On the chromatic number of random geometric graphs

Colin McDiarmid*and Tobias Müller†

University of Oxford and Technische Universiteit Eindhoven

Version: May 2007

Abstract

Given independent random points $X_1, \dots, X_n \in \mathbb{R}^d$, drawn according to some probability distribution ν on $\mathbb{R}^d$, and a positive distance $r > 0$ we construct a random geometric graph G_n with vertex set $\{X_1, \dots, X_n\}$ and an edge $X_iX_j \in E(G_n)$ when $\|X_i - X_j\| < r$. Here $\|.\|$ may be an arbitrary norm on $\mathbb{R}^d$ and we allow any probability distribution ν with a bounded density function. We consider the chromatic number $\chi(G_n)$ of G_n and its relation to the clique number $\omega(G_n)$ as n grows. We extend results by the first author [12] and by Penrose [15]. In both [12] and [15] the chromatic number was considered in the range of r when $n^{-\varepsilon} \ll nr^d \ll \ln n$ for all $\varepsilon > 0$ and in the range $nr^d \gg \ln n$, and their results showed a dramatic difference between these two cases. Both authors asked for the behaviour of the chromatic number in the "phase change" range when $nr^d = \Theta(\ln n)$. Here we will determine constants $c(t)$ such that $\frac{\chi(G_n)}{nr^d} \to c(t)$ almost surely when $nr^d \sim t \ln n$ and we will sharpen and extend the results from [12, 15] on other ranges. A striking feature of our results is a "sharp threshold"; (except for some less interesting choices of $\|.\|$ – when the unit ball tiles d-space) there is a sequence $r_0 = r_0(n)$ such that if $r \leq r_0$ then $\frac{\chi(G_n)}{\omega(G_n)} \to 1$ almost surely, but if $r > (1+\varepsilon)r_0$ for some $\varepsilon > 0$ then the lim inf of the ratio $\frac{\chi(G_n)}{\omega(G_n)}$ is bounded away from 1 almost surely.

1 Introduction and statement of main results

To set the stage, we fix a positive integer d, and a norm $\|.\|$ on $\mathbb{R}^d$. Then we introduce a probability distribution ν with bounded density function, and consider a sequence $X_1, X_2, \dots$ of independent rv's each with the given distribution ν. Also we need a sequence

[1]Department of Statistics, 1 South Parks Road, Oxford, OX1 3TG, United Kingdom. Email address: cmcd@stats.ox.ac.uk

[2]EURANDOM, Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. Email address: t.muller@tue.nl. The research in this paper was conducted while this author was a research student at the University of Oxford. He was partially supported by Bekker-la-Bastide fonds, Hendrik Muller's Vaderlandsch fonds, EPSRC, Oxford University Department of Statistics and Prins Bernhard Cultuurfonds.

$r = (r(1), r(2), ..)$ of positive real numbers such that $r(n) \to 0$ as $n \to \infty$. The random geometric graph G_n has vertex-set $V(G_n) := \{X_1, \ldots, X_n\}$ and an edge $X_iX_j \in E(G_n)$ $(i \neq j)$ if $\|X_i - X_j\| < r$. For technical reasons we shall always assume that $r(n) \to 0$ as $n \to \infty$.

Recall that a k-colouring of a graph G is a map $c : V(G) \to \{1, \ldots, k\}$ such that $c(v) \neq c(w)$ whenever $vw \in E(G)$ and that the chromatic number $\chi(G)$ is the least k for which G admits a k-colouring. Also recall that a clique in a graph G is a set of vertices $C \subseteq V(G)$ with the property that $vw \in E(G)$ for all pairs $v \neq w \in C$, and the clique number $\omega(G)$ is the largest cardinality of a clique (note that $\omega(G)$ is a trivial lower bound for $\chi(G)$). In this paper we are interested in the behaviour of the chromatic number, $\chi(G_n)$, and its relation to the clique number, $\omega(G_n)$, of G_n as n grows large.

The distance $r = r(n)$ plays a role similar to that of the edge-probability $p(n)$ for Erdős-Renyi random graphs $G(n, p)$. Depending on the choice of $r(n)$ qualitatively different types of behaviour can be observed. The various cases are best described in terms of the quantity nr^d, which scales with the average degree of the graph (see appendix A of [13] for a precise result).

Before we we can state our first result we will need some further definitions. In the rest of the paper σ will denote the *essential supremum* of the probability density function f of ν, ie.:

$$\sigma := \sup\{t : \mathrm{vol}(\{x : f(x) > t\}) > 0\}.$$

Here and in the rest of the paper vol(.) denotes the d-dimensional volume (Lebesgue measure). We also need to define the 'packing density' δ. Informally δ is the greatest proportion of $\mathbb{R}^d$ that can be filled with disjoint translates of the unit ball $B := \{x \in \mathbb{R}^d : \|x\| < 1\}$ wrt the (arbitrary) norm we have equipped $\mathbb{R}^d$ with. For $K > 0$ let $N(K)$ be the maximum cardinality of a collection of pairwise disjoint translates of B with centers in $(0, K)^d$. The (translational) *packing density* δ of the unit ball B may be defined as

$$\delta := \lim_{K \to \infty} \frac{N(K)\,\mathrm{vol}(B)}{K^d}.$$

(This limit always exists.) For an overview of results on packing see for instance [16] or [14].

The first theorem concerns the case when $\sigma nr^d / \ln n \to$ a limit t as $n \to \infty$; and it asserts that then $\chi(G_n)/\sigma nr^d$ tends a.s. to a limit $c(t)$, and gives some properties of the limiting function c. We shall give a formula for $c(t)$ later, after introducing further definitions.

Theorem 1.1 *There is a function $c : (0, \infty] \to (0, \infty)$, given explicitly in Theorem 3.4 below, such that*

- **(i)** *c is continuous and non-increasing, and $c(t) \to c(\infty) = \frac{\mathrm{vol}(B)}{2^d \delta}$ as $t \to \infty$; and*
- **(ii)** *if $\sigma nr^d / \ln n \to t \in (0, \infty]$ as $n \to \infty$, then $\dfrac{\chi(G_n)}{\sigma nr^d} \to c(t)$ a.s.*

In the case when $nr^d \gg \ln n$ (ie. $t = \infty$ in the above theorem) this gives an improvement over a result in [15]. Penrose ([15]) gave an almost sure upper bound for $\limsup \frac{\chi(G_n)}{\sigma nr^d}$ of $\frac{\mathrm{vol}(B)}{2^d \delta_L}$ and an almost sure lower bound for $\liminf \frac{\chi(G_n)}{\sigma nr^d}$ of $\frac{\mathrm{vol}(B)}{2^d \delta}$, where δ_L is the *lattice packing density* of B (that is, the proportion of $\mathbb{R}^d$ that can be filled with disjoint translates of B whose centres are the integer linear combinations of some basis for $\mathbb{R}^d$). The paper [12] considers only the Euclidean norm in the plane, where δ and δ_L coincide. However, let us note that in general dimension the question of whether $\delta = \delta_L$ is open, even for the Euclidean norm, and a widely held conjecture is that $\delta > \delta_L$ for large dimensions d. Theorem 3.4 shows that in fact δ_L is not relevant here and the lower bound given by Penrose is always attained.

Putting the above theorem together with some further results including one of Penrose on the clique number $\omega(G_n)$, we can obtain a description of the limiting behaviour of the informative ratio $\chi(G_n)/\omega(G_n)$. Some results were already known about the 'sparse' case when $\sigma nr^d/\ln n \to 0$, and the 'dense' case when $\sigma nr^d/\ln n \to \infty$ (these results are described more fully below), but nothing was known about the more challenging intermediate case. One feature which we find is a striking threshold value t_0 (except when $\delta = 1$).

Theorem 1.2 *The following holds for the ratio $\chi(G_n)/\omega(G_n)$ as $n \to \infty$:*

- **(i)** $\limsup_{n\to\infty} \chi(G_n)/\omega(G_n) \leq \frac{1}{\delta}$ *a.s., so in particular if $\delta = 1$ then $\chi(G_n)/\omega(G_n) \to 1$ a.s.*

- **(ii)** *Assume now that $\delta < 1$. Then there is a constant t_0 with $0 < t_0 < \infty$ and a function $x : [0,\infty] \to (0,\infty)$, given explicitly by (10) below, such that*

  (a) x is continuous, $x(t) = 1$ for $t \leq t_0$, x is strictly increasing for $t \geq t_0$, and $x(t) \to x(\infty) = \frac{1}{\delta}$ as $t \to \infty$; and

  (b) if $\sigma nr^d/\ln n \to t \in [0,\infty]$ as $n \to \infty$, then $\dfrac{\chi(G_n)}{\omega(G_n)} \to x(t)$ a.s.

What is more, for the threshold behaviour we do not need to assume that $nr^d/\ln n$ tends to a limit:

Theorem 1.3 *The following holds for the ratio $\chi(G_n)/\omega(G_n)$ as $n \to \infty$:*

- **(i)** *If $\delta = 1$ or if $\delta < 1$ and $\limsup \sigma nr^d/\ln n \leq t_0$ then*

$$\frac{\chi(G_n)}{\omega(G_n)} \to 1 \text{ a.s.}$$

- **(ii)** *If $\delta < 1$ and $\liminf \sigma nr^d/\ln n \geq t_0 + \varepsilon$ for some $\varepsilon > 0$ then*

$$\liminf \frac{\chi(G_n)}{\omega(G_n)} \geq x(t_0 + \varepsilon) > 1 \text{ a.s.}$$

So there is a "sharp threshold" at $r_0 := (\frac{t_0 \ln n}{\sigma n})^{\frac{1}{d}}$.

Part of the proof of Theorem 1.2 and Theorem 1.3 will be to derive the following result which is also of independent interest.

Theorem 1.4 *If $nr^d \leq n^{-\alpha}$ for some $\alpha > 0$ then*

$$\mathbb{P}(\chi(G_n) = \omega(G_n) \text{ for all but finitely many } n) = 1.$$

So we see that for small r we can be very precise. In [12] (two dimensional, euclidean case) and in [15] (the case of arbitrary norm in arbitrary dimension) it was already shown that if $n^{-\varepsilon} \ll \ln n \ll \ln n$ then $\chi(G_n), \omega(G_n)$ and $\Delta(G_n)$ are all asymptotically equivalent to $k(n) := \ln n / \ln(\frac{\ln n}{nr^d})$ in the sense that the ratios $\frac{\chi(G_n)}{k(n)}, \frac{\omega(G_n)}{k(n)}, \frac{\Delta(G_n)}{k(n)}$ all tend to one in probability as $n \to \infty$. As part of proof of part **(i)** of Theorem 1.3 we will settle the minor technical point that the type of convergence can be strengthened to almost sure convergence (which settles a question posed in [12] and [15]).

2 Notation and preliminaries

We will often omit the argument or subscript n for the sake of readability. We will denote $B(x;\rho) := \{y : \|x - y\| < \rho\}$ and $\text{diam}(A) := \sup\{\|x - y\| : x, y \in A\}$, all wrt the given norm. Following Penrose [15] we set $\theta := \text{vol}(B)$, with $B := B(0;1)$ the unit ball (wrt $\|.\|$).

For $V \subseteq \mathbb{R}^d$ a set of points and $\rho > 0$ a positive number we will denote by $G(V, \rho)$ the graph with vertex set V and an edge $vw \in E(G(V, \rho))$ iff $\|v - w\| \leq \rho$. Notice that with probability one $G_n = G(\{X_1, \dots, X_n\}, r(n))$ (as the events $\|X_i - X_j\| = r$ have probability zero for all i, j). We have chosen to use $\leq$ instead of strict inequality in the definition here for technical reasons to do with the proof of Proposition 3.15 below.

For $V \subseteq \mathbb{R}^d$ a set of points and $\varphi : \mathbb{R}^d \to \mathbb{R}$ a nonnegative function we will denote:

$$M(V, \varphi) := \sup_{x \in \mathbb{R}^d} \sum_{v \in V} \varphi(v - x)$$

A central role in our analysis will be played by these quantities and by the random variables

$$M_\varphi = M_\varphi(n, r) := M(r^{-1}\{X_1, \dots, X_n\}, \varphi) = \sup_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \varphi\left(\frac{X_i - x}{r}\right).$$

In the special case when $\varphi = 1_W$ is the indicator function of some set W, we will also denote $M_W := M_\varphi$. The variable M_W is called the *scan statistic* (wrt the scanning set W). Notice that M_W is the maximum number of points in any translate of rW, ie.

$$M_W = \max_{x \in \mathbb{R}^d} |\{X_1, \dots, X_n\} \cap (x + rW)|.$$

For $A \subseteq \mathbb{R}^d$ we will sometimes denote the number of X_i that fall inside A by $\mathcal{N}(A) = \mathcal{N}_n(A)$, ie.

$$\mathcal{N}(A) := |A \cap \{X_1, \dots, X_n\}| = \sum_{i=1}^{n} 1_A(X_i).$$

So $M_W = \max_x \mathcal{N}(x + rW)$.

If A is an event then we say that A holds almost surely (a.s.) if $\mathbb{P}(A) = 1$, and if $A_1, A_2, \dots$ is a sequence of events then $\{A_n$ almost always $\}$ denotes the event that "all but finitely many A_n hold". In the rest of this paper we will frequently deal with the situation in which $\mathbb{P}(A_n$ almost always $) = 1$, which we shall denote by A_n a.a.a.s. (A_n almost always almost surely). We thereby hope to introduce a convenient shorthand whilst avoiding clashes with the many different existing notations for $\mathbb{P}(A_n) = 1+o(1)$ (a.a., a.a.s., whp.) that are in use in the random graphs literature. The reader should observe that A_n a.a.a.s. is a much stronger statement than A_n a.a.s.

In the rest of this paper H will denote the function $x \mapsto x \ln x - x + 1$ for $x > 0$. Observe that $H(1) = 0$ and that H is strictly increasing for $x > 1$.

For $0 < t < \infty$ and φ a non-negative, bounded, measurable function with $0 < \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x < \infty$ let us denote by $\xi(\varphi, t)$ the integral

$$\xi(\varphi, t) := \int_{\mathbb{R}^d} \varphi(x) e^{s\varphi(x)} dx,$$

where $s = s(\varphi, t)$ is the unique nonnegative solution to

$$\int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x = \frac{1}{t}. \tag{1}$$

We will also set $\xi(\varphi, \infty) = \int \varphi$.

We leave to the reader the elementary considerations showing that the left hand side of (1) equals 0 when $s = 0$ and is continuous and strictly increasing in s (and finite) for $s \geq 0$ under the stated conditions on φ – so that $s(\varphi, t)$ and $\xi(\varphi, t)$ are well defined. If $\int \varphi = 0$ (in which case $\varphi = 0$ almost everywhere) we will set $\xi(\varphi, t) = 0$ and if $\int \varphi = \infty$ we will set $\xi(\varphi, t) = \infty$ and $s(\varphi, t) = 0$. We remark that if $\varphi = 1_W$ for some set $W \subseteq \mathbb{R}^d$ with $0 < \mathrm{vol}(W) < \infty$ then (as can be seen by straightforward computations):

$$\xi(\varphi, t) = c(t) \,\mathrm{vol}(W), \tag{2}$$

where $c = c(t) \geq 1$ solves $H(c) = \frac{1}{t \,\mathrm{vol}(W)}$ if $t < \infty$ and $c = 1$ if $t = \infty$.

We will often omit the domain we are integrating over and simply write $\int \varphi$ instead of $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x$. All integrals in this paper are over $\mathbb{R}^d$ (and wrt the d-dimensional Lebesgue measure) unless explicitly stated otherwise. The following lemma lists a number of basic properties of $\xi(\varphi, t)$. We will make frequent use of these properties in the rest of the paper.

Lemma 2.1 *For any $t \in (0, \infty]$ and non-negative, bounded, measurable, integrable functions φ, ψ the following hold.*

(i) *If $\varphi \leq \psi$ then $\xi(\varphi, t) \leq \xi(\psi, t)$;*

(ii) $\xi(\varphi + \psi, t) \le \xi(\varphi, t) + \xi(\psi, t)$*;*

(iii) $\xi(\lambda\varphi, t) = \lambda\xi(\varphi, t)$ *for any* $\lambda > 0$*;*

(iv) *For* $0 < \lambda < 1$ *let* φ_λ *be given by* $\varphi_\lambda(x) = \varphi(\lambda x)$*. Then* $\xi(\varphi, t) \le \xi(\varphi_\lambda, t) \le \lambda^{-d}\xi(\varphi, t)$*;*

(v) $(\frac{t}{t+h})\xi(\varphi, t) \le \xi(\varphi, t + h) \le \xi(\varphi, t)$ *for* $h > 0$*;*

(vi) *Let* $\varphi_1, \varphi_2, \dots$ *be a sequence of nonnegative, measurable functions with pointwise limit* φ *and* $\varphi_n \le \psi$ *for all* n *where* ψ *satisfies* $0 < \int \psi < \infty$*. Then* $\lim_{n\to\infty} \xi(\varphi_n, t) = \xi(\varphi, t)$*;*

(vii) *If* $\int \varphi 1_{\{\varphi \ge a\}} \le \int \psi 1_{\{\psi \ge a\}}$ *for all* a *then* $\xi(\varphi, t) \le \xi(\psi, t)$*.*

The proofs (most of which are relatively straightforward) can be found in section 5.

We will say that a set $W \subseteq \mathbb{R}^d$ has a *small neighbourhood* if $\lim_{\varepsilon\to 0} \mathrm{vol}(W_\varepsilon) = \mathrm{vol}(W)$, where $W_\varepsilon = W + B(0; \varepsilon)$. Then for sets W with $\mathrm{vol}(W) < \infty$, W has a small neighbourhood if and only if W is bounded and $\mathrm{vol}(\mathrm{cl}(W)) = \mathrm{vol}(W)$, where $\mathrm{cl}(.)$ denotes closure. So in particular all compact sets and all bounded convex sets have small neighbourhoods. We will say that a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is *tidy* if it is measurable, bounded, nonnegative, has bounded support and the sets $\{x : \varphi(x) > a\}$ have small neighbourhoods for all $a > 0$. We will call a function *simple* if it takes only finitely many values.

Let us denote by $\mathcal{S}$ the collection of all sets $S \subseteq \mathbb{R}^d$ that satisfy $\|v - w\| > 1$ for all $v \ne w \in S$. We will call a nonnegative, measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ *(dual) feasible* if it satisfies the condition that $\sum_{v\in S} \varphi(v) \le 1$ for any set $S \in \mathcal{S}$. For example, $\varphi_0 = 1_{B(0,\frac{1}{2})}$ is feasible. We will denote the set of all feasible functions by $\mathcal{F}$.

3 Proofs of main results

The proofs in this section rely heavily on the following result. We postpone the proof until section 4.

Theorem 3.1 *If* $\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty]$ *as* $n \to \infty$ *and* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *is a tidy function then*

$$\frac{M_\varphi}{\sigma n r^d} \to \xi(\varphi, t) \ \textit{a.s.}$$

Theorem 3.1 allows us to give a short proof of the following theorem of Penrose, which we shall need for the proof of Theorem 1.2. Here we used (2) to rephrase the statement in [15] in terms of ξ, and we have removed the assumption in [15] that the density function has compact support.

Theorem 3.2 (Penrose [15]) *If* $\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty]$ *as* $n \to \infty$ *then*

$$\frac{\omega(G_n)}{\sigma n r^d} \to \xi(\varphi_0, t) \ \textit{a.s.}$$

where $\varphi_0 = 1_{B(0;\frac{1}{2})}$*.*

Proof of Theorem 3.2: First set $W := B(0; \frac{1}{2})$. Any set of points contained in a translate of rW is a clique of G_n, so that by Theorem 3.1:

$$\liminf_{n\to\infty} \frac{\omega(G_n)}{\sigma n r^d} \geq \lim_{n\to\infty} \frac{M_W}{\sigma n r^d} = \xi(\varphi_0, t) \text{ a.s.}$$

Let us now fix $\varepsilon > 0$ and let $A_1, \dots, A_m \subseteq \varepsilon\mathbb{Z}^d$ be all the subsets of $\varepsilon\mathbb{Z}^d$ that satisfy $0 \in A_i$ and $\operatorname{diam}(A_i) \leq 1 + 2\varepsilon\rho$, where $\rho := \operatorname{diam}([0,1]^d)$. Let $W_i := \operatorname{conv}(A_i)$. We now claim that $\omega(G_n) \leq \max_i M_{W_i}$. To see that this holds, suppose $X_{i_1}, \dots, X_{i_k}$ form a clique in G_n. Let us set $y_j := (X_{i_j} - X_{i_1})/r$ and $A := \{p \in \varepsilon\mathbb{Z}^d : \|p - y_i\| \leq \varepsilon\rho \text{ for some } 1 \leq i \leq k\}$. Observe that $0 = y_1 \in A$ and $\operatorname{diam}(A) \leq 1 + 2\varepsilon\rho$, so that $A = A_i$ for some $1 \leq i \leq m$. What is more $\{y_1, \dots, y_k\} \subseteq W := \operatorname{conv}(A)$. But this gives $\{X_{i_1}, \dots, X_{i_k}\} \subseteq X_{i_1} + rW$ by choice of y_i, and the claim follows. Thus Theorem 3.1 yields

$$\limsup_{n\to\infty} \frac{\omega(G_n)}{\sigma n r^d} \leq \lim_{n\to\infty} \max_i \frac{M_{W_i}}{\sigma n r^d} = \max_i \xi(\varphi_i, t) \text{ a.s.},$$

writing $\varphi_i = 1_{W_i}$. We will now need the following inequality (for a proof see for instance [5]):

Lemma 3.3 (Bieberbach inequality) *Let $A \subseteq \mathbb{R}^d$. If A' is a ball with $\operatorname{diam}(A) = \operatorname{diam}(A')$ then $\operatorname{vol}(A) \leq \operatorname{vol}(A')$.*

By the Bieberbach inequality $\operatorname{vol}(W_i) \leq \operatorname{vol}(B(0; \frac{1+2\varepsilon\rho}{2}))$, so that also $\int \varphi_i 1_{\{\varphi_i \geq a\}} \leq \int \psi 1_{\{\psi \geq a\}}$ for all a where ψ denotes $1_{B(0;\frac{1+2\varepsilon\rho}{2})}$. Also observe that $\psi(x) = \varphi_0(\frac{x}{1+2\varepsilon\rho})$. By parts **(vii)** and **(iv)** of Lemma 2.1 we therefore have $\max_i \xi(\varphi_i, t) \leq \xi(\psi, t) \leq (1 + 2\varepsilon\rho)^d \xi(\varphi_0, t)$. Sending $\varepsilon \to 0$ now gives the result. $\square$

3.1 Proof of Theorem 1.1

Recall that $\mathcal{F}$ denotes the set of all feasible functions. In this section we will show the following explicit version of Theorem 1.1:

Theorem 3.4 *Suppose that $\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty]$ as $n \to \infty$. Then*

$$\frac{\chi(G_n)}{\sigma n r^d} \to \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) \textit{ a.s.},$$

and the right-hand side has the properties claimed for $c(t)$ in Theorem 1.1.

We will prove Theorem 3.4 in a number of intermediate steps. Recall that a stable or independent set in G is a subset $S \subseteq V(G)$ with the property that $vw \notin E(G)$ for all pairs $v \neq w \in S$ and that the chromatic number $\chi(G)$ of a graph G corresponds to a natural integer linear program (ILP), expressing the fact that the chromatic number is the least number of stable sets needed to cover the vertices, as follows. Let A be the vertex-stable set incidence matrix of G, that is, the rows of A are indexed by the vertices v, the columns are indexed by the stable sets S, and $(A)_{v,S} = 1$ if $v \in S$ and $(A)_{v,S} = 0$ otherwise. Then $\chi(G)$ equals

$$
\begin{array}{rl}
\min & 1^T x \\
\\
\text{subject to} & Ax \geq 1, \\
& x \geq 0, x \text{ integral.}
\end{array}
\tag{3}
$$

The *fractional chromatic number* $\chi_f(G)$ of a graph G is the objective value of the LP-relaxation of (3) (ie. we drop the constraint that x be integral). By definition we have $\chi_f(G) \leq \chi(G)$. In general the difference can be large (see [17] for more background on the fractional chromatic number and related notions), but as we will see that is not the case for G_n. We start with a deterministic lemma on the fractional chromatic number of a geometric graph $G(V, r)$. Recall that if $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a function and $V \subseteq \mathbb{R}^d$ is a set of points then $M(V, \varphi) := \sup_{x \in \mathbb{R}^d} \sum_{v \in V} \varphi(v - x)$.

Lemma 3.5 *Let $V \subseteq \mathbb{R}^d$ be a finite set of points and consider the graph $G = G(V, 1)$. Then*

$$\chi_f(G) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi).$$

Proof: By LP-duality $\chi_f(G)$ also equals the objective value of the dual LP:

$$
\begin{array}{rl}
\max & 1^T y \\
\\
\text{subject to} & A^T y \leq 1, \\
& y \geq 0.
\end{array}
$$

For convenience let us write $V = \{v_1, \dots, v_n\}$ where v_i is the vertex corresponding to the i-th row of A (and thus the i-th column of A^T). Notice that a vector $y = (y_1, \dots, y_n)^T$ is feasible for the dual LP if and only if it attaches nonnegative weights to the vertices of G in such a way that each stable set has total weight at most one. There is a natural correspondence between such vectors y and certain feasible functions (hence our choice of the name 'feasible function').

Let φ be any feasible function and $x \in \mathbb{R}^d$ an arbitrary point. We claim that the vector

$$y = (\varphi(v_1 - x), \dots, \varphi(v_n - x))^T$$

is a feasible point of the dual LP given above. To see this note that each row of A^T is the incidence vector of some stable set S of G; and $\|(z - x) - (z' - x)\| = \|z - z'\| > 1$ for each $z \neq z' \in S$ since S is stable in G. Hence

$$(A^T y)_S = \sum_{z \in S} \varphi(z - x) \leq 1,$$

by feasibility of φ. This holds for all rows of A^T, so that y is indeed feasible for the dual LP as claimed. Also, notice that the objective function value $1^T y$ equals $\sum_{j=1}^n \varphi(v_j - x)$. This shows that

$$\chi_f(G) \geq \sup_{\varphi \in \mathcal{F}} \sup_{x \in \mathbb{R}^d} \sum_{j=1}^{n} \varphi(v_j - x) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi).$$

Conversely let the vector $y = (y_1, \ldots, y_n)^T$ be feasible for the dual LP. Define $\varphi(z) = \sum_{i=1}^n y_i 1_{z=v_i}$. Then φ is clearly a feasible function, and

$$1^T y = \sum_{j=1}^{n} \varphi(v_j) \leq \sup_{x \in \mathbb{R}^d} \sum_{j=1}^{n} \varphi(v_j - x) = M(V, \varphi),$$

so that $\chi_f(G) \leq \sup_{\varphi \in \mathcal{F}} M(V, \varphi)$. □

Recall that (with probability one) $G_n = G(\{X_1, \ldots, X_n\}, r) \cong G(r^{-1}\{X_1, \ldots, X_n\}, 1)$ (where $\cong$ denotes isomorphic to). Thus by the above lemma and rescaling we get

$$\chi(G_n) \geq \chi_f(G_n) = \sup_{\varphi \in \mathcal{F}} M_\varphi \text{ a.s.}$$

Let $\mathcal{F}^*$ be the collection of all $\varphi \in \mathcal{F}$ that are tidy. Then Theorem 3.1 allows us to conclude that

$$\liminf_{n \to \infty} \frac{\chi(G_n)}{\sigma n r^d} \geq \liminf_{n \to \infty} \frac{\chi_f(G_n)}{\sigma n r^d} \geq \sup_{\varphi \in \mathcal{F}^*} \xi(\varphi, t) \text{ a.s.} \tag{4}$$

The last term is not quite the expression we are aiming for in Theorem 3.4, but we will deal with that later, and show that $\sup_{\varphi \in \mathcal{F}^*} \xi(\varphi, t) = \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t)$.

Let us now turn our attention towards deriving an upper bound. We start with another deterministic lemma. Given $\alpha > 0$ we say that the function φ on $\mathbb{R}^d$ is α-feasible if the function $\varphi_\alpha(x) = \varphi(\alpha x)$ is feasible (ie. if $S \subseteq \mathbb{R}^d$ satisfies $\|s - s'\| > \alpha$ for all $s \neq s' \in S$ then $\sum_{s \in S} \varphi(s) \leq 1$). Thus 1-feasible means feasible.

Lemma 3.6 *For each $\varepsilon > 0$ there exists a positive integer m, $(1+\varepsilon)$-feasible, tidy functions $\varphi_1, \ldots, \varphi_m$, and a constant c such that:*

$$\chi(G(V, 1)) \leq (1 + \varepsilon) \max_{i=1,\ldots,m} M(V, \varphi_i) + c,$$

for any set $V \subseteq \mathbb{R}^d$.

Proof: Let $\varepsilon > 0$. Let us again set $\rho := \operatorname{diam}([0,1]^d)$ and let η be the smallest multiple of ε such that $\|y - z\| \geq 1 + \varepsilon\rho$ whenever $|y_i - z_i| \geq \eta$ for some coordinate $1 \leq i \leq d$. Let $K > 0$ be such that K/ε is an integer (think of K as large and ε as small), and let

$N = (\frac{2K}{\varepsilon})^d$. We shall show that there exist $(1+2\varepsilon\rho)$-feasible tidy functions $\varphi_1,\ldots,\varphi_N$ such that the following holds for any $V \subseteq \mathbb{R}^d$:

$$\chi(G(V,1)) \leq (1+\frac{\eta}{2K})^d \max_i M(V,\varphi) + N^2(1+\frac{\eta}{2K})^d. \tag{5}$$

This of course yields the lemma, by adjusting ε and taking K sufficiently large.

We partition $\mathbb{R}^d$ into hypercubes of side ε. Let Γ be the (infinite) graph with vertex set $\varepsilon\mathbb{Z}^d$ and an edge pq when $\|p-q\| < 1+\varepsilon\rho$. For each $q \in \varepsilon\mathbb{Z}^d$ let C^q denote the hypercube $q+[0,\varepsilon)^d$. Observe that the hypercubes C^q for $q \in \varepsilon\mathbb{Z}^d$ partition $\mathbb{R}^d$. Thus for each $z \in \mathbb{R}^d$ we may define $p(z)$ to be the unique $q \in \varepsilon\mathbb{Z}^d$ such that $z \in C^q$.

Now let $V_0 = [-K,K)^d \cap \varepsilon\mathbb{Z}^d$, and note that $|V_0| = N$. For each $p \in \varepsilon\mathbb{Z}^d$ let Γ^p be the subgraph of Γ induced on the vertex set $p+V_0$, that is by the vertices of Γ in $p+[-K,K)^d$. Observe that the graphs Γ^p are simply translated copies of Γ^0. Let B be the vertex-stable set incidence matrix of Γ^0.

Now let $V \subseteq \mathbb{R}^d$ be arbitrary. Given a subset S of $\mathbb{R}^d$, let us use the notation $\mathcal{N}(S)$ here to denote $|S \cap V|$. Let Γ_V be the graph we get by replacing each node q of Γ by a clique of size $\mathcal{N}(C^q)$ and adding all the edges between the cliques corresponding to $q,q' \in V_0$ if $qq' \in E(\Gamma^0)$. It is easy to see from the definition of the threshold distance in Γ that $G(V,1)$ is isomorphic to a subgraph of Γ_V. For each $p \in \varepsilon\mathbb{Z}^d$ let Γ_V^p be the subgraph of Γ_V corresponding to the vertices of Γ^p.

Consider some $p \in \varepsilon\mathbb{Z}^d$. Then $\chi(\Gamma_V^p)$ is the objective value of the integer LP:

$$\begin{array}{rl} \min & 1^T x \\ \\ \text{subject to} & Bx \geq b^p \\ & x \geq 0,\ x \text{ integral} \end{array} \tag{6}$$

where $b^p = (\mathcal{N}(C^{p+q}))_{q\in V_0}$ and the vector x is indexed by the stable sets in Γ^0. Here we are using the fact that Γ^p is a copy of Γ^0 and that the vertex corresponding to q has been replaced by a clique of size $\mathcal{N}(C^{p+q})$. By again considering the LP-relaxation and switching to the dual we find that $\chi_f(\Gamma_V^p)$ equals the objective value of the LP:

$$\begin{array}{rl} \max & (b^p)^T y \\ \\ \text{subject to} & B^T y \leq 1 \\ & y \geq 0. \end{array} \tag{7}$$

Notice that the vectors $y = (y_q)_{q\in V_0}$ attach nonnegative weights to the points q of V_0 is such a way that if $S \subseteq V_0$ corresponds to a stable set in Γ_0 then the sum of the weights $\sum_{q\in S} y_q$ is less than one. Note the important fact that the feasible region (ie. the set of all y that satisfy $B^T y \leq 1, y \geq 0$) here does not depend on p or V.

The vectors y that satisfy $B^T y \leq 1, y \geq 0$ correspond to 'nearly feasible' functions φ in a natural way, as follows. Observe that $x \in [-K,K)^d$ if and only if $p(x) \in V_0$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be defined by setting

$$\varphi(x) := \begin{cases} y_{p(x)} & \text{if } x \in [-K,K)^d, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\varphi(x) = \sum_{q \in V_0} 1_{C^q}(x) y_q$. Then for each $p \in \varepsilon\mathbb{Z}^d$

$$\begin{aligned} (b^p)^T y &= \textstyle\sum_{q \in V_0} \mathcal{N}(C^{p+q}) y_q = \sum_{q \in V_0} \sum_{v \in V} 1_{C^{p+q}}(v) y_q \\ &= \textstyle\sum_{v \in V} \sum_{q \in V_0} 1_{C^q}(v-p) y_q = \sum_{v \in V} \varphi(v-p) \\ &\leq M(V, \varphi). \end{aligned}$$

We claim next that the functions φ thus defined are $(1+2\varepsilon\rho)$-feasible; that is they satisfy $\sum_{j=1}^k \varphi(z_j) \leq 1$ for any $z_1, \dots, z_k$ such that $\|z_j - z_l\| > 1 + 2\varepsilon\rho$ for all $j \neq l$. To see this, pick such $z_1, \dots, z_k$. Since φ is 0 outside of $[-K,K)^d$ we may as well suppose that all the z_j lie inside $[-K,K)^d$. For all pairs $i \neq j$ we have $\|p(z_i) - p(z_j)\| \geq \|z_i - z_j\| - \varepsilon\rho > 1 + \varepsilon\rho$. Thus $p(z_1), \dots, p(z_k)$ are distinct and form a stable set S in Γ^0, and therefore correspond to one of the rows of B^T. The condition $B^T y \leq 1$ now yields

$$\varphi(z_1) + \cdots + \varphi(z_k) = y_{p(z_1)} + \cdots + y_{p(z_k)} = (B^T y)_S \leq 1.$$

This shows that φ is $(1+2\varepsilon\rho)$-feasible as claimed, and it can be readily seen from the definition of φ that it is tidy.

Recall that a basic feasible solution of an LP with k constraints has at most k nonzero elements and that, provided the optimum value is bounded, the optimum value of the LP is always attained at a basic feasible solution (see for instance [2]). Thus, noting that by rounding up all the variables in an optimum basic feasible solution x to the LP-relaxation of (6) we get a feasible solution of the ILP (6) itself, we see that:

$$\chi(\Gamma_V^p) \leq \chi_f(\Gamma_V^p) + N.$$

Now let $y^1, \dots, y^m$ be the vertices of the polytope $B^T y \leq 1, y \geq 0$ and let $\varphi_1, \dots, \varphi_m$ be the corresponding $(1+2\varepsilon\rho)$-feasible, tidy functions. As the optimum of the LP (7) corresponding to $\chi_f(\Gamma_V^p)$ is attained at one of these vertices we see that:

$$\max_{j=1,\dots,m} (b^p)^T y^j = \chi_f(\Gamma_V^p) \leq \chi(\Gamma_V^p) \leq \max_{j=1,\dots,m} (b^p)^T y^j + N \leq \max_{j=1,\dots,m} M(V, \varphi_j) + N. \qquad (8)$$

What is more, for each $p \in \varepsilon\mathbb{Z}^d$ we can colour any subgraph of $G(V,1)$ induced by the points in the set $W^p = p + [-K,K)^d + (2K+\eta)\mathbb{Z}^d$ with this many colours, since by the definition of η, W^p is the union of hypercubes of side $2K$ which are far enough apart for any two points of Γ in different hypercubes not to be joined by an edge.
Now let the set P be defined by

$$P = \varepsilon\mathbb{Z}^d \cap [-K, K+\eta)^d = \{(\varepsilon i_1, \dots, \varepsilon i_d) : \frac{-K}{\varepsilon} \leq i_j < \frac{K+\eta}{\varepsilon}\}.$$

Note that if p runs through the set P then each $q \in \varepsilon\mathbb{Z}^d$ is covered by exactly N of the sets W^p. If H^p is the graph we get by replacing every vertex q of Γ that lies in W^p by a clique

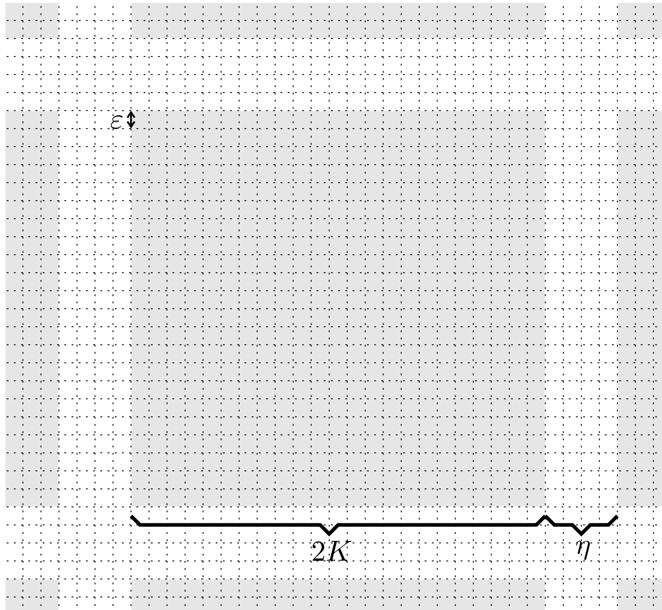

Figure 1: Depiction of a set W^p.

of size $\left\lceil \frac{\mathcal{N}(C^q)}{N} \right\rceil$ rather than one of size $\mathcal{N}(C^q)$ and removing any vertex that does not lie in W^p, then

$$\chi(H^p) \leq \frac{1}{N} \max_j M(V, \varphi_j) + N.$$

This is because we can consider the hypercubes of side $2K$ that make up W^p separately (and each of these corresponds to some Γ_V^q) and for each such constituent hypercube $q + [-K, K)^d$ all we need to do is replace the 'right hand side' vector b^q by $\frac{1}{N} b^q$ in the LP-relaxation of the ILP (6) (the rounding up of the variables will then take care of the rounding up in the constraints, as the entries of B are integers). Because each $q \in \varepsilon\mathbb{Z}^d$ is covered by exactly N of the sets W^p we can combine the $(\frac{2K+\eta}{\varepsilon})^d$ colourings of the graphs H^p for $p \in P$ to get a proper colouring of Γ_V with at most

$$(\frac{2K+\eta}{\varepsilon})^d \left(\frac{1}{N} \max_j M(V, \varphi_j) + N \right),$$

colours and the inequality (5) follows. □

Before we can finish the proof of Theorem 3.4 we must derive a lemma on the functions inside the class $\mathcal{F}$.

Lemma 3.7 $\sup_{\varphi \in \mathcal{F}} \int_{\mathbb{R}^d} \varphi(x) dx = \frac{\theta}{2^d \delta}$.

Proof: First note that the function φ_K which has the value $\frac{1}{N(2K)}$ on the hypercube $(0,K)^d$ and 0 elsewhere is feasible, giving that

$$\sup_{\varphi} \int_{\mathbb{R}^d} \varphi(x)dx \geq \lim_{K\to\infty} \frac{K^d}{N(2K)} = \frac{\theta}{2^d\delta},$$

by definition of δ. On the other hand, let φ be an arbitrary feasible function. Let $A \subseteq (0,K)^d$ with $|A| = N(2K)$ be a set of points satisfying $\|a-b\| > 1$ for all $a \neq b \in A$. If η is a constant such that $\|a-b\| > 1$ whenever $|(a)_i - (b)_i| > \eta$ for some $1 \leq i \leq d$, then the set $B := A + (K+\eta)\mathbb{Z}^d$ $(= \{a + (K+\eta)z : a \in A, z \in \mathbb{Z}^d\})$ also satisfies the condition that $\|a-b\| > 1$ for all $a \neq b \in B$. Set $\psi(x) := \sum_{b\in B} \varphi(b+x)$. Since φ is feasible we must have $\psi(x) \leq 1$ for all x. For $a \in A$ let us denote by B_a the "coset" $a + (K+\eta)\mathbb{Z}^d \subseteq B$, and let us set $\psi_a(x) := \sum_{b\in B_a} \varphi(b+x)$. We have that

$$(K+\eta)^d \geq \int_{[0,K+\eta)^d} \psi(x)dx = \sum_{a\in A} \int_{[0,K+\eta)^d} \psi_a(x)dx = N(2K) \int_{\mathbb{R}^d} \varphi(x)dx,$$

where the last equality follows because $\int_{[0,K+\eta)^d} \psi_a(x)dx = \sum_{b\in B_a} \int_{[0,K+\eta)^d} \varphi(b+x)dx = \sum_{b\in B_a} \int_{b+[0,K+\eta)^d} \varphi(x)dx$ and the sets $b + [0,K+\eta)^d$ with $b \in B_a$ form a dissection of $\mathbb{R}^d$. Thus we see that indeed for any feasible φ

$$\int_{\mathbb{R}^d} \varphi(x)dx \leq \lim_{K\to\infty} \frac{(K+\eta)^d}{N(2K)} = \frac{\theta}{2^d\delta}.$$

$\square$

From Lemma 3.7 we may also conclude:

Corollary 3.8 *For all $t \in (0,\infty]$:*

$$\frac{\theta}{2^d\delta} \leq \sup_{\varphi\in\mathcal{F}} \xi(\varphi,t) \leq c(t)\frac{\theta}{2^d\delta},$$

where $c(t) \geq 1$ solves $H(c) = \frac{2^d\delta}{\theta t}$ if $t < \infty$ and $c(\infty) = 1$.

The lower bound follows from Lemma 3.7 and the fact that $\xi(\varphi,t) \geq \int \varphi$ (as $s \geq 0$) for all φ. The upper bound follows from Lemma 3.7 together with (2) and part **(vii)** of Lemma 2.1 (if $\varphi \in \mathcal{F}$ and $W \subseteq \mathbb{R}^d$ has $\mathrm{vol}(W) = \frac{\theta}{2^d\delta}$ then $\int \varphi 1_{\{\varphi\geq a\}} \leq \int 1_W 1_{\{1_W\geq a\}}$ for all $a \in \mathbb{R}$ so that $\xi(\varphi,t) \leq \xi(1_W,t)$).
We may now complete the proof of Theorem 3.4.

Proof of Theorem 3.4: That $c(t) := \sup_{\varphi\in\mathcal{F}} \xi(\varphi,t)$ is non-increasing in t follows from the fact that $\xi(\varphi,t)$ is for each $\varphi \in \mathcal{F}$ separately. The bounds in Corollary 3.8 also give that $0 < c(t) < \infty$ for all $t > 0$ and $\lim_{t\to\infty} c(t) = \frac{\theta}{2^d\delta}$ as required. Furthermore, $c(t)$ is continuous, because by part **(v)** of Lemma 2.1, for any $h > 0$:

$$\left(\frac{t}{t+h}\right)\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)\leq\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t+h)\leq\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t).$$

Thus $c(t)$ is as required, and it only remains to show that $\chi(G_n)/\sigma nr^d \to c(t)$ a.s.

To this end, pick an arbitrary $\varepsilon > 0$ and let $\varphi_1,\ldots,\varphi_m$ and c be as in Lemma 3.6. Then by Theorem 3.1, and recalling that $G_n \cong G(r^{-1}\{X_1,\ldots,X_n\},1)$ with probability one, we have

$$\limsup_{n\to\infty}\frac{\chi(G_n)}{\sigma nr^d}\leq(1+\varepsilon)\max_j\xi(\varphi_j,t)\text{ a.s.}$$

For each j the function $\tilde{\varphi}_j$ given by $\tilde{\varphi}_j(x)=\varphi_j((1+\varepsilon)x)$ is feasible, and so by Lemma 2.1 part **(iv)** we have $\xi(\varphi_j,t)\leq(1+\varepsilon)^d\xi(\tilde{\varphi}_j,t)\leq(1+\varepsilon)^d\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)$. Hence

$$\limsup_{n\to\infty}\frac{\chi(G_n)}{\sigma nr^d}\leq(1+\varepsilon)^{d+1}\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)\text{ a.s.}$$

Putting together this last result and (4), it remains only to show that $\sup_{\varphi\in\mathcal{F}^*}\xi(\varphi,t)=\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)$. Let $\varphi\in\mathcal{F}$. We want to show that

$$\xi(\varphi,t)\leq\sup_{\psi\in\mathcal{F}^*}\xi(\psi,t). \tag{9}$$

We may assume that φ has bounded support, because (by Lemma 2.1, part **(vi)**) the sequence of functions $(\varphi_n)_n$ given by $\varphi_n=\varphi 1_{[-n,n]^d}$ satisfies $\lim_{n\to\infty}\xi(\varphi_n,t)=\xi(\varphi,t)$. Let $\varepsilon>0$ and for each $q\in\varepsilon\mathbb{Z}^d$ let $C^q:=q+[0,\varepsilon)^d$ as before. Define the function $\hat{\varphi}$ on $\mathbb{R}^d$ by setting $\hat{\varphi}(x)=\sup_{y\in C^{p(x)}}\varphi(y)$ (where again $p(x)$ is the unique $q\in\varepsilon\mathbb{Z}^d$ such that $x\in q+[0,\varepsilon)^d$). Clearly $\hat{\varphi}\geq\varphi$. Although $\hat{\varphi}$ is not necessarily feasible, the function φ' given by $\varphi'(x)=\hat{\varphi}((1+\varepsilon\rho)x)$ is. Also, φ' is tidy: clearly it is measurable, bounded, nonnnegative and has bounded support. That the set $\{\varphi'>a\}$ has a small neighbourhood for all $a>0$, follows from the fact that it is the union of finitely many hypercubes $(1+\varepsilon\rho)^{-1}C^q$. So $\varphi'\in\mathcal{F}^*$. We find that

$$\xi(\varphi,t)\leq\xi(\hat{\varphi},t)\leq(1+\varepsilon\rho)^d\xi(\varphi',t)\leq(1+\varepsilon\rho)^d\sup_{\psi\in\mathcal{F}^*}\xi(\psi,t),$$

using Lemma 2.1, parts **(i)** and **(iv)**, for the first and second inequalities respectively. Now we may send $\varepsilon\to 0$ to conclude the proof. □

3.2 Concerning $x(t)$ and t_0

As a consequence of Theorems 3.2 and 3.4 we see that for $t\in(0,\infty]$ the ratio $\chi(G_n)/\omega(G_n)$ tends almost surely to the limit:

$$x(t):=\frac{\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)}{\xi(\varphi_0,t)}, \tag{10}$$

where again $\varphi_0 := 1_{B(0;\frac{1}{2})}$. It is clear from the definition of ξ that $\xi(\varphi_0, t)$ is continuous in t (and positive for all $t > 0$) and we have already established that $c(t) := \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t)$ is continuous in t in the proof of Theorem 3.4. So $x(t)$ is continuous for $t > 0$ as claimed in Theorem 1.2. Combining the fact that $\lim_{t\to\infty} \xi(\varphi_0, t) = \int \varphi_0 = \frac{\theta}{2^d}$ (by definition of ξ), with the fact that $\lim_{t\to\infty} \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \frac{\theta}{2^d\delta}$ (by theorem 3.4) we see that $\lim_{t\to\infty} x(t) = \frac{1}{\delta}$ as claimed in Theorem 1.2.

Corollary 3.8 allows us to show $x(t) = 1$ for all $t > 0$ iff $\delta = 1$. To see this note that if $\delta = 1$ then the upper bound given in corollary 3.8 equals $\xi(\varphi_0, t)$ for all $t > 0$, whereas if $\delta < 1$, then $\xi(\varphi_0, t) < \frac{\theta}{2^d\delta}$ for sufficiently large t as $\lim_{t\to\infty} \xi(\varphi_0, t) = \int \varphi_0 = \frac{\theta}{2^d}$.

To finish the proof of Theorem 1.2, we still need to consider the case when $t = 0$, which we shall do later as an integral part of the proof of part **(i)** of Proposition 1.3, and we need to consider t_0 (show that it exists, in particular) and show that $x(t)$ is strictly increasing for $t > t_0$.

3.2.1 Concerning t_0

Let us set

$$t_0 := \inf\{t > 0 : x(t) \neq 1\}. \tag{11}$$

Our aim, in this section, will be to establish the following result:

Lemma 3.9 *$t_0 > 0$.*

First notice that from previous remarks it can be seen that $t_0 = \infty$ iff $\delta = 1$, so that we may assume $\delta < 1$ wlog in the rest of the section. The proof consists of a number of intermediate steps.

Lemma 3.10 *There is a $T > 0$ such that if $\frac{\sigma n r^d}{\ln n} \to t$ with $0 < t \leq T$ then a.a.a.s. there exists a subgraph H_n of G_n induced by the points in some ball of radius $2r$ such that $\chi(G_n) = \chi(H_n)$.*

Proof: If $\chi(G_n) = \omega(G_n)$ then clearly $\chi(G_n) = \chi(H_n)$ with H_n a subgraph contained in a ball of radius $2r$. If on the other hand $\chi(G_n) > \omega(G_n)$ and we remove all vertices from G_n whose degree is $< \omega(G_n)$ then the chromatic number does not change. We will show that when t is chosen sufficiently small, then a.a.a.s. any two vertices X_i, X_j of G_n with degrees $\geq \omega(G_n)$ will satisfy either $\|X_i - X_j\| < 2r$ or $\|X_i - X_j\| > 3r$. This implies that if we remove all vertices of degree $< \omega(G_n)$ then each of the components of the graph that remains will be contained in some ball of radius $2r$, which in turn yields the result. Let us set $W := B(0; 3)$. Note that if two vertices with $2r \leq \|X_i - X_j\| \leq 3r$ have degrees $\geq \omega(G_n)$ then there must exist a translate of rW containing at least $2\omega(G_n) + 2$ points. By Theorem 3.1 and Theorem 3.2 together with (2) we have

$$\frac{M_W}{\sigma n r^d} \to \xi(1_W, t) = c_1(t) w_1 \text{ a.s.},$$

$$\frac{\omega(G_n)}{\sigma n r^d} \to \xi(\varphi_0, t) = c_2(t) w_2 \text{ a.s.}$$

Here $w_1 = \text{vol}(W) = \theta 3^d, w_2 = \frac{\theta}{2^d}$, and $c_i(t) \geq 1$ solves $H(c_i) = \frac{1}{w_i t}$. We will now show that $\lim_{t\to 0} \frac{c_1(t)w_1}{c_2(t)w_2} = 1$, from which the proposition follows. First note that for $i = 1, 2$

$$\frac{1}{w_i t} = H(c_i) = c_i \ln c_i - c_i + 1 = (1 + o(1)) c_i \ln c_i, \tag{12}$$

because $c_i \to \infty$ as $t \to 0$. Taking logs on both sides of (12) gives $\ln c_i = (1 + o(1)) \ln(\frac{1}{t})$. This together with (12) gives:

$$c_i w_i = (1 + o(1)) \frac{1}{t \ln c_i} = (1 + o(1)) \left(\frac{1}{t}\right) / \ln \left(\frac{1}{t}\right).$$

The claim follows. □

As can be seen from the proof of Lemma 3.10, the conclusion of Lemma 3.10 holds if t satisfies

$$\xi(1_{B(0;3)}, t) < 2\xi(\varphi_0, t), \tag{13}$$

and there exists a $T > 0$ such that (13) holds for all $0 < t < T$. We will show that $t_0 \geq T$ by showing that $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \xi(\varphi_0, t)$ for all t that satisfy (13). Lemma 3.10 together with (fairly straightforward) adaptations of the proof of Theorem 3.4 gives:

Lemma 3.11 *Let $t > 0$ satisfy* (13)*. Then*

$$\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \sup_{\substack{\varphi\in\mathcal{F},\\ \text{supp}(\varphi)\subseteq B(0;2)}} \xi(\varphi, t).$$

Proof: Let r satisfy $\sigma n r^d \sim t \ln n$ and consider $\chi(G_n)$. We know that a.a.a.s. $\chi(G_n)$ equals the maximum over all $x \in \mathbb{R}^d$ of the chromatic number of the graph induced by the vertices in $B(x; 2r)$. Let us fix an $\varepsilon > 0$. Let us denote $V := r^{-1}\{X_1, \ldots, X_n\}$ and for $p \in \varepsilon\mathbb{Z}^d$ let Λ^p denote the subgraph of Γ induced by the points of $\varepsilon\mathbb{Z}^d$ inside $B(p, (2 + \varepsilon\rho))$, where again $\rho := \text{diam}([0, 1]^d)$, and let Λ_V^p be the corresponding subgraph of Γ_V^p (with $\Gamma, \Gamma^p, \Gamma_V, \Gamma_V^p$ as in the proof of Theorem 3.4). Since for every $x \in \mathbb{R}^d$ the subgraph of G_n induced by the vertices inside $B(x, 2r)$ is a subgraph of some Λ_V^p, we have:

$$\chi(G_n) \leq \max_p \chi(\Lambda_V^p) \leq \max_{i=1,\ldots,m} M_{\varphi_i} + c \text{ a.a.a.s.}, \tag{14}$$

where $\varphi_1, \ldots, \varphi_m$ are obtained from the ILP formulation of $\chi(\Lambda_V^p)$ via the same procedure we used in the proof of Theorem 3.4 (ie. the upper bound in (14) is the analogue of the upper bound in (8)) and $c = c(\varepsilon)$ is a constant that depends only on ε. By construction we have that $\mathrm{supp}(\varphi_i) \subseteq B(0, 2+2\varepsilon\rho)$ and that φ_i' given by $\varphi_i'(x) = \varphi_i((1+2\varepsilon\rho)x)$ is feasible. Notice that φ_i' also satisfies $\mathrm{supp}(\varphi_i') \subseteq B(0;2)$. Thus, (14) together with Theorem 3.4, Theorem 3.1 and part **(iv)** of Lemma 2.1 shows that

$$\sup_{\varphi\in\mathcal{F}} \xi(\varphi,t) \le \max_{i=1,\dots,m} \xi(\varphi_i,t) \le \max_{i=1,\dots,m} (1+2\varepsilon\rho)^d \xi(\varphi_i',t) \le (1+2\varepsilon\rho)^d \sup_{\substack{\varphi\in\mathcal{F},\\ \mathrm{supp}(\varphi)\subseteq B(0;2)}} \xi(\varphi,t).$$

The statement now follows by letting $\varepsilon \to 0$. □

Let us now fix a $t > 0$ that satisfies (13). If $\sup_{\varphi\in\mathcal{F}} \xi(\varphi,t) > \xi(\varphi_0,t)$ then there must also exist a feasible simple function $\psi := \sum_{k=1}^m \frac{k}{m} 1_{A_k}$ with $\mathrm{supp}(\psi) \subseteq B(0;2)$ such that $\xi(\psi,t) > \xi(\varphi_0,t)$, because (by Lemma 2.1, item **(vi)**) for any φ the increasing sequence of functions $(\varphi_n)_n$ given by $\varphi_n = \sum_{k=1}^{2^n} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \le \varphi < \frac{k+1}{2^n}\}}$ satisfies $\lim_{n\to\infty} \xi(\varphi_n,t) = \xi(\varphi,t)$.

So let $\psi = \sum_{i=1}^m \frac{i}{m} 1_{A_i}$ be a feasible simple function with $\xi(\psi,t) > \xi(\varphi_0,t)$ and $\mathrm{supp}(\psi) \subseteq B(0;2)$. We may suppose wlog that the A_k are disjoint and m is even. For $1 \le k \le \frac{m}{2}$ let ψ_k be the function which is $\frac{1}{2}$ on $\bigcup_{i=k}^{m-k} A_i$ and 1 on $\bigcup_{i>m-k} A_i$. We can write

$$\psi = \frac{2}{m} \sum_{k=1}^{m/2} \psi_k,$$

because for $x \in A_i$ with $i \le m/2$ we have $\frac{2}{m}\sum_{k=1}^{m/2} \psi_k(x) = i\frac{2}{m}\frac{1}{2} = \frac{i}{m}$, and if $x \in A_{m-i}$ with $i \le m/2$ then $\frac{2}{m}\sum_{k=1}^{m/2} \varphi_k(x) = 1 - i\frac{2}{m}\frac{1}{2} = \frac{m-i}{m}$.

Let us now observe that ξ is convex in its first argument, ie. for any two nonnegative, bounded, measurable functions σ, τ and any $t > 0$ and $\lambda \in [0,1]$:

$$\xi(\lambda\sigma + (1-\lambda)\tau, t) \le \lambda\xi(\sigma,t) + (1-\lambda)\xi(\tau,t).$$

This follows from parts **(iii)** and **(ii)** of Lemma 2.1. Because we have written ψ as a convex combination of the ψ_k, we must therefore have $\xi(\psi,t) \le \xi(\psi_k,t)$ for some k.

Let us first assume that $\{\psi_k = 1\} = \bigcup_{l>m-k} A_k = \emptyset$. Since $\mathrm{supp}(\psi) \subseteq B(0;2)$ we must have that $\psi_k \le \varphi'$, where φ' is the function which is $\frac{1}{2}$ on $B(0;3)$ and 0 elsewhere, and thus also $\xi(\psi,t) \le \xi(\psi_k,t) \le \xi(\varphi',t) = \frac{1}{2}\xi(1_{B(0;3)},t)$ (by choice of k and Lemma 2.1, items **(i)** and **(iii)**). But then (13) gives:

$$\xi(\psi,t) \le \frac{1}{2}\xi(1_{B(0;3)},t) < \xi(\varphi_0,t),$$

a contradiction.

So we must have $\{\psi_k = 1\} \ne \emptyset$. Let us denote by $C := \mathrm{cl}(B)$ the closed unit ball. Notice that

$$\mathrm{diam}(\{\psi_k = 1\}) \le 1,$$

$$\mathrm{supp}(\psi_k) \subseteq \bigcap_{x:\psi_k(x)=1} (x + C),$$

by feasibility of ψ (if $x \in \text{supp}(\psi_k), \psi_k(y) = 1$ then $\psi(x) + \psi(y) > \frac{k}{m} + \frac{m-k}{m} = 1$). Bieberbach's inequality tells us that $\text{vol}(\{\psi_k = 1\})$ cannot exceed $\frac{\theta}{2^d}$, the volume of a ball of diameter 1. Hence there is a $0 \le \beta \le 1$ with $\text{vol}(\{\psi_k = 1\}) = \text{vol}(B(0; \frac{1-\beta}{2})) = \frac{\theta}{2^d}(1-\beta)^d$. We will need another inequality, given by the following proposition:

Lemma 3.12 (K. Böröczky Jr., '05) *Let $C \subseteq \mathbb{R}^d$ be a compact, convex set. Let $A \subseteq \mathbb{R}^d$ be measurable and let A' be a homothet (ie. a scaled copy) of $-C$ with* $\text{vol}(A) = \text{vol}(A')$. *Then* $\text{vol}\left(\bigcap_{a \in A}(a + C)\right) \le \text{vol}\left(\bigcap_{a \in A'}(a + C)\right)$.

This is a generalisation of a result proved by the second author. With the kind permission of K. Böröczky Jr. we will present a proof in appendix A, because such a proof is not readily available elsewhere.

It follows from Lemma 3.12 that we must have

$$\text{vol}(\text{supp}(\psi_k)) \le \text{vol}\left(\bigcap_{x \in B(0; \frac{1-\beta}{2})} (x + C)\right) = \text{vol}\left(B(0; \frac{1+\beta}{2})\right) = \frac{\theta}{2^d}(1+\beta)^d.$$

Now let φ_β be the function which is 1 on $B(0; \frac{1-\beta}{2})$ and $\frac{1}{2}$ on $B(0; \frac{1+\beta}{2}) \setminus B(0; \frac{1-\beta}{2})$. We see that $\text{vol}(\{\psi_k = 1\}) = \text{vol}(\{\varphi_\beta = 1\})$ and $\text{vol}(\{\psi_k = \frac{1}{2}\}) \le \text{vol}(\{\varphi_\beta = \frac{1}{2}\})$. And thus we have $\int \psi_k 1_{\{\psi_k \ge a\}} \le \int \varphi_\beta 1_{\{\varphi_\beta \ge a\}}$ for all a, which gives $\xi(\psi_k, t) \le \xi(\varphi_\beta, t)$ by part **(vii)** of Lemma 2.1. We may conclude that if $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) > \xi(\varphi_0, t)$ and (13) holds then also $\xi(\varphi_\beta, t) > \xi(\varphi_0, t)$ for some $0 < \beta \le 1$. Let us set $\mu(\beta) := \xi(\varphi_\beta, t)$.

Lemma 3.13 $\max_{0 \le \beta \le 1} \mu(\beta) = \max(\mu(0), \mu(1))$.

Proof: Notice that for $0 \le \beta \le 1$

$$\mu(\beta) = \frac{\theta}{2^d}(\frac{1}{2}((1+\beta)^d - (1-\beta)^d)e^{s/2} + (1-\beta)^d e^s),$$

where $s = s(\beta)$ solves

$$\frac{\theta}{2^d}\left((1+\beta)^d - (1-\beta)^d)H(e^{s/2}) + (1-\beta)^d H(e^s)\right) = \frac{1}{t}. \tag{15}$$

The function $\mu(\beta)$ is continuous on $[0, 1]$. Differentiating equation (15) wrt β we see that for $0 < \beta < 1$

$$\begin{aligned} 0 \;=\; & d((1+\beta)^{d-1} + (1-\beta)^{d-1})H(e^{s/2}) + ((1+\beta)^d - (1-\beta)^d))\tfrac{s}{4}e^{s/2}s' - d(1-\beta)^{d-1}H(e^s) \\ & +(1-\beta)^d s e^s s', \end{aligned}$$

giving that

$$s'(\beta) = \frac{d(1-\beta)^{d-1}H(e^s) - d((1+\beta)^{d-1} + (1-\beta)^{d-1})H(e^{s/2})}{((1+\beta)^d - (1-\beta)^d)\frac{s}{4}e^{s/2} + (1-\beta)^d s e^s}.$$

Thus,

$$\begin{aligned}
\mu'(\beta) &= \tfrac{\theta}{2^d}\Big[d((1+\beta)^{d-1}+(1-\beta)^{d-1})\tfrac{1}{2}e^{s/2}-d(1-\beta)^{d-1}e^s \\
&\quad + s'(((1+\beta)^d-(1-\beta)^d)\tfrac{1}{4}e^{s/2}+(1-\beta)^d e^s)\Big] \\
&= \tfrac{\theta}{2^d}\Big[d((1+\beta)^{d-1}+(1-\beta)^{d-1})\tfrac{1}{2}e^{s/2}-d(1-\beta)^{d-1}e^s \\
&\quad +\tfrac{1}{s}(d(1-\beta)^{d-1}H(e^s)-d((1+\beta)^{d-1}+(1-\beta)^{d-1})H(e^{s/2}))\Big] \\
&= \tfrac{\theta}{2^d}\tfrac{1}{s}[d((1+\beta)^{d-1}+(1-\beta)^{d-1})(e^{s/2}-1)-d(1-\beta)^{d-1}(e^s-1)].
\end{aligned}$$

Clearly $\mu'(\beta)>0$ for β sufficiently close to 1, so that it suffices to show that (for any t) $\mu'(\beta)=0$ for no more than one $\beta\in(0,1)$. Note that $\mu'(\beta)=0$ if and only if

$$e^s-1=((\frac{1+\beta}{1-\beta})^{d-1}+1)(e^{s/2}-1).$$

Writing $a:=(\frac{1+\beta}{1-\beta})^{d-1}+1$ and $x:=e^{s/2}$ this translates into the quadratic $x^2-ax+(a-1)=0$, which has roots $1, a-1$. Now notice that $e^{s/2}=1$ would give $s(\beta)=0$, but this is never a solution of (15). So if $\mu'(\beta)=0$ for some $0<\beta<1$ then we must have $s(\beta)=2(d-1)\ln(\frac{1+\beta}{1-\beta})$. Notice that, as s cannot equal 0, this also shows that we must have $d\geq 2$ for $\mu'(\beta)=0$ to hold. The curve $u(\beta):=2(d-1)\ln(\frac{1+\beta}{1-\beta})$ has derivative

$$u'(\beta)=\frac{4(d-1)}{(1-\beta)(1+\beta)}.$$

On the other hand, for $0<\beta<1$

$$s'(\beta)<\frac{d(1-\beta)^{d-1}H(e^s)}{(1-\beta)^d s e^s}<\frac{d}{1-\beta}$$

We find $s'(\beta)<\frac{d}{1-\beta}<4(d-1)/(1+\beta)(1-\beta)=u'(\beta)$ for $0<\beta<1$ (recall $d\geq 2$ by a previous remark). We may conclude that the curves $u(\beta)$ and $s(\beta)$ meet in at most one point, as $u(\beta)-s(\beta)$ is strictly increasing on $(0,1)$. In other words, there is at most one $\beta\in(0,1)$ with $\mu'(\beta)=0$ and the proposition follows. □

Since we had chosen t so that (13) holds. parts **(iv)** and **(i)** of Lemma 2.1 tell us that:

$$\xi(\varphi_1,t)=\frac{1}{2}\xi(1_B,t)\leq\frac{1}{2}\xi(1_{B(0;3)},t)<\xi(\varphi_0,t).$$

Thus $\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)\leq\max_{0\leq\beta\leq 1}\mu(\beta)=\mu(0)=\xi(\varphi_0,t)$ after all. This concludes the proof of Lemma 3.9

3.2.2 The function $x(t)$ is strictly increasing for $t>t_0$

In this section we shall prove the following result:

Lemma 3.14 *The function $x(t)$ is strictly increasing for $t_0<t<\infty$.*

The proof makes use of the following observation, which is also of independent interest;

Proposition 3.15 *For each $t > 0$, either $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) = \frac{\theta}{2^d \delta}$ or the supremum is attained*

Proof of Lemma 3.14: First suppose that $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) = \frac{\theta}{2^d \delta}$. Notice that the lower bound in Corollary 3.8 then shows that also $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t') = \frac{\theta}{2^d \delta}$ for all $t' \geq t$, so that in this case $x(t') > x(t)$ for all $t' > t$ as $\xi(\varphi_0, t)$ is strictly decreasing in t.

Let us therefore assume $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) > \frac{\theta}{2^d \delta}$. By Proposition 3.15 there is a $\varphi \in \mathcal{F}$ s.t. the supremum equals $\xi(\varphi, t)$, where $0 < \int \varphi < \infty$. Observe that it would suffice to prove that for any $\lambda > 1$ there is at most one $t > 0$ that solves the equation $\xi(\varphi, t) = \lambda \xi(\varphi_0, t)$ (recall that $\xi(\varphi, t_0) \leq \xi(\varphi_0, t_0)$ so that this indeed gives that $\frac{\xi(\varphi,t)}{\xi(\varphi_0,t)}$ is increasing for every t where this ratio is > 1). Set $\psi := \lambda \varphi_0$ with $\lambda > 1$. By Lemma 2.1, part **(iii)**, $\xi(\psi, t) = \lambda \xi(\varphi_0, t)$, so that it suffices to show that the system of equations

$$\int_{\mathbb{R}^d} \varphi(x) e^{w\varphi(x)} \mathrm{d}x = \int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x, \tag{16}$$

$$\int_{\mathbb{R}^d} H(e^{w\varphi(x)}) \mathrm{d}x = \int_{\mathbb{R}^d} H(e^{s\psi(x)}) \mathrm{d}x \tag{17}$$

has at most one solution (w, s) with $w, s > 0$. For $s \in \mathbb{R}$ let $v(s)$ be the unique solution of (16) and let $u(s)$ be the unique non-negative solution of (17). Differentiating both sides of equation (16) wrt s we get

$$\int_{\mathbb{R}^d} v'(s) \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x = \int_{\mathbb{R}^d} \psi^2(x) e^{s\psi(x)} \mathrm{d}x,$$

where we have swapped integration wrt x and differentiation wrt s (this can be justified using for instance the fundamental theorem of calculus and Fubini's theorem for nonnegative functions[1]).

This implies

$$v'(s) = \frac{\int_{\mathbb{R}^d} \psi^2(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x} = \lambda \frac{\int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x} > \frac{\int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi(x) e^{v(s)\varphi(x)} \mathrm{d}x} = 1,$$

where we have used the specific form of ψ as a constant times an indicator function, the fact that $\varphi^2(x) \leq \varphi(x)$ (it's a $[0, 1]$ function) and the fact that $v(s)$ solves (16). Now not that $u(0) = 0$ and $u(s) > 0$ for $s > 0$. Differentiating (17) wrt s we get that for $s > 0$:

[1]Here we mean the following. If $g(x, u)$ denotes one of $\varphi(x)e^{u\varphi(x)}, \psi(x)e^{u\psi(x)}, H(e^{u\varphi(x)})$ or $H(e^{u\psi(x)})$ then $\int_{\mathbb{R}^d} g(x, u) - g(x, 0) \mathrm{d}x = \int_{\mathbb{R}^d} \int_0^u g_2(x, w) \mathrm{d}w \mathrm{d}x = \int_0^u \int_{\mathbb{R}^d} g_2(x, w) \mathrm{d}x \mathrm{d}w$, where g_2 denotes the derivative of g wrt the second argument, and we have used Fubini's theorem to switch the order of integration. Now the fundamental theorem of calculus shows that $\frac{\mathrm{d}}{\mathrm{d}u} \int_{\mathbb{R}^d} g(x, u) \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}u} \int_{\mathbb{R}^d} g(x, u) - g(x, 0) \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \int_{\mathbb{R}^d} g_2(x, w) \mathrm{d}x \mathrm{d}w = \int_{\mathbb{R}^d} g_2(x, u) \mathrm{d}x$.

$$u'(s) = \frac{s\int_{\mathbb{R}^d} \psi^2(x)e^{s\psi(x)}\mathrm{d}x}{u(s)\int_{\mathbb{R}^d} \varphi^2(x)e^{u(s)\varphi(x)}\mathrm{d}x}.$$

Let us first suppose that $v(0) \geq 0$. Since $v'(s) > 1$ for all $s \in \mathbb{R}$, we must also have that $v(s) > s$ for all $s > 0$. If $v(s) = u(s)$ for some $s > 0$ then

$$u'(s) = \frac{s\int_{\mathbb{R}^d} \psi^2(x)e^{s\psi(x)}\mathrm{d}x}{u(s)\int_{\mathbb{R}^d} \varphi^2(x)e^{u(s)\varphi(x)}\mathrm{d}x} = \frac{s}{v(s)}v'(s) < v'(s).$$

This shows that in any crossing of these two curves, $v(s)$ must come from below $u(s)$. But this means there can be at most one such crossing.

Now suppose that $v(0) < 0$ (recall that $u(0) = 0$). Let $s_1 > 0$ be the first solution of $v(s) = u(s)$ (supposing that such a solution even exists). As $v(s) < u(s)$ for $0 \leq s < s_1$ it must hold that $v'(s_1) \geq u'(s_1) = v'(s_1)\frac{s_1}{v(s_1)}$. This gives $v(s_1) \geq s_1$ and hence also $v(s) > s$ for all $s > s_1$ (as $v'(s) > 1$). Again there cannot be a second solution $v(s_2) = u(s_2)$ with $s_2 > s_1$ as for any such solution it would hold that $v'(s_2) > u'(s_2)$, while at the same time $v(s) > u(s)$ for $s_1 < s < s_2$. □

Proof of Proposition 3.15: Let us assume $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) > \frac{\theta}{2^d\delta}$ (otherwise there is nothing to prove) and let us consider a sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{F}$ s.t.

$$\lim_{n\to\infty} \xi(\varphi_n, t) = \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) \tag{18}$$

and let us suppose (wlog) that $\lim_n \int_B \varphi_n$ exists and is as large as possible subject to (18) (recall $B = B(0; 1)$ is the unit ball). We will first exhibit a subsequence $\varphi_{n_1}, \varphi_{n_2}, \ldots$ of $(\varphi_n)_n$ and a function $\psi \in \mathcal{F}$ s.t.

$$\limsup_{k\to\infty} \varphi_{n_k}(x) \leq \psi(x) \text{ for all } x \in \mathbb{R}^d. \tag{19}$$

We will then consider the "inner parts" $\psi_{k,i} := \varphi_{n_k}1_{B(0;R_k)}$ and the "outer parts" (where $(R_k)_k$ is a growing sequence, chosen in such a way that $\xi(\psi_{k,i}, t) \to \xi(\psi, t)$), and we will see that the outer part is negligible.

In order to construct ψ and the subsequence $(\varphi_{n_k})_k$, let $\mathcal{D}_k$ be the dissection $\{i + [0, 2^{-k})^d : i = (i_1, \ldots, i_d) \in 2^{-k}\mathbb{Z}^d\}$ of $\mathbb{R}^d$ into cubes of side 2^{-k} (observe that $\mathcal{D}_{k+1}$ refines $\mathcal{D}_k$). For $\sigma \in \mathcal{F}$ let us define the functions σ^k by setting:

$$\sigma^k(x) := \sup_{y\in C_{x,k}} \sigma(y),$$

where $C_{x,k}$ is the unique $C \in \mathcal{D}_k$ with $x \in C$. Let us now construct a nested sequence $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots$ of infinite subsets of $\{\varphi_1, \varphi_2, \ldots\}$ with the property that

$$|\sigma^k(x) - \tau^k(x)| \leq \frac{1}{k} \text{ for all } x \in [-k, k)^d \text{ and all } \sigma, \tau \in \mathcal{F}_k. \tag{20}$$

To see that this can be done, notice that the behaviour of σ^k on $[-k,k)^d$ is determined completely by $(\sigma^k(p_1),\dots,\sigma^k(p_K))$ where $p_1,\dots,p_K$ is some enumeration of $[-k,k)^d \cap 2^{-k}\mathbb{Z}^d$. Given $\mathcal{F}_{k-1}$ there must be intervals $I_1,\dots,I_K \subseteq [0,1]$ each of length $\frac{1}{k}$ such that the collection $\{\sigma \in \mathcal{F}_{k-1} : \sigma^k(p_i) \in I_i \text{ for all } 1 \le i \le K\}$ is infinite. So we can take $\mathcal{F}_k$ to be such an infinite collection. Let us now pick a subsequence $\varphi_{n_1},\varphi_{n_2},\dots$ of $(\varphi_n)_n$ with $\varphi_{n_k} \in \mathcal{F}_k$ and let the function ψ be defined by:

$$\psi(x) := \lim_{k\to\infty} \varphi_{n_k}^k(x). \tag{21}$$

To see that this limit exists for all x, notice that $\varphi_{n_l}^l(x) \le \varphi_{n_l}^k(x) \le \varphi_{n_k}^k(x) + \frac{1}{k}$ for all $l \ge k > \|x\|$. Thus,

$$\liminf_{k\to\infty} \varphi_{n_k}^k(x) \le \limsup_{k\to\infty} \varphi_{n_k}^k(x) \le \inf_{k>\|x\|} \varphi_{n_k}^k(x) + \frac{1}{k} \le \liminf_{k\to\infty} \varphi_{n_k}^k(x) + \frac{1}{k} = \liminf_{k\to\infty} \varphi_{n_k}^k(x).$$

We now claim that ψ and the sequence $(\varphi_{n_k})_k$ are as required (ie. $\psi \in \mathcal{F}$ and (19) holds). To see that (19) holds, notice that $\sup_{l\ge k} \varphi_{n_l}(x) \le \varphi_{n_k}^k(x) + \frac{1}{k}$ for any x and any $k > \|x\|$, so that $\limsup_k \varphi_{n_k}(x) \le \lim_k \varphi_{n_k}^k(x) = \psi(x)$.

To see that $\psi \in \mathcal{F}$, let $S = \{s_1,\dots,s_p\} \in \mathcal{S}$ be finite (observe it suffices to show $\sum_{x\in S} \psi(x) \le 1$ for all finite $S \in \mathcal{S}$). Since $\|s_i - s_j\| > 1$ for all $i \ne j$, there is a k_0 s.t. $\|s_i - s_j\| > 1 + 2^{-k_0}\rho$ for all $i \ne j$ where $\rho := \mathrm{diam}([0,1]^d)$. Thus if $k \ge k_0$ then

$$\varphi_{n_k}^k(s_1) + \cdots + \varphi_{n_k}^k(s_p) \le 1,$$

and hence the same must hold for ψ.

Also notice that the dominated convergence theorem (using $\psi, \varphi_{n_k} \le 1$) gives that

$$\int_B \psi(x)\mathrm{d}x = \int_B \lim_{k\to\infty} \varphi_{n_k}^k(x)\mathrm{d}x \ge \lim_{n\to\infty} \int_B \varphi_n(x)\mathrm{d}x. \tag{22}$$

Furthermore, for any fixed $R > 0$ we have that

$$\lim_{n\to\infty} \xi(\varphi_{n_k}^k 1_{B(0;R)}, t) = \xi(\psi 1_{B(0,R)}, t) \le \xi(\psi, t). \tag{23}$$

Here we have used parts **(i)** and **(vi)** of Lemma 2.1. Hence, there also is a sequence $(R_k)_k$ with R_k tending to infinity and $\limsup_{k\to\infty} \xi(\varphi_{n_k}^k 1_{B(0;R_k)}, t) \le \xi(\psi, t)$. To see this, notice that by (23) there exist $k_1 \le k_2 \le \dots$ such that $\xi(\varphi_{n_k}^k 1_{B(0;m)}, t) \le \xi(\psi, t) + \frac{1}{m}$ for all $k \ge k_m$. Thus, we may put $R_k := \max\{m : k_m \le k\}$.

Let us put

$$\psi_{k,i} := \varphi_{n_k} 1_{B(0;R_k)}, \quad \psi_{k,o} := \varphi_{n_k} 1_{\mathbb{R}^d \setminus B(0;R_k+1)}, \quad \psi_k := \psi_{k,i} + \psi_{k,o}.$$

We may assume wlog that R_k has been chosen in such a way that $\xi(\psi_k, t) = (1 + o(1))\xi(\varphi_{n_k}, t)$. To see this note that for $s = s(\varphi_{n_k}, t)$ there is an $\frac{R_k}{2} \le R' \le R_k$ s.t.

$$\int \varphi_{n_k} e^{s\varphi_{n_k}} 1_{B(0;R'+1)\setminus B(0;R')} \leq \frac{1}{\lfloor \frac{R_k}{2} \rfloor} \int \varphi_{n_k} e^{s\varphi_{n_k}}.$$

If we take such an R' and set $\psi'_k := \varphi_{n_k} 1_{\mathbb{R}^d \setminus B(0;R'+1)\cup B(0;R')}$ then $s(\psi'_k, t) \geq s(\varphi_{n_k}, t)$ (by the definition of s, as $\psi'_k \leq \varphi_{n_k}$) so that $\xi(\psi'_k, t) \geq (1 - \frac{1}{\lfloor \frac{R_k}{2} \rfloor})\xi(\varphi_{n_k}, t)$.

Let us define $\lambda(\varphi) := \sup_{S\in\mathcal{S}} \sum_{x\in S} \varphi(x)$. Clearly

$$\lambda(\psi_k) = \lambda(\psi_{k,i}) + \lambda(\psi_{k,o}) \leq 1. \tag{24}$$

For convenience let us write $\lambda_k := \lambda(\psi_{k,o})$. First let us suppose that $\lambda_k \to 0$. Notice that $\frac{1}{\lambda_k}\psi_{k,o} \in \mathcal{F}$, which implies

$$\xi(\psi_{k,o}, t) \leq \lambda_k \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = o(1),$$

using part **(iii)** of Lemma 2.1. As $\xi(\psi_{k,i}, t) \leq \xi(\psi_k, t) \leq \xi(\psi_{k,i}, t) + \xi(\psi_{k,o}, t)$ (by parts **(i)** and **(ii)** of Lemma 2.1) and $\xi(\psi_k, t) = \xi(\varphi_{n_k}, t) + o(1)$ it follows that

$$\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \lim_{k\to\infty} \xi(\varphi_{n_k}, t) = \lim_{k\to\infty} \xi(\psi_{k,i}, t) \leq \xi(\psi, t),$$

so that the proposition follows in the case when $\lambda_k \to 0$.

Now let us assume that $\limsup \lambda_k > 0$. We may assume for convenience that $\lim \lambda_k = \lambda > 0$ (by considering a subsequence if necessary). We first claim that $\mathrm{vol}(\{\psi_{k,o} \geq \varepsilon\}) \to 0$ for all $\varepsilon > 0$. To this end let us construct a new sequence of functions ψ'_k as follows. For each k pick an $x_k \in \mathbb{R}^d \setminus B(0; R_k + 1)$ that maximises $\int_{B(x_k;1)} \psi_{k,o}$.

To see that such an x_k exists, let us write $I(x) := \int_{B(x;1)} \psi_{k,o}$. Notice that I is continuous ($\psi_{k,o} \leq 1$ so that $|\psi_{k,o}(x) - \psi_{k,o}(y)| \leq \mathrm{vol}(B(x; 1) \setminus B(y; 1))$). Let us suppose that $c := \sup_{x\in\mathbb{R}^d\setminus B(0;R_k+1)} I(x) > 0$, for otherwise there is nothing to prove as any $x \in \mathbb{R}^d \setminus B(0; R_k + 1)$ will do. We first claim that the set $\{x \in \mathbb{R}^d \setminus B(0; R_k + 1) : I(x) > \frac{c}{2}\}$ can be covered by at most $\lfloor \frac{2\theta}{c} \rfloor$ balls of radius two. This is because if $I(x_1), \ldots, I(x_k) > \frac{c}{2}$ there must exist $y_i \in B(x_i; 1)$ for $1 \leq i \leq k$ with $\psi_{k,o}(y_i) > \frac{c}{2\theta}$. By feasability of $\psi_{k,o}$ we must have either $k < \frac{2\theta}{c}$ or $\|y_i - y_j\| \leq 1$ for some $i \neq j$. Thus, the y_i can be covered by at most $\frac{2\theta}{c}$ balls of radius one, and hence that x_i can be covered by at most $\frac{2\theta}{c}$ balls of radius two, as claimed. As I is continuous and we can restrict ourselves to a compact subset of $\mathbb{R}^d \setminus B(0; R_k + 1)$ we see that the supremum $c = \sup_{x\in\mathbb{R}^d\setminus B(0;R_k+1)} I(x)$ is indeed attained by some point $x_k \in \mathbb{R}^d \setminus B(0; R_k + 1)$.

Now let $\psi'_k := \psi_{k,i} + \psi_{k,o} \circ T_k$ where $T_k : y \mapsto y + x_k$ is the translation that sends 0 to x_k. By (24) we have $\psi'_k \in \mathcal{F}$. Notice that

$$\int \psi'_k 1_{\{\psi'_k \geq a\}} \geq \int \psi_k 1_{\{\psi_k \geq a\}}$$

for all a, because $\{\psi'_k \geq a\} \supseteq \{\psi_{k,i} \geq a\} \cup T_k^{-1}[\{\psi_{k,o} \geq a\}]$ so that

$$\begin{array}{rcl}\int \psi_k' 1_{\{\psi_k' \geq a\}} & \geq & \int \psi_{k,i} 1_{\{\psi_{k,i} \geq a\}} + \int (\psi_{k,o} \circ T_k) 1_{T_k^{-1}[\{\psi_{k,o} \geq a\}]} \\ & = & \int \psi_{k,i} 1_{\{\psi_{k,i} \geq a\}} + \int \psi_{k,o} 1_{\{\psi_{k,o} \geq a\}} \\ & = & \int \psi_k 1_{\{\psi_k \geq a\}}.\end{array}$$

Part **(vii)** of Lemma 2.1 therefore gives that

$$\xi(\psi_k', t) \geq \xi(\psi_k, t) = (1 + o(1))\xi(\varphi_{n_k}, t).$$

We therefore must have $\int_{B(x_k;1)} \psi_{k,o} \to 0$ for otherwise (a subsequence of) the ψ_k' would contradict the choice of $(\varphi_n)_n$. Now suppose that for some $\varepsilon > 0$ we have $\limsup_k \mathrm{vol}(\{\psi_{k,o} \geq \varepsilon\}) = c > 0$. Because $\psi_{k,o} \in \mathcal{F}$ we can cover $\{\psi_{k,o} \geq \varepsilon\}$ by at most $\lfloor \frac{1}{\varepsilon} \rfloor$ balls of radius 1. But this gives $\limsup_k \int_{B(x_k;1)} \psi_{k,o}(x)\mathrm{d}x \geq c\varepsilon$ and we know this cannot happen. The claim follows.

Recall that $\sigma_k := \frac{1}{\lambda_k}\psi_{k,o} \in \mathcal{F}$. Because $\lim_k \lambda_k = \lambda > 0$ the previous also gives $\lim_k \mathrm{vol}(\{\sigma_k \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. Let us fix $\varepsilon > 0$ for now and let $V_\varepsilon, W_{k,\varepsilon} \subseteq \mathbb{R}^d$ be disjoint sets with $\mathrm{vol}(V_\varepsilon) = \frac{\theta}{\varepsilon 2^d \delta}$, $\mathrm{vol}(W_{k,\varepsilon}) = \mathrm{vol}(\{\sigma_k \geq \varepsilon\})$. Let us set $\tau_k := \varepsilon 1_{V_\varepsilon} + 1_{W_{k,\varepsilon}}$. Then $\int \sigma_k 1_{\{\sigma_k \geq a\}} \leq \int \tau_k 1_{\{\tau_k \geq a\}}$ for all a (using Lemma 3.7), so that parts **(vii)**, **(iii)** and **(ii)** of Lemma 2.1 give:

$$\begin{array}{rcl}\xi(\sigma_k, t) & \leq & \xi(\tau_k, t) \leq \varepsilon\xi(1_{V_\varepsilon}, t) + \xi(1_{W_{k,\varepsilon}}, t) \\ & = & \varepsilon c(\varepsilon)\,\mathrm{vol}(V_\varepsilon) + d(k, \varepsilon)\,\mathrm{vol}(W_{k,\varepsilon}) \\ & = & c(\varepsilon)\frac{\theta}{2^d\delta} + d(k, \varepsilon)\,\mathrm{vol}(W_{k,\varepsilon}),\end{array}$$

where $c(\varepsilon) \geq 1$ solves $H(c) = \frac{1}{\mathrm{vol}(V_\varepsilon)t} = \varepsilon\frac{2^d\delta}{\theta t}$ and $d(k, \varepsilon) \geq 1$ solves $H(d) = \frac{1}{\mathrm{vol}(W_{k,\varepsilon})t}$. Observe that $\lim_{\varepsilon \to 0} c(\varepsilon) = 1$. For any fixed $\varepsilon > 0$ it also holds that

$$\lim_{k \to \infty} d(k, \varepsilon)\,\mathrm{vol}(W_{k,\varepsilon}) = 0. \tag{25}$$

To see this note that $d \to \infty$ and $H(d) \sim d \ln d$ as $k \to \infty$, so that $d \ll H(d) = \Theta(\mathrm{vol}(W_{k,\varepsilon})^{-1})$ which gives (25). We see that $\limsup_{k\to\infty} \xi(\sigma_k, t) \leq \frac{\theta}{2^d\delta}$. Since $\sigma_k' := \frac{1}{1-\lambda_k}\psi_{k,i} \in \mathcal{F}$ by (24) we thus have

$$\lim_{k\to\infty} \xi(\psi_k, t) = \lim_{k\to\infty} \xi(\lambda_k\sigma_k + (1 - \lambda_k)\sigma_k', t) \leq \lambda\frac{\theta}{2^d\delta} + (1 - \lambda)\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) < \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t),$$

using parts **(i)** and **(iii)** of Lemma 2.1. But this contradicts the fact that by construction $\lim_{k\to\infty} \xi(\psi_k, t) = \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t)$. So we must have $\lambda_k \to 0$ and the proposition follows from previous arguments. □

3.3 Proof of Theorem 1.3

To prove Theorem 1.3 and to finish the proof of Theorem 1.2 it suffices to prove the following lemma. The formulation in this lemma circumvents having to deal with the case $\delta = 1$ separately.

Lemma 3.16 *Pick* $0 < t < \infty$*.*

(i) *If* $\sigma n r^d / \ln n \leq t$ *then*

$$\limsup \frac{\chi(G_n)}{\omega(G_n)} \leq x(t) \text{ a.s.};$$

(ii) *If* $\sigma n r^d / \ln n \geq t$ *then*

$$x(t) \leq \liminf \frac{\chi(G_n)}{\omega(G_n)} \leq \limsup \frac{\chi(G_n)}{\omega(G_n)} \leq \frac{1}{\delta} \text{ a.s.}$$

In order to prove part **(i)** we will need to consider three ranges for the value of r, corresponding to $nr^d = \Theta(\ln n)$, $nr^d \leq n^{-\alpha}$ for some $\alpha > 0$, and intermediate values.

3.3.1 The proof of Theorem 1.4

As mentioned in the introduction, to deal with very small r we will need to prove Theorem 1.4.

Proof of Theorem 1.4: We will need the following lemma on the scan statistic, the proof of which can be found in section 4.

Lemma 3.17 *Let* $W \subseteq \mathbb{R}^d$ *be a measurable, bounded set with nonempty interior.*

(i) *If* $nr^d \leq n^{-\alpha}$ *with* $\alpha > \frac{1}{k}$ *then* $M_W \leq k$ *a.a.a.s.;*

(ii) *If* $nr^d \geq n^{-\beta}$ *with* $\beta < \frac{1}{k-1}$ *then* $M_W \geq k$ *a.a.a.s.*

To make use of this lemma, we will "split" r into subsequences. Let $K \in \mathbb{N}$ be such that $\frac{1}{K} < \alpha$. For $k = 1, \ldots, K$ set $a_k := \frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$ and set:

$$r_1(n) := \begin{cases} r(n) & \text{if } nr^d \leq n^{-a_1}, \\ n^{-\frac{2}{d}} & \text{otherwise} \end{cases},$$

and for $2 \leq k \leq K$:

$$r_k(n) := \begin{cases} r(n) & \text{if } n^{-a_{k-1}} \leq nr^d \leq n^{-a_k}, \\ n^{-\frac{k+1}{dk}} & \text{otherwise} \end{cases}.$$

Let us now put $G_n^{(k)} := G(\{X_1, \ldots, X_n\}, r_k(n))$ for $1 \leq k \leq K$. Observe that (with probability one) G_n is one of the $G_n^{(k)}$, but which k may vary with n. Thus it suffices to show that $\chi(G_n^{(k)}) = \omega(G_n^{(k)})$ a.a.a.s. for each k separately (as the intersection of finitely many events of probability one itself has probability one). Let us thus fix $1 \le k \le K$ and let us set $M_1 := \max_{x\in\mathbb{R}^d} \mathcal{N}(B(x; \frac{r_k}{2})), M_2 := \max_{x\in\mathbb{R}^d} \mathcal{N}(B(x; 100r_k))$. Observe that

$$M_1 \le \omega(G_n^{(k)}) \le \chi(G_n^{(k)}) \le \Delta(G_n^{(k)}) + 1 \le M_2. \tag{26}$$

Now notice that Lemma 3.17 shows that a.a.a.s. it holds that:

$$M_1, M_2 \in \{k, k+1\} \tag{27}$$

(in the case $k = 1$ we do not apply **(ii)**, but $M_1, M_2 \ge 1$ is trivially true). To finish the proof we will derive (deterministically) that if (27) holds then $\chi(G_n^{(k)}) = \omega(G_n^{(k)})$ must also hold. So let us assume that (27) holds. First note that (26) then implies that $\Delta(G_n^{(k)}) \in \{\omega(G_n^{(k)}) - 1, \omega(G_n^{(k)})\}$. If $\Delta(G_n^{(k)}) = \omega(G_n^{(k)}) - 1$ then we are done, so let us suppose $\Delta(G_n^{(k)}) = \omega(G_n^{(k)})$. In this case Brooks' lemma (see for instance [18]) tells us that $\chi(G_n^{(k)}) = \omega(G_n^{(k)})$ unless $\omega(G_n^{(k)}) = 2$ and $G_n^{(k)}$ contains an odd cycle. Let us therefore assume $\omega(G_n^{(k)}) = 2$. Then we must have $k \le 2$ and hence $M_2 \le 3$. But if $M_2 \le 3$ then $L(G_n^{(k)}) \le 3$ (where $L(G)$ denotes the order of the largest component of G). To see this note that if the subgraph induced by $X_{i_1} X_{i_2}, X_{i_3}, X_{i_4}$ is connected then $\|X_{i_1} - X_{i_j}\| < 3r_k$ for all $j \in \{1, \dots, 4\}$. But $L(G_n^{(k)}) \le 3$ means that the only odd cycles $G_n^{(k)}$ could possibly have are triangles. The existence of a triangle would however contradict $\omega(G_n^{(k)}) = 2$. Hence there are no odd cycles and $\chi(G_n^{(k)}) = 2$ as required. □

3.3.2 The proof of Lemma 3.16

We are now ready to combine some of the results in the previous sections to give a proof of Lemma 3.16:

Proof of Lemma 3.16, part (i): Let $\varepsilon > 0$ be arbitrary. It suffices to show that

$$\frac{\chi(G_n)}{\omega(G_n)} < (1+\varepsilon)x(t) \text{ a.a.a.s.} \tag{28}$$

It is clear that Theorem 1.4 takes care of the case when nr^d is bounded above by a negative power of n, because then $|\frac{\chi}{\omega} - 1| = 0$ a.a.a.s. (and $x(t) \ge 1$). To deal with larger r we will need the following lemma on the scan statistic, proved in section 4:

Lemma 3.18 *Let $W \subseteq \mathbb{R}^d$ be a measurable, bounded set with non-empty interior and $\varepsilon > 0$. Then there exists a $\beta = \beta(\varepsilon) > 0$ such that if $n^{-\beta} \le nr^d \le \beta \ln n$ then $(1-\varepsilon)k(n) \le M_W \le (1+\varepsilon)k(n)$ a.a.a.s. with $k(n) = \ln n / \ln(\frac{\ln n}{nr^d})$.*

Let us choose ε' such that $\frac{(1+\varepsilon')^2}{1-\varepsilon'} < 1+\varepsilon$. and let $\beta = \beta(\varepsilon')$ be the β we get in Lemma 3.18. Let $t_1 < \dots < t_m$ be chosen such that $t_1 = \beta(\varepsilon')/\sigma, t_m = t$ and $t_{i+1}/t_i < 1+\varepsilon'$. We will again "split" r into associated sequences $r_0, r_1, \dots, r_m$ where

$$r_0(n) := \begin{cases} r(n) & \text{if } nr^d \le n^{-\beta}, \\ n^{-\frac{\beta+1}{d}} & \text{otherwise} \end{cases}, \quad r_1(n) := \begin{cases} r(n) & \text{if } n^{-\beta} < nr^d \le \beta \ln n, \\ n^{-\frac{\beta+1}{d}} & \text{otherwise} \end{cases},$$

and for $2 \le i \le m$:

$$r_i(n) := \begin{cases} r(n) & \text{if } t_{i-1} \ln n < \sigma n r^d \le t_i \ln n, \\ (\frac{t_i \ln n}{\sigma n})^{\frac{1}{d}} & \text{otherwise} \end{cases}$$

For $0 \le i \le m$ set $G_n^{(i)} := G(\{X_1, \dots, X_n\}, r_i(n))$. So (with probability one) G_n is always one of the $G_n^{(i)}$, but which one is dependent on n. Thus, if we can show that $G^{(0)}, \dots, G^{(m)}$ all satisfy (28) then we are done. As mentioned before, Theorem 1.4 shows that $G^{(0)}$ satisfies (28). Next, let us consider $G^{(i)}$ with $2 \le i \le m$. We know that a.a.a.s. the following hold:

$$\begin{array}{l} \chi(G_n^{(i)}) \le (1+\varepsilon') \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t_i) \sigma t_i \ln n, \\ \omega(G_n^{(i)}) \ge (1-\varepsilon') \xi(\varphi_0, t_{i-1}) \sigma t_{i-1} \ln n, \end{array}$$

where we have used Theorem 3.2, Theorem 3.4, and the fact that both ω and χ are increasing with r. We have

$$\frac{\chi(G_n^{(i)})}{\omega(G_n^{(i)})} \le \frac{(1+\varepsilon')t_i}{(1-\varepsilon')t_{i-1}} \cdot \frac{\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t_i)}{\xi(\varphi_0, t_{i-1})} < (1+\varepsilon) x(t_i) \le (1+\varepsilon) x(t) \text{ a.a.a.s.}$$

Here we have used that $\xi(\varphi_0, t_{i-1}) > \xi(\varphi_0, t_i)$ by definition of ξ, and that x in non-decreasing.

Let us now consider $G^{(1)}$. Set $M_1 := \max_x \mathcal{N}(B(x; \frac{r_1}{2})), M_2 := \max_x \mathcal{N}(B(x; r_1))$. Then $M_1 \le \omega(G_n^{(1)}) \le \chi(G_n^{(1)}) \le \Delta(G_n^{(1)}) + 1 \le M_2$ and so

$$1 \le \frac{\chi(G_n^{(1)})}{\omega(G_n^{(1)})} \le \frac{M_2}{M_1}.$$

By Lemma 3.18 we have

$$\frac{M_2}{M_1} \le \frac{1+\varepsilon'}{1-\varepsilon'} < 1+\varepsilon \text{ a.a.a.s.},$$

which implies that $\frac{\chi(G^{(1)})}{\omega(G^{(1)})} < (1+\varepsilon)x(t)$ a.a.a.s. as required.

Proof of part (ii): This time is suffices to show

$$(1-\varepsilon)x(t) \le \frac{\chi(G_n)}{\omega(G_n)} \le \frac{1+\varepsilon}{\delta} \text{ a.a.a.s.}$$

We will need the following sharpening of the $t = \infty$ part of Theorem 3.1. The proof can again be found in section 4.

Lemma 3.19 *Let φ be a tidy function. For every $\varepsilon > 0$ there exists a $T = T(\varepsilon)$ such that if $\sigma n r^d \geq T \ln n$ then*

$$(1-\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k \text{ a.a.a.s.},$$

where $k = \sigma n r^d \int \varphi$.

Let us choose T large, to be determined later. Let us pick an arbitrary $\varepsilon' > 0$ such that $\frac{(1+\varepsilon')^2}{1-\varepsilon'} < 1+\varepsilon$ and $\frac{1-\varepsilon'}{1+\varepsilon'} > 1-\varepsilon$, and proceed as in the proof of the previous case, picking $t_1 < \cdots < t_m$ in such a way that $t_{i+1}/t_i < 1+\varepsilon'$, $t_1 = t, t_m = T$ and defining $G_n^{(i)}$ in the same way as before for $i = 2, \ldots, m$. Let $G_n^{(m+1)}$ be the graph with distance threshold

$$r_{m+1}(n) := \begin{cases} r(n) & \text{if } \sigma n r^d \geq t_m \ln n; \\ (\frac{t_m \ln n}{\sigma n})^{\frac{1}{d}} & \text{otherwise.} \end{cases}$$

Proceeding as in the previous part we know that $\frac{\chi(G_n^{(i)})}{\omega(G_n^{(i)})} \leq (1+\varepsilon)x(t_i) \leq \frac{1+\varepsilon}{\delta}$ a.a.a.s. for $2 \leq i \leq m$. We also have that

$$\begin{aligned} \chi(G_n^{(i)}) &\geq (1-\varepsilon') \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t_{i-1}) t_{i-1} \ln n \text{ a.a.a.s.}, \\ \omega(G_n^{(i)}) &\leq (1+\varepsilon') \xi(\varphi_0, t_{i+1}) t_{i+1} \ln n \text{ a.a.a.s. }, \end{aligned}$$

and hence:

$$\frac{\chi(G_n^{(i)})}{\omega(G_n^{(i)})} \geq \frac{(1-\varepsilon')t_{i-1}}{(1+\varepsilon')t_i} \cdot \frac{\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t_{i-1})}{\xi(\varphi_0, t_i)} > (1-\varepsilon)x(t_{i-1}) \geq (1-\varepsilon)x(t) \text{ a.a.a.s.},$$

using again the decreasingness of $\xi(\varphi_0, t)$ and the non-decreasingness of x. It remains to be seen the same is true for $G_n^{(m+1)}$ (for some choice of T).

By Lemma 3.6 for any $\varepsilon'' > 0$ there exists a constant $c = c(\varepsilon'')$ and tidy, $(1+\varepsilon'')$-feasible functions $\varphi_1, \ldots, \varphi_p$ such that

$$\chi(G_n) \leq (1+\varepsilon'')^d \max_i M_{\varphi_i} + c,$$

In fact this bound holds uniformly in $r > 0$. Now let us set $\varphi_i'(x) := \varphi_i((1+\varepsilon'')x)$ then $\int \varphi_i = (1+\varepsilon'')^d \int \varphi_i' \leq (1+\varepsilon'')^d \frac{\theta}{2^d \delta}$ by Lemma 3.7. Furthermore, from Theorem 3.1 and Theorem 3.4 (taking $t = \infty$) we see that we also must have $(1+\varepsilon'') \max_{i=1,\ldots,p} \int \varphi_i \geq \frac{\theta}{2^d \delta}$.

Thus, in view of Lemma 3.19, T, ε'' can be chosen such that $\sigma n r^d \geq T \ln n$ implies:

$$(1-\varepsilon')\frac{\theta}{2^d\delta} \leq \liminf \frac{\chi(G_n^{(m+1)})}{\sigma n r_{m+1}^d} \leq \limsup \frac{\chi(G_n^{(m+1)})}{\sigma n r_{m+1}^d} \leq (1+\varepsilon')\frac{\theta}{2^d\delta} \text{ a.s.},$$

$$(1-\varepsilon') \int \varphi_0 \leq \liminf \frac{\omega(G_n^{(m+1)})}{\sigma n r_{m+1}^d} \leq \limsup \frac{\omega(G_n^{(m+1)})}{\sigma n r_{m+1}^d} \leq (1+\varepsilon') \int \varphi_0 \text{ a.s.}$$

This concludes the proof. $\square$

4 Proofs of statements about M_W and M_φ

The proofs in this section make use of the following results on σ.

Proposition 4.1 ([13]) *Let $W \subseteq \mathbb{R}^d$ be bounded with positive Lebesgue measure and fix $\varepsilon > 0$. Then there exist $\Omega(r^{-d})$-many disjoint translates $x_1 + rW, \dots, x_N + rW$ of rW with $\nu(x_i + rW)/\operatorname{vol}(rW) \geq (1-\varepsilon)\sigma$.*

A proof can be found in appendix B of [13]. This last result extends to:

Corollary 4.2 *Fix $\varepsilon > 0$ and let $W \subseteq \mathbb{R}^d$ be bounded and let $W_1, \dots, W_k$ be a partition of W with $\operatorname{vol}(W_i) > 0$ for all i. Then there exist $\Omega(r^{-d})$-many points $x_1, \dots, x_N$ such that the sets $x_i + rW_j$ are pairwise disjoint and $\nu(x_i + rW_j)/\operatorname{vol}(rW_j) > (1-\varepsilon)\sigma$ for all $i = 1, \dots, N, j = 1, \dots, k$.*

Proof: Set $p_i := \frac{\operatorname{vol}(W_i)}{\operatorname{vol}(W)}, p := \min_i p_i$. By Proposition 4.1 there exist points $x_1, \dots, x_N$ with $N = \Omega(r^{-d})$ such that the sets $x_i + rW$ are disjoint and satisfying $\nu(x_i + rW) \geq (1 - p\varepsilon)\sigma \operatorname{vol}(W) r^d$. By construction the sets $x_i + rW_j$ are disjoint. We now observe that $\nu(x_i + rW_j)$ must be $\geq (1-\varepsilon)\sigma \operatorname{vol}(W_j) r^d$, because otherwise $\nu(x_i + rW) < (1 - p_j)\sigma \operatorname{vol}(W) r^d + (1-\varepsilon)\sigma p_j \operatorname{vol}(W) r^d = (1 - p_j\varepsilon)\sigma \operatorname{vol}(x_i + rW) \leq \nu(x_i + rW)$, a contradiction.
$\square$

For the proofs in this section we will also need some bounds on the binomial, Poisson and multinomial distributions. The following lemma is (one of) the so-called Chernoff-Hoeffding bound(s). A proof can for instance be found in [15].

Lemma 4.3 *Let Z be either binomial or Poisson with $\mu := \mathbb{E}Z > 0$. Then it holds that*

- **(i)** *If $k \geq \mu$ then $\mathbb{P}(Z \geq k) \leq e^{-\mu H(\frac{k}{\mu})}$;*
- **(ii)** *If $k \leq \mu$ then $\mathbb{P}(Z \leq k) \leq e^{-\mu H(\frac{k}{\mu})}$.*

Often the upper bound given by Lemma 4.3 is quite close to the truth. The following lemma gives a lower bound on $\mathbb{P}(\operatorname{Po}(\mu) \geq k)$ which is sufficiently sharp for our purposes (see [15] for a proof).

Lemma 4.4 *For $k, \mu > 0$ it holds that $\mathbb{P}(\operatorname{Po}(\mu) = k) \geq \frac{e^{-\frac{1}{12k}}}{\sqrt{2\pi k}} e^{-\mu H(\frac{k}{\mu})}$.*

A direct corollary of lemmas 4.3 and 4.4 is the following result:

Lemma 4.5 *For $\alpha > 1$ it holds that $\mathbb{P}(\operatorname{Po}(\mu) > \alpha\mu) = e^{-\mu H(\alpha) + o(\mu)}$.*

Another bound on the binomial and Poisson that will be useful in the sequel is the following standard elementary result (see for instance [12]).

Lemma 4.6 *Let Z be either binomial or Poisson and $k \geq \mu := \mathbb{E}Z$. Then*

$$(\frac{\mu}{ek})^k \leq \mathbb{P}(Z \geq k) \leq (\frac{e\mu}{k})^k$$

We will also need the following result, which is due to [9]:

Lemma 4.7 *Let $(Z_1, \dots, Z_m) \sim \text{mult}(n; p_1, \dots, p_m)$. Then*

$$\mathbb{P}(Z_1 \leq k_1, \dots, Z_m \leq k_m) \leq \Pi_{i=1}^m \mathbb{P}(Z_i \leq k_i).$$

Proof of Lemma 3.17, part (i): We first remark that it suffices to prove the result for W a ball. This is because W is bounded and hence we must have $W \subseteq B(0; R)$ for some $R > 0$ (and hence also $M_W \leq M_{B(0;R)}$, so that $M_{B(0;R)} \leq k$ implies $M_W \leq k$). Furthermore, since W is a ball it is clear that M_W is non-decreasing in r and we may assume without loss of generality that r is chosen such that $nr^d = n^{-\alpha}$. If some translate of rW contains $k+1$ points, then some X_i has at least k other points at distance $\leq 2Rr$. Hence

$$\mathbb{P}(M_W \geq k+1) \leq \mathbb{P}(\exists i : \mathcal{N}(B(X_i; 2Rr) \geq k+1) \leq n\mathbb{P}(\mathcal{N}(B(X_1; 2Rr)) \geq k+1).$$

Note that

$$\begin{array}{rcl} \mathbb{P}(\mathcal{N}(B(X_1; 2Rr)) \geq k+1) & \leq & \mathbb{P}(\text{Bi}(n, \sigma\theta 2^d R^d r^d) \geq k) \leq (\frac{e\sigma\theta 2^d R^d n r^d}{k})^k \\ & = & O(n^{-k\alpha}), \end{array}$$

where we have used Lemma 4.6. As $\alpha > \frac{1}{k}$, we have $\alpha' := k\alpha - 1 > 0$. We find

$$\mathbb{P}(M_W \geq k+1) = O(n^{-\alpha'}).$$

Unfortunately this expression is not necessarily summable in n so we cannot apply the Borel-Cantelli lemma directly. However, setting $K := \lceil \frac{1}{\alpha'} \rceil + 1$, we may conclude that

$$\mathbb{P}(M_W(m^K, r(m^K)) \leq k \text{ for all but finitely many } m) = 1,$$

because $\sum_m (m^K)^{-\alpha'} < \infty$. We now claim that from this it can be deduced that $M_W \leq k$ a.a.a.s. Note that

$$\lim_{m \to \infty} \frac{r((m-1)^K)}{r(m^K)} = 1,$$

because $nr^d = n^{-\alpha}$. Consequently $\gamma := \sup_m \frac{r((m-1)^K)}{r(m^K)} < \infty$. By the previous we may also conclude that

$$\mathbb{P}(M_{\gamma W}(m^K, r(m^K)) \leq k \text{ for all but finitely many } m) = 1.$$

Let n, m be such that $(m-1)^K < n \leq m^K$. Note that for any $x \in \mathbb{R}^d$ it holds that $x + r(n)W \subseteq x + \gamma r(m^K)W$ as $\gamma r(m^K) \geq r((m-1)^K) > r(n)$. In other words if $M_W(n) \geq k+1$ then also $M_{\gamma W}(m^K) \geq k+1$. Thus it follows that

$$\mathbb{P}(M_W(n, r(n)) \leq k \text{ for all but finitely many } n) = 1,$$

as required.

Proof of (ii): We may again assume that W is a ball. This is because W has non-empty interior and it must therefore contain some ball B, so that it suffices to show $M_B \geq k$ a.a.a.s. Again, by the fact that M_W is non-decreasing in r (when W is a ball) we may assume wlog that r is chosen such that $nr^d = n^{-\beta}$. By Proposition 4.1 we can find disjoint translates $W_1, \ldots, W_N$ of rW satisfying $\nu(W_i) \geq (1-\varepsilon)\sigma \operatorname{vol}(W) r^d$ where $N = \Omega(r^{-d})$. Now notice that the joint distribution of $(\mathcal{N}(W_1), \ldots, \mathcal{N}(W_N), \mathcal{N}(\mathbb{R}^d \backslash \cup_i W_i))$ is multinomial, so that we can apply Lemma 4.7 to see that

$$\mathbb{P}(M_W \leq k-1) \leq \mathbb{P}(\mathcal{N}(W_1) \leq k-1, \ldots, \mathcal{N}(W_N) \leq k-1) \leq \Pi_{i=1}^N \mathbb{P}(\mathcal{N}(W_i) \leq k-1).$$

The (marginal) distribution of $\mathcal{N}(W_i)$ is $\operatorname{Bi}(n, \nu(W_i))$, so that Lemma 4.6 tells us that

$$\mathbb{P}(\mathcal{N}(W_i) \geq k) \geq (\frac{n(1-\varepsilon)\sigma \operatorname{vol}(W) r^d}{ek})^k = cn^{-k\beta}.$$

Thus

$$\mathbb{P}(M_W \leq k-1) \leq (1 - cn^{-k\beta})^N \leq \exp[-cn^{-k\beta}N].$$

As $nr^d = n^{-\beta}$ we have $r^{-d} = n^{1+\beta}$. As $\beta < \frac{1}{k-1}$ we also have $\beta' := 1 + \beta - k\beta > 0$, so that $n^{-k\beta}N = \Omega(n^{\beta'})$. Thus

$$\mathbb{P}(M_W \leq k-1) \leq \exp[-\Omega(n^{\beta'})],$$

which is summable in n. It follows from the Borel-Cantelli lemma that $M_W \geq k$ a.a.a.s. as required. $\square$

Proof of Lemma 3.18: As in the proof of the previous lemma we may again assume that W is a ball. Set $k(n) := \ln n / \ln(\frac{\ln n}{nr^d})$. Let us first consider the lower bound. Completely analogously to the proof of Lemma 3.17, item **(ii)**, we have

$$\begin{aligned} \mathbb{P}(M_W \leq (1-\varepsilon)k) &\leq (1 - \mathbb{P}(\operatorname{Bi}(n, Cr^d) \geq (1-\varepsilon)k))^{\Omega(r^{-d})} \\ &\leq \exp[-\Omega(r^{-d}(\frac{Cnr^d}{e(1-\varepsilon)k})^{(1-\varepsilon)k})] \\ &= \exp[-\Omega(r^{-d}\exp[-(1-\varepsilon)k(\ln(\frac{k}{nr^d}) + D)])], \end{aligned} \tag{29}$$

with $C := (1-\varepsilon)\sigma \operatorname{vol}(W), D := \ln(\frac{e(1-\varepsilon)}{C})$. By choice of k:

$$\begin{aligned} k(\ln(\frac{k}{nr^d}) + D) &= \frac{\ln n}{\ln(\frac{\ln n}{nr^d})}\left[\ln(\frac{\ln n}{nr^d}) - \ln(\ln(\frac{\ln n}{nr^d})) + D\right] \\ &= \ln n\left[1 - \ln(\ln(\frac{\ln n}{nr^d}))/\ln(\frac{\ln n}{nr^d}) + D/\ln(\frac{\ln n}{nr^d})\right]. \end{aligned} \tag{30}$$

If $n^{-\beta} \leq nr^d \leq \beta \ln n$ then $\frac{\ln n}{nr^d} \geq \frac{1}{\beta}$. Thus, if $\beta = \beta(\varepsilon) > 0$ is chosen small enough then:

$$k(\ln(\frac{k}{nr^d}) + D) = \left(1 + \frac{D - \ln(\ln(\frac{\ln n}{nr^d}))}{\ln(\frac{\ln n}{nr^d})}\right)\ln n \leq \ln n. \tag{31}$$

Also note that $r^{-d} \geq \frac{n}{\beta \ln n} = n^{1+o(1)}$. Combining this with (29) and (31), we get

$$\begin{aligned} \mathbb{P}(M_W \leq (1-\varepsilon)k) &\leq \exp[-\Omega(r^{-d}e^{-(1-\varepsilon)\ln n})] = \exp[-\Omega(r^{-d}n^{-1+\varepsilon+o(1)})] \\ &\leq \exp[-n^{\varepsilon+o(1)}]. \end{aligned}$$

This last expression sums in n, so we may conclude that $M_W \geq (1-\varepsilon)k$ a.a.a.s. if $n^{-\beta} \leq nr^d \leq \beta \ln n$ for $\beta = \beta(\varepsilon) > 0$ sufficiently small.

Let us now shift attention to the upper bound. As in the proof of item **(i)** of Lemma 3.17 the obvious upper bound on $\mathbb{P}(M_W \geq (1+\varepsilon)k)$ does not sum in n. Unfortunately the trick we applied there does not seem to work here and we are forced to use a more elaborate method. For $s > 0$ let us set

$$M(n,s) := \max_{x \in \mathbb{R}^d} \mathcal{N}(x + sW), \quad k(n,s) := \ln n / \ln(\frac{\ln n}{ns^d}).$$

Note that $k(n,s)$ is increasing in n and s and so is $M(n,s)$ (because W is a ball). The rough idea for the rest of the proof is as follows. We fix a (large) constant K. Given n we appoximate n by m^K, chosen to satisfy $(m-1)^K < n \leq m^K$, and we approximate r by $\tilde{s} \geq r$, which is one of $O(\ln m)$ candidate values $s_1, \dots, s_{N(m)}$, in such a way that

(a) $M(m^K, \tilde{s}) \leq (1+\frac{\varepsilon}{2})k(m^K, \tilde{s})$ a.a.a.s.

(b) $(1+\frac{\varepsilon}{2})k(m^K, \tilde{s}) \leq (1+\varepsilon)k(n,r)$;

Note that $M(m^K, \tilde{s}) \geq M(n, r(n))$ because $m^K \geq n, \tilde{s} \geq r$ and W is a ball, and that combining this with items (a) and (b) will indeed show that $M(n,r) \leq (1+\varepsilon)k$ a.a.a.s. The reason we have chosen this setup is that if the constant K is chosen sufficiently large we will be able to use the Borel-Cantelli lemma to establish (a), making use of the fact that we are only considering a subsequence of $\mathbb{N}$ and $\tilde{s}$ is one of $O(\ln m)$ candidate values.

Let us pick $s_1(n) < s_2(n) < \dots$ such that $k(n, s_i(n)) = i$. Let us denote by $A(n)$ the event

$$A(n) := \{M(n, s_i(n)) > (1+\frac{\varepsilon}{2})i \text{ for some } 1 \leq i < I(n)\},$$

with $I(n) := \ln n / \ln(\frac{1}{2\beta})$, the value of $k(n,s)$ corresponding to $ns^d = 2\beta \ln n$, where $\beta = \beta(\varepsilon)$ is to be determined later (note that $k(n,s) = i$ implies that $\ln(\frac{\ln n}{ns^d}) = \frac{\ln n}{i}$). By computations done in the proof of **(i)** of Lemma 3.17 we know that

$$\mathbb{P}(M(n,s) > (1+\frac{\varepsilon}{2})k(n,s)) \leq n \left(\frac{Cns^d}{k(n,s)}\right)^{(1+\frac{\varepsilon}{2})k(n,s)} = ne^{-(1+\frac{\varepsilon}{2})k(n,s)(\ln(\frac{k(n,s)}{ns^d})+D)}, \quad (32)$$

for appropriately chosen constants C, D. We may assume wlog that $D \leq 0$. By (30) we have that

$$k(n,s)(\ln\left(\frac{k(n,s)}{ns^d}\right) + D) \geq \ln n(1 + \frac{D}{\ln(\frac{\ln n}{ns^d})}).$$

If $s_1 \leq s \leq s_{\lfloor I \rfloor}$ then $\ln n/(ns^d) \geq \frac{1}{2\beta}$. Hence, by taking $\beta = \beta(\varepsilon)$ sufficiently small we can guarantee that for $s_1 \leq s \leq s_{\lfloor I \rfloor}$:

$$(1+\frac{\varepsilon}{2})(1+\frac{D}{\ln(\frac{\ln n}{ns^d})}) \geq (1+\frac{\varepsilon}{2})(1+\frac{D}{\ln(\frac{1}{2\beta})}) > 1,$$

as we assumed wlog that $D \leq 0$. Let us write $1 + c := (1+\frac{\varepsilon}{2})(1+\frac{D}{\ln(\frac{1}{2\beta})})$. By (32) we have that for $s_1(n) \leq s \leq s_{\lfloor I(n) \rfloor}(n)$:

$$\mathbb{P}(M(n,s) \geq (1+\frac{\varepsilon}{2})k(n,s)) \leq ne^{-(1+\frac{\varepsilon}{2})k(n,s)(\ln(\frac{k(n,s)}{ns^d})+D)} \leq n^{-c+o(1)}.$$

It also follows that

$$\mathbb{P}(A(n)) \leq I(n) n^{-c+o(1)} = n^{-c+o(1)}.$$

This last expression does not necessarily sum in n, but if we take K such that $Kc > 1$ then we can apply Borel-Cantelli to deduce that (a) holds, ie.:

$$\mathbb{P}(A(m^K) \text{ holds for at most finitely many } m) = 1.$$

Now let $n \in \mathbb{N}$ be arbitrary and let m be such that $(m-1)^K < n \leq m^K$. Let i be such that $s_i(m^K) \leq r(n) < s_{i+1}(m^K)$. We first remark that if $n^{-\beta} \leq nr^d \leq \beta \ln n$ then $(m^K)^{-\beta} \leq m^K r^d \leq (\frac{m}{m-1})^K \beta \ln(m^K)$, giving:

$$\frac{1+o(1)}{\beta} \leq k(m^K, r) \leq -(1+o(1))\frac{\ln(m^K)}{\ln(\beta)}.$$

So for n sufficiently large, we must have $\frac{1}{2\beta} < i < I(m^K)$. To complete the proof we now aim to show that (for n large enough)

$$\begin{gathered}\{M(n,r(n)) \geq (1+\varepsilon)k(n,r(n))\} \\ \Downarrow \\ \{M(m^K, s_{i+1}(m^K)) \geq (1+\tfrac{\varepsilon}{2})k(m^K, s_{i+1}(m^K))\}.\end{gathered}$$

It suffices to show that $(1+\varepsilon)k(n,r(n)) \geq (1+\frac{\varepsilon}{2})k(m^K, s_{i+1}(m^K))$, because $M(n,r(n)) \leq M(m^K, s_{i+1}(m^K))$ (W is a ball). To this end, notice that

$$\begin{array}{rcl} k(n,r(n)) & \geq & k((m-1)^K, s_i(m^K)) \\ & = & \ln((m-1)^K)/\ln(\frac{\ln((m-1)^K}{(m-1)^K s_i(m^K)}) \\ & = & \left(\ln(m^K) + \ln((\frac{m-1}{m})^K)\right) / \left(\ln(\frac{\ln(m^K)}{m^K s_i(m^K)}) + \ln(\frac{\ln(m-1)^K m^K}{\ln(m^K)(m-1)^K})\right) \\ & = & \left(\ln(m^K) + O(\frac{1}{m})\right) / \left(\frac{\ln(m^K)}{i} + O(\frac{1}{m})\right) \\ & = & \ln(m^K)(1+o(1))/\frac{\ln(m^K)}{i}(1+o(1)) \\ & = & i(1+o(1)), \end{array}$$

where in the fourth line we've used that $\ln(1+x) = O(x)$, the definition of $s_i(m^K)$ and that $1 \leq \frac{\ln(m-1)^K m^K}{\ln(m^K)(m-1)^K} \leq (\frac{m}{m-1})^K$, and in the fifth line we've used that $\frac{\ln(m^K)}{i} \geq \frac{\ln(m^K)}{I(m^K)} = \ln(\frac{1}{2\beta})$.

So if n is sufficiently large, and $\beta = \beta(\varepsilon) > 0$ is chosen sufficiently small then

$$\begin{array}{rcl} (1+\varepsilon)k(n,r(n)) & \geq & (1+\varepsilon)(1+o(1))i \geq (1+\frac{\varepsilon}{2})(1+2\beta)i \\ & \geq & (1+\frac{\varepsilon}{2})(1+\frac{1}{i})i = (1+\frac{\varepsilon}{2})(i+1) \\ & = & (1+\frac{\varepsilon}{2})k(m^K, s_{i+1}(m^K)), \end{array}$$

as required, where we've used that $i \geq \frac{1}{2\beta}$. $\square$

Lemma 3.18 also allows us to deduce the following corollary, which is of independent interest and extends lemma 5.3 of [12].

Corollary 4.8 *Let $W \subseteq \mathbb{R}^d$ be bounded with non-empty interior. If $n^{-\varepsilon} \ll nr^d \ll \ln n$ for all $\varepsilon > 0$ then*

$$\frac{M_W}{k(n)} \to 1 \ a.s.,$$

with $k(n) := \ln n/\ln(\frac{\ln n}{nr^d})$.

Proof of Lemma 3.19: Let us first observe that it suffices to prove the result for φ a simple function, because the functions φ we are considering can be well approximated by the functions $\varphi_m^{\text{lower}}, \varphi_m^{\text{upper}}$ defined by:

$$\varphi_m^{\text{lower}} := \sum_{k=1}^{\lceil m\cdot\max\varphi\rceil} (\frac{k-1}{m}) 1_{\{\frac{k-1}{m} < \varphi \leq \frac{k}{m}\}}, \quad \varphi_m^{\text{upper}} := \sum_{k=1}^{\lceil m\cdot\max\varphi\rceil} (\frac{k}{m}) 1_{\{\frac{k-1}{m} < \varphi \leq \frac{k}{m}\}},$$

Here we mean by "well approximated" that $\varphi_m^{\text{lower}} \leq \varphi \leq \varphi_m^{\text{upper}}$ for all m and

$$\lim_{m\to\infty} \int \varphi_m^{\text{lower}} = \lim_{m\to\infty} \int \varphi_m^{\text{upper}} = \xi(\varphi, t). \tag{33}$$

Observe that (33) follows from the dominated convergence theorem (φ is bounded and has bounded support). Also observe that the sets $\{\varphi_m^{\text{upper}} > a\} = \{\varphi > \frac{\lfloor am \rfloor}{m}\}$ and $\{\varphi_m^{\text{lower}} > a\} = \{\varphi > \frac{\lceil am \rceil}{m}\}$ have a small neighbourhood for all a.

Clearly $M_{\varphi_m^{\text{lower}}} \leq M_\varphi \leq M_{\varphi_m^{\text{upper}}}$. Thus the result for non-simple functions will follow from the result for simple functions by taking m such that $\int \varphi_m^{\text{upper}} - \int \varphi_m^{\text{lower}} < \frac{\varepsilon}{3} \int \varphi$ and setting $T := \max(T_1, T_2)$, where T_1 is the value of $T(\frac{\varepsilon}{3})$ we get from the result for simple functions applied to φ_m^{upper} and T_2 is the value of $T(\frac{\varepsilon}{3})$ we get from the result for simple functions applied to φ_m^{lower}.

In the remainder of the proof we will always assume that $\varphi = \sum_{i=1}^m a_i 1_{A_i}$ is a simple function with the sets A_i disjoint and bounded and that $\{\varphi > a\}$ has a small neighbourhood for all a. Let us set

$$k := \sigma n r^d \int \varphi. \tag{34}$$

It remains to show that $(1-\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k$ a.a.a.s., whenever $\sigma n r^d \geq T \ln n$ for some sufficiently large $T = T(\varepsilon)$.

Proof of lower bound: Let $N \sim \text{Po}((1-\frac{\varepsilon}{100})n)$ be independent from $X_1, X_2, \dots$. It will be useful to consider $X_1, \dots, X_N$ (rather than $X_1, \dots, X_n$), because they constitute the points of a Poisson process with intensity function $(1-\frac{\varepsilon}{100})nf$ (where f is the probability density function of ν), see for instance [7].

By corollary 4.2 there are $\Omega(r^{-d})$ points $x_1, \dots, x_K$ such that $\nu(x_i + rA_j) \geq (1 - \frac{\varepsilon}{100})\sigma \operatorname{vol}(A_j) r^d$ for all $1 \leq i \leq K, 1 \leq j \leq m$ and the sets $x_i + rA_j$ are disjoint. For the current proof we will only need that $K \geq 1$, but the fact that $K = \Omega(r^{-d})$ will be needed for the proof of Theorem 3.1, which proceeds along similar lines as the current proof. Let us set $M_i := \sum_{j=1}^N \varphi(\frac{X_j - x_i}{r})$, so that

$$M_i = a_1 \mathcal{N}_N(x_i + rA_1) + \cdots + a_m \mathcal{N}_N(x_i + rA_m),$$

where $\mathcal{N}_N(B) := |\{X_1, \dots, X_N\} \cap B|$ denotes the number of points of the Poisson process in B. Note that $\mathcal{N}_N(x_i + rA_j)$ is a Poisson random variable with mean at least

$$\mu_j := (1 - \frac{\varepsilon}{100})^2 \sigma \operatorname{vol}(A_j) n r^d.$$

Setting

$$M'_\varphi := \sup_{x \in \mathbb{R}^d} \sum_{i=1}^N \varphi(\frac{X_i - x}{r}),$$

we have

$$\mathbb{P}(M'_\varphi \leq (1-\varepsilon)k) \leq \mathbb{P}(M_1 \leq (1-\varepsilon)k, \dots, M_K \leq (1-\varepsilon)k) = \Pi_{i=1}^K \mathbb{P}(M_i \leq (1-\varepsilon)k),$$

where in the last equality we have used that distinct M_i depend on the points of a Poisson process in disjoint areas of $\mathbb{R}^d$ and hence the M_i are independent. If $Z = a_1 Z_1 + \cdots + a_m Z_m$ with the Z_j independent Poisson variables satisfying $\mathbb{E} Z_j = \mu_j$ then M_i stochastically dominates Z, so that:

$$\mathbb{P}(M_\varphi \le (1-\varepsilon)k) \le \mathbb{P}(M'_\varphi \le (1-\varepsilon)k) + \mathbb{P}(N > n) \le \mathbb{P}(Z \le (1-\varepsilon)k)^K + \mathbb{P}(N > n),$$

and consequently, by Lemma 4.3

$$\mathbb{P}(M_\varphi < (1-\varepsilon)k) \le \mathbb{P}(Z < (1-\varepsilon)k) + \mathbb{P}(N > n) \le \mathbb{P}(Z < (1-\varepsilon)k) + e^{-\alpha n},$$

where $\alpha := (1-\frac{\varepsilon}{100})H(\frac{1}{1-\frac{\varepsilon}{100}})$ (and we've used that $K = \Omega(r^{-d})$ is ≥ 1 for n sufficiently large). On the other hand (using Lemma 4.3):

$$\begin{array}{rcl} \mathbb{P}(Z \le (1-\varepsilon)k) & \le & \sum_{i=1}^m \mathbb{P}(Z_i \le \frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\mu_i) \\ & \le & m \cdot \max_i \mathbb{P}(\mathrm{Po}(\mu_i) \le \frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\mu_i) \\ & \le & m \cdot \exp[-\min_i \mu_i H\left(\frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\right)]. \end{array}$$

Now suppose that T has been chosen in such a way that (and we may suppose this)

$$T \cdot (1-\frac{\varepsilon}{100})^2 \cdot \min_i \mathrm{vol}(A_i) \cdot H\left(\frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\right) \ge 2.$$

It follows that $\sum_n \mathbb{P}(M_\varphi < (1-\varepsilon)k) \le m\sum_n n^{-2} + \sum_n e^{-\alpha n} < \infty$, which concludes the proof of the lower bound.

Proof of upper bound: We may assume wlog that $a_1 > a_2 > \cdots > a_m > 0$ and that the sets A_i are disjoint (note the A_i are bounded by assumption). Recall that A_η denotes $A + B(0;\eta) = \cup_{a\in A} B(a;\eta)$. For $\eta > 0$ let φ_η be defined by

$$\varphi_\eta(x) := \left\{ \begin{array}{ll} a_i & \text{if } x \in (A_i)_\eta \setminus \bigcup_{j<i}(A_j)_\eta, \\ 0 & \text{if } x \notin (A_i)_\eta \text{ for all } 1 \le i \le m. \end{array} \right.,$$

and let η be chosen such that $\int \varphi_\eta \le (1+\frac{\varepsilon}{100})\int\varphi$. This can be done, because:

$$\begin{array}{rcl} \mathrm{vol}((A_i)_\eta \setminus \bigcup_{j<i}(A_j)_\eta) - \mathrm{vol}(A) & = & \mathrm{vol}((\bigcup_{j\le i} A_j)_\eta \setminus (\bigcup_{j<i} A_j)_\eta) - \mathrm{vol}((\bigcup_{j\le i} A_j) \setminus (\bigcup_{j<i} A_j)) \\ & = & (\mathrm{vol}((\bigcup_{j\le i} A_j)_\eta) - \mathrm{vol}(\bigcup_{j\le i} A_j)) \\ & & -(\mathrm{vol}((\bigcup_{j<i} A_j)_\eta) - \mathrm{vol}(\bigcup_{j<i} A_j)), \end{array}$$

and this can be made arbitrarily small by taking $\eta > 0$ small, since the sets $\bigcup_{j\le i} A_j = \{\varphi > a_{i+1}\}$ and $\bigcup_{j<i} A_j = \{\varphi > a_i\}$ have small neighbourhoods. Thus we can choose η so that $\mathrm{vol}((A_i)_\eta \setminus \bigcup_{j<i}(A_j)_\eta) \le (1+\frac{\varepsilon}{100})\,\mathrm{vol}(A_i)$ for all i, and then we also have $\int\varphi_\eta = \sum_i a_i \,\mathrm{vol}((A_i)_\eta \setminus \bigcup_{j<i}(A_j)_\eta) \le (1+\frac{\varepsilon}{100})\sum_i a_i\,\mathrm{vol}(A_i) = (1+\frac{\varepsilon}{100})\int\varphi$. Clearly $\varphi_\eta(x) \ge \varphi(x)$ for all x giving $M_{\varphi_\eta} \ge M_\varphi$.

Similarly to what we did for the lower bound, let $N \sim \mathrm{Po}((1+\frac{\varepsilon}{100})n)$ be independent of the X_i and set

$$M'_\varphi := \max_{x\in\mathbb{R}^d} \sum_{j=1}^N \varphi(\frac{X_j - x}{r}).$$

We have

$$\mathbb{P}(M_\varphi > (1+\varepsilon)k) \le \mathbb{P}(M'_\varphi > (1+\varepsilon)k) + \mathbb{P}(N < n) \le \mathbb{P}(M'_\varphi > (1+\varepsilon)k) + e^{-\alpha n}, \quad (35)$$

for some $\alpha > 0$ (where we have used the Lemma 4.3). Again the points $X_1, \dots, X_N$ are the points of a Poisson process, this time with intensity function $(1 + \frac{\varepsilon}{100})nf$.

Let $R > 0$ be a fixed constant such that the support of φ_η is contained in $[\frac{-R}{2}, \frac{R}{2})^d$ (R exists because we assumed the A_i are bounded). Let U be uniform on $[0, rR)^d$ and let $\Gamma(U)$ be the random set of points $U + rR\mathbb{Z}^d$ ($= \{U + rRz : z \in \mathbb{Z}^d\}$). For $x \in \mathbb{R}^d$ let M_x be the random variable given by $\sum_{j=1}^N \varphi_\eta(\frac{X_j - x}{r})$. Let us define

$$M(U) := \max_{z\in\Gamma(U)} M_z.$$

If $\|p - q\| \le \eta r$ then $\varphi_\eta(\frac{x-p}{r}) \ge \varphi(\frac{x-q}{r})$ for all x by definition of φ_η. For any $q \in \mathbb{R}^d$, the probability that some point of $\Gamma(U)$ lies in $B(q; \eta r)$ equals

$$\mathbb{P}(\Gamma(U) \cap B(q; \eta r) \ne \emptyset) = \frac{\theta \eta^d}{R^d}.$$

(We may assume wlog that R is much larger than η.) Because $\sum_{j=1}^N \varphi(\frac{X_j - x}{r}) \le \sum_{j=1}^N \varphi_\eta(\frac{X_j - y}{r})$ whenever $\|x - y\| < \eta r$, this gives the following inequality:

$$\mathbb{P}(M(U) \ge (1+\varepsilon)k | M'_\varphi \ge (1+\varepsilon)k) \ge \frac{\theta\eta^d}{R^d}.$$

We find:

$$\mathbb{P}(M'_\varphi \ge (1+\varepsilon)k) \le \frac{R^d}{\theta\eta^d} \mathbb{P}(M(U) \ge (1+\varepsilon)k). \quad (36)$$

Let us now bound $\mathbb{P}(M(U) \ge (1+\varepsilon)k)$. To do this we will condition on $U = u$ and give a uniform bound on $\mathbb{P}(M(u) \ge (1+\varepsilon)k)$. The random variables M_z, $z \in \Gamma(u)$ can be written as $a_1 M_{z,1} + \dots + a_m M_{z,m}$ with the $M_{z,i}$ independent Poisson variables with means

$$\mathbb{E}M_{z,i} \le (1 + \frac{\varepsilon}{100})^2 \operatorname{vol}(A_i) \sigma n r^d =: \mu_i.$$

Let us partition $\Gamma(u)$ into subsets $\Gamma_1, \dots, \Gamma_K$ with $K = O(r^{-d})$ such that

$$\sum_{z\in\Gamma_j} \mathbb{E}M_{z,i} \le \mu_i \text{ for all } i \in \{1, \dots, m\}. \quad (37)$$

To see that this can be done, notice we can inductively choose maximal subsets $\Gamma_j \subseteq \Gamma(u) \setminus \bigcup_{j'<j} \Gamma_{j'}$ with the property $\sum_{z\in\Gamma_j} \mathbb{E}M_{z,i} \leq \mu_i$ for all $i \in \{1,\dots,m\}$ (where by maximal we mean that the addition to Γ_j of any $z \notin \bigcup_{j'\leq j} \Gamma_{j'}$ would violate this last property). With the Γ_j chosen in this way, we must have that $\Gamma_j \cup \{z\}$ violates one of the constraints (37) for any $z \in \Gamma_{j+1}$. Thus, in particular $\sum_{i=1}^m \sum_{z\in\Gamma_j\cup\Gamma_{j+1}} \mathbb{E}M_{z,i} > \min_i \mu_i$ if $\Gamma_{j+1} \neq \emptyset$. Consequently, if we were able to select K subsets Γ_j we must have

$$\lfloor \frac{K-1}{2} \rfloor \min_i \mu_i \leq \sum_{j=1}^K \sum_{i=1}^m \sum_{z\in\Gamma_j} \mathbb{E}M_{z,i} \leq (1+\frac{\varepsilon}{100})n,$$

where the second inequality follows because the $M_{z,i}$ correspond to the number of points of a Poisson process of total intensity $(1+\frac{\varepsilon}{100})n$ in disjoint regions of $\mathbb{R}^d$. So we must indeed have $K = O(r^{-d})$, and that the process of selecting Γ_j must have stopped after $O(r^{-d})$ many Γ_j were selected.

Set $M_{\Gamma_j} := \sum_{z\in\Gamma_j} M_z$. As $\Gamma(u) = \bigcup_j \Gamma_j$ we have

$$M(u) = \max_{z\in\Gamma(u)} M_z \leq \max_j M_{\Gamma_j}.$$

Note the M_{Γ_j} are stochastically dominated by $Z = a_1Z_1 + \cdots + a_mZ_m$, where the Z_i are independent with $Z_i \sim \mathrm{Po}(\mu_i)$. Thus

$$\mathbb{P}(M(u) \geq (1+\varepsilon)k) \leq K\mathbb{P}(Z \geq (1+\varepsilon)k).$$

Because this bound does not depend on the choice of u we can also conclude

$$\begin{aligned} \mathbb{P}(M(U) \geq (1+\varepsilon)k) &= \int_{[0,rR)^d} \mathbb{P}(M(u) \geq (1+\varepsilon)k) f_U(u)\mathrm{d}u \\ &\leq K\mathbb{P}(Z \geq (1+\varepsilon)k), \end{aligned} \tag{38}$$

where f_U is the probability density function of U. We then have:

$$\begin{aligned} \mathbb{P}(Z \geq (1+\varepsilon)k) &= \mathbb{P}(\sum a_iZ_i \geq \tfrac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2} \sum_i a_i\mu_i) \leq \sum_{i=1}^m \mathbb{P}(Z_i \geq \tfrac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\mu_i) \\ &\leq m \cdot \exp[-\min_i \mu_i H\left(\tfrac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right)], \end{aligned}$$

using Lemma 4.3. Now suppose that $T = T(\varepsilon)$ has been in such a way that (and we may suppose this):

$$T \cdot (1+\frac{\varepsilon}{100})^2 \cdot \min_i \mathrm{vol}(A_i) \cdot H\left(\frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right) \geq 3,$$

so that

$$\exp[-\min_i \mu_i H\left(\frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right)] \leq n^{-3},$$

whenever $\sigma n r^d \geq T \ln n$. Because $K = O(r^{-d})$ and $\sigma n r^d \geq T \ln n$, we have that $K = O(n)$. By (38) we then also have $\mathbb{P}(M(U) \geq (1+\varepsilon)k) = O(n^{-2})$. Combining this with (35) and (36) we find

$$\mathbb{P}(M_\varphi \geq (1+\varepsilon)k) = O(n^{-2}).$$

The Borel-Cantelli lemma now gives the result. $\square$

Our next target will be to prove Theorem 3.1. We will do this along the lines of the proof of Lemma 3.19. We will however need a generalisation of the Chernoff bound to weighted sums of Poisson variables, which is given by the following lemma.

Lemma 4.9 *Let $X_1, \ldots, X_m$ be independent Poisson variables with $X_i \sim Poi(\lambda_i \mu)$ where $\lambda_i > 0$ is fixed, and set $Z := a_1 X_1 + \cdots + a_m X_m$ with $a_1, \ldots, a_m > 0$ fixed. Then for $s > 0$ (fixed):*

$$\mathbb{P}(Z \geq \mu \sum_i \lambda_i a_i e^{a_i s}) = \exp[-\mu \sum_i \lambda_i H(e^{a_i s}) + o(\mu)]$$

Proof: The moment generating function of Z (evaluated at s) is

$$\mathbb{E} e^{sZ} = \Pi_i \mathbb{E} e^{a_i s X_i} = \exp[\sum_i \lambda_i \mu (e^{a_i s} - 1)].$$

Hence Markov's inequality gives

$$\begin{aligned} \mathbb{P}(Z > \mu \textstyle\sum_i \lambda_i a_i e^{a_i s}) &= \mathbb{P}(e^{sZ} > e^{\mu s \sum_i \lambda_i a_i e^{a_i s}}) \leq \exp[\textstyle\sum_i \mu \lambda_i (e^{a_i s} - 1) - \mu s \sum_i \lambda_i a_i e^{a_i s}] \\ &= \exp[-\mu \textstyle\sum_i \lambda_i (a_i s e^{a_i s} - e^{a_i s} + 1)] = \exp[-\mu \sum_i \lambda_i H(e^{a_i s}).] \end{aligned}$$

On the other hand,

$$\mathbb{P}(Z > \mu \sum_i \lambda_i a_i e^{a_i s}) \geq \mathbb{P}(X_1 \geq \mu \lambda_1 e^{a_1 s}, \ldots, X_m \geq \mu \lambda_m e^{a_m s}) = \exp[-\mu \sum_i \lambda_i H(e^{a_i s}) + o(\mu)],$$

using Lemma 4.5. $\square$

Proof of Theorem 3.1: The case when $t = \infty$ follows from Lemma 3.19, so we only need to consider $t < \infty$ here. We shall proceed as in the proof of Lemma 3.19. Again it suffices to prove Theorem 3.1 for φ a simple function, because the functions φ considered can be well approximated by the functions $\varphi_m^{\text{lower}}, \varphi_m^{\text{upper}}$ defined in the proof of Lemma 3.19, where this time we mean by "well approximated" that

$$\lim_{m \to \infty} \xi(\varphi_m^{\text{lower}}, t) = \lim_{m \to \infty} \xi(\varphi_m^{\text{upper}}, t) = \xi(\varphi, t). \tag{39}$$

Observe that (39) follows from part **(vi)** of Lemma 2.1 (φ is bounded and has bounded support). So the result for non-simple functions will follow from the result for simple functions by noticing that $M_{\varphi_m^{\text{lower}}} \leq M_\varphi \leq M_{\varphi_m^{\text{upper}}}$ for all m and taking $m \to \infty$. We remark that if $\varphi = \sum_{i=1}^m a_i 1_{A_i}$ is a simple function with the sets A_i disjoint, then

$$\begin{array}{l} \int_{\mathbb{R}^d} \varphi(x) e^{s\varphi(x)} \mathrm{d}x = \sum_{i=1}^m a_i e^{s a_i} \operatorname{vol}(A_i), \\ \int_{\mathbb{R}^d} H(e^{s\varphi(x)}) \mathrm{d}x = \sum_{i=1}^m H(e^{s a_i}) \operatorname{vol}(A_i). \end{array}$$

Let us set

$$k := \xi(\varphi, t) \sigma n r^d. \tag{40}$$

Again it suffices to prove that $(1+\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k$ a.a.a.s., for any $\varepsilon > 0$.
Proof of lower bound: We will proceed as in the proof of the lower bound in Lemma 3.19. We again have that:

$$\mathbb{P}(M_\varphi \leq (1-\varepsilon)k) \leq \mathbb{P}(Z \leq (1-\varepsilon)k)^K + e^{-\alpha n}, \tag{41}$$

where $\alpha > 0$ is a fixed constant, $K = \Omega(r^{-d})$, and $Z = a_1 Z_1 + \cdots + a_m Z_m$ with the Z_i independent $\mathrm{Po}(\mu_i)$-random variables, where $\mu_i := (1 - \frac{\varepsilon}{100})^2 \sigma n r^d \operatorname{vol}(A_j)$. We can write

$$\mathbb{P}(Z \leq (1-\varepsilon)k) = \mathbb{P}(Z \leq \frac{(1-\varepsilon)}{(1-\frac{\varepsilon}{100})^2} \sum_{i=1}^m a_i e^{s a_i} \mu_i) = \mathbb{P}(Z \leq \sum_{i=1}^m a_i e^{s' a_i} \mu_i),$$

where $s' = s'(t, \varepsilon)$ solves $\sum_{i=1}^m a_i e^{s' a_i} \operatorname{vol}(A_i) = \frac{(1-\varepsilon)}{(1-\frac{\varepsilon}{100})^2} \sum_{i=1}^m a_i e^{s a_i} \operatorname{vol}(A_i)$. Note $s' < s$ and (provided ε is small enough) also $s' > 0$. Lemma 4.9 now gives:

$$1 - \mathbb{P}(Z \leq (1-\varepsilon)k) = \mathbb{P}(Z > (1-\varepsilon)k) = \exp[-(1-\frac{\varepsilon}{100})^2 \sigma n r^d (\sum_{i=1}^m H(e^{a_i s'}) \operatorname{vol}(A_i) + o(1))]$$

As $0 < s' < s$ we have that $\sum_{i=1}^m H(e^{a_i s'}) \operatorname{vol}(A_i) < \sum_{i=1}^m H(e^{a_i s}) \operatorname{vol}(A_i) = \frac{1}{t}$. Consequently there is a constant $c = c(t, \varepsilon) > 0$ such that

$$\mathbb{P}(Z > (1-\varepsilon)k) = \exp[-(1 - c + o(1)) \ln n] = n^{-1+c+o(1)}.$$

It follows that

$$\mathbb{P}(Z \leq (1-\varepsilon)k)^K \leq (1 - n^{-1+c+o(1)})^K \leq \exp[-K n^{-1+c+o(1)}] \leq \exp[-n^{c+o(1)}],$$

using that K is at least $n^{1+o(1)}$ (as $K = \Omega(r^{-d})$ and $r^{-d} \sim \frac{n}{t \ln n}$), we see that the right hand side of (41) sums in n, so that we may conclude that $M_\varphi \geq (1-\varepsilon)k$ a.a.a.s. by Borel-Cantelli.

Proof of upper bound: Let $N, M'_\varphi, \eta, \varphi_\eta, M(U)$ be as in the proof of the upper bound in Lemma 3.19. We again have

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k) \leq K \mathbb{P}(Z \geq (1+\varepsilon)k),$$

where $K = O(r^{-d})$ and $Z = a_1 Z_1 + \cdots + a_m Z_m$, with the Z_i independent $\mathrm{Po}(\mu_i)$ random variables, where $\mu_i := (1+\frac{\varepsilon}{100})^2 \operatorname{vol}(A_i)\sigma n r^d$. We now have

$$\mathbb{P}(Z \geq (1+\varepsilon)k) = \mathbb{P}(Z \geq \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\sum_i a_i e^{sa_i}\mu_i) = \mathbb{P}(Z \geq \sum_i a_i e^{s'a_i}\mu_i),$$

where $s' = s'(\varepsilon, t)$ is such that $\sum_{i=1}^m a_i e^{s'a_i} \operatorname{vol}(A_i) = \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\sum_{i=1}^m a_i e^{sa_i} \operatorname{vol}(A_i)$. Note that $s' > s$, giving $\sum_i H(e^{s'a_i}) \operatorname{vol}(A_i) > \sum_i H(e^{sa_i}) \operatorname{vol}(A_i) = \frac{1}{t}$, and consequently

$$\sum_i H(e^{s'a_i})\mu_i = (1+\frac{\varepsilon}{100})^2 \sigma n r^d \sum_i H(e^{s'a_i}) \operatorname{vol}(A_i) = (1+c+o(1))\ln n,$$

for some $c = c(\varepsilon, t) > 0$. Since $K = O(r^{-d}) \leq n$ for n large enough we find that:

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k) \leq n\mathbb{P}(Z > (1+\varepsilon)k) = n\exp[-(1+c+o(1))\ln n] = n^{-c+o(1)}. \quad (42)$$

Unfortunately this does not necessarily sum in n, so we will have to use a more elaborate method than the one used in Lemma 3.19. Note that for any $0 < \eta' < \eta$ we have, completely analogously to (36):

$$\mathbb{P}(M'_{\varphi_{\eta'}} \geq (1+\varepsilon)k) \leq \frac{R^d}{\theta(\eta-\eta')^d}\mathbb{P}(M(U) \geq (1+\varepsilon)k), \quad (43)$$

By (43) and (35) we also have that for all $0 \leq \eta' < \eta$:

$$\mathbb{P}(M_{\varphi_{\eta'}} \geq (1+\varepsilon)k) \leq n^{-c+o(1)} + e^{-\alpha n} = n^{-c+o(1)}.$$

Although the right hand side does not necessarily sum in n, it does hold that if $L > 0$ is such that $cL > 1$ then we can apply the Borel-Cantelli lemma to show that

$$\mathbb{P}(M_{\varphi_{\eta'}}(m^L, r(m^L)) < (1+\varepsilon)k(m^L) \text{ for all but finitely many } m) = 1. \quad (44)$$

We now claim that from this we can conclude that in fact $M_\varphi \leq (1+2\varepsilon)k$ a.a.a.s. To this end, let $n \in \mathbb{N}$ be arbitrary and let $m = m(n)$ be such that $(m-1)^L < n \leq m^L$. The claim follows if we can show that (for n sufficiently large)

$$\{M_{\varphi_{\eta'}}(m^L, r(m^L)) \leq (1+\varepsilon)k(m^L)\} \Rightarrow \{M_\varphi(n) \leq (1+2\varepsilon)k(n)\}. \quad (45)$$

To this end we will first establish that (for n sufficiently large and) for any x, y:

$$\varphi(\frac{y-x}{r(n)}) \leq \varphi_{\eta'}(\frac{y-x}{r(m^L)}). \quad (46)$$

Since the support of φ is contained in $[\frac{-R}{2}, \frac{R}{2}]^d$ we are done if $\|\frac{y-x}{r(n)}\| > \operatorname{diam}([0, \frac{R}{2}]^d) =: \gamma$. If on the other hand $\|\frac{y-x}{r(n)}\| \leq \gamma$ then $\|\frac{y-x}{r(n)} - \frac{y-x}{r(m^L)}\| = |1 - \frac{r(n)}{r(m^L)}|\|\frac{y-x}{r(n)}\| \leq |1 - \frac{r(n)}{r(m^L)}|\gamma = o(1)$ (because $nr^d \sim t\ln n$ giving $r(n) = (1+o(1))r(m^l)$), so that for n sufficiently large this is

$< \eta'$ and thus (46) holds uniformly for all x, y (for such sufficiently large n), as required. Since we also have $k(n) = (1+o(1))k(m^L)$, equation (45) does indeed hold for n sufficiently large, which concludes the proof. $\square$

5 Proof of Lemma 2.1

The case $t = \infty$ is always trivial, so in the proofs we will only consider the case when $t < \infty$. On several occasions in the proof below we will differentiate an integral over $x \in \mathbb{R}^d$ wrt a parameter s and swap the order of integration. In all cases this can again be justified by means of the fundamental theorem of calculus and Fubini's theorem, as in the proof of Lemma 3.14.

Proof of (i): If we differentiate the equation $t \int H(e^{s\varphi}) = 1$ wrt t we find:

$$0 = \int H(e^{s\varphi}) + t \int s' s \varphi^2 e^{s\varphi} = \frac{1}{t} + s' st \int \varphi^2 e^{s\varphi},$$

which gives

$$s' = \frac{-1}{t^2 s \int \varphi^2 e^{s\varphi}}.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(\varphi, t) = \int s' \varphi^2 e^{s\varphi} = \frac{-1}{t^2 s}.$$

Now notice that $\varphi \leq \psi$ implies $s(\varphi, t) \geq s(\psi, t)$, so that for all $0 < t < \infty$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(\varphi, t) \geq \frac{\mathrm{d}}{\mathrm{d}t}\xi(\psi, t),$$

which implies that $\xi(\psi, t) - \xi(\varphi, t)$ is non-increasing. Finally, $\lim_{t\to\infty} \xi(\psi, t) - \xi(\varphi, t) = \int \psi - \int \varphi \geq 0$, so that we must have $\xi(\psi, t) \geq \xi(\varphi, t)$ for all $t > 0$.

Proof of (iii): We must have $s(\lambda\varphi, t) = s(\varphi, t)/\lambda$ as $\int H(e^{s(\varphi,t)\varphi}) = \int H(e^{s(\lambda\varphi,t)\lambda\varphi}) = \frac{1}{t}$. So indeed $\xi(\lambda\varphi, t) = \int \lambda\varphi e^{s(\lambda\varphi,t)\lambda\varphi} = \lambda \int \varphi e^{s(\varphi,t)\varphi} = \lambda\xi(\varphi, t)$.

Proof of (iv): Note that the substitution $y = \lambda x$ gives that:

$$\frac{1}{t} = \int_{\mathbb{R}^d} H(e^{s\phi(\lambda x)})dx = \lambda^{-d} \int_{\mathbb{R}^d} H(e^{s\varphi(y)})dy,$$

so that $s(\varphi_\lambda, t) = s(\varphi, \lambda^{-d}t)$. Using the same substitution we get

$$\xi(\varphi_\lambda, t) = \lambda^{-d} \int_{\mathbb{R}^d} \varphi(y) e^{s(\varphi,\lambda^{-d}t)\varphi(y)} dy = \lambda^{-d}\xi(\varphi, \lambda^{-d}t).$$

The upper bound now follows from the fact that $\lambda^{-d} > 1$ and that $s(\varphi, t)$ is decreasing in t. The lower bound follows from part **(vii)**, to be proved independently below, because $\int \varphi_\lambda 1_{\{\varphi_\lambda \geq a\}} = \lambda^{-d} \int \varphi 1_{\{\varphi \geq a\}}$ for all a (again by the substitution $y = \lambda x$).
Proof of (vi): For any fixed $s \geq 0$ the dominated convergence theorem (using that $H(e^{s\varphi_n}) \leq H(e^{s\psi})$ and $\int H(e^{s\psi}) < \infty$) gives

$$\lim_{n\to\infty} \int H(e^{s\varphi_n}) = \int H(e^{s\varphi}),$$

which shows that $\lim_{n\to\infty} s(\varphi_n, t) = s(\varphi, t)$. Thus, for all $\varepsilon > 0$ and n sufficiently large:

$$\int \varphi_n e^{(s(\varphi,t)-\varepsilon)\varphi_n} \leq \xi(\varphi_n, t) \leq \int \varphi_n e^{(s(\varphi,t)+\varepsilon)\varphi_n} \leq \int \psi e^{(s(\varphi,t)+\varepsilon)\psi}.$$

As $\int \psi e^{(s(\varphi,t)+\varepsilon)\psi} < \infty$ the dominated convergence theorem also gives that

$$\int \varphi e^{(s(\varphi,t)-\varepsilon)\varphi} \leq \liminf \xi(\varphi_n, t) \leq \limsup \xi(\varphi_n, t) \leq \int \varphi e^{(s(\varphi,t)+\varepsilon)\varphi}.$$

Two more applications of the the dominated convergence theorem now yield

$$\lim_{\varepsilon\to 0} \int \varphi e^{(s(\varphi,t)-\varepsilon)\varphi} = \lim_{\varepsilon\to 0} \int \varphi e^{(s(\varphi,t)+\varepsilon)\varphi} = \xi(\varphi, t),$$

giving the result.
Proof of (ii): Now that **(vi)** has been established, we see that it suffices to take φ, ψ simple functions. If $\varphi = \sum_{i=1}^m a_i 1_{A_i}$ (with the A_i disjoint) then $\xi(\varphi, t)$ depends only on the values a_i and $\mathrm{vol}(A_i)$. Similarly if $\psi := \sum_{i=1}^k b_i 1_{B_i}$ (with the B_i disjoint) then $\xi(\psi, t)$ depends only on the values b_i, $\mathrm{vol}(B_i)$ and similarly $\xi(\varphi+\psi, t)$ is determined by the numbers $\mathrm{vol}(A_i \cap B_j), \mathrm{vol}(A_i \setminus \bigcup_{j=1}^k B_j), \mathrm{vol}(B_j \setminus \bigcup_{i=1}^m A_i)$ and the a_i, b_j. We may therefore assume that all these sets have small neighbourhoods (by taking two different functions if needed). This means we are assuming φ, ψ and $\varphi+\psi$ all satisfy the conditions of Theorem 3.1. Now notice that

$$\begin{aligned} M_{\varphi+\psi} &= \sup_x \textstyle\sum_{i=1}^n (\varphi(\frac{X_i - x}{r}) + \psi(\frac{X_i - x}{r})) \\ &\leq \sup_x \textstyle\sum_{i=1}^n \varphi(\frac{X_i - x}{r}) + \sup_y \sum_{i=1}^n \psi(\frac{X_i - y}{r})) = M_\varphi + M_\psi. \end{aligned}$$

The result now follows from Theorem 3.1.
Proof of (v): Reasoning as in the proof of part **(ii)** we see that we may assume wlog that φ satisfies the conditions of Theorem 3.1. Let r satisfy $nr^d \sim t \ln n$. Let us put $\lambda = \left(\frac{t}{t+h}\right)^{\frac{1}{d}}$. By Theorem 3.1 we have:

$$\lim_{n\to\infty} \frac{M_\varphi(n, \lambda^{-1} r)}{\sigma \lambda^{-d} n r^d} = \xi(\varphi, t+h) \text{ a.s.} \tag{47}$$

Now observe that:

$$M_\varphi(n,\lambda^{-1}r)=\sup_{x\in\mathbb{R}^d}\sum_{j=1}^n\varphi\left(\frac{X_j-x}{\lambda^{-1}r}\right)=\sup_{y\in\mathbb{R}^d}\sum_{j=1}^n\varphi_\lambda\left(\frac{X_j-y}{r}\right)=M_{\varphi_\lambda}(n,r).$$

Now φ_λ also satisfies the conditions of Theorem 3.1 (as φ does) so that we see that:

$$\lim_{n\to\infty}\frac{M_\varphi(n,\lambda^{-1}r)}{\sigma n r^d}=\xi(\varphi_\lambda,t)\text{ a.s.}\tag{48}$$

Combining (47) and (48) it follows that $\xi(\varphi,t+h)=(\frac{t}{t+h})\xi(\varphi_\lambda,t)$. The result now follows from part **(iv)**.

Proof of (vii): It suffices to show that $s(\varphi,t)\geq s(\psi,t)$ for all t, because then the argument given in the proof of part **(i)** will give the result. Therefore it also suffices to show that

$$\int_{\mathbb{R}^d}\left(H(e^{s\psi(x)})-H(e^{s\varphi(x)})\right)\mathrm{d}x\geq 0,\tag{49}$$

for all $s\geq 0$. Equation (49) is certainly true for $s=0$, and it is thus enough to show that

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\left[\int_{\mathbb{R}^d}\left(H(e^{s\psi(x)})-H(e^{s\varphi(x)})\right)\mathrm{d}x\right]=s\int_{\mathbb{R}^d}\left(\psi^2(x)e^{s\psi(x)}-\varphi^2(x)e^{s\varphi(x)}\right)\mathrm{d}x\geq 0\tag{50}$$

for $s\geq 0$. Let us set $F(s):=\int_{\mathbb{R}^d}\left(\psi^2(x)e^{s\psi(x)}-\varphi^2(x)e^{s\varphi(x)}\right)\mathrm{d}x$. To show that $F(s)\geq 0$ for $s\geq 0$, it suffices[2] to show that $F^{(k)}(0)\geq 0$ for all $k\geq 0$. To see that this holds, note that $F^{(k)}(s)=\int_{\mathbb{R}^d}\left(\psi^{k+2}(x)e^{s\psi(x)}-\varphi^{k+2}(x)e^{s\varphi(x)}\right)\mathrm{d}x$ and that for any $k\geq 1$:

$$\begin{aligned}\textstyle\int_{\mathbb{R}^d}\psi^k(x)\mathrm{d}x &= \textstyle\int_{\mathbb{R}^d}\psi(x)\int_0^{\psi(x)}\dots\int_0^{\psi(x)}1\mathrm{d}u_1\dots\mathrm{d}u_{k-1}\mathrm{d}x\\ &= \textstyle\int_0^\infty\dots\int_0^\infty\int_{\mathbb{R}^d}\psi(x)1_{\psi(x)\geq\max(u_1,\dots,u_{k-1})}\mathrm{d}x\mathrm{d}u_1\dots\mathrm{d}u_{k-1}\\ &\geq \textstyle\int_0^\infty\dots\int_0^\infty\int_{\mathbb{R}^d}\varphi(x)1_{\varphi(x)\geq\max(u_1,\dots,u_{k-1})}\mathrm{d}x\mathrm{d}u_1\dots\mathrm{d}u_{k-1}\\ &= \textstyle\int_{\mathbb{R}^d}\varphi^k(x)\mathrm{d}x.\end{aligned}$$

Here we have used Fubini's lemma for nonnegative functions to change the order of integration and the fact that $\int\varphi 1_{\{\varphi\geq a\}}\leq\int\psi 1_{\{\psi\geq a\}}$ for all a. $\square$

[2]To be completely rigorous, we should remark that we are using that the infinite Taylor series of F about 0 converges to F for all s. Taylors theorem applies because $F^{(k)}(s)=\int_{\mathbb{R}^d}\psi^{k+2}(x)e^{s\psi(x)}-\varphi^{k+2}(x)e^{s\varphi(x)}\mathrm{d}x$ is continuous in s for all k. The error terms $R_{k+1}(s)=\frac{F^{(k+1)}(c)}{(k+1)!}s^{k+1}$ (with $0\leq c\leq s$) tend to zero, because $F^{(k)}(c)\leq C^{k+1}e^{cC}\int\psi$ if C is chosen s.t. $\varphi,\psi\leq C$ (recall φ,ψ are bounded).

6 Concluding remarks

In this paper we have proved a number of almost sure convergence results on the chromatic number of the random geometric graph and we have investigated its relation to the clique number. Amongst other things we have set out to describe the "phase change" regime when $nr^d = \Theta(\ln n)$. An important shift in the behaviour of the chromatic number occurs in this range of r (except in the less interesting case $\delta = 1$). We have seen that (except when $\delta = 1$) there exists a finite constant t_0 such that if $\sigma nr^d \leq t_0 \ln n$ then the chromatic number and the clique number of the random geometric graph are essentially equal in the sense that

$$\frac{\chi(G_n)}{\omega(G_n)} \to 1 \text{ a.s.},$$

and if on the other hand $nr^d \geq (t_0 + \varepsilon) \ln n$ for some fixed (but arbitrarily small) $\varepsilon > 0$ then the lim inf of this ratio is bounded away from 1 almost surely. Moreove, if $nr^d \gg \ln n$ then

$$\frac{\chi(G_n)}{\omega(G_n)} \to \frac{1}{\delta} \text{ a.s.},$$

We have also given expressions for the almost sure limit $c(t)$ of $\frac{\chi(G_n)}{\sigma nr^d}$ and the almost sure limit $x(t)$ of $\frac{\chi(G_n)}{\omega(G_n)}$ if $\sigma nr^d \sim t \ln n$ for some $t > 0$. Furthermore we have shown that if nr^d is bounded above by a negative power of n then with probability one $\chi(G_n) = \omega(G_n)$ for all but finitely many n. An interesting observation is that t_0 and the limiting constants $c(t), x(t)$ do not depend on the choice of probability measure ν, and that the only feature of the probability measure that plays any role in the proofs and results in this paper is σ, the essential supremum of the probability density. We have not spelled this out, but it is quite straightforward to combine and adapt some of the proofs given in this paper to show that $\frac{\chi(G_n)}{\chi_f(G_n)} \to 1$ almost surely, for any sequence r with $r \to 0$.

It should be mentioned that considering the ratio $\frac{\chi(G_n)}{\omega(G_n)}$, apart from the fact that it provides an easy to state summary of the results, can also be motivated by the fact that while colouring unit disk graphs (non-random geometric graphs when $d = 2$ and $\|.\|$ is the Euclidean norm) is NP-hard [3, 4] finding their clique number is in P [3], unlike finding the clique number in general graphs. In fact finding the clique number of a unit disk graph is in P even if an embedding (ie. an explicit representation with points on the plane) is not given [10]. Thus, the results given here suggest that even though finding the chromatic number of a unit disk graph is NP-hard, the polynomial approximation of finding the clique number and multiplying this by $\frac{1}{\delta}$ (which equals $\frac{2\sqrt{3}}{\pi} \approx 1.103$ for the Euclidean norm in the plane) might work quite well in practice.

It is instructive to consider what happens to the ratio of the chromatic number to the clique number in the Erdős-Rényi model for comparison. Let us consider $p = p(n)$ bounded away from one. Results of Bollobás [1] and Łuczak [8] show that

$$\chi(G(n,p)) \sim \frac{n\ \ln(\frac{1}{1-p})}{2\ln n} \text{ whp}$$

as long as $np \to \infty$; and (see for example [6]) for such p

$$\omega(G(n,p)) \sim \frac{2\ln n}{\ln(\frac{1}{p})} \text{ whp.}$$

Thus

$$\frac{\chi(G(n,p))}{\omega(G(n,p))} \sim \frac{n\ \ln(\frac{1}{p})\ln(\frac{1}{1-p})}{4\ln^2 n} \text{ whp}$$

and the last quantity tends to infinity if $np/\ln n \to \infty$. Thus, the results on the chromatic number given here and in [12, 15] highlight a dramatic difference between the Erdős-Rényi model on the one hand and the random geometric model on the other hand.

Although we have presented substantial progress on the current state of knowledge on the chromatic number of random geometric graphs in this paper, several questions remain. Our proofs for instance do not yield an explicit expression for t_0 (when $\delta < 1$) and it would certainly be of interest to find such an expression or to give some (numerical) procedure to determine it. More generally, it is far from trivial to extract information from the expression for $x(t)$ we have given in Theorem 3.4. At present we are still lacking a good understanding of the class of functions $\mathcal{F}$ and the behaviour of ξ on this class.

A question that has not been addressed at all in this paper is the probability distribution of $\chi(G_n)$. In a recent paper by the second author [13] it was shown that when $nr^d \ll \ln n$ then $\chi(G_n)$ is two-point concentrated, in the sense that

$$\mathbb{P}(\chi(G_n) \in \{k(n), k(n)+1\}) \to 1,$$

as $n \to \infty$ for some sequence $k(n)$. Analogous results were also shown to hold for the clique number $\omega(G_n)$, the maximum degree $\Delta(G_n)$ and the degeneracy $\delta^*(G_n)$. For other choices of r the distribution of $\chi(G_n)$ and $\omega(G_n)$ and $\delta^*(G_n)$ is not known. However, it is possible to extend an argument in [15] to show that if ν is uniform and $\ln n \ll nr^2 \ll (\ln n)^d$ then $\Delta(G_n)$ is approximately doubly exponential and if $nr^d = \Theta(\ln n)$ then $\Delta(G_n)$ is not finitely concentrated and the distribution does not tend to a nice limiting distribution.

7 Acknowledgments

The authors would like to thank Joel Spencer for helpful discussions and e-mail correspondence related to the paper. We would also like to thank Gregory McColm, Mathew Penrose, Alex Scott, Miklos Simonovits and Dominic Welsh and for helpful discussions related to the paper, and Karolyi Böröczky Jr. both for helpful discussions and for allowing him to reproduce the proof of Lemma 3.12 in appendix A below.

References

[1] B. Bollobás. The chromatic number of random graphs. *Combinatorica*, 8(1):49–55, 1988.

[2] V. Chvátal. *Linear Programming*. W. H. Freedman and Company, New York, 1983.

[3] B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Math.*, 86(1-3):165–177, 1990.

[4] A. Gräf, M. Stumpf, and G. Weißenfels. On coloring unit disk graphs. *Algorithmica*, 20(3):277–293, 1998.

[5] P. M. Gruber and J. M. Wills. *Handbook of Convex Geometry*. North-Holland, Amsterdam, 1993.

[6] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[7] J. Kingman. *Poisson Processes*. Oxford University Press, Oxford, 1993.

[8] T. Łuczak. The chromatic number of random graphs. *Combinatorica*, 11(1):45–54, 1991.

[9] C. L. Mallows. An inequality involving multinomial probabilities. *Biometrika*, 55(2):422–424, 1968.

[10] M. V. Marathe, H. Breu, H. B. Hunt, III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(2):59–68, 1995.

[11] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[12] C. J. H. McDiarmid. Random channel assignment in the plane. *Random Structures Algorithms*, 22(2):187–212, 2003.

[13] T. Müller. Two-point concentration in random geometric graphs. Combinatorica, to appear.

[14] J. Pach and P. K. Agarwal. *Combinatorial geometry*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[15] M. D. Penrose. *Random Geometric Graphs*. Oxford University Press, Oxford, 2003.

[16] C. A. Rogers. *Packing and covering*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 54. Cambridge University Press, New York, 1964.

[17] E. R. Scheinerman and D. H. Ullman. *Fractional graph theory.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1997.

[18] J. H. van Lint and R. M. Wilson. *A course in combinatorics.* Cambridge University Press, Cambridge, second edition, 2001.

A The proof of Lemma 3.12

In this appendix we will give a proof of Lemma 3.12 above. The proof is due to K. Böröczky Jr. and we reproduce it here with his kind permission, as it is not readily available from other sources. The proposition is a generalisation of a statement proved by the second author.

Proof of Lemma 3.12: Let us set $I := \cap_{a \in A}(a + C)$. Then I is compact and convex. We may suppose wlog that $\mathrm{vol}(I) > 0$ (otherwise there is nothing to prove). Let us remark that

$$I + \mathrm{cl}(-A) \subseteq C.$$

This is because for any $x \in I$ and $a \in A$ there exists a $c \in C$ such that $x = c + a$, by definition of I. Hence, for any $x \in I, a \in A$ we have $x - a \in C$. In other words $I + (-A) \subseteq C$. This also gives that $I + \mathrm{cl}(-A) = \mathrm{cl}(I + (-A)) \subseteq C$ as C is closed. We will need the following result (see chapter 12 of [11] for a very readable proof).

Theorem A.1 (Brunn-Minkowski inequality) *Let $A, B \subseteq \mathbb{R}^d$ be nonempty and compact. Then* $\mathrm{vol}(A + B) \geq \left(\mathrm{vol}(A)^{\frac{1}{d}} + \mathrm{vol}(B)^{\frac{1}{d}}\right)^d$.

The Brunn-Minkowski inequality gives that

$$\mathrm{vol}(I)^{\frac{1}{d}} + \mathrm{vol}(\mathrm{cl}(-A))^{\frac{1}{d}} \leq \mathrm{vol}(I + \mathrm{cl}(-A))^{\frac{1}{d}} \leq \mathrm{vol}(C)^{\frac{1}{d}}. \tag{51}$$

Thus:

$$\mathrm{vol}(I) \leq \left(\mathrm{vol}(C)^{\frac{1}{d}} - \mathrm{vol}(A)^{\frac{1}{d}}\right)^d. \tag{52}$$

The proposition will now follow by showing that if A is of the form $A = \lambda(-C)$ for some $\lambda > 0$ then equality holds in (52). Let us thus suppose that $A = \lambda(-C)$ for some $0 \leq \lambda < 1$ (note $\lambda \geq 1$ would contradict $\mathrm{vol}(I) > 0$). We claim that in this case

$$(1 - \lambda)C \subseteq I. \tag{53}$$

Observe that this will prove that equality holds in (52) (as in this case $\mathrm{vol}(I) \geq (1 - \lambda)^d \mathrm{vol}(C)$ and $\mathrm{vol}(A) = \lambda^d \mathrm{vol}(C)$), so that it only remains to establish (53). Pick $x \in$

$(1-\lambda)C$, and let $a \in A = \lambda(-C)$ be arbitrary. We can write $x = (1-\lambda)c_1, a = -\lambda c_2$ for some $c_1, c_2 \in C$. Because C is convex, $c_3 := (1-\lambda)c_1 + \lambda c_2 \in C$ and thus $x = a + c_3 \in (a + C)$. As $a \in A$ was arbitrary this gives $x \in I$ as required. $\square$