A new graph parameter related to bounded rank positive semidefinite matrix completions

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Abstract. The Gram dimension $\mathrm{gd}(G)$ of a graph G is the smallest integer $k \geq 1$ such that any partial real symmetric matrix, whose entries are specified on the diagonal and at the off-diagonal positions corresponding to edges of G, can be completed to a positive semidefinite matrix of rank at most k (assuming a positive semidefinite completion exists). For any fixed k the class of graphs satisfying $\mathrm{gd}(G) \leq k$ is minor closed, hence it can characterized by a finite list of forbidden minors. We show that the only minimal forbidden minor is K_{k+1} for $k \leq 3$ and that there are two minimal forbidden minors: K_5 and $K_{2,2,2}$ for $k = 4$. We also show some close connections to Euclidean realizations of graphs and to the graph parameter $\nu^=(G)$ of [21]. In particular, our characterization of the graphs with $\mathrm{gd}(G) \leq 4$ implies the forbidden minor characterization of the 3-realizable graphs of Belk and Connelly [8,9] and of the graphs with $\nu^=(G) \leq 4$ of van der Holst [21].

1 Introduction

Given a graph $G = (V = [n], E)$, a *G-partial matrix* is a real symmetrix $n \times n$ matrix whose entries are specified on the diagonal and at the off-diagonal positions corresponding to the edges of G. The problem of completing a partial matrix to a full positive semidefinite (psd) matrix is one of the most extensively studied matrix completion problems. A particular instance is the completion problem for correlation matrices (where all diagonal entries are equal to 1) arising in probability and statistics, and it is also closely related to the completion problem for Euclidean distance matrices with applications, e.g., to sensor network localization and molecular conformation in chemistry. We give definitions below and refer, e.g., to [12,24] and further references therein for additional details.

Among all psd completions of a partial matrix, the ones with the lowest possible rank are of particular importance. Indeed the rank of a matrix is often a good measure of the complexity of the data it represents. As an example, it is well known that the minimum dimension of a Euclidean embedding of a finite metric space can be expressed as the rank of an appropriate psd matrix (see e.g. [12]). Moreover, in applications, one is often interested in embeddings in low dimension, say 2 or 3. The problem of computing (approximate) low rank psd (or Euclidean) completions of a partial matrix is a challenging non-continuous,

non-convex problem which, due to its great importance, has been extensively studied (see, e.g., [1,2,33], the recent survey [22] and further references therein).

The following basic questions arise about psd matrix completions: Decide whether a given partial rational matrix has a psd completion, what is the smallest rank of a completion, and if so find an (approximate) one (of smallest rank). This leads to hard problems and of course the answer depends on the actual values of the entries of the partial matrix.

However, taking a combinatorial approach to the problem and looking at the structure of the graph G of the specified entries, one can sometimes get tractable instances. For instance, when the graph G is chordal (i.e., has no induced circuit of length at least 4), the above questions are fully answered in [18,25] (see also the proof of Lemma 2 below): There is a psd completion if and only if each fully specified principal submatrix is psd, the minimum possible rank is equal to the largest rank of the fully specified principal submatrices, and such a psd completion can be found in polynomial time (in the bit number model). Further combinatorial characterizations (and some efficient algorithms for completions – in the real number model) exist for graphs with no K_4-minor (more generallly when excluding certain splittings of wheels), see [6,23,25].

In the present paper we focus on the question of existence of low rank psd completions. Our approach is combinatorial, so we look for conditions on the graph G of specified entries permitting to guarantee the existence of low rank completions. This is captured by the notion of *Gram dimension* of a graph which we introduce in Definition 1 below.

We use the following notation: $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices and $\mathcal{S}_+^n$ (resp., $\mathcal{S}_{++}^n$) is the subset of all positive semidefinite (psd) (resp., positive definite) matrices. For a matrix $X \in \mathcal{S}^n$, the notation $X \succeq 0$ means that X is psd. Given a graph $G = (V = [n], E)$, it will be convenient to identify V with the set of diagonal pairs, i.e., to set $V = \{(i, i) \mid i \in [n]\}$. Then, a G-partial matrix corresponds to a vector $a \in \mathbb{R}^{V \cup E}$ and π_{VE} denotes the projection from $\mathcal{S}^n$ onto the subspace $\mathbb{R}^{V \cup E}$ indexed by the diagonal entries and the edges of G.

Definition 1. *The Gram dimension* $\mathrm{gd}(G)$ *of a graph* $G = ([n], E)$ *is the smallest integer* $k \geq 1$ *such that, for any matrix* $X \in \mathcal{S}_+^n$, *there exists another matrix* $X' \in \mathcal{S}_+^n$ *with rank at most k and such that* $\pi_{VE}(X) = \pi_{VE}(X')$.

Hence, if a G-partial matrix admits a psd completion, it also has one of rank at most $\mathrm{gd}(G)$. This motivates the study of bounds for the graph parameter $\mathrm{gd}(G)$. As we will see in Section 2.1, for any fixed k the class of graphs with $\mathrm{gd}(G) \leq k$ is closed under taking minors, hence it can be characterized by a finite list of forbidden minors. Our main result is such a characterization for each integer $k \leq 4$.

Main Theorem. *For $k \leq 3$, $\mathrm{gd}(G) \leq k$ if and only if G has no K_{k+1} minor. For $k = 4$, $\mathrm{gd}(G) \leq 4$ if and only if G has no K_5 and $K_{2,2,2}$ minors.*

An equivalent way of rephrasing the notion of Gram dimension is in terms of ranks of feasible solutions to semidefinite programs. Indeed, the Gram dimension of a graph $G = (V, E)$ is at most k if and only if the set

$$S(G, a) = \{X \succeq 0 \mid X_{ij} = a_{ij} \text{ for } ij \in V \cup E\}$$

contains a matrix of rank at most k for all $a \in \mathbb{R}^{V \cup E}$ for which $S(G, a)$ is not empty. The set $S(G, a)$ is a typical instance of spectrahedron. Recall that a *spectrahedron* is the convex region defined as the intersection of the positive semidefinite cone with a finite set of affine hyperplanes, i.e., the feasibility region of a semidefinite program in canonical form:

$$\max\langle A_0, X\rangle \text{ subject to } \langle A_j, X\rangle = b_j, \ (j = 1, \ldots, m), \qquad X \succeq 0. \qquad (1)$$

If the feasibility region of (1) is not empty, it follows from well known geometric results that it contains a matrix X of rank k satisfying $\binom{k+1}{2} \leq m$ (see e.g. [7]). Applying this to the spectahedron $S(G, a)$, we obtain the bound

$$\mathrm{gd}(G) \leq \left\lfloor \frac{\sqrt{1 + 8(|V| + |E|)} - 1}{2} \right\rfloor.$$

For the complete graph $G = K_n$ the upper bound is equal to n, so it is tight. As we will see one can get other bounds depending on the structure of G; for instance, $\mathrm{gd}(G)$ is at most the tree-width plus 1 (cf. Lemma 3).

As an application, the Gram dimension can be used to bound the rank of optimal solutions to semidefinite programs. Namely, consider a semidefinite program in canonical form (1). Its *aggregated sparsity pattern* is the graph G with node set $[n]$ and whose edges are the pairs corresponding to the positions where at least one of the matrices A_j $(j \geq 0)$ has a nonzero entry. Then, whenever (1) attains its maximum, it has an optimal solution of rank at most $\mathrm{gd}(G)$. Results ensuring existence of low rank solutions are important, in particular, for approximation algorithms. Indeed semidefinite programs are widely used as convex tractable relaxations to hard combinatorial problems. Then the rank one solutions typically correspond to the desired optimal solutions of the discrete problem and low rank solutions can sometimes lead to improved performance guarantees (see, e.g., the result of [4] for max-cut and the result of [10] for maximum stable sets).

As an illustration, consider the max-cut problem for graph G and its standard semidefinite programming relaxation:

$$\max \frac{1}{4}\langle L_G, X\rangle \text{ subject to } X_{ii} = 1 \ (i = 1, \ldots, n), \quad X \succeq 0, \qquad (2)$$

where L_G denotes the Laplacian matrix of G. Clearly, G is the aggregated sparsity pattern of the program (2). In particular, our main Theorem implies that if G is K_5 and $K_{2,2,2}$ minor free, then (2) has an optimal solution of rank at most four. (On the other hand recall that the max-cut problem can be solved in polynomial time for K_5 minor free graphs [5]).

In a similar flavor, for a graph $G = ([n], E)$ with weights $w \in \mathbb{R}_+^{V \cup E}$, the authors of [17] study semidefinite programs of the form

$$\max \sum_{i=1}^{n} w_i X_{ii} \text{ s.t. } \sum_{i,j=1}^{n} w_i w_j X_{ij} = 0, \ X_{ii} + X_{ij} - 2X_{ij} \leq w_{ij} \ (ij \in E), \ X \succeq 0,$$

and show the existence of an optimal solution of rank at most the tree-width of G plus 1. There is a large literature on dimensionality questions for various geometric representations of graphs. We refer, e.g., to [15,16,19,27,29] for results and further references. We will point out links to the parameter $\nu^=(G)$ of [20,21] in Section 2.4.

Yet another, more geometrical, way of interpreting the Gram dimension is in terms of isometric embeddings in the spherical metric space [12]. For this, consider the unit sphere $\mathbf{S}^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}$, equipped with the distance

$$d_{\mathbf{S}}(x, y) = \arccos(x^T y) \text{ for } x, y \in \mathbf{S}^{k-1}.$$

Here, $\|x\|$ denotes the usual Euclidean norm. Then $(\mathbf{S}^{k-1}, d_{\mathbf{S}})$ is a metric space, known as the *spherical metric space.* A graph $G = ([n], E)$ has Gram dimension at most k if and only if, for any assignment of vectors $p_1, \ldots, p_n \in \mathbf{S}^h$ (for some $h \geq 1$), there exists another assignment $q_1, \ldots, q_n \in \mathbf{S}^{k-1}$ such that

$$d_{\mathbf{S}}(p_i, p_j) = d_{\mathbf{S}}(q_i, q_j), \text{ for } ij \in E.$$

In other words, this is the question of deciding whether a partial matrix can be realized in the $(k-1)$-dimensional spherical space. The analogous question for the Euclidean metric space $(\mathbb{R}^k, \|\cdot\|)$ has been extensively studied. In Section 2.3 we will establish close connections with the notion of k-realizability of graphs introduced in [8,9] and to the corresponding graph parameter $\text{ed}(G)$.

Complexity issues concerning the parameter $\text{gd}(G, x)$ are discussed in [14]. Specifically, given a graph G and a rational vector in $\mathcal{E}(G)$, the problem of deciding whether $\text{gd}(G, x) \leq k$ is proven to be NP-hard for every fixed $k \geq 2$ [14].

Contents of the paper. In Section 2.1 we determine basic properties of the graph parameter $\text{gd}(G)$ and in Section 2.2 we reduce the proof of our main Theorem to the problem of computing the Gram dimension of the two graphs V_8 and $C_5 \times C_2$. In Sections 2.3 and 2.4 we investigate the links of $\text{gd}(G)$ with the graph parameters $\text{ed}(G)$ and $\nu^=(G)$, respectively. Section 3 introduces the main ingredients for our proof: In Section 3.1 we discuss some genericity assumptions we can make, in Section 3.2 we show how to use semidefinite programming, in Section 3.3 we establish a number of useful lemmas, and in Section 3.4 we show that $\text{gd}(V_8) = 4$. Section 4 is dedicated to proving that $\text{gd}(C_5 \times C_2) = 4$ – this is the most technical part of the paper. Lastly, in Section 5 we conclude with some comments and open problems.

Note. Part of this work will appear as an extended abstract in the proceedings of ISCO 2012 [26].

2 Preliminaries

2.1 Basic definitions and properties

For a graph $G = (V = [n], E)$ let $\mathcal{S}_+(G) = \pi_{VE}(\mathcal{S}_+^n) \subseteq \mathbb{R}^{V \cup E}$ denote the projection of the positive semidefinite cone onto $\mathbb{R}^{V \cup E}$, whose elements can be seen as the G-partial matrices that can be completed to a psd matrix. Let $\mathcal{E}_n$ denote the set of matrices in $\mathcal{S}_+^n$ with an all-ones diagonal (aka the correlation matrices), and let $\mathcal{E}(G) = \pi_E(\mathcal{E}_n) \subseteq \mathbb{R}^E$ denote its projection onto the edge subspace $\mathbb{R}^E$, known as the *elliptope* of G; we only project on the edge set since all diagonal entries are implicitly known and equal to 1 for matrices in $\mathcal{E}_n$.

Definition 2. *Given a graph $G = (V, E)$ and a vector $a \in \mathbb{R}^{V \cup E}$, a Gram representation of a in $\mathbb{R}^k$ consists of a set of vectors $p_1, \dots, p_n \in \mathbb{R}^k$ such that*

$$p_i^T p_j = a_{ij} \ \forall ij \in V \cup E.$$

The Gram dimension of $a \in \mathcal{S}_+(G)$, denoted as $\mathrm{gd}(G, a)$, is the smallest integer k for which a has a Gram representation in $\mathbb{R}^k$.

Definition 3. *The Gram dimension of a graph $G = (V, E)$ is defined as*

$$\mathrm{gd}(G) = \max_{a \in \mathcal{S}_+(G)} \mathrm{gd}(G, a). \tag{3}$$

Clearly, the maximization in (3) can be restricted to be taken over all vectors $a \in \mathcal{E}(G)$ (where all diagonal entries are implicitly taken to be equal to 1). We denote by $\mathcal{G}_k$ the class of graphs G for which $\mathrm{gd}(G) \le k$.

As a warm-up example, $\mathrm{gd}(K_n) = n$: The upper bound is clear as $|V(K_n)| = n$ and the lower bound follows by considering, e.g., $a = \pi_{V \cup E}(I_n)$.

We now investigate the behavior of the graph parameter $\mathrm{gd}(G)$ under some simple graph operations. Recall that $G \backslash e$ (resp., G/e) denotes the graph obtained from G by deleting (resp., contracting) the edge e. A graph H is a minor of G (denoted as $H \preceq G$) if H can be obtained from G by successively deleting and contracting edges and deleting nodes.

Lemma 1. *The graph parameter $\mathrm{gd}(G)$ is monotone nonincreasing with respect to edge deletion and contraction: $\mathrm{gd}(G \backslash e), \mathrm{gd}(G/e) \le \mathrm{gd}(G)$ for any edge $e \in E$.*

Proof. Let $G = ([n], E)$ and $e \in E$. It follows directly from the definition that $\mathrm{gd}(G \backslash e) \le \mathrm{gd}(G)$. We show that $\mathrm{gd}(G/e) \le \mathrm{gd}(G)$. Say e is the edge $(1, n)$ and $G/e = ([n-1], E')$. Consider $X \in \mathcal{S}_+^{n-1}$; we show that there exists $X' \in \mathcal{S}_+^{n-1}$ with rank at most $k = \mathrm{gd}(G)$ and such that $\pi_{E'}(X) = \pi_{E'}(X')$. For this, extend X to the matrix $Y \in \mathcal{S}_+^n$ defined by $Y_{nn} = X_{11}$ and $Y_{in} = X_{1i}$ for $i \in [n-1]$. By assumption, there exists $Y' \in \mathcal{S}_+^n$ with rank at most k such that $\pi_E(Y) = \pi_E(Y')$. Hence $Y'_{1i} = Y'_{ni}$ for all $i \in [n]$, so that the principal submatrix X' of Y' indexed by $[n-1]$ has rank at most k and satisfies $\pi_{E'}(X') = \pi_{E'}(X)$. $\square$

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs, where $V_1 \cap V_2$ is a clique in both G_1 and G_2. Their *clique sum* is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$, also called their *clique k-sum* when $|V_1 \cap V_2| = k$. The following result follows from well known arguments (cf. e.g. [18]; a proof is included for completeness). For a matrix X indexed by V and a subset $U \subseteq V$, $X[U]$ denotes the principal submatrix of X indexed by U.

Lemma 2. *If G is the clique sum of two graphs G_1 and G_2, then*

$$\mathrm{gd}(G) = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2)\}.$$

Proof. The proof relies on the following fact: Two psd matrices X_i indexed by V_i ($i = 1, 2$) such that $X_1[V_1 \cap V_2] = X_2[V_1 \cap V_2]$ admit a common psd completion X indexed by $V_1 \cup V_2$ with rank $\max\{\dim(X_1), \dim(X_2)\}$. Indeed, let $u_j^{(i)}$ ($j \in V_i$) be a Gram representation of X_i and let U an orthogonal matrix mapping $u_j^{(1)}$ to $u_j^{(2)}$ for $j \in V_1 \cap V_2$, then the Gram representation of $U u_j^{(1)}$ ($j \in V_1$) together with $u_j^{(2)}$ ($j \in V_2 \setminus V_1$) is such a common completion. $\square$

Recall that the *tree-width* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum integer k for which G is contained (as a subgraph) in a clique sum of copies of K_{k+1}. As a direct application of Lemmas 1 and 2 we obtain the following bound:

Lemma 3. *For any graph G, $\mathrm{gd}(G) \leq \mathrm{tw}(G) + 1$.*

In view of Lemma 1, the class $\mathcal{G}_k$ of graphs with Gram dimension at most k is closed under taking minors. Hence, by the celebrated graph minor theorem of [34], it can be characterized by finitely many minimal forbidden minors.

Clearly, K_n is a minimal forbidden minor for $\mathcal{G}_{n-1}$ for all n, since contracting an edge yields a graph with $n-1$ nodes and deleting an edge yields a graph with tree-width at most $n - 2$.

It follows by its definition that the tree-width of a graph is a minor-monotone graph parameter. One can easily verify that $\mathrm{tw}(G) \leq 1 \iff K_3 \npreceq G$ and it is known that $\mathrm{tw}(G) \leq 2 \iff K_4 \npreceq G$ [13]. Combining these two facts with Lemma 3 yields the full list of forbidden minors for the class $\mathcal{G}_k$ when $k \leq 3$.

Theorem 1. *For $k \leq 3$, $\mathrm{gd}(G) \leq k$ if and only if G has no minor K_{k+1}.*

2.2 Characterizing graphs with Gram dimension at most 4

The next natural question is to characterize the class $\mathcal{G}_4$. Clearly, K_5 is a minimal forbidden minor for $\mathcal{G}_4$. We now show that this is also the case for the complete tripartite graph $K_{2,2,2}$.

Lemma 4. *The graph $K_{2,2,2}$ is a minimal forbidden minor for $\mathcal{G}_4$.*

Proof. First we construct $a \in \mathcal{E}(K_{2,2,2})$ with $\text{gd}(K_{2,2,2}, a) \geq 5$, thus implying $\text{gd}(K_{2,2,2}) \geq 5$. For this, let $K_{2,2,2}$ be obtained from K_6 by deleting the edges $(1,4), (2,5)$ and $(3,6)$. Let $e_1, \ldots, e_5$ denote the standard unit vectors in $\mathbb{R}^5$, let X be the Gram matrix of the vectors e_1, e_2, e_3, e_4, e_5 and $(e_1 + e_2)/\sqrt{2}$ labeling the nodes $1, \ldots, 6$, respectively, and let $a \in \mathcal{E}(K_{2,2,2})$ be the projection of X. We now verify that X is the unique psd completion of a which shows that $\text{gd}(K_{2,2,2}, a) \geq 5$. Indeed the chosen Gram labeling of the matrix X implies the following linear dependency: $X[\cdot, 6] = (X[\cdot, 4] + X[\cdot, 5])/\sqrt{2}$ among its columns $X[\cdot, i]$ indexed respectively by $i = 4, 5, 6$; this implies that the unspecified entries X_{14}, X_{25}, X_{36} are uniquely determined in terms of the specified entries of X.

On the other hand, one can easily verify that $K_{2,2,2}$ is a partial 4-tree, therefore $\text{gd}(K_{2,2,2}) \leq 5$. Moreover, deleting or contracting an edge in $K_{2,2,2}$ yields a partial 3-tree, thus with Gram dimension at most 4. □

By Lemma 3, all graphs with tree-width at most three belong to $\mathcal{G}_4$. Moreover, these graphs can be characterized in terms of forbidden minors as follows:

Theorem 2. *[3] A graph G has $\text{tw}(G) \leq 3$ if and only if G does not have $K_5, K_{2,2,2}, V_8$ and $C_5 \times C_2$ as a minor.*

The graphs V_8 and $C_5 \times C_2$ are shown in Figures 1 and 2 below, respectively. These four graphs are natural candidates for being forbidden minors for the class $\mathcal{G}_4$. We have already seen that for K_5 and $K_{2,2,2}$ this is indeed the case. However, this is not true for V_8 and $C_5 \times C_2$. Both belong to $\mathcal{G}_4$, this will be proved in Section 3.4 for V_8 (Theorem 10) and in Section 4 for $C_5 \times C_2$ (Theorem 11). These two results form the main technical part of the paper. Using them, we can complete our characterization of the class $\mathcal{G}_4$.

Theorem 3. *For a graph G, $\text{gd}(G) \leq 4$ if and only if G does not have K_5 or $K_{2,2,2}$ as a minor.*

Proof. Necessity follows from Lemmas 1 and 4. Sufficiency follows from the following graph theoretical result, obtained by combining Theorem 2 with Seymour's splitter theorem (for a self-contained proof see [20]): every graph with no K_5 and $K_{2,2,2}$ minors can be obtained as a subgraph of a clique k-sum ($k \leq 2$) of copies of graphs with tree-width at most 3, V_8 and $C_5 \times C_2$. Combining this fact with Theorems 10, 11 and Lemmas 2, 3 the claim follows. □

2.3 Links to Euclidean graph realizations

In this section we investigate the links between the Gram dimension and the notion of k-realizability of graphs introduced in [8,9]. We start the discussion with some necessary definitions.

Recall that a matrix $D = (d_{ij}) \in \mathcal{S}^n$ is a *Euclidean distance matrix* (EDM) if there exist vectors $p_1, \ldots, p_n \in \mathbb{R}^k$ (for some $k \geq 1$) such that $d_{ij} = \|p_i - p_j\|^2$ for all $i, j \in [n]$. Then EDM_n denotes the cone of all $n \times n$ Euclidean distance matrices and, for a graph $G = ([n], E)$, $\text{EDM}(G) = \pi_E(\text{EDM}_n)$ is the set of G-partial matrices that can be completed to a Euclidean distance matrix.

Definition 4. *Given a graph $G = ([n], E)$ and $d \in \mathbb{R}_+^E$, a Euclidean (distance) representation of d in $\mathbb{R}^k$ consists of a set of vectors $p_1, \ldots, p_n \in \mathbb{R}^k$ such that*

$$\|p_i - p_j\|^2 = d_{ij} \ \forall ij \in E.$$

Then, $\mathrm{ed}(G, d)$ is the smallest integer k for which d has a Euclidean representation in $\mathbb{R}^k$ and the graph parameter $\mathrm{ed}(G)$ is defined as

$$\mathrm{ed}(G) = \max_{d \in \mathrm{EDM}(G)} \mathrm{ed}(G, d). \tag{4}$$

In the terminology of [8,9] a graph G satisfying $\mathrm{ed}(G) \leq k$ is called k-*realizable*. It is easy to verify that the graph parameter $\mathrm{ed}(G)$ is minor monotone. Hence for any fixed $k \geq 1$ the class of graphs satisfying $\mathrm{ed}(G) \leq k$ can be characterized by a finite list of minimal forbidden minors. For $k \leq 2$ the only forbidden minor is K_{k+2}. Belk and Connelly [8,9] have determined the list of forbidden minors for $k = 3$.

Theorem 4. *[8,9] For a graph G, $\mathrm{ed}(G) \leq 3$ if and only if G does not have K_5 and $K_{2,2,2}$ as minors.*

The hard part of the proof of [8,9] is to prove sufficiency, i.e., that if a graph G has no K_5 and $K_{2,2,2}$ minors then $\mathrm{ed}(G) \leq 3$. We will obtain this result as a corollary of our main theorem (cf. Corollary 1). To this end, we have to establish some connections between the graphs parameters $\mathrm{ed}(G)$ and $\mathrm{gd}(G)$.

There is a well known correspondence between psd and EDM completions (for details and references see, e.g., [12]). Namely, for a graph G, let ∇G denote its *suspension graph*, obtained by adding a new node (the *apex* node, denoted by 0), adjacent to all nodes of G. Consider the one-to-one map $\phi : \mathbb{R}^{V \cup E(G)} \mapsto \mathbb{R}_+^{E(\nabla G)}$, which maps $x \in \mathbb{R}^{V \cup E(G)}$ to $d = \phi(x) \in \mathbb{R}_+^{E(\nabla G)}$ defined by

$$d_{0i} = x_{ii} \ (i \in [n]), \quad d_{ij} = x_{ii} + x_{jj} - 2x_{ij} \ (ij \in E(G)).$$

Then the vectors $u_1, \ldots, u_n \in \mathbb{R}^k$ form a Gram representation of x if and only if the vectors $u_0 = 0, u_1, \ldots, u_n$ form a Euclidean representation of $d = \phi(x)$ in $\mathbb{R}^k$. This shows:

Lemma 5. *Let $G = (V, E)$ be a graph. Then, $\mathrm{gd}(G, x) = \mathrm{ed}(\nabla G, \phi(x))$ for any $x \in \mathbb{R}^{V \cup E}$ and thus $\mathrm{gd}(G) = \mathrm{ed}(\nabla G)$.*

For the Gram dimension of a graph one can show the following property:

Lemma 6. *Consider a graph $G = (V = [n], E)$ and its suspension graph $\nabla G = ([n] \cup \{0\}, E \cup F)$, where $F = \{(0, i) \mid i \in [n]\}$. Given $x \in \mathbb{R}^E$, its 0-extension is the vector $y = (x, 0) \in \mathbb{R}^{E \cup F}$. If $x \in \mathcal{S}_+(G)$, then $y \in \mathcal{S}_+(\nabla G)$ and $\mathrm{gd}(\nabla G, y) = \mathrm{gd}(G, x) + 1$. Moreover, $\mathrm{gd}(\nabla G) = \mathrm{gd}(G) + 1$.*

Proof. The first part is clear and implies $\mathrm{gd}(\nabla G) \geq \mathrm{gd}(G) + 1$. Set $k = \mathrm{gd}(G)$; we show the reverse inequality $\mathrm{gd}(\nabla G) \leq k + 1$. For this, let $X \in \mathcal{S}_+^{n+1}$, written in block-form as $X = \begin{pmatrix} \alpha\ a^T \\ a\ A \end{pmatrix}$, where $A \in \mathcal{S}_+^n$ and the first row/column is indexed by the apex node 0 of ∇G. If $\alpha = 0$ then $a = 0$, $\pi_{VE}(A)$ has a Gram representation in $\mathbb{R}^k$ and thus $\pi_{V(\nabla G)E(\nabla G)}(X)$ too. Assume now $\alpha > 0$ and without loss of generality $\alpha = 1$. Consider the Schur complement Y of X with respect to the entry $\alpha = 1$, given by $Y = A - aa^T$. As $Y \in \mathcal{S}_+^n$, there exists $Z \in \mathcal{S}_+^n$ such that $\mathrm{rank}(Z) \leq k$ and $\pi_{VE}(Z) = \pi_{VE}(Y)$. Define the matrix

$$X' := \begin{pmatrix} 1\ a^T \\ a\ aa^T \end{pmatrix} + \begin{pmatrix} 0\ 0 \\ 0\ Z \end{pmatrix}.$$

Then, $\mathrm{rank}(X') = \mathrm{rank}(Z) + 1 \leq k + 1$. Moreover, X' and X coincide at all diagonal entries as well as at all entries corresponding to edges of ∇G. This concludes the proof that $\mathrm{gd}(\nabla G) \leq k + 1$. $\qquad\square$

We do not know whether the analogous property is true for the graph parameter $\mathrm{ed}(G)$. On the other hand, the following partial result holds, whose proof was communicated to us by A. Schrijver.

Theorem 5. *For a graph G, $\mathrm{ed}(\nabla G) \geq \mathrm{ed}(G) + 1$.*

Proof. Set $\mathrm{ed}(\nabla G) = k$; we show $\mathrm{ed}(G) \leq k - 1$. We may assume that G is connected (else deal with each connected component separately). Let $d \in \mathrm{EDM}(G)$ and let $p_1 = 0, p_2, \ldots, p_n$ be a Euclidean representation of d in $\mathbb{R}^h$ ($h \geq 1$). Extend the p_i's to vectors $\widehat{p}_i = (p_i, 0) \in \mathbb{R}^{h+1}$ by appending an extra coordinate equal to zero, and set $\widehat{p}_0(t) = (0, t) \in \mathbb{R}^{h+1}$ where t is any positive real scalar. Now consider the distance $\widehat{d}(t) \in \mathrm{EDM}(\nabla G)$ with Euclidean representation $\widehat{p}_0(t), \widehat{p}_1, \ldots, \widehat{p}_n$.

As $\mathrm{ed}(\nabla G) = k$, there exists another Euclidean representation of $\widehat{d}(t)$ by vectors $q_0(t), q_1(t), \ldots, q_n(t)$ lying in $\mathbb{R}^k$. Without loss of generality, we can assume that $q_0(t) = \widehat{p}_0(t) = (0, t)$ and $q_1(t)$ is the zero vector; for $i \in [n]$, write $q_i(t) = (u_i(t), a_i(t))$, where $u_i(t) \in \mathbb{R}^{k-1}$ and $a_i(t) \in \mathbb{R}$. Then $\|q_i(t)\| = \|\widehat{p}_i\| = \|p_i\|$ whenever node i is adjacent to node 1 in G. As the graph G is connected, this implies that, for any $i \in [n]$, the scalars $\|q_i(t)\|$ ($t \in \mathbb{R}_+$) are bounded. Therefore there exists a sequence $t_m \in \mathbb{R}_+$ ($m \in \mathbb{N}$) converging to $+\infty$ and for which the sequence $(q_i(t_m))_m$ has a limit. Say $q_i(t_m) = (a_i(t_m), u_i(t_m))$ converges to $(u_i, a_i) \in \mathbb{R}^k$ as $m \to +\infty$, where $u_i \in \mathbb{R}^{k-1}$ and $a_i \in \mathbb{R}$. The condition $\|q_0(t) - q_i(t)\|^2 = \widehat{d}(t)_{0i}$ implies that $\|p_i\|^2 + t^2 = \|u_i(t)\|^2 + (a_i(t) - t)^2$ and thus

$$a_i(t_m) = \frac{a_i^2(t_m) + \|u_i(t_m)\|^2 - \|p_i\|^2}{2t_m} \quad \forall m \in \mathbb{N}.$$

Taking the limit as $m \to \infty$ we obtain that $\lim_{m \to \infty} a_i(t_m) = 0$ and thus $a_i = 0$. Then, for $i, j \in [n]$, $d_{ij} = \widehat{d}(t_m)_{ij} = \|(a_i(t_m), u_i(t_m)) - (a_j(t_m), u_j(t_m))\|^2$ and taking the limit as $m \to +\infty$ we obtain that $d_{ij} = \|u_i - u_j\|^2$. This shows that the vectors $u_1, \ldots, u_n$ form a Euclidean representation of d in $\mathbb{R}^{k-1}$. $\qquad\square$

Combining Lemma 5 with Theorem 5 we obtain the following inequality relating the parameters $\mathrm{ed}(G)$ and $\mathrm{gd}(G)$.

Theorem 6. *For any graph G we have that $\mathrm{ed}(G) \leq \mathrm{gd}(G) - 1$.*

Combining Theorem 6 with our main theorem we can recover sufficiency in Theorem 4.

Corollary 1. *For a graph G, if G has no K_5 and $K_{2,2,2}$ minors then $\mathrm{ed}(G) \leq 3$.*

2.4 Relation with the graph parameter $\nu^=(G)$

In this section we investigate the relation between the Gram dimension of a graph and the graph parameter $\nu^=(G)$ introduced in [20,21]. Recall that the *corank* of a matrix $M \in \mathbb{R}^{n \times n}$ is the dimension of its kernel. Consider the cone

$$\mathcal{C}(G) = \{M \in \mathcal{S}_+^n : M_{ij} = 0 \ \text{ for all distinct } i, j \in V \text{ with } (i,j) \notin E\}$$

which, as is well known, can be seen as the dual cone of the cone $\mathcal{S}_+(G)$. We now introduce the graph parameter $\nu^=(G)$.

Definition 5. *Given a graph $G = ([n], E)$ the parameter $\nu^=(G)$ is defined as the maximum corank of a matrix $M \in \mathcal{C}(G)$ satisfying the following property:*

$$\forall X \in \mathcal{S}^n \quad MX = 0, \ X_{ii} = 0 \ \forall i \in V, \ X_{ij} = 0 \ \forall (i,j) \in E \implies X = 0,$$

known as the strong Arnold property.

It is proven in [20,21] that $\nu^=(G)$ is a minor monotone graph parameter. Hence for any fixed integer $k \geq 1$ the class of graphs with $\nu^=(G) \leq k$ can be characterized by a finite family of minimal forbidden minors. For $k \leq 3$ the only forbidden minor is K_{k+1}. Van der Holst [20,21] has determined the list of forbidden minors for $k = 4$.

Theorem 7. *[20,21] For a graph G, $\nu^=(G) \leq 4$ if and only if G does not have K_5 and $K_{2,2,2}$ as minors.*

By relating the two parameters $\mathrm{gd}(G)$ and $\nu^=(G)$ we can derive sufficiency in Theorem 7 from our main Theorem.

Theorem 8. *For any graph G, $\mathrm{gd}(G) \geq \nu^=(G)$.*

Proof. Let $k = \nu^=(G)$ be attained by some matrix $M \in \mathcal{S}_+^n$. Write $M = \sum_{i=1}^n \lambda_i v_i v_i^T$, where $\lambda_i \geq 0$, $\{v_1, \ldots, v_n\}$ is an orthonormal base of eigenvectors of M, and $\{v_1, \ldots, v_k\}$ spans the kernel of M. Consider the matrix $X = \sum_{i=1}^k v_i v_i^T$ and its projection $a = \pi_{E \cup V}(X) \in \mathcal{S}_+(G)$. By construction, $\mathrm{rank}(X) = k$. Hence it is enough to show that a has a unique psd completion, which will imply $\mathrm{gd}(G) \geq \mathrm{gd}(G, a) = k$.

For this let $Y \in \mathcal{S}_+^n$ be another psd completion of a. Hence the matrix $X - Y$ has zero entries at all positions $(i, j) \in V \cup E$. Since the matrix M has zero entries at all off-diagonal positions corresponding to non-edges of G, we deduce that $\langle M, X - Y \rangle = 0$. On the other hand, $\langle M, X \rangle = \sum_{i=1}^k \lambda_i v_i^T M v_i = 0$. Therefore, $\langle M, Y \rangle = 0$. As M, X, Y are psd, the conditions $\langle M, X \rangle = \langle M, Y \rangle = 0$ imply that $MX = MY = 0$ and thus $M(X - Y) = 0$. Now we can apply the assumption that the matrix M satisfies the strong Arnold property and deduce that $X = Y$. □

Combining Theorem 8 with our main theorem we can recover sufficiency in Theorem 7.

Corollary 2. *For a graph G, if G does not have K_5 and $K_{2,2,2}$ as minors then $\nu^=(G) \leq 4$.*

Colin de Verdière [11] studies the graph parameter $\nu(G)$, defined as the maximum corank of a matrix M satisfying the strong Arnold property and such that, for any $i, j \in V$, $M_{ij} = 0 \iff (i, j) \notin E$. In particular he shows that $\nu(G)$ is unbounded for the class of planar graphs. As $\nu(G) \leq \nu^=(G) \leq \mathrm{gd}(G)$, we obtain as a direct application:

Corollary 3. *The graph parameter $\mathrm{gd}(G)$ is unbounded for the class of planar graphs.*

3 Bounding the Gram dimension

In this section we sketch our approach to show that $\mathrm{gd}(V_8) = \mathrm{gd}(C_5 \times C_2) = 4$.

Definition 6. *Given a graph $G = (V = [n], E)$, a configuration of G is an assignment of vectors $p_1, \ldots, p_n$ (in some space) to the nodes of G; the pair $(G, \mathbf{p})$ is called a framework. We use the notation $\mathbf{p} = \{p_1, \ldots, p_n\}$ and, for a subset $T \subseteq V$, $\mathbf{p}_T = \{p_i \mid i \in T\}$. Thus $\mathbf{p} = \mathbf{p}_V$ and we also set $\mathbf{p}_{-i} = \mathbf{p}_{V \setminus \{i\}}$.*
Two configurations $\mathbf{p}, \mathbf{q}$ of G (not necessarily lying in the same space) are said to be equivalent if $p_i^T p_j = q_i^T q_j$ for all $ij \in V \cup E$.

Our objective is to show that the two graphs $G = V_8$, $C_5 \times C_2$ belong to $\mathcal{G}_4$. That is, we must show that, given any $a \in \mathcal{S}_+(G)$, one can construct a Gram representation $\mathbf{q}$ of (G, a) lying in the space $\mathbb{R}^4$.

Along the lines of [8] (which deals with Euclidean distance realizations), our strategy to achieve this is as follows: First, we construct a 'flat' Gram representation $\mathbf{p}$ of (G, a) obtained by maximizing the inner product $p_{i_0}^T p_{j_0}$ along a given pair (i_0, j_0) which is not an edge of G. As suggested in [31] (in the context of Euclidean distance realizations), this configuration $\mathbf{p}$ can be obtained by solving a semidefinite program; then $\mathbf{p}$ corresponds to the Gram representation of an optimal solution X to this program.

In general we cannot yet claim that $\mathbf{p}$ lies in $\mathbb{R}^4$. However, we can derive useful information about $\mathbf{p}$ by using an optimal solution Ω (which will correspond

to a 'stress matrix') to the dual semidefinite program. Indeed, the optimality condition $X\Omega = 0$ will imply some linear dependencies among the p_i's that can be used to show the existence of an equivalent representation $\mathbf{q}$ of (G, a) in low dimension. Roughly speaking, most often, these dependencies will force the majority of the p_i's to lie in $\mathbb{R}^4$, and one will be able to rotate each remaining vector p_j about the space spanned by the vectors labeling the neighbors of j into $\mathbb{R}^4$. Showing that the initial representation $\mathbf{p}$ can indeed be 'folded' into $\mathbb{R}^4$ as just described makes up the main body of the proof.

Before going into the details of the proof, we indicate some additional genericity assumptions that can be made w.l.o.g. on the vector $a \in \mathcal{S}_+(G)$. This will be particularly useful when treating the graph $C_5 \times C_2$.

3.1 Genericity assumptions

By definition, $\mathrm{gd}(G)$ is the maximum value of $\mathrm{gd}(G, a)$ taken over all $a \in \mathcal{E}(G)$. Clearly we can restrict the maximum to be taken over all a lying in a dense subset of $\mathcal{E}(G)$. For instance, the set $\mathcal{D}$ consisting of all $x \in \mathcal{E}(G)$ that admit a positive definite completion in $\mathcal{E}_n$ is dense in $\mathcal{E}(G)$. We next identify a smaller dense subset $\mathcal{D}^*$ of $\mathcal{D}$ which will we use in our study of the Gram dimension of $C_5 \times C_2$.

We start with a useful lemma, which characterizes the vectors $a \in \mathcal{E}(C_n)$ admitting a Gram realization in $\mathbb{R}^2$. Here C_n denotes the cycle on n nodes.

Lemma 7. *Consider the vector $a = (\cos\vartheta_1, \cos\vartheta_2, \ldots, \cos\vartheta_n) \in \mathbb{R}^{E(C_n)}$, where $\vartheta_1, \ldots, \vartheta_n \in [0, \pi]$. Then $\mathrm{gd}(C_n, a) \leq 2$ if and only if there exist $\epsilon \in \{\pm 1\}^n$ and $k \in \mathbb{Z}$ such that $\sum_{i=1}^n \epsilon_i \vartheta_i = 2k\pi$.*

Proof. We prove the 'only if' part. Assume that $u_1, \ldots, u_n \in \mathbb{R}^2$ are unit vectors such that $u_i^T u_{i+1} = \cos\vartheta_i$ for all $i \in [n]$ (setting $u_{n+1} = u_1$). We may assume that $u_1 = (1, 0)^T$. Then, $u_1^T u_2 = \cos\vartheta_1$ implies that $u_2 = (\cos(\epsilon_1\vartheta_1), \sin(\epsilon_1\vartheta_1))^T$ for some $\epsilon_1 \in \{\pm 1\}$. Analogously, $u_2^T u_3 = \cos\vartheta_2$ implies $u_3 = (\cos(\epsilon_1\vartheta_1 + \epsilon_2\vartheta_2), \sin(\epsilon_1\vartheta_1 + \epsilon_2\vartheta_2))^T$ for some $\epsilon_2 \in \{\pm 1\}$. Iterating, we find that there exists $\epsilon \in \{\pm 1\}^n$ such that $u_i = (\cos(\sum_{j=1}^{i-1} \epsilon_i\vartheta_i), \sin(\sum_{j=1}^{i-1} \epsilon_i\vartheta_i))^T$ for $i = 1, \ldots, n$. Finally, the condition $u_n^T u_1 = \cos\vartheta_n = \cos(\sum_{i=1}^{n-1} \epsilon_i\vartheta_i)$ implies $\sum_{i=1}^n \epsilon_i\vartheta_i \in 2\pi\mathbb{Z}$. The arguments can be reversed to show the 'if part'. $\square$

Lemma 8. *Let $\mathcal{D}^*$ be the set of all $a \in \mathcal{E}(G)$ that admit a positive definite completion in $\mathcal{E}_n$ satisfying the following condition: For any circuit C in G, the restriction $a_C = (a_e)_{e \in C}$ of a to C does not admit a Gram representation in $\mathbb{R}^2$. Then the set $\mathcal{D}^*$ is dense in $\mathcal{E}(G)$.*

Proof. We show that $\mathcal{D}^*$ is dense in $\mathcal{D}$. Let $a \in \mathcal{D}$ and set $a = \cos\vartheta$, where $\vartheta \in [0, \pi]^E$. Given a circuit C in G (say of length p), it follows from Lemma 7 that a_C has a Gram realization in $\mathbb{R}^2$ if and only if $\sum_{i=1}^p \epsilon_i\vartheta_i = 2k\pi$ for some $\epsilon \in \{\pm 1\}^p$ and $k \in \mathbb{Z}$ with $|k| \leq p/2$. Let $\mathcal{H}_C$ denote the union of the hyperplanes in $\mathbb{R}^{E(C)}$ defined by these equations. Therefore, $a \notin \mathcal{D}^*$ if and only if $\vartheta \in \cup_C \mathcal{H}_C$, where the union is taken over all circuits C of G. Clearly we can find a sequence

$\vartheta^{(i)} \in [0, \pi]^E \setminus \cup_C \mathcal{H}_C$ converging to ϑ as $i \to \infty$. Then the sequence $a^{(i)} :=$ $\cos \vartheta^{(i)}$ tends to a as $i \to \infty$ and, for all i large enough, $a^{(i)} \in \mathcal{D}^*$. This shows that $\mathcal{D}^*$ is a dense subset of $\mathcal{D}$ and thus of $\mathcal{E}(G)$. □

Corollary 4. *For any graph* $G = ([n], E)$, $\mathrm{gd}(G) = \max \mathrm{gd}(G, a)$, *where the maximum is over all* $a \in \mathcal{E}(G)$ *admitting a positive definite completion and whose restriction to any circuit of* G *has no Gram representation in the plane.*

3.2 Semidefinite programming formulation

We now describe how to model the 'flattening' procedure using semidefinite programming (sdp) and how to obtain a 'stress matrix' using sdp duality.

Let $G = (V = [n], E)$ be a graph and let $e_0 = (i_0, j_0)$ be a non-edge of G (i.e., $i_0 \neq j_0$ and $e_0 \notin E$). Let $a \in \mathcal{S}_+(G)$ be a partial positive semidefinite matrix for which we want to show the existence of a Gram representation in a small dimensional space. For this consider the semidefinite program:

$$\max \langle E_{i_0 j_0}, X \rangle \ \text{ s.t. } \langle E_{ij}, X \rangle = a_{ij} \ (ij \in V \cup E), \ \ X \succeq 0, \tag{5}$$

where $E_{ij} = (e_i e_j^T + e_j e_i^T)/2$ and $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$. The dual semidefinite program of (5) reads:

$$\min \sum_{ij \in V \cup E} w_{ij} a_{ij} \ \text{ s.t. } \ \Omega = \sum_{ij \in V \cup E} w_{ij} E_{ij} - E_{i_0 j_0} \succeq 0. \tag{6}$$

Theorem 9. *Consider a graph* $G = ([n], E)$, *a pair* $e_0 = (i_0, j_0) \notin E$, *and let* $a \in \mathcal{S}_{++}(G)$. *Then there exists a Gram realization* $\mathbf{p} = (p_1, \ldots, p_n)$ *in* $\mathbb{R}^k$ *(for some* $k \geq 1$*) of* (G, a) *and a matrix* $\Omega = (w_{ij}) \in \mathcal{S}_+^n$ *satisfying*

$$w_{i_0 j_0} \neq 0, \tag{7}$$

$$w_{ij} = 0 \ \text{ for all } ij \notin V \cup E \cup \{e_0\}, \tag{8}$$

$$w_{ii} p_i + \sum_{j \mid ij \in E \cup \{e_0\}} w_{ij} p_j = 0 \ \text{for all } i \in [n], \tag{9}$$

$$\dim \langle p_i, p_j \rangle = 2 \ \text{ for all } ij \in E. \tag{10}$$

We refer to equation (9) as the equilibrium condition *at vertex* i.

Proof. Consider the sdp (5) and its dual program (6). By assumption, a has a positive definite completion, hence the program (5) is strictly feasible. Clearly, the dual program (6) is also strictly feasible. Hence there is no duality gap and the optimal values are attained in both programs. Let (X, Ω) be a pair of primal-dual optimal solutions. Then (X, Ω) satisfies the optimality condition: $\langle X, \Omega \rangle = 0$ or, equivalently, $X\Omega = 0$. Say X has rank k and let $\mathbf{p} = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^k$ be a Gram realization of X. Now it suffices to observe that the condition $X\Omega = 0$ can be reformulated as the equilibrium conditions (9). The conditions (7) and (8) follow from the form of the dual program (6), and (10) follows from the assumption $a \in \mathcal{S}_{++}(G)$. □

Note that, using the following variant of Farkas' lemma for semidefinite programming, one can show the existence of a nonzero positive semidefinite matrix $\Omega = (w_{ij})$ satisfying (8) and the equilibrium conditions (9) also in the case when the sdp (5) is not strictly feasible, however now with $w_{i_0 j_0} = 0$. This remark will be useful in the exceptional case considered in Section 4.5 where we will have to solve again a semidefinite program of the form (5); however this program will have additional conditions imposing that some of the p_i's are pinned so that one cannot anymore assume strict feasibility (see the proof of Lemma 20).

Lemma 9. (Farkas' lemma for semidefinite programming) *(see [28]) Let $b \in \mathbb{R}^m$ and let $A_1, \ldots, A_m \in \mathcal{S}^n$ be given. Then exactly one of the following two assertions holds:*

(i) *Either there exists $X \in \mathcal{S}^n_{++}$ such that $\langle A_j, X \rangle = b_j$ for $j = 1, \ldots, m$.*
(ii) *Or there exists a vector $y \in \mathbb{R}^m$ such that $\Omega := \sum_{j=1}^m y_j A_j \succeq 0$, $\Omega \neq 0$ and $b^T y \leq 0$.*

Moreover, for any $X \succeq 0$ satisfying $\langle A_j, X \rangle = b_j$ $(j = 1, \ldots, m)$, we have in (ii) $\langle X, \Omega \rangle = b^T y = 0$ and thus $X\Omega = 0$.

Proof. Clearly, if (i) holds then (ii) does not hold. Conversely, assume (i) does not hold, i.e., $\mathcal{S}^n_{++} \cap \mathcal{L} = \emptyset$, where $\mathcal{L} = \{X \in \mathcal{S}^n \mid \langle A_j, X \rangle = b_j \ \forall j\}$. Then there exists a separating hyperplane, i.e., there exists a nonzero matrix $\Omega \in \mathcal{S}^n$ and $\alpha \in \mathbb{R}$ such that $\langle \Omega, X \rangle \geq \alpha$ for all $X \in \mathcal{S}^n_{++}$ and $\langle \Omega, X \rangle \leq \alpha$ for all $X \in \mathcal{L}$. This implies $\Omega \succeq 0$, $\Omega \in \mathcal{L}^\perp$, and $\alpha \leq 0$, so that (ii) holds and the lemma follows. □

3.3 Useful lemmas

We start with some definitions about stressed frameworks and then we establish some basic tools that we will repeatedly use later in our proof for V_8 and $C_5 \times C_2$. For a matrix $\Omega \in \mathcal{S}^n$ its *support graph* is the graph $\mathcal{S}(\Omega)$ is the graph with node set $[n]$ and with edges the pairs (i, j) with $\Omega_{ij} \neq 0$.

Definition 7. (Stressed framework $(H, \mathbf{p}, \Omega)$) *Consider a framework $(H = (V = [n], F), \mathbf{p})$. A nonzero matrix $\Omega = (w_{ij}) \in \mathcal{S}^n$ is called a* stress matrix *for the framework $(H, \mathbf{p})$ if its support graph $\mathcal{S}(\Omega)$ is contained in H (i.e., $w_{ij} = 0$ for all $ij \notin V \cup F$) and Ω satisfies the equilibrium condition*

$$w_{ii}p_i + \sum_{j:(i,j)\in F} w_{ij}p_j = 0 \quad \forall i \in V. \tag{11}$$

Then the triple $(H, \mathbf{p}, \Omega)$ is called a stressed framework*, and a* psd stressed framework *if moreover $\Omega \succeq 0$.*

We let V_Ω denote the set of nodes $i \in V$ for which $\Omega_{ij} \neq 0$ for some $j \in V$. A node $i \in V$ is said to be a 0-node when $w_{ij} = 0$ for all $j \in V$. Hence, $V \setminus V_\Omega$ is the set of all 0-nodes and, when $\Omega \succeq 0$, i is a 0-node if and only if $w_{ii} = 0$.

The support graph $\mathcal{S}(\Omega)$ of Ω is called the stressed graph*; its edges are called the* stressed edges *of H and the nodes $i \in V_\Omega$ are called the* stressed nodes*.*

Given an integer $t \geq 1$, a node $i \in V$ is said to be a t-node *if its degree in the stressed graph $\mathcal{S}(\Omega)$ is equal to t.*

Throughout we will deal with stressed frameworks $(H, \mathbf{p}, \Omega)$ obtained by applying Theorem 9. Hence the graph H arises by adding a new edge e_0 to a given graph G, which we then denote as $H = \widehat{G}$, as indicated below.

Definition 8. (Extended graph $\widehat{G}$) *Given a graph $G = (V = [n], E)$ and a fixed pair $e_0 = (i_0, j_0)$ not belonging to E, we set $\widehat{G} = (V, \widehat{E} = E \cup \{e_0\})$.*

We now group some useful lemmas which we will use in order to show that a given framework $(H, \mathbf{p})$ admits an equivalent configuration in lower dimension.

Clearly, the stress matrix provides some linear dependencies among the vectors p_i labeling the stressed nodes, but it gives no information about the vectors labeling the 0-nodes. However, if we have a set S of 0-nodes forming a stable set, then we can use the following lemma in order to 'fold' the corresponding vectors p_i ($i \in S$) in a lower dimensional space.

Lemma 10. (Folding the vectors labeling a stable set) *Let $(H = (V, F), \mathbf{p})$ be a framework and let $T \subseteq V$. Assume that $S = V \setminus T$ is a stable set in H, that each node $i \in S$ has degree at most $k - 1$ in H, and that $\dim\langle \mathbf{p}_T \rangle \leq k$. Then there exists a configuration $\mathbf{q}$ of H in $\mathbb{R}^k$ which is equivalent to $(H, \mathbf{p})$.*

Proof. Fix a node $i \in S$. Let $N[i]$ denote the closed neighborhood of i in H consisting of i and the nodes adjacent to i. By assumption, $|N[i]| \leq k$ and both sets of vectors $\mathbf{p}_T$ and $\mathbf{p}_{N[i]}$ have rank at most k. Hence one can find an orthogonal matrix P mapping all vectors p_j ($j \in T \cup N[i]$) into the space $\mathbb{R}^k$. Repeat this construction with every other node of S. As no two nodes of S are adjacent, this produces a configuration $\mathbf{q}$ in $\mathbb{R}^k$ which is equivalent to $(H, \mathbf{p})$. $\square$

The next lemma uses the stress matrix to upper bound the dimension of a given stressed configuration.

Lemma 11. (Bounding the dimension) *Let $(H = (V = [n], F), \mathbf{p}, \Omega)$ be a psd stressed framework. Then $\dim\langle \mathbf{p}_V \rangle \leq n - 2$, except $\dim\langle \mathbf{p}_V \rangle \leq n - 1$ if $\mathcal{S}(\Omega)$ is a clique.*

Proof. Let X denote the Gram matrix of the p_i's, so that $\text{rank}(X) = \dim\langle \mathbf{p}_V \rangle$. By assumption, $X\Omega = 0$. This implies that $\text{rank}(X) \leq n - 1$. Moreover, if $\mathcal{S}(\Omega)$ is not a clique, then $\text{rank}(\Omega) \geq 2$ and thus $\text{rank}(X) \leq n - 2$. $\square$

The next lemma indicates how 1-nodes can occur in a stressed framework.

Lemma 12. *Let $(H = (V, F), \mathbf{p}, \Omega)$ be a stressed framework. If node i is a 1-node in the stressed graph $\mathcal{S}(\Omega)$, i.e., there is a unique edge $ij \in F$ such that $w_{ij} \neq 0$, then $\dim\langle p_i, p_j \rangle \leq 1$.*

Proof. Directly, using the equilibrium condition (11) at node i. $\square$

We now consider 2-nodes in a stressed framework. First recall the notion of Schur complement. For a matrix $\Omega = (w_{ij}) \in \mathcal{S}^n$ and $i \in [n]$ with $w_{ii} \neq 0$, the *Schur complement* of Ω with respect to its (i,i)-entry is the matrix, denoted as $\Omega_{-i} = (w'_{jk})_{j,k \in [n] \setminus \{i\}} \in \mathcal{S}^{n-1}$, with entries $w'_{jk} = w_{jk} - w_{ik}w_{jk}/w_{ii}$ for $i, j \in [n] \setminus \{i\}$. As is well known, $\Omega \succeq 0$ if and only if $w_{ii} > 0$ and $\Omega_{-i} \succeq 0$. We also need the following notion of 'contracting a degree two node' in a graph.

Definition 9. *Let $H = (V, F)$ be a graph, let $i \in V$ be a node of degree two in H which is adjacent to nodes $i_1, i_2 \in V$. The graph obtained by* contracting *node i in H is the graph H/i with node set $V \setminus \{i\}$ and with edge set $F/i = F \setminus \{(i, i_1), (i, i_2)\} \cup \{(i_1, i_2)\}$ (ignoring multiple edges).*

Lemma 13. (Contracting a 2-node) *Let $(H = (V, F), \mathbf{p}, \Omega)$ be a psd stressed framework, let $i \in V$ be a 2-node in the stressed graph $\mathcal{S}(\Omega)$ and set $N(i) = \{i_1, i_2\}$. Then $p_i \in \langle p_{i_1}, p_{i_2} \rangle$ and thus $\dim\langle \mathbf{p} \rangle = \dim\langle \mathbf{p}_{-i} \rangle$.*

Moreover, if the stressed graph $\mathcal{S}(\Omega)$ is not the complete graph on $N[i] = \{i, i_1, i_2\}$, then $(H/i, \mathbf{p}_{-i}, \Omega_{-i})$ is a psd stressed framework.

Proof. The equilibrium condition at node i implies $p_i \in \langle p_{i_1}, p_{i_2} \rangle$. Note that the Schur complement Ω_{-i} of Ω with respect to the (i, i)-entry w_{ii} has entries $w'_{i_1 i_2} = w_{i_1 i_2} - w_{ii_1} w_{ii_2}/w_{ii}$, $w'_{i_r i_r} = w_{i_r i_r} - w_{ii_r}^2/w_{ii}$ for $r = 1, 2$, and $w'_{jk} = w_{jk}$ for all other edges jk of H/i. As $\Omega \succeq 0$ we also have $\Omega_{-i} \succeq 0$. Moreover, $\Omega_{-i} \neq 0$. Indeed, $w'_{i_1 i_2} \neq 0$ if $(i_1, i_2) \notin F$; otherwise, as $\mathcal{S}(\Omega)$ is not the clique on $N[i]$, there is another edge jk of H/i in the support of Ω so that $w'_{jk} = w_{jk} \neq 0$.

In order to show that Ω_{-i} is a stress matrix for $(H/i, \mathbf{p}_{-i})$, it suffices to check the stress equilibrium at the nodes i_1 and i_2. To fix ideas consider node i_1. Then we can rewrite $w'_{i_1 i_1} p_{i_1} + w'_{i_1 i_2} p_{i_2} + \sum_{j \in N(i_1) \setminus \{i_2\}} w'_{i_1 j} p_j$ as $\left(\sum_j w_{i_1 j} p_j\right) - (w_{ii} p_i + w_{ii_1} p_{i_1} + w_{ii_2} p_{i_2}) w_{ii_1}/w_{ii}$, where both terms are equal to 0 using the equilibrium conditions of $(\Omega, \mathbf{p})$ at nodes i_1 and i. $\qquad\square$

We will apply the above lemma iteratively to contract a set I containing several 2-nodes. Of course, in order to obtain useful information, we want to be able to claim that, after contraction, we obtain a stressed framework $(H/I, \mathbf{p}_{V \setminus I}, \Omega_{-I})$, i.e., with $\Omega_{-I} \neq 0$. Problems might occur if at some step we get a stressed graph which is a clique on 3 nodes. Note that this can happen only when a connected component of the stressed graph is a circuit. However, when we will apply this operation of contracting 2-nodes to the case of $G = C_5 \times C_2$, we will make sure that this situation cannot happen; that is, we will show that we may assume that the stressed graph does not have a connected component which is a circuit (see Remark 1 in Section 4.1).

3.4 The graph V_8 has Gram dimension 4

Let $V_8 = (V = [8], E)$ be the graph shown in Figure 1. In this section we use the tools developed above to show that V_8 has Gram dimension 4.

Theorem 10. *The graph V_8 has Gram dimension 4.*

Proof. Set $G = V_8 = ([8], E)$. Clearly $\mathrm{gd}(G) \geq 4$ since K_4 is a minor of G. Fix $a \in \mathcal{S}_{++}(G)$; we show that (G, a) has a Gram realization in $\mathbb{R}^4$. For this we first apply Theorem 9. As stretched edge e_0, we choose the pair $e_0 = (1, 4)$ and we denote by $\widehat{G} = ([8], \widehat{E} = E \cup \{(1, 4)\})$ the extended graph obtained by adding the stretched pair $(1, 4)$ to G. Let $\mathbf{p}$ be the initial Gram realization of (G, a) and let $\Omega = (w_{ij})$ be the corresponding stress matrix obtained by applying Theorem
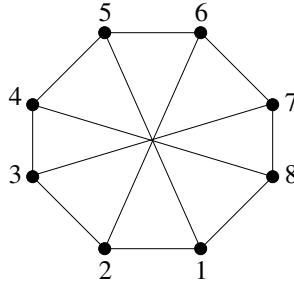
Fig. 1. The graph V_8.

9. We now show how to construct from $\mathbf{p}$ an equivalent realization q of (G, a) lying in $\mathbb{R}^4$.

In view of Lemma 10, we know that we are done if we can find a subset $S \subseteq V$ which is stable in the graph G and satisfies $\dim\langle\mathbf{p}_{V\setminus S}\rangle \leq 4$. This permits to deal with 1-nodes. Indeed suppose that there is a 1-node in the stressed graph $\mathcal{S}(\Omega)$. In view of Lemma 12 and (10), this can only be node 1 (or node 4) (i.e., the end points of the stretched pair) and $\dim\langle p_1, p_4\rangle \leq 1$. Then, choosing the stable set $S = \{2, 5, 7\}$, we have $\dim\langle\mathbf{p}_{V\setminus S}\rangle \leq 4$ and we can conclude using Lemma 10. Hence we can now assume that there is no 1-node in the stressed graph $\mathcal{S}(\Omega)$.

Next, observe that we are done in any of the following two cases:

(i) There exists a set $T \subseteq V$ with $|T| = 4$ and $\dim\langle\mathbf{p}_T\rangle \leq 2$.
(ii) There exists a set $T \subseteq V$ of cardinality $|T| = 3$ such that T does not consist of three consecutive nodes on the circuit $(1, 2, \ldots, 8)$ and $\dim\langle\mathbf{p}_T\rangle \leq 2$.

Indeed, in case (i) (resp., case (ii)), there is a stable set $S \subseteq V \setminus T$ of cardinality $|S| = 2$ (resp., $|S| = 3$), so that $|V \setminus (S \cup T)| = 2$ and thus $\dim\langle\mathbf{p}_{V\setminus S}\rangle \leq \dim\langle\mathbf{p}_T\rangle + \dim\langle\mathbf{p}_{V\setminus(S\cup T)}\rangle \leq 2 + 2 = 4$.

Hence we may assume that we are not in the situation of cases (i) and (ii).

Assume first that one of the nodes in $\{5, 6, 7, 8\}$ is a 0-node. Then all of them are 0-nodes. Indeed, if (say) 5 is a 0-node and 6 is not a 0-node then the equilibrium equation at node 6 implies that $\dim\langle p_6, p_7, p_2\rangle \leq 2$ so that we are in the situation of case (ii). As nodes 1, 4 are not 1-nodes, the stressed graph $\mathcal{S}(\Omega)$ is the circuit $(1, 2, 3, 4)$. Using Lemma 13, we deduce that $\dim\langle p_1, p_2, p_3, p_4\rangle \leq 2$ and thus we are in the situation of case (i) above.

Assume now that none of the nodes in $\{5, 6, 7, 8\}$ is a 0-node but one of the nodes in $\{2, 3\}$ is a 0-node. Then both nodes 2 and 3 are 0-nodes (else we are in the situation of case (ii)). Therefore, both nodes 6 and 7 are 2-nodes. Applying Lemma 13, after contracting both nodes 6,7, we obtain a stressed framework on $\{1, 4, 5, 8\}$ and thus $\dim\langle\mathbf{p}_{V\setminus\{2,3\}}\rangle = \dim\langle p_1, p_4, p_5, p_8\rangle$. Using Lemma 11, we deduce that $\dim\langle p_1, p_4, p_5, p_8\rangle \leq 3$. Therefore, $\dim\langle\mathbf{p}_{V\setminus\{3\}}\rangle \leq 4$ and one can find a new realization $\mathbf{q}$ in $\mathbb{R}^4$ equivalent to $(G, \mathbf{p})$ using Lemma 10.

Finally assume that none of the nodes in $\{2, 3, 5, 6, 7, 8\}$ is a 0-node. We show that $\langle \mathbf{p} \rangle = \langle p_2, p_3, p_6, p_7 \rangle$. The equilibrium equation at node 6 implies that $\dim\langle p_2, p_5, p_6, p_7 \rangle \leq 3$. Moreover, $\dim\langle p_2, p_6, p_7 \rangle = 3$ (else we are in case (ii) above). Hence $p_5 \in \langle p_2, p_6, p_7 \rangle$. Analogously, the equilibrium equations at nodes 7,2,3 give that $p_8, p_1, p_4 \in \langle p_2, p_3, p_6, p_7 \rangle$, respectively. □

4   The graph $C_5 \times C_2$ has Gram dimension 4

This section is devoted to proving that the graph $C_5 \times C_2$ has Gram dimension 4. The analysis is considerably more involved than the analysis for V_8. Figure 2 shows two drawings of $C_5 \times C_2$, the second one making its symmetries more apparent.

Theorem 11. *The graph $C_5 \times C_2$ has Gram dimension 4.*

Throughout this section we set $G = C_5 \times C_2 = (V = [10], E)$. Clearly, $\mathrm{gd}(G) \geq 4$ since K_4 is a minor of G. In order to show that $\mathrm{gd}(G) \leq 4$, we must show that $\mathrm{gd}(G, a) \leq 4$ for any $a \in \mathcal{S}_{++}(G)$. Moreover, in view of Corollary 4, it suffices to show this for all $a \in \mathcal{S}_{++}(G)$ satisfying the following 'genericity' property: For any Gram realization $\mathbf{p}$ of (G, a),

$$\dim\langle \mathbf{p}_C \rangle \geq 3 \ \text{ for any circuit } C \text{ in } G. \tag{12}$$

From now on, we fix $a \in \mathcal{S}_{++}(G)$ satisfying this genericity property. Our objective is to show that there exists a Gram realization of (G, a) in $\mathbb{R}^4$.

Again we use Theorem 9 to construct an initial Gram realization $\mathbf{p}$ of (G, a). As stretched edge e_0, we choose the pair $e_0 = (3, 8)$ and we denote by $\widehat{G} = ([8], \widehat{E} = E \cup \{(3, 8)\})$ the extended graph obtained by adding the stretched pair $(3, 8)$ to G. By Theorem 9, we also have a stress matrix Ω so that $(\widehat{G}, \mathbf{p}, \Omega)$ is a psd stressed framework. Our objective is now to construct from $\mathbf{p}$ another Gram realization $\mathbf{q}$ of (G, a) lying in $\mathbb{R}^4$.

4.1   Additional useful lemmas

First we deal with the case when $\dim\langle p_i, p_j \rangle = 1$ for some pair (i, j) of distinct nodes. As $a \in \mathcal{S}_{++}(G)$, this can only happen when $(i, j) \notin E$.

Lemma 14. *If $\dim\langle p_i, p_j \rangle = 1$ for some pair $(i, j) \notin E$, then there is a configuration in $\mathbb{R}^4$ equivalent to $(G, \mathbf{p})$.*

Proof. By assumption, $p_i = \epsilon p_j$ for some scalar $\epsilon \neq 0$. Up to symmetry there are two cases to consider: (i) $(i, j) = (1, 5)$ (two nodes at distance 2 in G), or (ii) $(i, j) = (1, 6)$ (two nodes at distance 3). Consider first case (i) when $(i, j) = (1, 5)$, so $p_1 = \epsilon p_5$. Set $V' = V \setminus \{1\}$. Let $G' = (V', E')$ be the graph on V' obtained from G by deleting node 1 and adding the edges $(2, 5)$ and $(5, 9)$ (in other words, get G' by identifying nodes 1 and 5 in G). Let X' be the Gram matrix of the
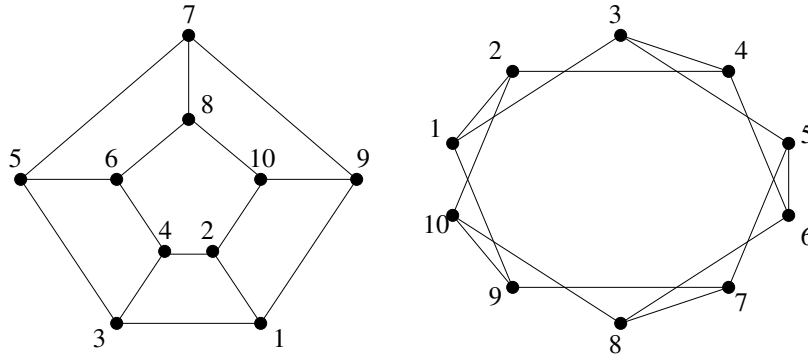
Fig. 2. Two drawings of the graph $C_5 \times C_2$.

vectors p_i ($i \in V'$) and define $a' = (X'_{jk})_{jk \in V' \cup E'} \in \mathcal{S}_+(G')$. First we show that (G', a') has a Gram realization in $\mathbb{R}^4$. For this, consider the graph H obtained from G by deleting both nodes 1 and 5. Then G' is a subgraph of ∇H and thus $\mathrm{gd}(G') \le \mathrm{gd}(\nabla H) = \mathrm{gd}(H) + 1$. As $\mathrm{tw}(H) \le 2$ it follows that $\mathrm{gd}(H) \le 3$ and thus $\mathrm{gd}(G') \le 4$. Finally, if $\mathbf{q}_{V'}$ is a Gram realization in $\mathbb{R}^4$ of (G', a') then, setting $q_1 = \epsilon q_5$, we obtain a Gram realization $\mathbf{q}$ of (G, a) in $\mathbb{R}^4$.

Case (ii) is analogous, based on the fact that the graph H obtained from G by deleting nodes 1 and 6 is a partial 2-tree. □

We now consider the case when the stressed graph might have a circuit as a connected component.

Lemma 15. *Let C be a circuit in $\widehat{G}$. If C is a connected component of $\mathcal{S}(\Omega)$, then $\dim\langle \mathbf{p}_C \rangle \le 2$.*

Proof. Directy, using Lemma 13 combined with Lemma 11. □

Therefore, in view of the genericity assumption (12), if a circuit C is a connected component of the stressed graph, then C cannot be a circuit in G and thus C must contain the stretched pair $e_0 = (3, 8)$. The next result is useful to handle this case, treated in Corollary 5 below.

Lemma 16. *Let $N_2(i)$ be the set of nodes at distance 2 from a given node i in G. If $\dim\langle \mathbf{p}_{N_2(i)} \rangle \le 3$, then there is a configuration equivalent to $(G, \mathbf{p})$ in $\mathbb{R}^4$.*

Proof. Say, $i = 1$ so that $N_2(1) = \{4, 5, 7, 10\}$, cf. Figure 3. Consider the set $S = \{2, 3, 6, 9\}$ which is stable in G. Let H denote the graph obtained from G in the following way: For each node $i \in S$, delete i and add the clique on $N(i)$. One can verify that H is contained in the clique 4-sum of the two cliques H_1 and H_2 on the node sets $V_1 = \{1, 4, 5, 7, 10\}$ and $V_2 = \{4, 5, 7, 8, 10\}$, respectively. By assumption, $\dim\langle \mathbf{p}_{V_1} \rangle \le 4$ and $\dim\langle \mathbf{p}_{V_2} \rangle \le 4$. Therefore, one can apply an orthogonal transformation and find vectors $q_i \in \mathbb{R}^4$ ($i \in V_1 \cup V_2$) such that $\mathbf{p}_{V_r}$ and $\mathbf{q}_{V_r}$ have the same Gram matrix, for $r = 1, 2$. Finally, as $V_1 \cup V_2 = V \setminus S$

and the set S is stable in G, one can extend to a configuration $\mathbf{q}_V$ equivalent to $\mathbf{p}_V$ by applying Lemma 10. $\qquad\square$

Corollary 5. *If there is a circuit C in $\widehat{G}$ containing the (stretched) edge $(3,8)$ such that $\dim\langle\mathbf{p}_C\rangle \leq 2$, then there is a configuration equivalent to $(G,\mathbf{p})$ in $\mathbb{R}^4$.*

Proof. If $|C| \geq 7$, pick $i \in V \setminus C$ and note that $\dim\langle\mathbf{p}_{-i}\rangle \leq 4$. If $|C| = 6$, pick a subset $S \subseteq V \setminus C$ of cardinality 2 which is stable in G, so that $\dim\langle\mathbf{p}_{V \setminus S}\rangle \leq 4$. In both cases we can conclude using Lemma 10. Assume now that $|C| = 4$ or 5. In view of Lemma 16, it suffices to check that there exists a node i for which $|C \cap N_2(i)| = 3$. For instance, for $C = (3,8,7,5)$, this holds for node $i = 9$, and for $C = (3,8,10,9,1)$ this holds for $i = 2$. Then, Lemma 16 implies that $\dim\langle\mathbf{p}_{N_2(i)}\rangle \leq 3$. $\qquad\square$

Remark 1. From now on, we will assume that $\dim\langle p_i, p_j\rangle = 2$ for all $i \neq j \in V$ (by Lemma 14). Hence there is no 1-node in the stressed graph. Moreover, we will assume that no circuit C of $\widehat{G}$ satisfies $\dim\langle\mathbf{p}_C\rangle \leq 2$. Therefore, the stressed graph does not have a connected component which is a circuit (by (12), Lemma 15 and Corollary 5). Hence we are guaranteed that after contracting several 2-nodes we do obtain a stressed framework (i.e, with a nonzero stress matrix).

The next two lemmas settle the case when there are sufficiently many 2-nodes.

Lemma 17. *If there are at least four 2-nodes in the stressed graph $\mathcal{S}(\Omega)$, then there is a configuration equivalent to $(G,\mathbf{p})$ in $\mathbb{R}^4$.*

Proof. Let I be a set of four 2-nodes in $\mathcal{S}(\Omega)$. Hence, $\mathbf{p}_I \subseteq \langle\mathbf{p}_{V \setminus I}\rangle$ and thus it suffices to show that $\dim\langle\mathbf{p}_{V \setminus I}\rangle \leq 4$.

After contracting each of the four 2-nodes of I, we obtain a psd stressed framework $(\widehat{G}/I, \mathbf{p}_{V \setminus I}, \Omega')$. Indeed, we can apply Lemma 13 and obtain a nonzero psd stress matrix Ω' in the contracted graph (recall Remark 1). If the support graph of Ω' is not a clique, Lemma 11 implies that $\dim\langle\mathbf{p}_{V \setminus I}\rangle \leq |V \setminus I| - 2 = 4$.

Assume now that $\mathcal{S}(\Omega')$ is a clique on $T \subseteq V \setminus I$. Then $\dim\langle\mathbf{p}_T\rangle \leq t - 1$, $|V \setminus (I \cup T)| = 6 - t$, and $t = |T| \in \{3,4,5\}$. Indeed one cannot have $t = 6$ since, after contracting the four 2-nodes, at least 4 edges are lost so that there remains at most $16 - 4 = 12 < 15$ edges. It suffices now to show that we can partition $V \setminus (I \cup T)$ as $S \cup S'$, where S is stable in G and $|S'| + t - 1 \leq 4$. Indeed, we then have $\dim\langle\mathbf{p}_{V \setminus S}\rangle = \dim\langle\mathbf{p}_{T \cup S'}\rangle \leq t - 1 + |S'| \leq 4$ and we can conclude using Lemma 10. If $t = 5$, then $|V \setminus (I \cup T)| = 1$ and choose $S' = \emptyset$. If $t = 4$, then choose $S' \subseteq V \setminus (I \cup T)$ of cardinality 1. If $t = 3$, then one can choose a stable set of cardinality 2 in $V \setminus (I \cup T)$ and $|S'| = 1$. $\qquad\square$

Lemma 18. *If there is at least one 0-node and at least three 2-nodes in the stressed graph $\mathcal{S}(\Omega)$, then there is a configuration equivalent to $(G,\mathbf{p})$ in $\mathbb{R}^4$.*

Proof. For $r = 0,2$, let V_r denote the set of r-nodes and set $n_r = |V_r|$. By assumption, $n_0 \geq 1$ and we can assume $n_2 = 3$ (else apply Lemma 17). Set $W = V \setminus (V_0 \cup V_2)$. After contracting the three 2-nodes in the stressed framework

$(\widehat{G}, \mathbf{p}, \Omega)$, we get a stressed framework $(H, \mathbf{p}_W, \Omega')$ on $|W| = 7 - n_0$ nodes. Hence $n_0 \leq 4$ and $\mathbf{p}_{V_2} \subseteq \langle \mathbf{p}_W \rangle$.

Assume first that $\mathcal{S}(\Omega')$ is not a clique. Then $\dim\langle \mathbf{p}_W \rangle \leq |W| - 2 = 5 - n_0$ by Lemma 11. Now we can conclude using Lemma 10 since in each of the cases: $n_0 = 1, 2, 3, 4$, one can find a stable set $S \subseteq V_0$ such that $\dim\langle \mathbf{p}_{W \cup (V_0 \setminus S)} \rangle \leq 4$.

Assume now that $\mathcal{S}(\Omega')$ is a clique. Then $\dim\langle \mathbf{p}_W \rangle \leq |W| - 1 = 6 - n_0$ by Lemma 11. Note first that $n_0 \neq 1, 2$. Indeed, if $n_0 = 1$ then, after deleting the 0-node and contracting the three 2-nodes, we have lost at least $3 + 3 = 6$ edges. Hence there remains at most $16 - 6 = 10$ edges in the stressed graph $\mathcal{S}(\Omega')$, which therefore cannot be a clique on six nodes. If $n_0 = 2$ then, after deleting the two 0-nodes and contracting the three 2-nodes, we have lost at least $5 + 3 = 8$ edges. Hence there remain at most $16 - 8 = 8$ edges in the stressed graph $\mathcal{S}(\Omega')$, which therefore cannot be a clique on five nodes. In each of the two cases $n_0 = 3, 4$, one can find a stable set $S \subseteq V_0$ of cardinality 2 and thus $\dim\langle \mathbf{p}_{W \cup (V_0 \setminus S)} \rangle \leq (6 - n_0) + (n_0 - 2) = 4$. Again conclude using Lemma 10. □

4.2   Sketch of the proof

In the proof we distinguish two cases: (i) when there is no 0-node, and (ii) when there is at least one 0-node, which are considered, respectively, in Sections 4.3 and 4.4. In both cases the tools developed in the preceding section permit to conclude, except in one exceptional situation, occurring in case (ii). This execptional situation is when nodes 1,2,9 and 10 are 0-nodes and all edges of $\widehat{G} \setminus \{1, 2, 9, 10\}$ are stressed. This situation needs a specific treatment which is done in Section 4.5.

4.3   There is no 0-node in the stressed graph

In this section we consider the case when each node is stressed in $\mathcal{S}(\Omega)$, i.e., $w_{ii} \neq 0$ for all $i \in [n]$.

Lemma 19. *Assume that all vertices are stressed in the stressed graph $\mathcal{S}(\Omega)$ and that there exists a circuit C of length 4 in G such that all edges in the cut $\delta(C)$ are stressed, i.e., $w_{ij} \neq 0$ for all edges $ij \in \widehat{E}$ with $i \in C$ and $j \in V \setminus C$. Then $\dim\langle \mathbf{p}_V \rangle \leq 4$.*

Proof. Up to symmetry, there are three types of circuits C of length 4 to consider: (i) C does not meet $\{3, 8\}$, i.e., $C = (1, 2, 10, 9)$; or (ii) C contains one of the two nodes 3,8, say node 8, and it contains a node adjacent to the other one, i.e., node 3, like $C = (5, 6, 8, 7)$; or (iii) C contains one of 3,8 but has no node adjacent to the other one, like $C = (7, 8, 10, 9)$.

Consider first the case (i), when $C = (1, 2, 10, 9)$. We show that the set $\mathbf{p}_C$ spans $\mathbf{p}_V$. Using the equilibrium conditions at the nodes 1,2,9,10, we find that $p_3, p_4, p_7, p_8 \in \langle \mathbf{p}_C \rangle$. As 6 is not a 0-node, $w_{6i} \neq 0$ for some $i \in \{4, 8\}$. Then, the equilibrium condition at node i implies that $p_6 \in \langle \mathbf{p}_C \rangle$. Analogously for node 5.

Case (ii) when $C = (5, 6, 8, 7)$ can be treated in analogous manner. Just note that the equilibrium conditions applied to nodes 7,5,6 and 8 respectively, imply that $p_9, p_3, p_4, p_{10} \in \langle \mathbf{p}_C \rangle$.

We now consider case (iii) when $C = (7, 8, 10, 9)$. Then one sees directly that $p_1, p_2, p_5 \in \langle \mathbf{p}_C \rangle$. If $w_{24} \neq 0$, then the equilibrium conditions at nodes 2,3,6 imply that $p_4, p_3, p_6 \in \langle \mathbf{p}_C \rangle$ and thus $\langle \mathbf{p}_C \rangle = \langle \mathbf{p}_V \rangle$. Assume now that $w_{24} = 0$, which implies $w_{34}, w_{46} \neq 0$. If $w_{13} \neq 0$, then the equilibrium conditions at nodes 1,3,4 imply that p_C spans p_3, p_4, p_6 and we are done. Assume now that $w_{24} = w_{13} = 0$, so that 1,2,4 are 2-nodes. If there is one more 2-node then we are done by Lemma 17. Hence we can now assume that $w_{ij} \neq 0$ whenever $(i, j) \neq (2, 4)$ or $(1, 3)$. After contracting the three 2-nodes 1,2,4 in the psd stressed framework $(\widehat{G}, \mathbf{p}, \Omega)$, we obtain a new psd stressed framework on $V \setminus \{1, 2, 4\}$ where nodes 9, 10 have again degree 2. So contract these two nodes and get another psd stressed framework on $V \setminus \{1, 2, 4, 9, 10\}$. Finally this implies $\dim \langle \mathbf{p}_V \rangle = \dim \langle \mathbf{p}_{V \setminus \{1,2,4,9,10\}} \rangle \leq 4$.   □

In view of Lemma 19, we can now assume that, for each circuit C of length 4 in G, there is at least one edge $ij \in \delta(C)$ which is not stressed, i.e., $w_{ij} = 0$. It suffices now to show that this implies the existence of at least four 2-nodes, as we can then conclude using Lemma 17.

For this let us enumerate the cuts $\delta(C)$ of the 4-circuits C in G:
- For $C = (1, 2, 10, 9)$, $\delta(C) = \{(1, 3), (2, 4), (7, 9), (8, 10)\}$.
- For $C = (7, 9, 10, 8)$, $\delta(C) = \{(1, 9), (2, 10), (5, 7), (6, 8)\}$.
- For $C = (5, 6, 8, 7)$, $\delta(C) = \{(7, 9), (8, 10), (3, 5), (4, 6)\}$.
- For $C = (3, 5, 6, 4)$, $\delta(C) = \{(1, 3), (2, 4), (5, 7), (6, 8)\}$.
- For $C = (1, 3, 4, 2)$, $\delta(C) = \{(3, 5), (4, 6), (1, 9), (2, 10)\}$.

For instance, $w_{24} = 0$ implies that both 2 and 4 are 2-nodes, while $w_{13} = 0$ implies that 1 is a 2-node. One can easily verify that there are at least four 2-nodes in $\mathcal{S}(\Omega)$.

4.4   There is at least one 0-node in the stressed graph

Note that the mapping $\sigma : V \to V$ that permutes each of the pairs $(1, 10)$, $(4, 7)$, $(5, 6)$, $(2, 9)$ and $(3, 8)$ is an automorphism of G. This can be easily seen using the second drawing of $C_5 \times C_2$ in Figure 2. Hence, as nodes 3 and 8 are not 0-nodes, up to symmetry, it suffices to consider the following three cases:
- Node 1 is a 0-node.
- Nodes 1, 10 are not 0-nodes and node 4 is a 0-node.
- Nodes 1, 10, 4, 7 are not 0-nodes and one of 5 or 2 is a 0-node.

Node 1 is a 0-node. It will be useful to use the drawing of $\widehat{G}$ from Figure 3. There, the thick edge (3,8) is known to be stressed, the dotted edges are known to be non-stressed (i.e., $w_{ij} = 0$), while the other edges could be stressed or not. In view of Lemma 18, we can assume that there are at most two 2-nodes (else we are done).
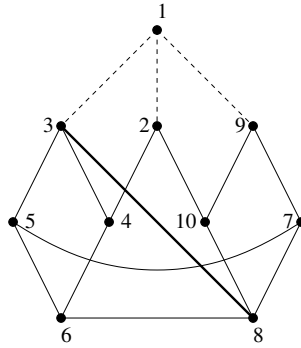
Fig. 3. A drawing of $\widehat{C_5 \times C_2}$ with 1 as the root node.

Assume first that both nodes 2 and 9 are 0-nodes. Then node 10 too is a 0-node and each of nodes 4 and 7 is a 0- or 2-node. If both 4,7 are 2-nodes, then all edges in the graph $G \backslash \{1, 2, 9, 10\}$ are stressed. Hence we are in the *exceptional case,* which we will consider in Section 4.5 below. If 4 is a 0-node and 7 is a 2-node, then 3,7 must be the only 2-nodes and thus 6 is a 0-node. Hence, the stressed graph is the circuit $C = (3, 8, 5, 7)$, which implies $\dim \langle \mathbf{p}_C \rangle \leq 2$ and thus we can conclude using Corollary 5. If 4 is a 2-node and 7 is a 0-node, then we find at least two more 2-nodes. Finally, if both 4,7 are 0-nodes, then the stressed graph is the circuit $C = (3, 8, 6, 5)$ and thus we can again conclude using Corollary 5.

We can now assume that at least one of the two nodes 2,9 is a 2-node. Then, node 3 has degree 3 in the stressed graph. (Indeed, if 3 is a 2-node, then 10 must be a 0-node (else we have three 2-nodes), which implies that 2,9 are 0-nodes, a contradiction.) If exactly one of nodes 2,9 is stressed, one can easily see that there should be at least three 2-nodes. Finally consider the case when both nodes 2,9 are stressed. Then they are the only 2-nodes which implies that all edges of $G \backslash 1$ are stressed. Set $I = \{4, 5, 8\}$. We show that $\mathbf{p}_I$ spans $\mathbf{p}_{V \backslash \{1\}}$, so that $\mathbf{p}_{\{1,4,5,8\}}$ spans $\mathbf{p}_V$. Indeed, the equilibrium conditions at 3 and 6 imply that $p_3, p_6 \in \langle \mathbf{p}_I \rangle$. Next, the equilibrium conditions at $4, 5, 2, 9$ imply, respectively, that $p_2 \in \langle p_3, p_4, p_6 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_7 \in \langle p_3, p_5, p_6 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_{10} \in \langle p_2, p_4 \rangle \subseteq \langle \mathbf{p}_I \rangle$, and $p_9 \in \langle p_7, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$. This concludes the proof.

Nodes 1, 10 are not 0-nodes and node 4 is a 0-node. It will be useful to use the drawing of $\widehat{G}$ from Figure 4. We can assume that node 2 is a 2-node and that node 3 has degree 3 in the stressed graph, since otherwise one would find at least three 2-nodes. Consider first the case when 6 is a 2-node.

Then nodes 2 and 6 are the only 2-nodes which implies that all edges in the graph $G \backslash 4$ are stressed. Set $I = \{3, 5, 7, 10\}$. We show that p_I spans $p_{V \backslash \{4\}}$, and then we can conclude using Lemma 10. Indeed, the equilibrium conditions applied, respectively, to nodes 5,6,3,1,2 imply that $p_6 \in \langle \mathbf{p}_I \rangle$, $p_8 \in \langle p_5, p_6 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_1 \in \langle p_3, p_5, p_8 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_9 \in \langle p_1, p_7, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_2 \in \langle p_1, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$.
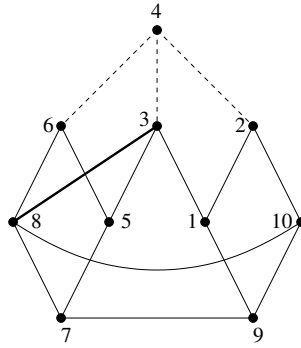
Fig. 4. A drawing of $\widehat{C_5 \times C_2}$ with 2 as the root node.

Consider now the case when 6 is a 0-node. Then 2 and 5 are the only 2-nodes so that all edges in the graph $G\backslash\{4,6\}$ are stressed. Set $I = \{3,7,10\}$. We show that $\mathbf{p}_I$ spans $\mathbf{p}_{V\backslash\{4,6\}}$, and then we can again conclude using Lemma 10. Indeed the equilibrium conditions applied, respectively, at nodes 5,8,3,2,1 imply that $p_5, p_8 \in \langle \mathbf{p}_I \rangle$, $p_1 \in \langle p_3, p_5, p_8 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_2 \in \langle p_1, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_9 \in \langle p_2, p_8, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$.

Nodes 1, 4, 7, 10 are not 0-nodes and node 5 or 2 is a 0-node. It will be useful to use the drawing of $\widehat{G}$ from Figure 5. We assume that nodes 1,4,7,10 are not 0-nodes. Consider first the case when node 5 is a 0-node. Then node 7 is a 2-node.
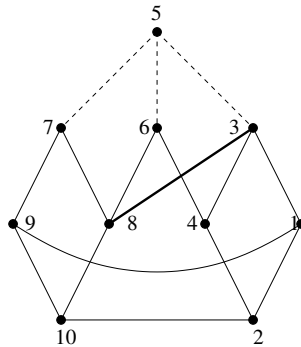
Fig. 5. A drawing of $\widehat{C_5 \times C_2}$ with 3 as the root node.

If node 6 is a 2-node, then 6 and 7 are the only 2-nodes and thus all edges of the graph $G\backslash 5$ are stressed. Setting $I = \{1,2,4,8\}$, one can verify that $\mathbf{p}_I$ spans $\mathbf{p}_{V\backslash\{5\}}$ and then one can conclude using Lemma 10.

If node 6 is a 0-node, then nodes 4 and 7 are the only 2-nodes and thus all edges in the graph $G \backslash \{5, 6\}$ are stressed. Setting $I = \{2, 3, 9\}$, one can verify that $\mathbf{p}_I$ spans $\mathbf{p}_{V \backslash \{5,6\}}$. Thus $\mathbf{p}_{\{2,3,9,6\}}$ spans $\mathbf{p}_{V \backslash \{5\}}$ and one can again conclude using Lemma 10.

Consider finally the case when nodes 1,4,7,10, 5 and 6 are not 0-nodes and node 2 is a 0-node. As node 2 is adjacent to nodes 1, 4 and 10 in G, we find three 2-nodes and thus we are done.

4.5 The exceptional case

In this section we consider the following case which was left open in the first case considered in Section 4.4: Nodes 1, 2, 9 and 10 are 0-nodes and all edges of the graph $\widehat{G} \backslash \{1, 2, 9, 10\}$ are stressed.

Then, nodes 4 and 7 are 2-nodes in the stressed graph. After contracting both nodes 4,7, we obtain a stressed graph which is the complete graph on 4 nodes. Hence, using Lemma 11, we can conclude that $\dim \langle \mathbf{p}_{V_1} \rangle \leq 3$, where $V_1 = V \setminus \{1, 2, 9, 10\}$. Among the nodes 1, 2, 9 and 10, we can find a stable set of size 2. Hence, if $\dim \langle \mathbf{p}_{V_1} \rangle \leq 2$ then, using Lemma 10, we can find an equivalent configuration in dimension $2 + 2 = 4$ and we are done. From now on we assume that

$$\dim \langle \mathbf{p}_{V_1} \rangle = 3. \tag{13}$$

In this case it is not clear how to fold $\mathbf{p}$ in $\mathbb{R}^4$. In order to settle this case, we proceed as in Belk [8]: We fix (or *pin*) the vectors p_i labeling the nodes $i \in V_1$ and we search for another set of vectors p'_i labeling the nodes $i \in V_2 = V \setminus V_1 = \{1, 2, 9, 10\}$ so that $\mathbf{p}_{V_1} \cup \mathbf{p}'_{V_2}$ can be folded into $\mathbb{R}^4$. Again, our starting point is to get such new vectors p'_i ($i \in V_2$) which, together with $\mathbf{p}_{V_1}$, provide a Gram realization of (G, a), by stretching along a second pair e'; namely we stretch the pair $e' = (4, 9) \in V_1 \times V_2$. As in So and Ye [31], this configuration $\mathbf{p}'_{V_2}$ is again obtained by solving a semidefinite program; details are given below.

Computing $\mathbf{p}'_{V_2}$ via semidefinite programming. Let $E[V_2]$ denote the set of edges of G contained in V_2 and let $E[V_1, V_2]$ denote the set of edges $(i, j) \in E$ with $i \in V_1$, $j \in V_2$. Moreover, set $|V_1| = n_1 \geq |V_2| = n_2$, so the configuration $\mathbf{p}_{V_1}$ lies in $\mathbb{R}^{n_1}$. (Here $n_1 = 6$, $n_2 = 4$). We now search for a new configuration $\mathbf{p}'_{V_2}$ by stretching along the pair $(4, 9)$. For this we use the following semidefinite program:

$$\max \langle F_{49}, Z \rangle \text{ such that } \begin{array}{ll} \langle F_{ij}, Z \rangle = a_{ij} \ \forall ij \in E[V_1, V_2] \\ \langle E_{ij}, Z \rangle = a_{ij} \ \forall ij \in V_2 \cup E[V_2] \\ \langle E_{ij}, Z \rangle = 0 \quad \forall i < j, i, j \in V_1 \\ \langle E_{ii}, Z \rangle = 1 \quad \forall i \in V_1 \\ Z \succeq 0. \end{array} \tag{14}$$

Here, $E_{ij} = (e_i e_j^T + e_j e_i^T)/2 \in \mathcal{S}^{n_1 + n_2}$, where e_i ($i \in [n_1 + n_2]$) are the standard unit vectors in $\mathbb{R}^{n_1 + n_2}$. Moreover, for $i \in V_1$, $j \in V_2$, $F_{ij} = (p'_i e_j^T + e_j (p'_i)^T)/2$, after setting $p'_i = (p_i, 0) \in \mathbb{R}^{n_1 + n_2}$.

Consider now a matrix Z feasible for (14). Then Z can be written in the block form $Z = \begin{pmatrix} I_{n_1} & Y \\ Y^T & X \end{pmatrix}$, and let $y_i \in \mathbb{R}^{n_1}$ $(i \in V_2)$ denote the columns of Y. The condition $Z \succeq 0$ is equivalent to $X - Y^T Y \succeq 0$. Say, $X - Y^T Y$ is the Gram matrix of the vectors $z_i \in \mathbb{R}^{n_2}$ $(i \in V_2)$. For $i \in V_2$, set $p_i' = (y_i, z_i) \in \mathbb{R}^{n_1+n_2}$. Then X is the Gram matrix of the vectors p_i' $(i \in V_2)$.

For $i \in V_1$, $j \in V_2$ we have that $\langle F_{ij}, Z \rangle = (p_i, 0)^T(y_j, z_j) = (p_i')^T p_j'$. Moreover, for $i, j \in V_2$, we have that $\langle E_{ij}, Z \rangle = X_{ij} = (p_i')^T p_j'$.

Therefore, the linear conditions $\langle F_{ij}, Z \rangle = a_{ij}$ for $ij \in E[V_1, V_2]$ and $\langle E_{ij}, Z \rangle = a_{ij}$ for $ij \in V_2 \cup E[V_2]$ imply that the vectors p_i' $(i \in V_1 \cup V_2)$ form a Gram realization of (G, a).

We now consider the dual semidefinite program of (14) which, as we see in Lemma 20 below, will give us some equilibrium conditions on the new vectors p_i' $(i \in V_2)$. The dual program involves scalar variables w_{ij}' (for $ij \in E[V_1, V_2] \cup V_2 \cup E[V_2]$) and a matrix $U' = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$, and it reads:

$$
\begin{aligned}
&\min && \langle I_{n_1}, U \rangle + \sum_{ij \in E[V_1, V_2]} w_{ij}' a_{ij} + \sum_{ij \in V_2 \cup E[V_2]} w_{ij}' a_{ij} \\
&\text{such that} && \Omega' = -F_{49} + U' + \sum_{ij \in E[V_1, V_2]} w_{ij}' F_{ij} + \sum_{ij \in V_2 \cup E[V_2]} w_{ij}' E_{ij} \succeq 0.
\end{aligned}
\tag{15}
$$

Since the primal program (14) is bounded and the dual program (15) is strictly feasible it follows that program (14) has an optimal solution Z. Let $p_i' \in \mathbb{R}^{n_1+n_2}$ $(i \in V_1 \cup V_2)$ be the vectors as defined above, which thus form a Gram realization of (G, a).

Lemma 20. *There exists a nonzero matrix $\Omega' = (w_{ij}') \succeq 0$ satisfying the optimality condition $Z\Omega' = 0$ and the following conditions on its support:*

$$
\begin{aligned}
w_{ij}' &= 0 \quad \forall (i, j) \in (V_1 \times V_2) \setminus (E[V_1, V_2] \cup \{(4, 9)\}), \\
w_{ij}' &= 0 \quad \forall i \neq j \in V_2, (i, j) \notin E[V_2].
\end{aligned}
\tag{16}
$$

Moreover, the following equilibrium conditions hold:

$$
w_{ii}' p_i' + \sum_{j \in V_1 \cup V_2 \mid ij \in E \cup \{(4,9)\}} w_{ij}' p_j' = 0 \quad \forall i \in V_2
\tag{17}
$$

and $w_{ij}' \neq 0$ for some $ij \in V_2 \cup E[V_2]$. Furthermore, a node $i \in V_2$ is a 0-node, i.e., $w_{ij}' = 0$ for all $j \in V_1 \cup V_2$, if and only if $w_{ii}' = 0$.

Proof. If the primal program (14) is strictly feasible, then (15) has an optimal solution Ω' which satisfies $Z\Omega' = 0$ and (16) (with $w_{49}' = -1$). Otherwise, if (14) is feasible but not strictly feasible then, using Farkas' lemma (Lemma 9), we again find a matrix $\Omega' \succeq 0$ satisfying $Z\Omega' = 0$ and (16) (now with $w_{49}' = 0$). We now indicate how to derive (17) from the condition $Z\Omega' = 0$.

For this write the matrices Z and Ω' in block form

$$Z = \begin{pmatrix} I_{n_1} & Y \\ Y^T & X \end{pmatrix}, \quad \Omega' = \begin{pmatrix} \Omega'_1 & \Omega'_{12} \\ (\Omega'_{12})^T & \Omega'_2 \end{pmatrix}.$$

From $Z\Omega' = 0$, we derive $Y^T\Omega'_{12} + X\Omega'_2 = 0$ and $\Omega'_{12} + Y\Omega'_2 = 0$. First this implies $(X - Y^TY)\Omega'_2 = 0$ which in turn implies that the V_2-coordinates of the vectors on the left hand side in (17) are equal to 0. Second, the condition $\Omega'_{12} + Y\Omega'_2 = 0$ together with expressing $\Omega'_{12} = \sum_{ij \in E[V_1,V_2] \cup \{(4,9)\}} w'_{ij} p'_i e_j^T$, implies that the V_1-coordinates of the vectors on the left hand side in (17) are 0. Thus (17) holds. Finally, we verify that $\Omega'_2 \neq 0$. Indeed, $\Omega'_2 = 0$ implies $\Omega'_{12} = 0$ and thus $\Omega' = 0$ since $0 = \langle Z, \Omega' \rangle = \langle I_{n_1}, \Omega'_1 \rangle$. □

Folding p′ into $\mathbb{R}^4$. We now use the above configuration p' and the equilibrium conditions (17) at the nodes of V_2 to construct a Gram realization of (G, a) in $\mathbb{R}^4$. By construction, $p'_i = (p_i, 0)$ for $i \in V_1$. Note that no node $i \in V_2$ is a 1-node with respect to the new stress Ω' (recall Lemma 14). Let us point out again that Lemma 20 does not guarantee that $w'_{49} \neq 0$ (as opposed to relation (7) in Theorem 9).

By assumption nodes 1,2,9 and 10 are 0-nodes and all other edges of the graph $\widehat{G} \setminus \{1, 2, 9, 10\}$ are stressed w.r.t. the old stress matrix Ω. We begin with the following easy observation about $\mathbf{p}'_{V_1}$.

Lemma 21. $\dim\langle p'_4, p'_7, p'_8 \rangle = \dim\langle p'_3, p'_4, p'_8 \rangle = 3$.

Proof. It is easy to see that each of these sets spans $\mathbf{p}'_{V_1}$ and $\dim\langle \mathbf{p}'_{V_1} \rangle = \dim\langle \mathbf{p}_{V_1} \rangle = 3$ by (13). □

As an immediate corollary we may assume that

$$p'_i \notin \langle \mathbf{p}'_{V_1} \rangle \ \forall i \in V_2 \tag{18}$$

Indeed, if there exists $i \in V_2$ satisfying $p'_i \in \langle \mathbf{p}'_{V_1} \rangle$ then we can find a stable set of size two in $V_2 \setminus \{i\}$ and using Lemma 10 we can construct an equivalent configuration in $\mathbb{R}^4$. Therefore, we can assume that at most two nodes in V_2 are 0-nodes in $\mathcal{S}(\Omega')$ since, by construction, for the new stress matrix Ω' there exists $ij \in V_2 \cup E[V_2]$ such that $w'_{ij} \neq 0$. This guides our discussion below. Figure 6 shows the graph containing the support of the new stress matrix Ω'.

There are two 0-nodes in V_2. The cases when either 2,9, or 1,10, are 0-nodes are excluded (since then one would have a 1-node). If nodes 1 and 9 are 0-nodes, then the equilibrium conditions at nodes 2 and 10 imply that $p'_4, p'_8 \in \langle p'_2, p'_{10} \rangle$ and by Lemma 21 we have that $\langle p'_2, p'_{10} \rangle = \langle p'_4, p'_8 \rangle \subseteq \langle \mathbf{p}'_{V_1} \rangle$, contradicting (18). The case when nodes 9,10 are 0-nodes is similar.

Finally assume that nodes 1,2 are 0-nodes (the case when 2,10 are 0-nodes is analogous). As $w'_{8,10} \neq 0$, the equilibrium condition at node 10 implies that $p'_8 \in \langle p'_9, p'_{10} \rangle$. If $w'_{49} = 0$ then the equilibrium condition at node 9 implies that $p'_7 \in \langle p'_9, p'_{10} \rangle$. Hence $\langle p'_7, p'_8 \rangle \subseteq \langle p'_9, p'_{10} \rangle$, thus equality holds, contradicting (18). If $w'_{49} \neq 0$, then $p'_4 \in \langle p'_7, p'_9, p'_{10} \rangle$ and thus $\langle p'_4, p'_7, p'_8 \rangle \subseteq \langle p'_7, p'_9, p'_{10} \rangle$. Hence equality holds (by Lemma 21), contradicting again (18).
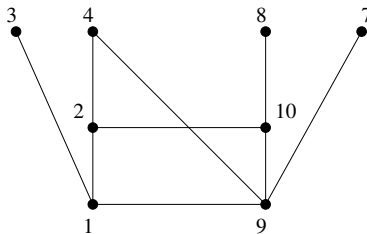
Fig. 6. The graph containing the support of the new stress matrix Ω'

There is one 0-node in V_2. Suppose first 9 is the only 0-node in V_2. The equilibrium conditions at nodes 1 and 10 imply that $p'_1 \in \langle p'_3, p'_2 \rangle$ and $p'_{10} \in \langle p'_2, p'_8 \rangle$. Hence $\langle p'_1, p'_{10} \rangle \subseteq \langle p'_{V_1}, p'_2 \rangle$ and thus $\dim \langle \mathbf{p}'_{V \setminus \{9\}} \rangle = 4$. Then conclude using Lemma 10.

Suppose now that node 1 is the only 0-node (the cases when 2 or 10 is 0-node are analogous). The equilibrium conditions at nodes 2 and 9 imply that $p'_2 \in \langle p'_4, p'_{10} \rangle$ and $p'_9 \in \langle p'_4, p'_7, p'_{10} \rangle$. Hence, $\langle p'_2, p'_9 \rangle \subseteq \langle p'_{V_1}, p'_{10} \rangle$ and we can conclude using Lemma 10.

There is no 0-node in V_2. We can assume that $w'_{ij} \neq 0$ for some $(i,j) \in V_1 \times V_2$ for otherwise we get the stressed circuit $C = (1, 2, 10, 9)$, thus with $\dim \langle \mathbf{p}'_C \rangle = 2$, contradicting Corollary 4. We show that $\dim \langle \mathbf{p}'_V \rangle = 4$. For this we discuss according to how many parameters are equal to zero among $w'_{13}, w'_{24}, w'_{8,10}$. If none is zero, then the equilibrium conditions at nodes 1,2 and 10 imply that $p'_3, p'_4, p'_8 \in \langle \mathbf{p}'_{V_2} \rangle$ and thus Lemma 21 implies that $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 3$. Therefore, $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}'_{V_2} \rangle = \dim \langle \mathbf{p}'_{V_1} \rangle + \dim \langle \mathbf{p}'_{V_2} \rangle - \dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \leq 3 + 4 - 3 = 4$.

Assume now that (say) $w'_{13} = 0$, $w'_{24}, w'_{8,10} \neq 0$. Then $\dim \langle \mathbf{p}'_{V_2} \rangle \leq 3$ (using the equilibrium condition at node 1). As $w'_{24}, w'_{8,10} \neq 0$, we know that $p'_4, p'_8 \in \langle \mathbf{p}'_{V_2} \rangle$. Hence $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 2$ and thus $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}'_{V_2} \rangle \leq 3 + 3 - 2 = 4$.

Assume now (say) that $w'_{13} = w'_{24} = 0$, $w'_{8,10} \neq 0$. Then the equilibrium conditions at nodes 1 and 2 imply that $\dim \langle \mathbf{p}'_{V_2} \rangle \leq 2$. Moreover, $p'_8 \in \langle \mathbf{p}'_{V_2} \rangle$. Hence $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 1$ and thus $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}'_{V_2} \rangle \leq 3 + 2 - 1 = 4$.

Finally assume now that $w'_{13} = w'_{24} = w'_{8,10} = 0$. Then $\dim \langle \mathbf{p}'_{V_2} \rangle = 2$. Moreover, at least one of w'_{49}, w'_{79} is nonzero. Hence $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 1$ and thus $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}_{V_2} \rangle \leq 3 + 2 - 1 = 4$.

5 Concluding remarks

One of the main contributions of this paper is the proof that $\mathrm{gd}(C_5 \times C_2) \leq 4$, an inequality which underlies the characterization of graphs with Gram dimension at most four. As already explained we obtain as corollaries the inequalities $\mathrm{ed}(C_5 \times C_2) \leq 3$ of [8] and $\nu^=(C_5 \times C_2) \leq 4$ of [21].

Although our proof of the inequality $\text{gd}(C_5 \times C_2) \leq 4$ goes roughly along the same lines as the proof of the inequality $\text{ed}(C_5 \times C_2) \leq 3$ given in [8], there are important differences and we believe that our proof is simpler. This is due in particular to the fact that we introduce a number of new auxiliary lemmas (cf. Sections 3.3 and 4.1) that enable us to deal more efficiently with the case checking which constitutes the most tedious part of the proof. Furthermore, the use of semidefinite programming to construct a stress matrix permits to eliminate some case checking since, as was already noted in [31] (in the context of Euclidean realizations), the stress is nonzero along the stretched pair of vertices. Additionally, our analysis complements and at several occasions even corrects the proof in [8]. As an example, the case when two vectors labeling two non-adjacent nodes are parallel in not discussed in [8]; this leads to some additional case checking which we address in Lemma 14.

As the class of graphs with $\text{gd}(G) \leq k$ is closed under taking minors, it follows from the general theory of Robertson and Seymour [34] that there exists a polynomial time algorithm for testing $\text{gd}(G) \leq k$. For $k \leq 4$ the forbidden minors for $\text{gd}(G) \leq k$ are known, hence one can make this polynomial time algorithm explicit. We refer to [30, §4.2.5] for details on how to check $\text{gd}(G) \leq 4$ or, equivalently, $\text{ed}(G) \leq 3$.

The next algorithmic question is how to construct a Gram representation in $\mathbb{R}^4$ of a given partial matrix $a \in \mathcal{S}_+(G)$ when G has Gram dimension 4. As explained in [30, §4.2.4,§4.2.5], the first step consists of finding a graph G' containing G as a subgraph and such that G' is a clique sum of copies of V_8, $C_5 \times C_2$ and chordal graphs with tree-width at most 3. Then, if a psd completion A of a is available, it suffices to deal with each of these components separately. Such a psd completion can be computed approximately by solving a semidefinite program. Chordal components are easy to deal with in view of the general results on psd completions in the chordal case. For the components of the form V_8 or $C_5 \times C_2$ one has to go through the steps of the proof to get a new Gram representation in $\mathbb{R}^4$. The basic ingredient of our proof is the *existence* of a primal-dual pair of optimal solutions to the programs (5) and (6). Under appropriate genericity assumptions, the existence of such a pair of optimal solutions is guaranteed by standard results of semidefinite programming duality theory (cf. Section 3.2). Also in the case when the primal program is not strictly feasible, we can still guarantee the existence of a psd stress matrix; our proof uses Farkas' lemma and is simpler than the proof in [30] of the analogous result in the context of Euclidean realizations. However, in the case of the graph $C_5 \times C_2$, we must make an additional genericity assumption on the vector $a \in \mathcal{S}_{++}(G)$ (namely, that the configuration restricted to any circuit is not coplanar). This is problematic since the folding procedure apparently breaks down for non-generic configurations; note that this issue also arises in the case of Euclidean embeddings since an analogous genericity assumption is made in [8], although it is not discussed in the algorithmic approach of [30,31]. Moreover, the above procedure relies on solving several semidefinite programs, which however cannot be solved exactly in general, but only to some given precision. This thus excludes the possibility of turning

the proof into an efficient algorithm for computing exact Gram representations in $\mathbb{R}^4$.

We conclude with the following question about the relation between the two parameters $\mathrm{gd}(G)$ and $\mathrm{ed}(G)$, which has been left open: Prove or disprove the inequality:

$$\mathrm{ed}(\nabla G) \leq \mathrm{ed}(G) + 1.$$

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