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The regulator problem with robust stability

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Abstract

The design of a controller such that the closed-loop system will track reference signals or reject disturbance signals from a specified class is known as the 'servomechanism problem' or the 'regulator problem'. For the regulator problem to be solvable with robust closed-loop stability, the plant obviously needs to be such that the regulation problem and the robust stabilization problem are solvable separately. In this paper we determine the extra conditions that are necessary and sufficient for the two problems to be solved simultaneously. It turns out that these conditions can be given a simple geometric interpretation in terms of a multivariable version of the Nyquist curve of the plant.

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1 Introduction

In classical control theory, perhaps the most central issue is the reconciliation of various design objectives. Modern control theory, on the other hand, has tended to isolate specific aspects of design and to provide separate solutions for the associated problems. While the modern approach has brought much progress, recent ('postmodern') research has emphasized the need for a study of the trade-offs between various design objectives in order to work towards a unification of the classical and the modern theory. Various approaches have been suggested, including realizability constraints [14], optimization methods [7], and loop shaping [27, 12]. In this paper we study the interaction between robust stability requirements and regulation requirements. It turns out that this particular interaction can be described in a remarkably simple way.

By a *regulation requirement*, we understand in this paper a requirement on the closed-loop systems to reject or follow a signal produced by an 'exosystem' of the form $\dot{z} = Fz$, $d = Hz$, where the eigenvalues of the matrix F are located on the imaginary axis. Signals that can be described in this way include steps, ramps, and sinusoids of fixed frequency. In particular the rejection of constant disturbances under closed-loop stability is one of the most classical problems in control theory [26]. The regulator problem has been extensively studied from various points of view during the seventies and early eighties; see the references in [37] and [6].

The eighties also saw new developments in the theory of *robust stabilization*. Among the nonparametric perturbation models, the one based on normalized coprime factorizations drew considerable attention, especially after it was shown by Glover and McFarlane that the problem of designing an optimally robust controller with respect to this perturbation class has a relatively straightforward solution. We shall use the same perturbation model in this paper.

The main subject of the paper will be to combine the regulation requirement with the robust stability requirement (in the sense of coprime factor perturbations). A first concern is to express the two requirements in a common framework. For this we use the formulation in terms of subspace-valued functions, which can be traced back to [25, 8]. It has already been demonstrated [30, 32] that subspace-valued functions are excellently suited to describe robust stability properties. In this paper we employ the same framework for the regulator problem; more specifically, we show that the regulator problem can be formulated as an interpolation problem for subspace-valued functions. Finite-dimensional geometry then readily leads to necessary conditions for the solvability of the regulator problem when a stability margin γ is imposed. We finally show that these conditions are also sufficient if two other (obvious) conditions are satisfied, namely that the regulation problem and the robust stabilization problem are solvable separately.

The paper is organized as follows. In the next section, we give precise formulations of the problem we want to solve. In section 3, it is shown that the regulator problem may be viewed as an interpolation problem, and we derive necessary conditions for solvability of the problem. In section 4, we obtain a parametrization of all solutions of the regulator problem. The solution of the regulator problem with robust stability is given in section 5, and is followed by an example in section 6. Conclusions are stated in section 7. A number of lemmas used in the main text are relegated to an appendix which appears as section 8.

2 Problem formulation and preliminaries

The regulator problem may be formulated as follows, assuming that the observed outputs coincide with the regulated outputs (cf. for instance [6, p. 317]). Consider a finite-dimensional linear time-invariant system of the following form:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \tag{2.1}$$

$$\dot{x}_2(t) = A_{22}x_2(t) \tag{2.2}$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{2.3}$$

The interpretation is as follows: x_1 denotes the state of the plant, whereas x_2 is the state of an 'exosystem' that generates signals which can be disturbances or references. The matrix A_{22} has its eigenvalues on the imaginary axis, allowing the reference/disturbance signals to be steps, ramps, sinusoids, etc. The variable $y(t)$ should converge to zero, irrespective of the presence of the signals generated by the exosystem. This is to be achieved by a linear time-invariant compensators of the form

$$\dot{z}(t) = Fz(t) + Gy(t) \tag{2.4}$$

$$u(t) = Hz(t) + Jy(t). \tag{2.5}$$

For ease of reference, we list here a number of assumptions that will be used in this paper. Not all of these are used in all results; we shall mention each time explicitly under which assumptions a particular result is valid.

Assumptions The system (2.1–2.3) satisfies

(A1) the pair (A_{11}, B_1) is stabilizable;

(A2) the pair (C, A) given by

$$C = [C_1 \;\; C_2], \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{2.6}$$

is detectable;

(A3) all eigenvalues of A_{22} are on the imaginary axis;

(A4) for every eigenvalue λ of A_{22}, the matrix

$$\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix}$$

has full column rank.

Assumption (A1) is necessary for the plant to be stabilizable by a feedback compensator, and so this is a natural assumption to make. Detectability of the pair (C_1, A_{11}) is necessary as well for closed-loop stability to be achieved by a compensator of the form (2.4–2.5); assumption (A2) requires a bit more, however. It can be argued that (A2) may be assumed without essential loss of generality in the regulator problem (cf. [37, §8.1]). Instead of (A3), the usual assumption is that the exosystem poles are in the closed right half plane (cf. for instance [13]); although (A3) is of course stronger, it represents hardly a restriction from the applications point of view. Assumption (A4) is not quite so harmless, because it implies that the number of outputs is at least equal to the number of inputs, whereas it is well-known [37, Ch. 8] that the regulator problem can only be 'well-posed' if the number of outputs is at most equal to the number of inputs. One may therefore say that (A4) essentially limits one to the case in which the number of control inputs is equal to the number of regulated outputs. The assumption requires that the plant zeros do not coincide with the exosystem poles, which is a well-known condition in connection with the regulator problem [37, Thm. 8.3], [6, Cor. 5.2-2]. The assumption (A4) is used for convenience in this paper; it can however be dispensed with, as will be shown in future work.

An important role in our analysis will be played by certain subspace-valued functions associated to plant and controller. With the plant given by the triple (A_{11}, B_1, C_1) we associate the function

$$\mathcal{P}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists x \text{ s.t.} \begin{bmatrix} sI - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\}, \quad \mathcal{P}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.7}$$

To the full system (2.1–2.3) we associate

$$\mathcal{M}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists x_1, x_2 \text{ s.t.} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 & -B_1 \\ 0 & sI - A_{22} & 0 & 0 \\ C_1 & C_2 & -I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \\ u \end{bmatrix} = 0 \right\},$$

$$\mathcal{M}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.8}$$

In the same way, we finally associate to the controller the subspace-valued function

$$\mathcal{C}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists z \text{ s.t.} \begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0 \right\}, \quad \mathcal{C}(\infty) = \operatorname{im} \begin{bmatrix} I \\ J \end{bmatrix}. \tag{2.9}$$

Note that all functions take values in the set of subspaces of the product space $\mathcal{Y} \times \mathcal{U}$, which is an $(m+p)$-dimensional space if m is the number of inputs and p is the number of outputs. The functions above may be considered as functions on the extended complex plane $\mathbb{C} \cup \{\infty\}$, but we shall usually consider them as functions on the closed right half plane

$$\mathbb{C}^+ \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\} \cup \{\infty\}. \tag{2.10}$$

The *closed-loop system* takes the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) = A_e \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) \tag{2.11}$$

$$y(t) = [C_1 \quad 0 \quad C_2]\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) \tag{2.12}$$

where

$$A_e = \begin{bmatrix} A_{11}+B_1JC_1 & B_1H & A_{12} \\ GC_1 & F & GC_2 \\ 0 & 0 & A_{22} \end{bmatrix}. \tag{2.13}$$

The compensator is said to satisfy the *internal stability requirement* if the closed-loop system is stable when $x_2(t) = 0$, that is, if the matrix

$$\begin{bmatrix} A_{11}+B_1JC_1 & B_1H \\ GC_1 & F \end{bmatrix}$$

is stable.

In order to define a requirement for *robust* stability, it is of interest to consider an equivalent formulation based on the subspace-valued functions (2.7) and (2.9), and on the notion of the minimal angle between subspaces. The minimal angle between two subspaces $\mathcal{Y}$ and $\mathcal{Z}$ of a unitary space $\mathcal{X}$ is defined as follows (see for instance [19, p. 339]):

$$\sin\phi(\mathcal{Y},\mathcal{Z}) = \min\{\|y-z\| \mid y\in\mathcal{Y},\ \ z\in\mathcal{Z},\ \ \|y\|=1\}, \qquad 0\le\phi\le\tfrac{1}{2}\pi. \tag{2.14}$$

Note that the minimal angle is nonzero if and only if the two subspaces intersect only in 0. If this condition holds, another formula for the minimal angle is given by (see again for instance [19, p. 339])

$$\sin\phi(\mathcal{Y},\mathcal{Z}) = \|\Pi_{\mathcal{Z}}^{\mathcal{Y}}\|^{-1} \tag{2.15}$$

where $\Pi_{\mathcal{Z}}^{\mathcal{Y}}$ denotes the skew projection along $\mathcal{Y}$ onto $\mathcal{Z}$, defined on $\mathcal{Y}+\mathcal{Z}$.

LEMMA 2.1 *The closed-loop system formed by the plant* (A_{11}, B_1, C_1) *and the compensator (2.4–2.5) is stable if and only if*

$$\min_{s\in\mathbb{C}^+}\sin\phi(\mathcal{P}(s),\mathcal{C}(s)) > 0. \tag{2.16}$$

PROOF First assume that the closed-loop system is stable. According to Lemma 8.2 in the Appendix, this implies that $\sin\phi(\mathcal{P}(s),\mathcal{C}(s))$ is positive for all s with $\operatorname{Re}s\ge 0$. From (2.7) and (2.9), we see that $\sin\phi(\mathcal{P}(\infty),\mathcal{C}(\infty))$ is positive as well. It follows from [25] and Lemma 8.4 that the functions $s\mapsto\mathcal{P}(s)$ and $s\mapsto\mathcal{C}(s)$ are continuous mappings from $\mathbb{C}^+$ to the Grassmannian manifolds $G^m(\mathcal{Y}\times\mathcal{U})$ and $G^p(\mathcal{Y}\times\mathcal{U})$ respectively. It is then seen from [11, Lemma 2.4] that the function $s\mapsto\sin\phi(\mathcal{P}(s),\mathcal{C}(s))$ is continuous. Because $\mathbb{C}^+$ is compact, it follows that this function indeed assumes a minimum on $\mathbb{C}^+$, which must be positive by the assumption of closed-loop stability. The converse is immediate from Lemma 8.2. □

Another way to express the above result is that $\mathcal{P}(s)$ and $\mathcal{C}(s)$ should be complementary at each point $s\in\mathbb{C}^+$. It has been shown in [32] that the minimal angle is the appropriate measure of the robustness of complementarity of two subspaces $\mathcal{Y}$ and $\mathcal{Z}$, in the sense that it gives exactly the distance (in the sense of the gap) of $\mathcal{Y}$ to the set of subspaces $\mathcal{Y}'$ that are not complementary to $\mathcal{Z}$. As a measure of robustness of stability, we shall therefore take

$$\sin\phi(\mathcal{P},\mathcal{C}) \stackrel{\text{def}}{=} \min{}_{s\in\mathbb{C}^+} \sin\phi(\mathcal{P}(s),\mathcal{C}(s)). \tag{2.17}$$

The minimum is actually achieved on the imaginary axis or at infinity. The above measure can also be motivated in other ways and has been used for instance in [35, 18, 16, 36]. As the notation suggests, the expression $\phi(\mathcal{P},\mathcal{C})$ can be interpreted as an angle between linear spaces associated with plant and controller [28, 32].

Now consider the following problems.

PROBLEM 1 (regulator problem with internal stability: RPIS)
Given the plant and exosystem (2.1–2.3), find a compensator of the form (2.4–2.5) such that the closed-loop system (2.11–2.13) is internally stable and satisfies the *regulation requirement*

$$\mathcal{X}_+(A_e) \subset \ker\,[C_1 \;\; 0 \;\; C_2] \tag{2.18}$$

where $\mathcal{X}_+(A_e)$ denotes the unstable subspace of A_e.

PROBLEM 2 (robust stabilization problem with margin γ: RSP(γ))
Given the plant (2.1–2.2) and γ with $0 < \gamma < 1$, find a compensator of the form (2.4–2.5) that satisfies the *robust stability requirement*

$$\min{}_{s\in\mathbb{C}^+} \sin\phi(\mathcal{P}(s),\mathcal{C}(s)) > \gamma. \tag{2.19}$$

PROBLEM 3 (regulator problem with robust stability margin γ: RPRS(γ))
Given the plant and exosystem (2.1–2.3) and γ with $0 < \gamma < 1$, find a compensator of the form (2.4–2.5) such that both the regulation property (2.18) and the robust stability property (2.19) hold.

Necessary and sufficient conditions for RPIS and RSP(γ) to be solvable, along with synthesis procedures to obtain a suitable compensator, are well-known; for RPIS, see [37, 6] and the references therein, and for RSP(γ), see [35, 18, 27]. More specifically, see [37, Thm. 8.1] and [27, Thm. 4.14]. Our purpose in this paper is to get the same results for RPRS(γ).

The subspace-valued functions that will be central in our discussion below were defined in state space terms above. Other popular representations include, of course, matrix fraction descriptions and the transfer matrix. *Under minimality assumptions*, the relation between these representations and the associated subspace-valued function as in (2.7) is given by the formulas (see Lemma 8.4)

$$\mathcal{P}(s) \;=\; \operatorname{im}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \;=\; \ker\,[\tilde{D}(s) \;\; -\tilde{N}(s)] \;=\; \operatorname{im}\begin{bmatrix} G(s) \\ I \end{bmatrix} \tag{2.20}$$

where $N(s)D^{-1}(s)$ and $\tilde{D}^{-1}(s)\tilde{N}(s)$ are right and left coprime factorizations respectively of the transfer matrix $G(s)$, and where the rightmost equality is understood to hold for all s that are not poles of $G(s)$. If $P(s)$ and $\tilde{P}(s)$ are any matrix functions of full generic column and row rank respectively, and

$$\mathcal{P}(s) \;=\; \operatorname{im} P(s) \;=\; \ker \tilde{P}(s) \tag{2.21}$$

then we shall call $P(s)$ an *image representation* and $\tilde{P}(s)$ a *kernel representation* of $\mathcal{P}(s)$. As is seen from the above, kernel representations can be seen as left factorizations and image representations as right factorizations; coprimeness corresponds to the representations having full rank everywhere on their domains of definition. By putting the subspace-valued functions at center stage rather than their representations, we emphasize a geometric viewpoint.

As already mentioned, we shall usually consider subspace-valued functions as being defined on the closed right half plane $\mathbb{C}^+$. It can readily be seen (cf. [11]) that it is actually sufficient to give the values of $\mathcal{P}(s)$ on the extended imaginary axis, by the uniqueness of analytic continuation into the right half plane. The curve $\mathcal{P}(i\omega)$ traced out as ω traverses the real line may reasonably be called the *Nyquist curve* of the system that gives rise to $\mathcal{P}(s)$. Indeed, the usual Nyquist curve for single-input-single-output systems is obtained via the standard identification of the Grassmannian manifold $G^1(\mathbb{C}^2)$ with the extended complex plane by the mapping im $\left[\begin{smallmatrix} s \\ 1 \end{smallmatrix}\right] \mapsto s$, im $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \mapsto \infty$.

3 Regulation as an interpolation problem

In this section we shall show how the regulator problem can be viewed as an interpolation problem, and we shall give a parametrization for all internally stabilizing controllers that achieve regulation. An important role is played by the relation between the subspace-valued functions $\mathcal{M}(s)$ and $\mathcal{P}(s)$ that were introduced in (2.8) and (2.7). Note that $\mathcal{M}(\lambda) = \mathcal{P}(\lambda)$ for all λ that are not eigenvalues of A_{22} (i. e. poles of the exosystem), and that in general we have $\mathcal{P}(s) \subset \mathcal{M}(s)$. Unlike $\mathcal{P}(s)$, the function $\mathcal{M}(s)$ has singularities, in the sense that it is not of constant dimension on the complex plane. In particular it can therefore not be considered as a mapping from the complex plane to any Grassmannian. The way in which $\mathcal{M}(s)$ plays a role in describing the regulation property is most easily seen in the case in which the eigenvalues of A_{22} are simple (i. e. when A_{22} is diagonalizable). We shall treat this case first in a proposition, and then make the necessary adjustments to handle the general case.

Proposition 3.1 *In the regulator problem as defined in the previous section, assume that A_{22} is diagonalizable. A controller is then a solution to the regulator problem with internal stability if and only if the associated subspace-valued function $\mathcal{C}(s)$ satisfies*

$$\mathcal{C}(\lambda) \cap \mathcal{M}(\lambda) \subset \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid y = 0 \right\} \quad \forall \lambda \in \sigma(A_{22}), \tag{3.1}$$

and

$$\mathcal{C}(\lambda) \oplus \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U} \quad \forall \lambda \in \mathbb{C}^+. \tag{3.2}$$

Proof By Lemma 8.2, the condition (3.2) is equivalent to internal stability of the combination of plant and compensator. If internal stability holds, the unstable eigenvalues of the closed-loop system matrix A_e must coincide with the eigenvalues of A_{22}. The regulation property will be satisfied if and only if the characteristic modes corresponding to these eigenvalues have zero output values associated to them. Because of the assumption that A_{22} has only simple eigenvalues, it suffices to consider solutions of the form $x(t) = x_0 e^{\lambda t}$, $z(t) = z_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$, $u(t) = u_0 e^{\lambda t}$. Substituting the assumed solutions in (2.1–2.5) and equating the coefficients of $e^{\lambda t}$ results in the equations

$$\begin{bmatrix} \lambda I - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ u_0 \end{bmatrix} = 0 \tag{3.3}$$

$$\begin{bmatrix} \lambda I - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z_0 \\ y_0 \\ u_0 \end{bmatrix} = 0. \tag{3.4}$$

These equations do indeed imply $y_0 = 0$ if condition (3.1) holds. Conversely, if the regulation property is not satisfied, then there exists a solution of equations (3.3–3.4) with $y_0 \neq 0$, and it follows that in this case condition (3.1) is not satisfied. □

In order to handle the case of higher multiplicities, it is convenient to introduce the concept of the 'blow-up' of a matrix function. A definition of this notion can be given in coordinate-free terms, in the following way. Consider an analytic function $M(s)$ defined on some domain Ω of the complex plane and taking values in the set of linear mappings from a linear space $\mathcal{X}$ to a linear space $\mathcal{Y}$. If

$x(s)$ is an analytic vector-valued function taking values in $\mathcal{X}$, then the first r coefficients in the Taylor series development of $M(s)x(s)$ around any point $\lambda \in \Omega$ are determined by the first r coefficients in the Taylor series development of $x(s)$ around λ. The dependence is of course linear and we denote the associated mapping by $M^{[r]}(\lambda)$, which is a linear mapping from the r-fold product X^r to the r-fold product Y^r. By repeating this construction at every $\lambda \in \Omega$ we obtain a new operator-valued function $M^{[r]}(s)$, which we shall call the *r-fold blow-up* of $M(s)$. An explicit expression for $M^{[r]}(s)$ in terms of $M(s)$ is given by

$$M^{[r]}(s) = \begin{bmatrix} M(s) & 0 & \cdots & \cdots & 0 \\ M'(s) & M(s) & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{(r-1)!}M^{(r-1)}(s) & \cdots & \cdots & M'(s) & M(s) \end{bmatrix}. \tag{3.5}$$

This clearly shows that $M^{[r]}(s)$ will again be an analytic operator-valued function. In the Appendix, we collect a number of useful properties of the blow-up. We shall sometimes use the notation $[M(s)]^{[r]}$ instead of $M^{[r]}(s)$, in particular when $M(s)$ is a partitioned matrix, and in such cases even write $[M(s)]^{[r]}(\lambda)$ instead of $M^{[r]}(\lambda)$.

In addition to the blow-ups of matrix functions, we shall also need blown-up versions of the various subspace-valued functions that were introduced above. For the functions $\mathcal{P}(s)$ and $\mathcal{C}(s)$ defined in (2.7) and (2.9) respectively, these can be defined via either image or kernel representations as follows:

$$\mathcal{P}^{[r]}(s) = \ker \tilde{P}^{[r]}(s) = \operatorname{im} P^{[r]}(s) \tag{3.6}$$

$$\mathcal{C}^{[r]}(s) = \ker \tilde{C}^{[r]}(s) = \operatorname{im} C^{[r]}(s). \tag{3.7}$$

It follows from Lemmas 8.7 and 8.8 in the Appendix that this definition is unambiguous. The subspace-valued function $\mathcal{M}(s)$ defined in (2.8) requires more care because it has singularities. Note that we may write

$$\mathcal{M}(s) = \Pi \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \tag{3.8}$$

where Π denotes the natural projection from $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ to $\mathcal{Y} \times \mathcal{U}$. We now define $\mathcal{M}^{[r]}(s)$ by

$$\mathcal{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]}, \quad \mathcal{M}^{[r]}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}^{[r]}. \tag{3.9}$$

A matrix function $\tilde{M}(s)$ will be called a *kernel representation* of the *sequence* of subspace-valued functions $\mathcal{M}^{[r]}(s)$ if $\ker \tilde{M}^{[r]}(s) = \mathcal{M}^{[r]}(s)$ for all s. It is shown in Lemma 8.13 that such representations do indeed exist.

We can now proceed to the general (higher-multiplicity) version of the above proposition. For ease of notation, we introduce

$$\mathcal{K} = \{ \begin{bmatrix} y \\ u \end{bmatrix} \mid y = 0 \} \tag{3.10}$$

and denote the natural projection from $\mathcal{Y} \times \mathcal{U}$ to $\mathcal{Y}$ by $\tilde{K} = [I \;\; 0]$, so that

$$\mathcal{K} = \ker \tilde{K} = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{3.11}$$

Regarding $\tilde{K}$ as a constant matrix-valued function, we can also consider $\tilde{K}^{[r]}$ which is simply a block diagonal matrix with $\tilde{K}$ on the diagonal entries, and $\mathcal{K}^{[r]} = \ker \tilde{K}^{[r]}$. By the *multiplicity* of an eigenvalue of a matrix we mean the length of the longest Jordan chain associated with that eigenvalue.

Theorem 3.2 *A controller of the form (2.4–2.5) is a solution to the regulator problem with internal stability if and only if the associated subspace-valued function $\mathcal{C}(s)$ satisfies*

$$\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda) \subset \mathcal{K}^{[r]} \qquad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r \tag{3.12}$$

and

$$\mathcal{C}(\lambda) \oplus \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U} \qquad \forall \lambda \in \mathbb{C}^{+}. \tag{3.13}$$

Proof The analysis is the same as in the previous proposition, except that we now have to take into account (for an eigenvalue λ of A_{22} of multiplicity r) solutions of the form

$$x(t) = (x_0 + x_1 t + \cdots + x_{r-1} t^{r-1}) e^{\lambda t}$$

and similarly for $z(t)$, $y(t)$, and $u(t)$. Substituting these solutions in (2.1–2.5) and equating the coefficients of $t^k e^{\lambda t}$ for $k = 0, 1, \cdots, r-1$ results in the following equations, where $x^r = \operatorname{col}(x_0, \cdots, x_{r-1})$ and y^r and u^r are defined likewise, and where we use the mingling operator introduced in the Appendix (see (8.15)):

$$\begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]} (\lambda)\ Mi \begin{bmatrix} x^r \\ y^r \\ u^r \end{bmatrix} = 0, \tag{3.14}$$

$$\begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix}^{[r]} (\lambda)\ Mi \begin{bmatrix} z^r \\ y^r \\ u^r \end{bmatrix} = 0. \tag{3.15}$$

The regulation property holds if the above equations imply that $y_0 = \cdots = y_{r-1} = 0$, that is, if (3.12) holds. Conversely, if (3.12) is not satisfied, then it follows as in the proof of Prop. 3.1 that the given controller does not solve the regulator problem. □

The above formulation of the regulator problem shows that a necessary condition for the problem to be solvable is that at each exosystem pole λ, there should exist a subspace $\mathcal{C}$ complementary to $\mathcal{P}(\lambda)$ which moreover should be such that $\mathcal{C} \cap \mathcal{M} \subset \mathcal{K}$. We shall therefore analyze the implied geometric problem in some more detail. First we note that assumption (A4), which states that no exosystem poles coincide with plant zeros, can be expressed geometrically as follows.

Lemma 3.3 *Consider the system (2.1–2.3), and assume that the pair (C_1, A_{11}) is detectable, and that all eigenvalues of A_{22} are in the closed right half plane. Under these conditions, which are in particular satisfied if assumptions (A2) and (A3) hold, assumption (A4) holds if and only if*

$$\mathcal{P}(\lambda) \cap \mathcal{K} = \{0\}. \tag{3.16}$$

Proof Take an eigenvalue λ of A_{22}. First suppose that (3.16) holds, and let x and u be such that

$$\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0. \tag{3.17}$$

We then obviously have

$$\begin{bmatrix} \lambda I - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ u \end{bmatrix} = 0 \tag{3.18}$$

which implies that $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right] \in \mathcal{P}(\lambda)$. By (3.16) it then follows that $u = 0$, and by the detectability assumption we then also have $x = 0$ from (3.18). The converse is proved by reversing this reasoning. □

LEMMA 3.4 *Let $\mathcal{W}$ be a unitary space with given subspaces $\mathcal{M}$, $\mathcal{P}$, and $\mathcal{K}$. Assume that $\mathcal{P} \subset \mathcal{M}$ and $\mathcal{P} \cap \mathcal{K} = \{0\}$. Under these conditions, there exists a subspace $\mathcal{C}$ such that*

$$\mathcal{C} \oplus \mathcal{P} = \mathcal{W} \text{ and } \mathcal{C} \cap \mathcal{M} \subset \mathcal{K} \tag{3.19}$$

if and only if $\mathcal{M} = \mathcal{P} + (\mathcal{K} \cap \mathcal{M})$.

PROOF First assume that there exists a subspace $\mathcal{C}$ satisfying the stated conditions. We then have $\mathcal{M} = \mathcal{M} \cap (\mathcal{C} + \mathcal{P}) = (\mathcal{M} \cap \mathcal{C}) + \mathcal{P} \subset (\mathcal{K} \cap \mathcal{M}) + \mathcal{P}$, whereas the reverse inclusion is immediate from the assumption $\mathcal{P} \subset \mathcal{M}$. Now, assume that the condition $\mathcal{M} = \mathcal{P} + (\mathcal{K} \cap \mathcal{M})$ holds. Let $\mathcal{T}$ be any complement of $\mathcal{M}$ in $\mathcal{W}$, and take $\mathcal{C} = (\mathcal{K} \cap \mathcal{M}) \oplus \mathcal{T}$. We have to show that $\mathcal{C}$ is complementary to $\mathcal{P}$, and that $\mathcal{C} \cap \mathcal{M} \subset \mathcal{K}$. The first claim is immediate by noting that the assumptions imply that $\mathcal{K} \cap \mathcal{M}$ is a direct complement of $\mathcal{P}$ in $\mathcal{M}$. For the second claim, suppose that $w \in \mathcal{C} \cap \mathcal{M}$. We can write $w = w_1 + w_2$ with $w_1 \in \mathcal{K} \cap \mathcal{M}$ and $w_2 \in \mathcal{T}$. Because $w \in \mathcal{M}$ and $w_1 \in \mathcal{M}$, it follows that $w_2 \in \mathcal{M} \cap \mathcal{T} = \{0\}$ so that $w = w_1 \in \mathcal{K}$. □

Because of the robustness criterion that we adopted, it is of interest to see what the maximum value is of $\phi(\mathcal{P}, \mathcal{C})$ under the constraint (3.19).

LEMMA 3.5 *Let $\mathcal{W}$ be a unitary space with given subspaces $\mathcal{M}$, $\mathcal{P}$, and $\mathcal{K}$. Assume that $\mathcal{P} \subset \mathcal{M}$, $\mathcal{P} \cap \mathcal{K} = \{0\}$, and $\mathcal{M} = \mathcal{P} + (\mathcal{K} \cap \mathcal{M})$. Under these conditions, we have*

$$\phi(\mathcal{P}, \mathcal{C}) \; \leq \; \phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M}) \tag{3.20}$$

for all subspaces $\mathcal{C}$ satisfying (3.19), and equality is achieved for instance for $\mathcal{C} = (\mathcal{K} \cap \mathcal{M}) \oplus \mathcal{M}^{\perp}$.

PROOF By the assumptions, we have $\mathcal{C} \cap \mathcal{M} \subset \mathcal{K} \cap \mathcal{M}$ and $\dim \mathcal{C} \cap \mathcal{M} = \dim \mathcal{M} - \dim \mathcal{P} = \dim \mathcal{K} \cap \mathcal{M}$, so that actually $\mathcal{C} \cap \mathcal{M} = \mathcal{K} \cap \mathcal{M}$. From this it is clear that the inequality (3.20) holds. The fact that equality holds for $\mathcal{C} = (\mathcal{K} \cap \mathcal{M}) \oplus \mathcal{M}^{\perp}$ is immediate from the definition of the minimal angle. □

To get a formula for the upper bound appearing in (3.20), assume that we have *normalized* kernel and image representations for the subspace $\mathcal{P}$, so

$$\mathcal{P} = \operatorname{im} P = \ker \tilde{P}, \qquad P^* P = I, \qquad \tilde{P} \tilde{P}^* = I. \tag{3.21}$$

Also take an image representation C for $\mathcal{C}$. Because $\mathcal{C}$ and $\mathcal{P}$ are complementary, the matrix $\tilde{P}C$ must be invertible, and since image representations are only determined up to right multiplication by nonsingular matrices, we may as well assume that

$$\tilde{P} C = I. \tag{3.22}$$

Under the assumptions we have made, the projection along $\mathcal{P}$ onto $\mathcal{K} \cap \mathcal{M} = \mathcal{C} \cap \mathcal{M}$ in $\mathcal{M}$ is given by $C\tilde{P}|_{\mathcal{M}}$ (cf. Lemma 8.14), and so by (2.15) we have $\sin \phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M}) = ||C\tilde{P}|_{\mathcal{M}}||^{-1}$. The latter expression can be further evaluated as follows, using the fact that the matrix $[P \;\; \tilde{P}^*]$ is unitary:

$$||C\tilde{P}|_{\mathcal{M}}|| = ||C|_{\tilde{P}\mathcal{M}}|| = || \begin{bmatrix} P^* \\ \tilde{P} \end{bmatrix} C|_{\tilde{P}\mathcal{M}}|| = (1 + ||P^* C|_{\tilde{P}\mathcal{M}}||^2)^{\frac{1}{2}}. \tag{3.23}$$

In all, we get (under the assumptions (3.21–3.22))

$$\sin \phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M}) \; = \; (1 + ||P^* C|_{\tilde{P}\mathcal{M}}||^2)^{-\frac{1}{2}}. \tag{3.24}$$

4 Parametrization of all regulators

Our aim in this section is to obtain a parametrization of all controllers (2.4–2.5) that solve the regulator problem RPIS for a given system (2.1–2.2); this will be instrumental in optimizing with respect to the robustness of stability. The parametrization will be given through an image representation for $\mathcal{C}(s)$. First, let $\tilde{P}(s)$ be a kernel representation for $\mathcal{P}(s)$. Since $\tilde{P}(s)$ has full row rank everywhere on $\mathbb{C}^+$, we can find a matrix $\tilde{P}'(s)$ such that $[\begin{smallmatrix}\tilde{P}(s)\\ \tilde{P}'(s)\end{smallmatrix}]$ is RH_∞-unimodular. Write

$$\begin{bmatrix}\tilde{P}(s)\\ \tilde{P}'(s)\end{bmatrix}^{-1} = [P'(s) \quad P(s)]; \tag{4.1}$$

then $P(s)$ is an image representation of $\mathcal{P}(s)$. A matrix $C(s)$ is an image representation for a stabilizing compensator $\mathcal{C}(s)$ if and only if $\tilde{P}(s)C(s)$ is RH_∞-unimodular, and since an image representation is only determined up to right multiplication by unimodular matrices, we may without loss of generality even require that $\tilde{P}(s)C(s) = I$. Let $C_0(s)$ be a particular solution to this equation, and let $C(s)$ be any solution; then $\tilde{P}(s)(C(s) - C_0(s)) = 0$ so

$$C(s) - C_0(s) \;=\; [P'(s) \quad P(s)]\begin{bmatrix}\tilde{P}(s)\\ \tilde{P}'(s)\end{bmatrix}(C(s) - C_0(s)) \;=\; P(s)\tilde{P}'(s)(C(s) - C_0(s)) \tag{4.2}$$

which shows that $C(s)$ is of the form $C_0(s) - P(s)Q(s)$ for some RH_∞-matrix $Q(s)$. Conversely we see that any matrix of this form satisfies the equation $\tilde{P}(s)C(s) = I$. Here we have, of course, the Kučera-Youla parametrization of all stabilizing compensators [24, 38]. We now want to refine this parametrization in order to find all stabilizing compensators that solve the regulation problem.

In view of Lemma 8.14, the regulation requirement (3.12) may be written in the form

$$\Pi^{\mathcal{P}^{[r]}}_{\mathcal{C}^{[r]}}\mathcal{M}^{[r]} \subset \mathcal{K}^{[r]} \tag{4.3}$$

where $\Pi^{\mathcal{P}^{[r]}}_{\mathcal{C}^{[r]}}$ denotes the projection along $\mathcal{P}^{[r]}$ onto $\mathcal{C}^{[r]}$. If $C(s)$ is chosen such that $\tilde{P}(s)C(s) = I$, then

$$\Pi^{\mathcal{P}^{[r]}}_{\mathcal{C}^{[r]}} = [C\tilde{P}]^{[r]} \tag{4.4}$$

and so we can write equation (4.3) in the form

$$[C\tilde{P}]^{[r]}\ker \tilde{M}^{[r]} \subset \ker \tilde{K}^{[r]}. \tag{4.5}$$

At this point, the following lemma is important.

Lemma 4.1 *Let $\tilde{P}(s)$ be a kernel representation of the subspace-valued function $\mathcal{P}(s)$ defined in (2.7), and let $\tilde{M}(s)$ be a kernel representation of the sequence of subspace-valued functions $\mathcal{M}^{[r]}(s)$ defined in (3.9). Under the assumption (A2), there exists a square and nonsingular RH_∞-matrix function $\tilde{H}(s)$ such that*

$$\tilde{M}(s) \;=\; \tilde{H}(s)\tilde{P}(s). \tag{4.6}$$

Moreover, the nontrivial elementary divisors of $\tilde{H}(s)$ are the same as those of $sI - A_{22}$.

Proof A kernel representation for the sequence $\mathcal{M}^{[r]}(s)$ is constructed as follows (cf. the proof of Lemma 8.13). By the detectability assumption, we can find RH_∞-matrices $\tilde{N}_1(s)$, $\tilde{N}_2(s)$, and $\tilde{D}(s)$ such that

$$\ker[-\tilde{N}_1(s) \;\; \tilde{D}(s) \;\; -\tilde{N}_2(s)] \;=\; \mathrm{im}\begin{bmatrix} sI - A_{11} & -A_{12}\\ -C_1 & -C_2\\ 0 & sI - A_{22}\end{bmatrix} \quad \forall s \in \mathbb{C}^+. \tag{4.7}$$

We then set

$$\tilde{M}(s) = [-\tilde{N}_1(s)\ \ \tilde{D}(s)\ \ -\tilde{N}_2(s)] \begin{bmatrix} 0 & -B_1 \\ I & 0 \\ 0 & 0 \end{bmatrix} = [-\tilde{N}_1(s)\ \ \tilde{D}(s)] \begin{bmatrix} 0 & -B_1 \\ I & 0 \end{bmatrix}. \tag{4.8}$$

On the other hand, a kernel representation $\tilde{P}(s)$ is constructed by finding $\tilde{N}_0(s)$ and $\tilde{D}_0(s)$ such that

$$\ker\,[-\tilde{N}_0(s)\ \ \tilde{D}_0(s)] \ = \ \mathrm{im} \begin{bmatrix} sI - A_{11} \\ -C_1 \end{bmatrix} \quad \forall s \in \mathbb{C}^+ \tag{4.9}$$

and setting

$$\tilde{P}(s) = [-\tilde{N}_0(s)\ \ \tilde{D}_0(s)] \begin{bmatrix} 0 & -B_1 \\ I & 0 \end{bmatrix}. \tag{4.10}$$

It follows from Lemma 8.15 in the Appendix that there exists an RH_∞-matrix $\tilde{H}(s)$ with the properties as stated in the lemma such that

$$[-\tilde{N}_1(s)\ \ \tilde{D}(s)] \ = \ \tilde{H}(s)[-\tilde{N}_0(s)\ \ \tilde{D}_0(s)]. \tag{4.11}$$

From this together with (4.8) and (4.10), the claim in the lemma follows for the matrix functions $\tilde{M}(s)$ and $\tilde{P}(s)$ constructed above. Lemma 8.13 in the Appendix shows that the same conclusion must hold for any representations $\tilde{M}(s)$ and $\tilde{P}(s)$ that satisfy the specified conditions. □

Using this lemma, we can rewrite (4.5) as

$$[C\tilde{P}]^{[r]}\ker\,[\tilde{H}\tilde{P}]^{[r]} \ \subset \ \ker \tilde{K}^{[r]}. \tag{4.12}$$

Because $\tilde{P}(s)$ has full row rank everywhere on $\mathbb{C}^+$, the same holds for $\tilde{P}^{[r]}(s)$ (Lemma 8.8) and so (4.12) is equivalent to

$$C^{[r]}\ker \tilde{H}^{[r]} \ \subset \ \ker \tilde{K}^{[r]} \tag{4.13}$$

which is the same as

$$(\tilde{K}C)^{[r]}|_{\ker \tilde{H}^{[r]}} \ = \ 0. \tag{4.14}$$

Because the matrix function $\tilde{H}(s)$ is nonsingular, the same holds for $\tilde{H}^{[r]}(s)$ and so the subspace-valued function $\ker \tilde{H}^{[r]}(s)$ takes the value $\{0\}$ almost everywhere on $\mathbb{C}^+$. Consequently, the inclusion (4.13) is trivial almost everywhere. The only interesting points are those at which $\tilde{H}(s)$ has a zero, which by the lemma above are exactly the exosystem poles. The lemma also guarantees that the multiplicities of the zeros of $\tilde{H}(s)$ are the same as the multiplicities of the exosystem poles, so that we may reformulate the condition (4.14) as follows:

$$(\tilde{K}C)^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \quad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{4.15}$$

Now, assume that the regulator problem with internal stability is solvable and let $C_0(s)$ be an image representation of the subspace-valued function associated to a particular solution. We know from the Kučera-Youla parametrization that any controller achieving internal stability can be represented by $C(s) = C_0(s) - P(s)Q(s)$ where $Q(s)$ is an arbitrary RH_∞-matrix of the appropriate size. It is clear from (4.15) that such a controller will also be a solution to the regulator problem if and only if

$$(\tilde{K}PQ)^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \quad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{4.16}$$

If we assume now that assumption (A4) holds, so that $\tilde{K}P(\lambda)$ is injective (cf. Lemma 3.3), then the same holds for $(\tilde{K}P)^{[r]}(\lambda)$ and the condition (4.16) simplifies to

$$Q^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} = 0 \quad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{4.17}$$

But then we also have

$$\ker \tilde{H}^{[r]}(s) \subset \ker Q^{[r]}(s) \quad \forall s \in \mathbb{C}^+ \tag{4.18}$$

since the inclusion is trivial for those s that are not eigenvalues of A_{22}. It is shown in the Appendix (Lemma 8.9) that (4.18) implies that

$$Q(s) = \Psi(s)\tilde{H}(s) \tag{4.19}$$

for some RH_∞-matrix $\Psi(s)$. Conversely, it is clear that any matrix of the form $C_0(s) - P(s)\Psi(s)\tilde{H}(s)$ provides a solution to the regulator problem. Therefore, we have proved the main result of this section which gives a parametrizations of all controllers of the form (2.4–2.5) that achieve regulation with internal stability.

THEOREM 4.2 *Consider the system (2.1–2.3) under the assumptions (A1–A4). Let $P(s)$ and $\tilde{P}(s)$ denote image and kernel representations respectively for the subspace-valued function $\mathcal{P}(s)$ associated to the plant as defined by (2.7). Assume that the regulator problem with internal stability is solvable, and let $C_0(s)$ be an image representation of the function $\mathcal{C}(s)$ associated as in (2.9) to a particular solution, normalized such that $\tilde{P}(s)C_0(s) = I$. Let $\tilde{H}(s)$ be as in Lemma 4.1. Under these conditions, the general form of an image representation $C(s)$ of a solution of the regulator problem with internal stability is given by*

$$C(s) = C_0(s) - P(s)\Psi(s)\tilde{H}(s) \tag{4.20}$$

where $\Psi(s)$ is an arbitrary element of $RH_\infty^{m\times p}$.

For other parametrizations of all solutions to the regulator problem, see for instance [9, 31, 1]. The parametrization given above turns out to be particularly useful in connection with the robust stabilization problem.

5 Solution of the regulator problem with robust stability

In this section, unlike the previous one, we shall take norm constraints into account. Therefore it will be convenient to always work with *normalized* image and kernel representations for the plant; that is, we shall require these representations to satisfy

$$P^*(s)P(s) = I, \qquad \tilde{P}(s)\tilde{P}^*(s) = I \tag{5.1}$$

where $M^*(s)$, for a real rational matrix $M(s)$, denotes $M^T(-s)$. For the controller we shall always work with image representations $C(s)$ normalized such that $\tilde{P}(s)C(s) = I$. Due to these normalizations, the stability margin achieved by a controller represented by $C(s)$ is

$$\sin\phi(\mathcal{P},\mathcal{C}) \;=\; ||C\tilde{P}||_\infty^{-1} \;=\; ||C||_\infty^{-1}. \tag{5.2}$$

In terms of the parametrization (4.20) obtained in the previous section, we should therefore aim at minimizing $||C_0 - P\Psi\tilde{H}||$ over the RH_∞-matrices $\Psi(s)$. Applying the same trick as in the derivation of (3.24), we can also write

$$\sin\phi(\mathcal{P},\mathcal{C}) \;=\; ||C_0 - P\Psi\tilde{H}||_\infty^{-1} \;=\; (1 + ||P^*C_0 - \Psi\tilde{H}||_\infty^2)^{-\frac{1}{2}} \tag{5.3}$$

and so we may as well minimize $||P^*C_0 - \Psi\tilde{H}||_\infty$.

There are two obvious lower bounds for this problem. First of all, since $\Psi(s)\tilde{H}(s)$ is an RH_∞-matrix, it follows from Nehari's theorem that a lower bound for the minimization problem is given by the norm of the Hankel operator associated with P^*C_0:

$$||P^*C_0 - \Psi\tilde{H}||_\infty \;\geq\; ||\Gamma_{P^*C_0}||. \tag{5.4}$$

In the case in which we have no regulation constraints and so we have a pure robust stabilization problem, we can take $\tilde{H}(s) = I$ and then the Hankel norm is an exact lower bound [27]. At first sight the bound may seem to depend on the choice of the particular stabilizing compensator $C_0(s)$; however, another compensator $C(s) = C_0(s) - P(s)Q(s)$ would produce the symbol $P^*C = P^*C_0 - Q$ which differs only by an H_∞-matrix from P^*C_0 so that the Hankel norm would not be affected. Glover and McFarlane [18] give an expression that is explicitly independent of the choice of the compensator: they show that

$$(1 + ||\Gamma_{P^*C_0}||^2)^{-\frac{1}{2}} \;=\; (1 - ||\Gamma_{\tilde{P}^*}||^2)^{\frac{1}{2}}. \tag{5.5}$$

The second lower bound that is immediately seen to hold is

$$||P^*C_0 - \Psi\tilde{H}||_\infty \;\geq\; ||P^*(\lambda)C_0(\lambda)|_{\ker \tilde{H}(\lambda)}|| \quad \text{for all zeros } \lambda \in i\mathbb{R} \text{ of } \tilde{H}(s). \tag{5.6}$$

From the analysis in the previous section, we see that this inequality is really of a geometric nature. Indeed, since $\tilde{P}(\lambda)$ is surjective for all $\lambda \in i\mathbb{R}$, we have $\tilde{P}(\lambda)\ker\tilde{M}(\lambda) = \tilde{P}(\lambda)\ker\tilde{H}(\lambda)\tilde{P}(\lambda) = \ker\tilde{H}(\lambda)$, so it follows from (3.24) that the above inequality may also be written as

$$\phi(\mathcal{P}, \mathcal{C}) \;\leq\; \phi(\mathcal{P}(\lambda), \mathcal{K} \cap \mathcal{M}(\lambda)). \tag{5.7}$$

Our goal in this section will be to show that the actual situation is as good as one might hope on the basis of the above two inequalities, namely that RPRS(γ) is solvable whenever RPIS is solvable and we have both

$$\gamma < (1 + ||\Gamma_{P^*C_0}||^2)^{-\frac{1}{2}} \tag{5.8}$$

and

$$\gamma < \sin\phi(\mathcal{P}(\lambda), \mathcal{K} \cap \mathcal{M}(\lambda)) \quad \forall\lambda \in \sigma(A_{22}). \tag{5.9}$$

Our strategy to show this will be as follows. First, we use the well-known parametrization of all suboptimal solutions to the Nehari problem in terms of a norm-bounded parameter in order to transform the problem into a boundary Nevanlinna-Pick problem; this will require, of course, a translation of the interpolation data for the Nehari problem into interpolation constraints on the parameter. We then show that the Nevanlinna-Pick problem is solvable by proving that an associated Pick matrix is positive definite. It should be noted that alternative approaches would be possible, for instance by adapting the method of [21] to the case at hand. We believe that the derivation below provides a reasonably transparent route.

First we introduce some convenient notation and a rescaling. We shall write

$$R(s) = P^*(s)C_0(s) \tag{5.10}$$

and

$$W(s) = R(s) - \Psi(s)\tilde{H}(s), \tag{5.11}$$

so we let $W(s)$ play the role of a parameter rather than $\Psi(s)$. By a suitable rescaling, we may assume that the given bound is 1. Therefore, what we need to prove can be formulated as follows (compare [21, Thm. 3]).

THEOREM 5.1 *Let $R(s)$ be a given matrix in $RL_\infty^{m\times p}$, and let $\tilde{H}(s)$ be a given nonsingular matrix in $RH_\infty^{p\times p}$ having zeros only on the finite imaginary axis. Assume that*

$$||\Gamma_R|| < 1 \tag{5.12}$$

and that

$$||R(\lambda)|_{\ker \tilde{H}(\lambda)}|| < 1 \tag{5.13}$$

for all zeros λ *of* $\tilde{H}(s)$. *Under these conditions, there exists a matrix* $W \in RL_{\infty}^{m\times p}$ *such that the following conditions hold:*

$$W - R \in RH_{\infty}^{m\times p}, \tag{5.14}$$

$$W^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} \;=\; R^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} \tag{5.15}$$

for all zeros λ *of* $\tilde{H}(s)$ *of multiplicity* r, *and*

$$||W||_{\infty} \;\leq\; 1. \tag{5.16}$$

The proof of the theorem will proceed through a number of lemmas. In the first of these, we replace the interpolation constraints (5.15) by a stronger version, which will be convenient below. Of course the replacement has to be done in such a way that in particular the norm constraint (5.16) can still be satisfied, and this is the main point of the lemma. In the scalar case, the lemma is trivial.

Lemma 5.2 *In the situation of Thm. 5.1, let* λ *be a zero of* $\tilde{H}(s)$ *of multiplicity* r. *There exist matrices* $W_0, \cdots, W_{r-1}$ *with* $||W_0|| < 1$ *such that the interpolation constraint (5.15) holds for any matrix* $W(s) \in RL_{\infty}^{m\times p}$ *such that*

$$\frac{1}{j!}W^{(j)}(\lambda) = W_j \qquad (j = 0, \cdots, r-1). \tag{5.17}$$

Proof For this proof we use the representation of interpolation data by means of right null chains as in [2, § 1.2]. Choose a canonical set of right null chains for $\tilde{H}(s)$ at λ

$$x_{10}, \cdots, x_{1,r_1-1}; x_{20}, \cdots, x_{2,r_2-1}; \;\cdots\; ; x_{k0}, \cdots, x_{k,r_k-1}$$

with $r_1 \geq \cdots \geq r_k$. Introduce the indices $\mu_j = \max\{\ell \mid r_\ell > j\}$, and for $0 \leq i \leq j \leq r-1$ form the matrices

$$X_{ij} \;=\; [x_{1i} \;\cdots\; x_{\mu_j i}].$$

A matrix function $W(s)$ will satisfy the interpolation constraints (5.15) if it satisfies (5.17) and

$$W_0 X_{00} \;=\; R(\lambda)X_{00} \tag{5.18}$$

$$W_1 X_{01} + W_0 X_{11} \;=\; R'(\lambda)X_{01} + R(\lambda)X_{11} \tag{5.19}$$

$$\vdots$$

$$W_{r-1}X_{0,r-1} + \cdots + W_0 X_{r-1,r-1} \;=\; \frac{1}{(r-1)!}R^{(r-1)}(\lambda)X_{0,r-1} + \cdots + R(\lambda)X_{r-1,r-1}. \tag{5.20}$$

The vectors $x_{10}, \cdots, x_{k0}$ are linearly independent and span the space $\ker \tilde{H}(\lambda)$ [2, Prop. 1.2.2]. The matrix X_{00} is therefore a basis matrix for $\ker \tilde{H}(\lambda)$ and because of the assumption (5.13) we can find a matrix W_0 of norm less than one such that (5.18) holds. The matrices $X_{01}, \cdots, X_{0,r-1}$ are of full column rank as well and so we can solve the equations (5.19–5.20) recursively to get $W_1, \cdots, W_{r-1}$. The matrices so obtained satisfy the conditions of the lemma. □

The solutions to the Nehari problem can be parametrized as

$$W = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1} \tag{5.21}$$

where the matrix

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix} \tag{5.22}$$

can be explicitly constructed from state space data for $R(s)$ [17, 4, 5]. Our next step will be to give interpolation data on the parameter G in order for W as determined by (5.21) to satisfy (5.17). We do this first without regard to the norm constraints.

Lemma 5.3 *Let matrices $W_0, \cdots, W_{r-1}$ of size $m \times p$ be given, and let a matrix function $\Theta(s)$ of size $(m+p) \times (m+p)$ be given as in (5.22). Define matrices W^r and F^r of size $rp \times m$ by*

$$W^r = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_{r-1} \end{bmatrix}, \qquad F^r = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let $\lambda \in \mathbb{C}$ be given, and assume that $\Theta(\lambda)$ is invertible. Define matrices N^r and D^r by

$$Mi \begin{bmatrix} N^r \\ D^r \end{bmatrix} = (\Theta^{[r]}(\lambda))^{-1} Mi \begin{bmatrix} W^r \\ F^r \end{bmatrix}. \tag{5.23}$$

If now $G(s)$ is an $m \times p$ matrix function such that $\Theta_{21}(s)G(s) + \Theta_{22}(s)$ is nonsingular and

$$G^{[r]}(\lambda) D^r = N^r, \tag{5.24}$$

then the matrix function $W(s)$ defined by (5.21) satisfies

$$\frac{1}{j!} W^{(j)}(\lambda) = W_j \quad (j = 0, 1, \cdots, r-1). \tag{5.25}$$

Proof The equation (5.21) may be written in the form

$$\begin{bmatrix} W \\ I \end{bmatrix} = \Theta \begin{bmatrix} G \\ I \end{bmatrix} (\Theta_{21} G + \Theta_{22})^{-1} \tag{5.26}$$

and (5.24) implies

$$\begin{bmatrix} G \\ I \end{bmatrix}^{[r]} (\lambda)\, D^r = Mi \begin{bmatrix} G^{[r]}(\lambda) \\ I \end{bmatrix} D^r = Mi \begin{bmatrix} N^r \\ D^r \end{bmatrix}. \tag{5.27}$$

Therefore, we can rewrite (5.23) as

$$\begin{aligned} Mi \begin{bmatrix} W^r \\ F^r \end{bmatrix} &= \Theta^{[r]}(\lambda)\, Mi \begin{bmatrix} N^r \\ D^r \end{bmatrix} \\ &= \Theta^{[r]}(\lambda) \begin{bmatrix} G \\ I \end{bmatrix}^{[r]} (\lambda) D^r \qquad (5.28) \\ &= \begin{bmatrix} W \\ I \end{bmatrix}^{[r]} (\lambda)\, (\Theta_{21} G + \Theta_{22})^{[r]}(\lambda)\, D^r \\ &= Mi \begin{bmatrix} W^{[r]}(\lambda) \\ I \end{bmatrix} (\Theta_{21} G + \Theta_{22})^{[r]}(\lambda)\, D^r. \qquad (5.29) \end{aligned}$$

From the resulting equations

$$W^r = W^{[r]}(\lambda)(\Theta_{21} G + \Theta_{22})^{[r]}(\lambda) D^r \tag{5.30}$$
$$F^r = (\Theta_{21} G + \Theta_{22})^{[r]}(\lambda) D^r \tag{5.31}$$

we immediately get

$$W^r = W^{[r]}(\lambda) F^r \tag{5.32}$$

which is the same as (5.25). □

In connection with the norm constraint in the parametrization of the suboptimal solutions of the Nehari problem, the matrix $\Theta(s)$ is required to be J-unitary, where

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

In the next lemma we translate the assumption $||W_0|| < 1$ on the interpolation data for the parameter W to an assumption on the interpolation data for the parameter G. The argument in the proof is a standard one.

Lemma 5.4 *In the situation of the previous lemma, suppose that $\Theta(s)$ is J-unitary and that $||W_0|| < 1$. We then have*

$$N_0^* N_0 < D_0^* D_0, \tag{5.33}$$

where N_0 and D_0 denote the upper blocks in the matrices N^r and D^r respectively.

Proof From the definition (5.23), it follows that in particular

$$\begin{bmatrix} W_0 \\ I \end{bmatrix} = \Theta(\lambda) \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}, \tag{5.34}$$

so that

$$\begin{aligned} N_0^* N_0 - D_0^* D_0 &= \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}^* J \begin{bmatrix} N_0 \\ D_0 \end{bmatrix} = \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}^* \Theta^*(\lambda) J \Theta(\lambda) \begin{bmatrix} N_0 \\ D_0 \end{bmatrix} \\ &= \begin{bmatrix} W_0 \\ I \end{bmatrix}^* J \begin{bmatrix} W_0 \\ I \end{bmatrix} = W_0^* W_0 - I < 0. \end{aligned} \tag{5.35}$$

□

The suboptimal solutions of the Nehari problem are obtained by using a parameter G in (5.21) that satisfies the norm constraint $||G||_\infty < 1$. After the reformulations of the preceding lemmas, we see that we can get a solution of the original problem if we can find a matrix function $G(s)$ in $RH_\infty^{m \times p}$ such that $||G||_\infty < 1$ and G satisfies the interpolation constraints (5.24) at a number of points λ on the imaginary axis, where we may assume that (5.33) holds. This is the *boundary Nevanlinna-Pick* problem. The fact that the boundary NP problem comes up in connection with regulation constraints in an H_∞-context has been recognized before by Sugie and Hara [33]. The following lemma can be seen as an extension of [33, Lemma B].

Lemma 5.5 *Let there be given numbers λ_i $(i = 1, \cdots, n)$, all on the imaginary axis, and matrices*

$$D_i = \begin{bmatrix} D_{i0} \\ D_{i1} \\ \vdots \\ D_{i,r_i-1} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{i0} \\ N_{i1} \\ \vdots \\ N_{i,r_i-1} \end{bmatrix}, \quad i = 1, \cdots, n. \tag{5.36}$$

There exists an RH_∞-matrix $G(s)$ such that $||G||_\infty < 1$ and

$$G^{[r_i]}(\lambda_i) D_i = N_i \quad (i = 1, \cdots, n) \tag{5.37}$$

if and only if

$$N_{i0}^* N_{i0} < D_{i0}^* D_{i0} \quad (i = 1, \cdots, n). \tag{5.38}$$

PROOF We only sketch the proof here; a detailed proof is provided in the Appendix (§8.6). Consider the (more demanding) problem of finding a function $G(s)$ that is analytic and less than one in modulus on a region $\{s \in \mathbb{C} \mid \operatorname{Re} s \geq -\varepsilon\}$ $(\varepsilon > 0)$ and that satisfies the interpolation constraints. This problem is no longer a *boundary* NP problem since the interpolation points are now inside the region of analyticity, and so one can form the Pick matrix which of course depends on ε. The problem is solvable if and only if this Pick matrix, which we shall denote by $P(\varepsilon)$, is positive definite. Upon examining the behavior of the elements of $P(\varepsilon)$ as ε tends to zero, one finds that the diagonal elements tend to $+\infty$ whereas the off-diagonal elements remain bounded. Therefore, $P(\varepsilon)$ is guaranteed to be positive definite for sufficiently small ε and the problem is solved. □

Putting all the lemmas together, it is now easy to get a proof of the theorem.

PROOF (of Thm. 5.1) For each zero λ of $\tilde{H}(s)$ of multiplicity r, construct matrices $W_0, \cdots, W_{r-1}$ as in Lemma 5.2. From these, construct interpolation data for the parameter $G(s)$ as in Lemma 5.3. It follows from Lemma 5.5 and Lemma 5.4 that these interpolation data can be satisfied by an RH_∞-matrix $G(s)$ of H_∞-norm less than one. By the parametrization of solutions to the suboptimal Nehari problem (see for instance [5]), the matrix $W(s)$ given by (5.21) satisfies the conditions of the theorem. □

This leads to the main result of the paper.

THEOREM 5.6 *Consider the problems RPIS, RSP(γ), and RPRS(γ), as described in section 2. Define subspace-valued functions $\mathcal{P}(s)$ and $\mathcal{M}(s)$ by (2.7) and (2.8) respectively, and define $\mathcal{K}$ by (3.10). Under assumptions (A1–A4), the problem RPRS(γ) is solvable if and only if the following conditions hold:*

(i) RPIS is solvable;

(ii) RSP(γ) is solvable;

(iii) $\gamma < \sin \phi(\mathcal{P}(\lambda), \mathcal{K} \cap \mathcal{M}(\lambda))$ *for all exosystem poles* λ.

PROOF The necessity of conditions (i) and (ii) is obvious from the problem formulation, and the necessity of (iii) was proven in section 3 (Thm. 3.2 and Lemma 3.5). Assume now that (i–iii) hold. Let $P(s)$ and $\tilde{P}(s)$ denote image and kernel representations for the function $\mathcal{P}(s)$, normalized as in (5.1), and let $C_0(s)$ be an image representation for a particular solution of RPIS, normalized such that $\tilde{P}(s)C_0(s) = I$. Construct $\tilde{M}(s)$ as in the proof of Lemma 4.1, and compute $\tilde{H}(s)$ such that $\tilde{M}(s) = \tilde{H}(s)\tilde{P}(s)$. Note that

$$\tilde{P}(\lambda)\mathcal{M}(\lambda) = \tilde{P}(\lambda) \ker \tilde{H}(\lambda)\tilde{P}(\lambda) = \ker \tilde{H}(\lambda) \tag{5.39}$$

for all $\lambda \in \mathbb{C}^+$, because $\tilde{P}(\lambda)$ is surjective for all such λ by the stabilizability and detectability assumptions. Define $\alpha = \frac{1}{\gamma}\sqrt{1-\gamma^2}$, and write $R(s) = \frac{1}{\alpha}P^*(s)C_0(s)$. It follows from (iii) with the formula (3.24) that $\|R(\lambda)|_{\ker \tilde{H}(\lambda)}\| < 1$ for all exosystem poles λ. Also, (ii) implies that $\|\Gamma_R\| < 1$. Compute the matrix $\Theta(s)$ in (5.22) from $R(s)$ as indicated for instance in [5]. For each zero λ of $\tilde{H}(s)$ of multiplicity r, compute matrices $W_0, \cdots, W_{r-1}$ as in the proof of Lemma 5.2, and from these compute interpolation data $(D_0, \cdots, D_{r-1})$ and $(N_0, \cdots, N_{r-1})$ as in Lemma 5.3. Collecting all the data from the various zeros of $\tilde{H}(s)$, compute a matrix $G(s)$ as in Lemma 5.5. Next, find $W(s)$ from $G(s)$ by (5.21). The matrix $W(s)$ will then satisfy conditions (5.14–5.16), and it follows that $C(s) = C_0(s) + \alpha P(s)(W(s) - R(s))$ provides an image representation for a solution of RPRS(γ). □

The sufficiency part of the proof is constructive. In the next section, we illustrate the computational procedure by an example.

6 Example

To illustrate the methods of this paper in the simplest possible context, let us consider a first-order system that is to be regulated against a constant disturbance. After scaling, the equations can be written in the form

$$\dot{x}_1 = ax_1 + x_2 + u \tag{6.1}$$
$$\dot{x}_2 = 0 \tag{6.2}$$
$$y = x_1. \tag{6.3}$$

The assumptions (A1–A4) are satisfied for all values of a. The subspace-valued function $\mathcal{M}(s)$ defined in (2.8) is given by

$$\mathcal{M}(s) = \ker\,[s(s-a) \quad -s]. \tag{6.4}$$

This expression can be obtained symbolically by eliminating x_1 and x_2 from the equations $(s-a)x_1 = x_2 + u$, $sx_2 = 0$, and $y = x_1$, considering s as a non-cancellable parameter. If we do allow cancellation, we get the expression for $\mathcal{P}(s)$ as in (2.7):

$$\mathcal{P}(s) = \ker\,[s-a \quad -1]. \tag{6.5}$$

The upper bound on the achievable robustness of stability imposed by the regulation constraint is obtained from (5.7), noting that $\mathcal{M}(0) = \mathbb{C}^2 = \mathcal{Y} \times \mathcal{U}$ and $\mathcal{K} = \mathrm{im}\,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$:

$$\sin\phi(\mathcal{P}(0), \mathcal{K}) = \sin\phi(\mathrm{im}\begin{bmatrix}-1\\a\end{bmatrix}, \mathrm{im}\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{\sqrt{1+a^2}}. \tag{6.6}$$

The achievable robustness of stability, not taking into account the regulation constraint, can be computed from either side of (5.5). In view of the fact that a particular stabilizing controller $C_0(s)$ will have to be computed anyway, there is perhaps no clear preference for either method of computation. First we need normalized image and kernel representations for $\mathcal{P}(s)$; these are given by

$$P(s) = \frac{1}{s+\sqrt{a^2+1}}\begin{bmatrix}1\\s-a\end{bmatrix} \tag{6.7}$$

and

$$\tilde{P}(s) = \frac{1}{s+\sqrt{a^2+1}}[s-a \quad -1]. \tag{6.8}$$

An image representation $C_0(s)$ for a stabilizing controller, normalized such that $\tilde{P}(s)C_0(s) = I$, is found by solving the equation

$$(s-a)c_{01}(s) - c_{02}(s) = s + \sqrt{a^2+1} \tag{6.9}$$

in the unknown RH_∞-functions $c_{01}(s)$ and $c_{02}(s)$. A simple solution is provided by

$$C_0(s) = \begin{bmatrix}1\\-a-\sqrt{a^2+1}\end{bmatrix}. \tag{6.10}$$

We get

$$(1 + ||\Gamma_{P^*C_0}||^2)^{-\frac{1}{2}} = (1 - ||\Gamma_{\tilde{P}^*}||^2)^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{2}(1 - \frac{a}{\sqrt{a^2+1}})^{\frac{1}{2}}. \tag{6.11}$$

A somewhat more attractive formulation is obtained if we re-parametrize by setting

$$a = \cot\theta, \qquad 0 < \theta < \pi. \tag{6.12}$$

Note that $a \to \infty$ as $\theta \downarrow 0$, and $a \to -\infty$ as $\theta \uparrow \pi$. After some goniometry, we find for the two upper bounds:

$$\sin \phi(\mathcal{P}(0), \mathcal{K}) \;=\; \sin \theta \tag{6.13}$$

$$(1 + \|\Gamma_{P^* C_0}\|^2)^{-\frac{1}{2}} \;=\; \sin \tfrac{1}{2}\theta. \tag{6.14}$$

Note that $\sin \theta > \sin \frac{1}{2}\theta$ for $0 < \theta < \frac{2}{3}\pi$, whereas the inequality is reversed for $\frac{2}{3}\pi < \theta < \pi$; the value $\theta = \frac{2}{3}\pi$ corresponds to $a = -\frac{1}{3}\sqrt{3}$. Taking into account that $-a^{-1}$ is the dc gain of the system (6.1) and that this quantity is not affected by the scaling we used to arrive at the form (6.1), we get the following conclusion:

> *for a first-order system regulated against a constant disturbance, the regulation requirement is restrictive with respect to the achievable robustness of stability if and only if the system is open-loop stable with a dc gain less than $\sqrt{3}$.*

Of course, similar rules may be derived for higher-order systems and for other types of regulation constraints.

Let us now proceed to the calculation of an actual controller. In order to simplify matters even further, we shall from now on assume that the parameter a in (6.1) is equal to zero. According to the rule given above, the regulation requirement is in this case not restrictive with respect to robustness of stability and so we should be able to get arbitrarily close to the optimal margin of stability while at the same time achieving regulation against constant disturbances. Specifically, the upper bound given by (6.6) is 1, whereas the one given by (6.11) is $\frac{1}{2}\sqrt{2}$. We shall compute a controller that achieves a robustness margin of at least γ, where γ is a given number less than $\frac{1}{2}\sqrt{2}$, and that at the same time satisfies the regulation requirement.

Taking the particular stabilizing controller $C_0(s)$ of (6.10), we get from the Kučera-Youla parametrization the following general form for an image representation of a stabilizing controller:

$$C(s) \;=\; \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{Q(s)}{s+1}\begin{bmatrix} 1 \\ s \end{bmatrix}, \tag{6.15}$$

where $Q(s)$ is an arbitrary RH_∞-function. The regulation requirement (3.12) is in the present case

$$\mathcal{C}(0) \;\subset\; \text{im}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.16}$$

which will be satisfied for a representation of the form (6.15) if and only if $Q(0) = 1$. Therefore a particular solution to RPIS is given (with a change of notation) by

$$C_0(s) \;=\; \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{s+1}\begin{bmatrix} 1 \\ s \end{bmatrix} \;=\; \frac{1}{s+1}\begin{bmatrix} s \\ -(2s+1) \end{bmatrix} \tag{6.17}$$

and the general solution to RPIS is

$$C(s) \;=\; \frac{1}{s+1}\begin{bmatrix} s \\ -(2s+1) \end{bmatrix} - \frac{s\Psi(s)}{s+1}\begin{bmatrix} 1 \\ s \end{bmatrix} \tag{6.18}$$

where $\Psi(s)$ is an arbitrary RH_∞-function. This is in line with the result in Thm. 4.2; note that the function $\tilde{H}(s)$ is in the present case given by $\tilde{H}(s) = s$ (see (6.4) and (6.5)). Now, define

$$\alpha = \frac{1}{\gamma}\sqrt{1-\gamma^2} > 1 \tag{6.19}$$

and write

$$R(s) \;=\; \frac{1}{\alpha}P^*(s)C_0(s) \;=\; \frac{1}{\alpha}\frac{2s}{1-s} \;=\; \frac{2}{\alpha}\frac{1}{1-s} - \frac{2}{\alpha}. \tag{6.20}$$

We are now looking for a Nehari extension $W(s)$ of $R(s)$ that satisfies the norm bound $||W||_\infty \leq 1$ and the interpolation constraint (5.15), which in this case comes down to

$$W(0) = 0. \tag{6.21}$$

All extensions satisfying the norm constraint are given by

$$W(s) \;=\; (\Theta_{11}(s)G(s) + \Theta_{12}(s))(\Theta_{21}(s)G(s) + \Theta_{22}(s))^{-1} \tag{6.22}$$

where $G(s)$ is an arbitrary RH_∞-function of norm less than 1 and where the matrix $\Theta(s)$ is computed from the formulas in [5]:

$$\Theta(s) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2}{(\alpha^2-1)(s^2-1)} \begin{bmatrix} -(s+1) & -\alpha(s+1) \\ \alpha(s-1) & s-1 \end{bmatrix}. \tag{6.23}$$

The next step is to translate the interpolation constraint on $W(s)$ into one on the parameter $G(s)$. The general procedure for doing this is given by Lemma 5.3; in the present case we obtain from

$$\begin{bmatrix} W(0) \\ 1 \end{bmatrix} \;=\; \Theta(0) \begin{bmatrix} G(0) \\ 1 \end{bmatrix} \tag{6.24}$$

the constraint

$$G(0) \;=\; \frac{-2\alpha}{\alpha^2+1}. \tag{6.25}$$

Now we have to solve the boundary Nevanlinna-Pick problem of finding an RH_∞-function of norm less than one that satisfies (6.25). This is not at all difficult; we can simply take the constant solution

$$G(s) \;=\; \frac{-2\alpha}{\alpha^2+1}. \tag{6.26}$$

From this we get

$$W(s) \;=\; \frac{s+1}{s-1}\,\frac{-2\alpha s}{(\alpha^2+1)s+\alpha^2-1} \tag{6.27}$$

which gives

$$R(s) - W(s) \;=\; \frac{2}{\alpha}\,\frac{s}{(\alpha^2+1)s+\alpha^2-1}. \tag{6.28}$$

The result is in RH_∞ and is a multiple of s, as it should be. The parameter $\Psi(s)$ in (6.18) becomes (see (5.11))

$$\Psi(s) \;=\; \frac{\alpha}{s}(R(s)-W(s)) \;=\; \frac{2}{(\alpha^2+1)s+\alpha^2-1}. \tag{6.29}$$

Inserting this in (6.18), we obtain

$$C(s) \;=\; \frac{1}{(\alpha^2+1)s+\alpha^2-1} \begin{bmatrix} (\alpha^2+1)s \\ -2\alpha^2 s - \alpha^2 + 1 \end{bmatrix}. \tag{6.30}$$

This is our final solution. The controller transfer function is given by

$$c(s) \;=\; \frac{-2\alpha^2 s - \alpha^2 + 1}{(\alpha^2+1)s} \tag{6.31}$$

and has a pole at 0, as it should have to satisfy the regulation requirement. The actual robustness margin achieved by the above controller is

$$\sin\phi(\mathcal{P},\mathcal{C}) \;=\; \left(1 + \left[\frac{2\alpha}{\alpha^2+1}\right]^2 \alpha^2\right)^{-\frac{1}{2}} \tag{6.32}$$

and this is indeed better than the required margin $\gamma = (1+\alpha^2)^{-\frac{1}{2}}$. If we let γ tend to its optimal value $\frac{1}{2}\sqrt{2}$ then α tends to 1 and the controller tends to $C(s) = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$; this controller optimizes the robustness margin, but it no longer satisfies the regulation constraint.

7 Conclusions

The problem of optimizing the robustness of stability with respect to coprime factor perturbations was posed by Vidyasagar and Kimura [35] and was reduced in that paper to a certain H_∞ optimization problem. Later on it was shown by Glover and McFarlane [18] that, if the perturbations are taken with respect to *normalized* coprime factors, an exact solution can be obtained in a relatively simple way. The fact that the optimal robust stabilization problem is such a nice one came as a surprise at the time. In the present paper we show that the problem even remains nice if we add regulation constraints to it; in view of the generally adverse behavior of optimization problems when side constraints are added, this outcome may be viewed as a new surprise.

Our techniques in this paper have relied in particular on the use of subspace-valued functions associated both with plant and controller. These can be seen as a multivariable generalization of the Nyquist curve, as may be argued in a mathematical sense using the identification of the extended complex plane with the Grassmannian $G^1(\mathbb{C}^2)$. The present paper however demonstrates more than that: it shows that the multivariable Nyquist curve continues to play the role of a mediator between various design objectives, as the scalar version does in classical control theory.

One modern approach towards integration of various design objectives is the loop shaping approach developed by Glover and co-workers [27], which already has seen several successful applications in particular to controller design for aircraft (see also [22, 29]). In this approach, the construction of robustly stabilizing controllers is used as the basic synthesis procedure. The present paper shows when and how it is possible to incorporate regulation requirements in this design method.

A noticeable difference between the scalar and the multivariable versions of the Nyquist curve is that geometry plays a much larger role in the latter than in the former. The reason for this is that the scalar version can be interpreted as a function whose values are one-dimensional subspaces of a two-dimensional space, and not much subspace geometry is possible in two dimensions; in particular there are no subspaces inclusions which are not trivial in one way or another. In the multivariable case, however, we do have a nontrivial setting, leading to formulations that would seem contrived in the scalar case. The role of finite-dimensional geometry is clear for instance in the characterization that we gave of the upper bound on the achievable robustness of stability due to regulation requirements.

We have not strived for maximal generality in this paper, in order to present the main ideas in a transparent way. It will be shown in future work that the full column rank assumption (A4) can be dispensed with. Also of interest is to consider the case in which the to-be-controlled outputs are not necessarily the same as the observed outputs. Another extension that suggests itself is to incorporate robustness of regulation ([37, Ch. 8], [34, §7.5]).

8 Appendix

This appendix contains a number of results that are ancillary to the main development of this paper; some of them may however be of independent interest. There are six subsections, devoted in order to stability, representations, the blow-up of a matrix function, a geometric lemma, a divisibility result, and the proof of Lemma 5.5.

8.1 Closed-loop stability

Since we start in this paper from a state space context, we insert the following lemma; compare [32] for a polynomial version. The proof uses the well-known fact (see for instance [23, p. 650]) that a square matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

in which the block A_{22} is invertible, is invertible itself if and only if the Schur complement $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is invertible.

LEMMA 8.1 *The closed-loop system consisting of a plant of the form*

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{8.1}$$

$$y(t) = Cx(t) \tag{8.2}$$

and a compensator of the form (2.4–2.5) is stable if and only if for each s in $\mathbb{C}$ with $\operatorname{Re} s \geq 0$ the two subspaces

$$\ker \begin{bmatrix} sI - A & 0 & 0 & -B \\ -C & 0 & I & 0 \end{bmatrix} \tag{8.3}$$

and

$$\ker \begin{bmatrix} 0 & sI - F & -G & 0 \\ 0 & -H & -J & I \end{bmatrix} \tag{8.4}$$

are complementary.

PROOF The closed-loop system matrix is

$$A_e = \begin{bmatrix} A + BJC & BH \\ GC & F \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} 0 & -B \\ -G & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -J & I \end{bmatrix}^{-1} \begin{bmatrix} -C & 0 \\ 0 & -H \end{bmatrix},$$

so that $sI - A_e$ is invertible for all s in the closed right half plane if and only if the matrix

$$\begin{bmatrix} sI - A & 0 & 0 & -B \\ 0 & sI - F & -G & 0 \\ -C & 0 & I & 0 \\ 0 & -H & -J & I \end{bmatrix}$$

has the same property. This in turn is equivalent to the condition in the statement of the lemma. □

Note that the subspace in (8.3) is entirely determined by the system parameters, whereas the one in (8.4) is determined by the compensator parameters. We now let $\mathcal{P}(s)$ and $\mathcal{C}(s)$ denote the projections of the two subspaces on the direct sum of the input space $\mathcal{U}$ and the output space $\mathcal{Y}$, that is to say:

$$\mathcal{P}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists x \text{ s.t. } \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\} \tag{8.5}$$

$$\mathcal{C}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists z \text{ s.t. } \begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0 \right\}. \tag{8.6}$$

Two alternative ways of writing $\mathcal{P}(s)$ are

$$\mathcal{P}(s) = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}^{-1} \operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \ker \,[sI - A \quad -B]. \tag{8.7}$$

In state space terms, the characterization of closed-loop stability in terms of complementarity is now proved as follows.

LEMMA 8.2 *Let a plant (8.1–8.2) and a compensator (2.4–2.5) be given, and suppose that both are stabilizable and detectable. Under these conditions, the closed-loop system is stable if and only if the subspaces $\mathcal{P}(s)$ and $\mathcal{C}(s)$ defined in (8.5) and (8.6) respectively are complementary for all s in $\mathbb{C}$ with $\operatorname{Re} s \geq 0$.*

PROOF It follows from Lemma 2.3 in [10] that $\dim \mathcal{P}(s) = \dim \mathcal{U}$ and $\dim \mathcal{C}(s) = \dim \mathcal{Y}$ for all s in the closed right half plane. To prove complementarity of the two subspaces, it therefore suffices to show that they intersect only in zero. Suppose to the contrary that, for some λ with $\mathrm{Re}\,\lambda \geq 0$, the intersection $\mathcal{P}(\lambda) \cap \mathcal{C}(\lambda)$ contains a nonzero vector $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. By definition, this means that there exists an x such that

$$\begin{bmatrix} \lambda I - A & 0 & -B \\ C & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \tag{8.8}$$

and a z such that

$$\begin{bmatrix} \lambda I - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0. \tag{8.9}$$

But then obviously

$$\begin{bmatrix} x \\ z \\ y \\ u \end{bmatrix} \in \ker \begin{bmatrix} \lambda I - A & 0 & 0 & -B \\ -C & 0 & I & 0 \end{bmatrix} \cap \ker \begin{bmatrix} 0 & \lambda I - F & -G & 0 \\ 0 & -H & -J & I \end{bmatrix} \tag{8.10}$$

which shows, by the previous lemma, that the closed-loop system is not stable. The converse part of the proof is obtained by reversing this reasoning. □

REMARK 8.3 If the plant is not strictly proper and is given by state space parameters (A, B, C, D), the description of $\mathcal{P}(s)$ is modified in the obvious way, and $\mathcal{P}(\infty)$ is given by $\ker\,[-I \quad D]$. The statement of the above lemma is then changed to: the closed-loop system is stable *and well-posed* if and only if the subspaces $\mathcal{P}(s)$ and $\mathcal{C}(s)$ are complementary for all s in the *extended* closed right half plane (cf. [32]).

8.2 Representations

Matrix fractional descriptions can be viewed as representations of subspace-valued functions. The relation with the state-space representation is given by the following lemma (see also [10, Lemma 2.4], where an alternative proof is given).

LEMMA 8.4 *Consider a set of state space parameters (A, B, C, D), and assume that (A, B) is stabilizable and that (C, A) is detectable. Let $N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ be respectively a right and a left coprime factorization over RH_∞ of the transfer matrix $G(s) = C(sI - A)^{-1}B + D$. Under these conditions, one has*

$$\mathrm{im} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \ker\,[\tilde{D}(s) \quad -\tilde{N}(s)] = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \ker\,[sI - A \quad -B] \tag{8.11}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$, and

$$\mathrm{im} \begin{bmatrix} N(\infty) \\ D(\infty) \end{bmatrix} = \ker\,[\tilde{D}(\infty) \quad -\tilde{N}(\infty)] = \mathrm{im} \begin{bmatrix} D \\ I \end{bmatrix}. \tag{8.12}$$

PROOF All functions appearing in (8.11) are continuous as mappings from the closed right half plane to the Grassmannian manifold of m-dimensional subspaces of $\mathcal{Y} \times \mathcal{U}$; the extension indicated in (8.12) even makes all functions continuous as mappings from the extended right half plane (including the point at infinity) to the Grassmannian. For the state space representation, this follows from the stabilizability and detectability assumptions, see [10]; concerning the image and kernel representations, confer [25]. For all points s in the right half plane that are not eigenvalues of A, it is easily seen that all entries in

(8.11) are just alternative ways of writing im $\begin{bmatrix} G(s) \\ I \end{bmatrix}$, so that equality holds in these points. But since A has only finitely many eigenvalues, equality must then by continuity hold everywhere in $\mathbb{C}^+$. □

The formulas (8.11) and (8.12) define a subspace-valued function associated to a system given either by a left coprime factorization or by a right coprime factorization or by state space parameters.

8.3 The blow-up

In this section we collect some simple properties of the blow-up of a matrix function that was introduced in section 3. We begin with a product formula.

LEMMA 8.5 *For any matrix functions $T(s) \in \mathbb{R}^{p\times m}(s)$ and $S(s) \in \mathbb{R}^{m\times \ell}(s)$ and any $r = 1, 2, \cdots$, one has*

$$(TS)^{[r]}(s) \;=\; T^{[r]}(s)S^{[r]}(s). \tag{8.13}$$

PROOF This is immediate from the definition, since $T(s)(S(s)x(s)) = (TS)(s)x(s)$. One may also give a more computational proof based on the expression (3.5), using the Leibniz rule for derivatives of products:

$$\frac{1}{k!}(TS)^{(k)}(s) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}T^{(j)}(s)S^{(k-j)}(s) = \sum_{j=0}^{k}\frac{1}{j!}T^{(j)}(s)\frac{1}{(k-j)!}S^{(k-j)}(s). \tag{8.14}$$

□

The blow-up does not commute with matrix partitioning; indeed, if A and B are linear mappings from $\mathcal{X}$ to $\mathcal{Z}$ and from $\mathcal{Y}$ to $\mathcal{Z}$ respectively, then $[A \;\; B]^{[r]}$ is a mapping from $(\mathcal{X}\times\mathcal{Y})^r$ to $\mathcal{Z}^r$, but $[A^{[r]} \;\; B^{[r]}]$ is a mapping from $\mathcal{X}^r \times \mathcal{Y}^r$ to $\mathcal{Z}^r$. To get a proper correspondence we need an operator from $\mathcal{X}^r \times \mathcal{Y}^r$ to $(\mathcal{X}\times\mathcal{Y})^r$ that we shall call the *mingling* operator. It is defined by

$$Mi : (x_1, \cdots, x_r, y_1, \cdots, y_r) \;\mapsto\; (x_1, y_1, \cdots, x_r, y_r). \tag{8.15}$$

We shall use the mingling operator between various spaces and even use its obvious generalization to products of more than two factors, employing the same symbol Mi every time; this severe abuse of notation should cause no confusion. The following lemma is given without proof.

LEMMA 8.6 *For matrix functions $A(s)$ and $B(s)$ with the same domain space, we have*

$$\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}^{[r]} \;=\; Mi\begin{bmatrix} A^{[r]}(s) \\ B^{[r]}(s) \end{bmatrix}. \tag{8.16}$$

For matrix functions $A(s)$ and $B(s)$ with the same codomain space, we have

$$[A(s) \;\; B(s)]^{[r]} \;=\; [A^{[r]}(s) \;\; B^{[r]}(s)]Mi^{-1}. \tag{8.17}$$

LEMMA 8.7 *Consider matrix functions $T(s)$ and $\tilde{T}(s)$ that are analytic on a neighborhood of $\lambda \in \mathbb{C}\cup\{\infty\}$. Let r be any positive integer. If $T(\lambda)$ has full column rank, then the same holds for $T^{[r]}(\lambda)$, and if $\tilde{T}(\lambda)$ has full row rank, then the same is true for $\tilde{T}^{[r]}(s)$. If moreover $\ker\tilde{T}(s) = \operatorname{im} T(s)$ for all s in a neighborhood of λ, then $\ker\tilde{T}^{[r]}(\lambda) = \operatorname{im} T^{[r]}(\lambda)$ for all $r\in\mathbb{N}$.*

PROOF The first claim is immediate from the matrix form of $T^{[r]}(s)$ and $\tilde{T}^{[r]}(s)$ (see (3.5)). If now $\ker\tilde{T}(s) = \operatorname{im} T(s)$ for all s in a neighborhood of λ, then $\tilde{T}(s)T(s) = 0$ so that $\tilde{T}^{[r]}(s)T^{[r]}(s) = 0$ which implies that $\operatorname{im} T^{[r]}(\lambda) \subset \ker\tilde{T}^{[r]}(\lambda)$. By the full rank assumptions and because $\dim\ker\tilde{T}(\lambda) = \dim\operatorname{im} T(\lambda)$, we also have $\dim\ker\tilde{T}^{[r]}(\lambda) = \dim\operatorname{im} T^{[r]}(\lambda)$ so that actually equality must hold. □

LEMMA 8.8 *Let $T_1(s)$ and $T_2(s)$ be RH_∞-matrices. If $\operatorname{im} T_1(s) = \operatorname{im} T_2(s)$ for $s \in \mathbb{C}^+$ and both $T_1(s)$ and $T_2(s)$ have full column rank everywhere on $\mathbb{C}^+$, then $\operatorname{im} T_1^{[r]}(s) = \operatorname{im} T_2^{[r]}(s)$ for all $s \in \mathbb{C}^+$. An analogous statement is true for kernel representations.*

PROOF Under the stated conditions, there exists an RH_∞-unimodular matrix $U(s)$ such that $T_1(s) = T_2(s)U(s)$ for all $s \in \mathbb{C}^+$ (this is essentially the standard uniqueness theorem for right coprime factorizations). From this we get $T_1^{[r]}(s) = T_2^{[r]}(s)U^{[r]}(s)$ where $U^{[r]}(s)$ is nonsingular for all $s \in \mathbb{C}^+$ by the previous lemma, and the claim follows. □

It is well-known that interpolation conditions can often be expressed as divisibility conditions (cf. for instance [2, Ch. 10]). The connection between blown-up matrix functions and divisibility is brought out by the following proposition.

PROPOSITION 8.9 *Let $Q(s) \in RH_\infty^{m\times p}$ and $H(s) \in RH_\infty^{p\times p}$, and suppose that $H(s)$ is nonsingular. Under these conditions, $Q(s)$ is right divisible by $H(s)$, in the sense that the matrix function $Q(s)H^{-1}(s)$ belongs to $RH_\infty^{m\times p}$, if and only if*

$$\ker Q^{[r]}(s) \supset \ker H^{[r]}(s) \tag{8.18}$$

for all $s \in \mathbb{C}^+$ and all $r \in \mathbb{N}$. The conclusion in fact already holds if the inclusion (8.18) is satisfied at each zero λ of $H(s)$ in $\mathbb{C}^+$, and with r equal to the multiplicity of that zero.

For the proof it is convenient to introduce the ring $A(\lambda)$ of functions analytic in a neighborhood of $\lambda \in \mathbb{C}$, and the $A(\lambda)$-module $Z_r(H;\lambda)$ defined by

$$Z_r(H;\lambda) \;=\; \{f \in A^p(\lambda) \mid (s-\lambda)^{-r}H(s)f(s) \in A^p(\lambda)\}. \tag{8.19}$$

We first prove the following lemma.

LEMMA 8.10 *In the situation of the above proposition, $Q(s)$ is right divisible by $H(s)$ if and only if*

$$Z_r(Q;\lambda) \supset Z_r(H;\lambda) \quad \forall \lambda \in \mathbb{C}^+, r \in \mathbb{N}. \tag{8.20}$$

PROOF It is clear that the condition is necessary. Assume now that (8.20) holds. We shall show that $Q(s)H^{-1}(s)f(s)$ belongs to RH_∞^m for every $f \in RH_\infty^p$. Take such an f, and suppose to the contrary that $Q(s)H^{-1}(s)f(s)$ would have a pole at some point $\lambda \in \mathbb{C}^+$. We can write $H^{-1}(s)f(s) = (s-\lambda)^{-r}g(s)$ for some $r \in \mathbb{N}$ and some $g \in RH_\infty^p$. Then $H(s)(s-\lambda)^{-r}g(s) = f(s)$ so that g belongs to $Z_r(H;\lambda)$ and hence to $Z_r(Q;\lambda)$ by (8.20). But then $Q(s)H^{-1}(s)f(s) = Q(s)(s-\lambda)^{-r}g(s)$ cannot have a pole at λ and we have a contradiction. □

The proof shows that it is sufficient to consider only the zeros of $H(s)$, and to take r equal to the multiplicity of the zero. The proof of the proposition is now easy.

PROOF (of Prop. 8.9) Given a matrix function $M(s)$, direct calculation shows that

$$Z_r(M;\lambda) \;=\; \{f = \sum_{j=0}^{\infty} f_j(s-\lambda)^j \in A(\lambda) \mid M^{[r]}(\lambda)\operatorname{col}(f_0, f_1, \cdots, f_{r-1}) = 0\}. \tag{8.21}$$

So the claim in the proposition is immediate from the above lemma. □

We note the following corollaries of the proposition.

COROLLARY 8.11 *Let $Q_1(s) \in RH_\infty^{m\times p}(s)$ and $Q_2(s) \in RH_\infty^{\ell\times p}(s)$, and suppose that $Q_2(s)$ has full generic row rank. Under these conditions, there exists a matrix function $F(s) \in RH_\infty^{p\times \ell}(s)$ such that $Q_1(s) = F(s)Q_2(s)$ if and only if*

$$\ker Q_1^{[r]}(s) \supset \ker Q_2^{[r]}(s) \tag{8.22}$$

for all $s \in \mathbb{C}^+$ *and* $r \in \mathbb{N}$.

Proof The necessity of the condition is immediate from Lemma 8.5. To show the sufficiency, write (after a column permutation, if necessary) $Q_2(s) = [Q_{21}(s) \quad Q_{22}(s)]$ where $Q_{21}(s)$ is nonsingular, and partition $Q_1(s)$ correspondingly as $[Q_{11}(s) \quad Q_{12}(s)]$. From (8.22) it follows that $\ker Q_{11}^{[r]}(s) \supset \ker Q_{21}^{[r]}(s)$. By the proposition, this implies that there exists a matrix function $F(s) \in RH_\infty^{m\times\ell}$ such that $Q_{11}(s) = F(s)Q_{12}(s)$; it remains to prove that also $Q_{12}(s) = Q_{22}(s)$. Take a rational vector $x_2(s)$ of length $\ell - p$, and define $x_1(s) = -Q_{21}^{-1}(s)Q_{22}(s)x_2(s)$. Applying (8.22) with $r = 1$, we then have $Q_{12}(s)x_2(s) = -Q_{11}(s)x_1(s) = -F(s)Q_{21}(s)x_1(s) = F(s)Q_{22}(s)x_2(s)$. Because $x_2(s)$ was arbitrary, the desired conclusion follows. □

Corollary 8.12 *Let* $Q_1(s) \in RH_\infty^{m\times p}(s)$ *and* $Q_2(s) \in RH_\infty^{\ell\times p}(s)$, *and suppose that both matrix functions have full generic row rank. Under these conditions, there exists an* RH_∞*-unimodular matrix function* $U(s)$ *such that* $Q_1(s) = U(s)Q_2(s)$ *if and only if* $m = \ell$ *and*

$$\ker Q_1^{[r]}(s) \;=\; \ker Q_2^{[r]}(s) \tag{8.23}$$

for all $s \in \mathbb{C}^+$ *and* $r \in \mathbb{N}$.

Proof The necessity follows from Lemma 8.5 and Lemma 8.7. Assume now that (8.23) holds. From the previous corollary it follows that there exist RH_∞-matrix functions $F_1(s)$ and $F_2(s)$ such that $Q_1(s) = F_2(s)Q_2(s)$ and $Q_2(s) = F_1(s)Q_1(s)$. We get $Q_2(s) = F_1(s)F_2(s)Q_2(s)$ and since $Q_2(s)$ is surjective as a mapping from $\mathbb{C}^p(s)$ to $\mathbb{C}^m(s)$, this implies that $F_1(s)F_2(s) = I$. In the same way we have $F_2(s)F_1(s) = I$ and it follows that both $F_1(s)$ and $F_2(s)$ are unimodular. □

Lemma 8.13 *Consider a set of state space parameters* $(\mathcal{X}, \mathcal{Y}, \mathcal{U}; A, B, C, D)$ *and suppose that the pair* (C, A) *is detectable. Let* Π *denote the natural projection from* $\mathcal{X}\times\mathcal{Y}\times\mathcal{U}$ *to* $\mathcal{Y}\times\mathcal{U}$. *For each* $r = 1, 2, \cdots$, *define a subspace-valued function* $\mathcal{M}^{[r]}(s)$ *by*

$$\mathcal{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & D \end{bmatrix}^{[r]}, \qquad \mathcal{M}^{[r]}(\infty) = \operatorname{im} \begin{bmatrix} D \\ I \end{bmatrix}^{[r]}. \tag{8.24}$$

Then we can find an RH_∞*-function* $\tilde{M}(s)$ *such that*

$$\mathcal{M}^{[r]}(s) = \ker \tilde{M}^{[r]}(s) \quad \forall s \in \mathbb{C}^+, r \in \mathbb{N}. \tag{8.25}$$

Moreover, if $\tilde{M}_1(s)$ *and* $\tilde{M}_2(s)$ *are both matrix functions of full generic row rank satisfying (8.25), then there exists an* RH_∞*-unimodular matrix* $U(s)$ *such that* $\tilde{M}_2(s) = U(s)\tilde{M}_1(s)$.

Proof Write $C(sI - A)^{-1} = \tilde{D}^{-1}(s)\tilde{N}(s)$ where $\tilde{D}(s)$ and $\tilde{N}(s)$ are left coprime matrices over $\mathbb{C}^+$. By the coprimeness and the detectability assumption, we have

$$\operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix} = \ker [-\tilde{N}(s) \quad \tilde{D}(s)] \tag{8.26}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$. Now define

$$\tilde{M}(s) = [-\tilde{N}(s) \quad \tilde{D}(s)] \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix} = [-\tilde{D}(s) \quad -\tilde{N}(s)B]. \tag{8.27}$$

Note that we may write

$$\mathcal{M}^{[r]}(s) = \left(\begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]} \right)^{-1} \operatorname{im} \begin{bmatrix} sI - A \\ C \end{bmatrix}^{[r]} \tag{8.28}$$

whereas it follows from (8.26) by Lemma 8.8 that

$$\operatorname{im}\begin{bmatrix} sI-A \\ C \end{bmatrix}^{[r]} = \ker\,[-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]}. \tag{8.29}$$

Therefore, we have

$$\begin{aligned} \mathcal{M}^{[r]}(s) &= \left(\begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]}\right)^{-1} \ker\,[-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]} \\ &= \ker\left([-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]} \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}^{[r]}\right) \\ &= \ker\left([-\tilde{N}(s) \quad \tilde{D}(s)] \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}\right)^{[r]} = \ker \tilde{M}^{[r]}(s) \end{aligned} \tag{8.30}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$. Concerning the point at infinity, we have

$$\ker\,[-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]} = \operatorname{im}\begin{bmatrix} I \\ 0 \end{bmatrix}^{[r]}. \tag{8.31}$$

This equality follows by taking limits in both sides of (8.29); note that the matrix $[-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]}$ has full row rank for all $s \in \mathbb{C}^+$ by Lemma 8.8, so that the subspace-valued function $\ker\,[-\tilde{N}(s) \quad \tilde{D}(s)]^{[r]}$ is continuous on $\mathbb{C}^+$. It is now immediate from the definition (8.27) that the equality (8.25) also holds at $s = \infty$. The final claim about the uniqueness of solutions is immediate from (8.25) by Cor. 8.12. □

8.4 A lemma in unitary space

LEMMA 8.14 *Let $\mathcal{W}$ be a unitary space, and let $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{M}$ be subspaces of $\mathcal{W}$ such that $\mathcal{P} \oplus \mathcal{C} = \mathcal{W}$ and $\mathcal{P} \subset \mathcal{M}$. Denoting the projection onto $\mathcal{C}$ along $\mathcal{P}$ by $\Pi_{\mathcal{C}}^{\mathcal{P}}$, we have*

$$\mathcal{C} \cap \mathcal{M} = \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}. \tag{8.32}$$

PROOF If $w \in \mathcal{C} \cap \mathcal{M}$, then $w = \Pi_{\mathcal{C}}^{\mathcal{P}} w \in \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}$. Conversely, suppose that $w \in \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}$. Then certainly $w \in \mathcal{C}$, and also there is an $x \in \mathcal{M}$ such that $w = \Pi_{\mathcal{C}}^{\mathcal{P}} x$. Because $(I - \Pi_{\mathcal{C}}^{\mathcal{P}})x \in \mathcal{P} \subset \mathcal{M}$, we have $w = x - (I - \Pi_{\mathcal{C}}^{\mathcal{P}})x \in \mathcal{M}$. □

8.5 A divisibility result

LEMMA 8.15 *Suppose that the matrix function $Q(s) \in RH_\infty^{(k+m)\times(\ell+m)}$ is of the form*

$$Q(s) = \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ 0 & Q_{22}(s) \end{bmatrix} \tag{8.33}$$

and has full column rank for all $s \in \mathbb{C}^+$, so that in particular the matrix $Q_{11}(s)$ has full column rank for all $s \in \mathbb{C}^+$. Let $P(s) = [P_1(s) \quad P_2(s)] \in RH_\infty^{(k-\ell)\times(k+m)}$ be such that

$$\ker\,[P_1(s) \quad P_2(s)] = \operatorname{im}\begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ 0 & Q_{22}(s) \end{bmatrix} \qquad \text{for all } s \in \mathbb{C}^+. \tag{8.34}$$

Also, let $P_{11}(s) \in RH_\infty^{(k-\ell)\times k}$ be such that

$$\ker P_{11}(s) = \operatorname{im} Q_{11}(s) \qquad \text{for all } s \in \mathbb{C}^+. \tag{8.35}$$

Under these conditions, there exists a square and nonsingular matrix function $H(s) \in RH_\infty^{(k-\ell)\times(k-\ell)}$ such that

$$P_1(s) \;=\; H(s)P_{11}(s). \tag{8.36}$$

Moreover, the nontrivial elementary divisors of $H(s)$ are the same as those of $Q_{22}(s)$.

Proof Because of the full column rank assumption on $Q(s)$, there exists a unimodular matrix $U(s)$ of size $k+m$ such that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ 0 & Q_{22}(s) \end{bmatrix} \;=\; \begin{bmatrix} I_{\ell+m} \\ 0 \end{bmatrix}. \tag{8.37}$$

Note that, in this partitioning, $U_{21}(s)$ has size $(k-\ell)\times k$. Because the matrix $[P_1(s) \quad P_2(s)]$ is determined by (8.34) up to left multiplication by an RH_∞-unimodular matrix, we may for the purposes of the proof set

$$[P_1(s) \quad P_2(s)] \;=\; [U_{21}(s) \quad U_{22}(s)]. \tag{8.38}$$

Now, let $Q_0(s)$ be such that $[Q_0(s) \quad Q_{11}(s)]$ is unimodular. Then there exists a unimodular matrix $V(s)$ such that

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} [Q_0(s) \quad Q_{11}(s)] \;=\; \begin{bmatrix} I_\ell & 0 \\ 0 & I_{k-\ell} \end{bmatrix} \tag{8.39}$$

and we may set

$$P_{11}(s) \;=\; V_1(s). \tag{8.40}$$

Define

$$H(s) \;=\; U_{21}(s)Q_0(s). \tag{8.41}$$

Because $U_{21}(s)Q_{11}(s)=0$ by (8.37), we then have

$$P_1(s) = U_{21}(s) = U_{21}(s)[Q_0(s) \quad Q_{11}(s)] \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = U_{21}(s)Q_0(s)V_1(s) = H(s)P_{11}(s). \tag{8.42}$$

Finally note that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} Q_0(s) & Q_{11}(s) & Q_{12}(s) \\ 0 & 0 & Q_{22}(s) \end{bmatrix} \;=\; \begin{bmatrix} U_{11}(s)Q_0(s) & I_{\ell+m} \\ U_{21}(s)Q_0(s) & 0 \end{bmatrix}. \tag{8.43}$$

The nontrivial elementary divisors of the left hand side are equal to those of $Q_{22}(s)$, since $[Q_0(s) \quad Q_{11}(s)]$ is unimodular, whereas on the right hand side they are equal to those of $U_{21}(s)Q_0(s) = H(s)$. □

8.6 Boundary Nevanlinna-Pick interpolation

This subsection provides a proof of Lemma 5.5. The main point of the proof is to show that the diagonal blocks in the Pick matrix, which are mentioned in the sketch of the proof given in the main text, tend to infinity in the appropriate sense.

Proof (of Lemma 5.5) The necessity of (5.38) is easy to see: since in particular $G(\lambda_i)D_{i0} = N_{i0}$, we have $\|N_{i0}x\| = \|G(\lambda_i)D_{i0}x\| < \|D_{i0}x\|$ for all x, which is the same as (5.38). To prove sufficiency, we turn the problem into an ordinary NP problem as described above for the scalar case. To allow application of the standard formulas, we shift the interpolation points to the right rather than the boundary to the left. Following [2, Ch. 18], introduce

$$A_i(\varepsilon) \;=\; (sI)^{[r_i]}(\lambda_i + \tfrac{1}{2}\varepsilon), \quad A = \operatorname{diag}(A_1, \cdots, A_n), \tag{8.44}$$

$$C_i \;=\; [D_{i,r_i-1} \;\cdots\; D_{i0}], \quad C = [C_1 \;\cdots\; C_n] \tag{8.45}$$

$$Z_i \;=\; [N_{i,r_i-1} \;\cdots\; N_{i0}], \quad Z = [Z_1 \;\cdots\; Z_n]. \tag{8.46}$$

The shifted interpolation problem can now be formulated as: find an RH_∞-matrix $G(s)$ with $\|G\|_\infty < 1$ such that

$$\sum_{\lambda\in\sigma(A(\varepsilon))} \mathrm{Res}_{s=\lambda} G(s)C(sI - A(\varepsilon))^{-1} \;=\; Z. \tag{8.47}$$

The Pick matrix associated with this interpolation problem is the solution $P(\varepsilon)$ of the Lyapunov equation

$$P(\varepsilon)A(\varepsilon) + A^*(\varepsilon)P(\varepsilon) \;=\; C^*C - Z^*Z \tag{8.48}$$

([2, Thm. 18.5.1]). Writing this out in blocks corresponding to the block structure of $A(\varepsilon)$, we get

$$P_{ij}(\varepsilon)A_j(\varepsilon) + A_i^*(\varepsilon)P_{ij}(\varepsilon) \;=\; C_i^*C_j - Z_i^*Z_j \quad (i,j = 1,\cdots,n). \tag{8.49}$$

As a consequence of the block diagonal structure of $A(\varepsilon)$, these equations are decoupled and can be solved separately. For $i \neq j$ (the off-diagonal blocks), the eigenvalues of $A_j(\varepsilon)$ and $-A_i^*(\varepsilon)$ are distinct for all ε and so the solution $P_{ij}(\varepsilon)$ of (8.49) tends to a finite limit as ε tends to zero [15, §VIII.3]. We claim that for the diagonal blocks there exists a positive constant c such that

$$P_{ii}(\varepsilon) \;\geq\; \frac{c}{\varepsilon} I \tag{8.50}$$

for all sufficiently small ε. If this holds true, then we can write

$$P(\varepsilon) \;\geq\; \frac{1}{\varepsilon}(cI + \varepsilon\hat{P}(\varepsilon)) \tag{8.51}$$

where $\hat{P}(\varepsilon)$ tends to a finite limit as ε tends to zero; surely then the expression between brackets will be positive definite for sufficiently small ε, so that $P(\varepsilon)$ will then be positive definite and the problem is solved. An explicit expression for the interpolants is given in [2, Thm. 18.5.1]. (Note that it is not sufficient to show that $\|P_{ii}(\varepsilon)\|$ tends to infinity, as suggested in the proof of Lemma B in [33]; we need that *all* eigenvalues of $P_{ii}(\varepsilon)$ tend to infinity, not just the largest one. Of course this difficulty does not occur in the scalar case [2, §21.1], [3, p. 154].)

It remains to verify the claim (8.50). Since the problems for different values of the index are decoupled, we may as well drop the index and consider just one interpolation point. Moreover, since the imaginary part of the point λ drops out of the equation (8.49) for $j = i$, it is no restriction of generality to assume that this interpolation point is the origin of the complex plane. In order to avoid unwieldy notation, we shall present the proof for the case that the multiplicity of the interpolation r is equal to 2; the argument that we indicate, however, is valid in the general case.

The situation now comes down to the following. For positive ε, define a Hermitian matrix $P(\varepsilon)$ by

$$P(\varepsilon)(N + \tfrac{1}{2}\varepsilon I) + (N + \tfrac{1}{2}\varepsilon I)^* P(\varepsilon) \;=\; M, \tag{8.52}$$

where

$$N = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \tag{8.53}$$

and

$$M \;=\; M^* \;=\; \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{8.54}$$

with $M_{22} > 0$ (due to (5.38) and the definitions (8.45–8.46)). We want to show that $P(\varepsilon)$ satisfies an estimate of the form (8.50). The solution $P(\varepsilon)$ of (8.52) can be written explicitly as

$$P(\varepsilon) \;=\; \begin{bmatrix} \frac{2}{\varepsilon^3}M_{22} - \frac{1}{\varepsilon^2}(M_{12}+M_{21}) + \frac{1}{\varepsilon}M_{11} & -\frac{1}{\varepsilon^2}M_{22} + \frac{1}{\varepsilon}M_{12} \\ -\frac{1}{\varepsilon^2}M_{22} + \frac{1}{\varepsilon}M_{21} & \frac{1}{\varepsilon}M_{22} \end{bmatrix}. \tag{8.55}$$

The function $\varepsilon^3 P(\varepsilon)$ is a Hermitian matrix-valued function depending analytically on ε. Therefore it follows from Rellich's theorem (see for instance [20, Thm. S6.3]) that there exists a basis of normalized eigenvectors of $P(\varepsilon)$ depending analytically on ε. Since all singular values of $P(\varepsilon)$ can be expressed in the form $x^* P(\varepsilon) x$ for some normalized eigenvector x, it will therefore be sufficient if we can prove that there exists a positive constant c such that

$$\lim_{\varepsilon \downarrow 0} x(\varepsilon)^* \varepsilon P(\varepsilon) x(\varepsilon) \;\geq\; c \tag{8.56}$$

for all *analytic* vector-valued functions $x(\varepsilon)$ such that $\|x(\varepsilon)\| = 1$. Take such an $x(\varepsilon)$, and write

$$x(\varepsilon) \;=\; \begin{bmatrix} x_1(\varepsilon) \\ x_2(\varepsilon) \end{bmatrix} \;=\; \begin{bmatrix} x_{10} + x_{11}\varepsilon + x_{12}\varepsilon^2 + \cdots \\ x_{20} + x_{21}\varepsilon + x_{22}\varepsilon^2 + \cdots \end{bmatrix}. \tag{8.57}$$

If $x_{10} \neq 0$, the leading term in the Laurent series expansion of $\varepsilon x(\varepsilon)^* P(\varepsilon) x(\varepsilon)$ around 0 is $\frac{2}{\varepsilon^2} x_{10}^* M_{22} x_{10}$ where $x_{10}^* M_{22} x_{10} > 0$ because M_{22} is positive definite. In this case, the limit in the left hand side of (8.56) will be $+\infty$ and so (8.56) certainly holds. So now assume that $x_{10} = 0$. Note that this implies that $\|x_{20}\| = 1$, because we must have $\|x(0)\| = 1$. It is seen from (8.55) that the constant term is now the leading one. This term equals

$$\begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}^* \begin{bmatrix} 2M_{22} & -M_{22} \\ -M_{22} & M_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix} \;\stackrel{\text{def}}{=}\; \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}^* P' \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}. \tag{8.58}$$

The matrix P' appearing here is strictly positive definite, since it is the solution of the Lyapunov equation (8.52) with $\varepsilon = 1$ and M replaced by

$$\begin{bmatrix} 0 & 0 \\ 0 & M_{22} \end{bmatrix} \;=\; L^* L$$

where $L = [0 \;\; M_{22}^{\frac{1}{2}}]$; note that the pair $(L, -N - \frac{1}{2}I)$ is observable, and of course $-N - \frac{1}{2}I$ is stable. Moreover, we know that $\| \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix} \| \geq 1$ because $\|x_{20}\| = 1$ and so the expression in (8.58) is at least equal to the smallest singular value of P', which is positive. This completes the proof. □

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