymmetric finite difference scheme, both schemes are described in Günter et al[4].

**Results**

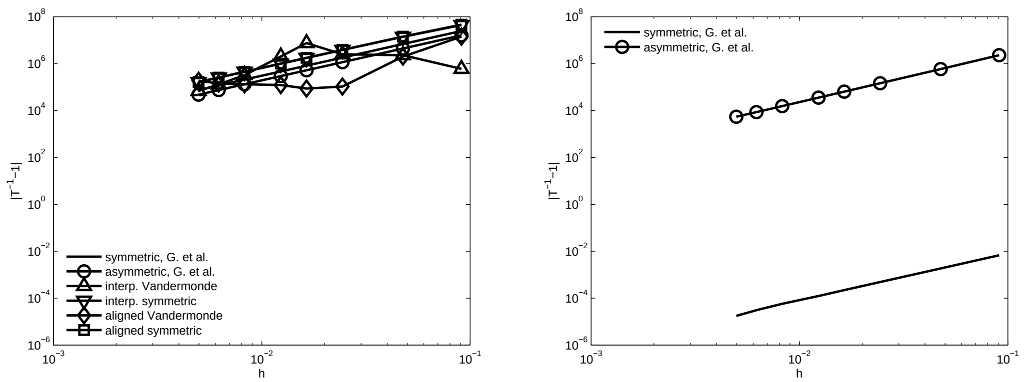
As an example we show the $\varepsilon_\infty$-error convergence for a diffusion test with as exact function; $T = 1 - (x^2 + y^2)^{3/2}, \quad x, y \in -0.5, 0.5$ and an anisotropy ratio of $10^9$. The $\varepsilon_\infty$-error norm is given by $|T - T_e|_{max}/|T_e|_{max}$. Here the field lines are tangent to the temperature contourlines.




(c) (d)

Figure 1: $\varepsilon_\infty$-error of the temperature, anisotropy $\varsigma = 10^9$

From the convergence plot (see figure 1) we can see that our aligned scheme and interpolation scheme are competitive with existing schemes. This was confirmed also for higher anisotropy ratios. In another test case by Sovinec et al [5] we specifically look at the error of the perpendicular diffusion, the exact solution is given by $T = \frac{1}{D_\perp}\psi, \quad f = 2\pi^2\psi, \quad \psi = \cos(\pi x)\cos(\pi y)$ and the error is given by $|T(0,0)^{-1} - D_\perp|$.




(a) $\varsigma = 10^9$, co-located (b) $\varsigma = 10^9$, staggered

Figure 2: Error in perpendicular diffusion $|T^{-1} - 1|$

Here only the symmetric scheme on a staggered grid is able to capture the perpendicular diffusion accurately and practically independent of the level of anisotropy. Our aligned schemes maintain $2^{nd}$ order convergence independent of the anisotropy.

**Discussion**

We have used a new differencing method on a colocated grid that implements the concept of following the field line track within the stencil area to obtain the differencing points that are finally used in the approximation. The aligned scheme can be applied to the non-linear heat diffusion problem, performance is comparable to the symmetric scheme by Günter et al as it conserves the order of accuracy independent of the anisotropy.

**Acknowledgments**

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