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# Continuity of Singular Perturbations in the Graph Topology

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**Abstract**

For a certain model for singular perturbations in control systems, which we motivate by a simple example, we show that under weak assumptions continuity in the graph topology holds as the perturbation parameter tends to zero. This may be contrasted with a result by Cobb, who considered a different model for singular perturbations and who found a strong condition to be necessary for continuity in that model. Our proof techniques are based on the characterization (due to Qiu and Davison) of the graph topology as a topology of uniform convergence.

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## 1 Introduction

Singular perturbations may be used to describe a situation that often occurs in the modeling of physical systems. For instance, an engineer who is modeling a mechanical structure will frequently discard some of the flexibility that is in principle present in all parts of the structure. A sturdy beam may be modeled as being rigid, but every beam has some flexibility and therefore could be modeled more accurately by a differential equation, even though the resulting motions would be strongly damped and would have very high frequencies associated to them. By nevertheless modeling the beam as a rigid connection, the engineer replaces certain differential equations by algebraic constraints, thereby lowering the dynamic order of the model. The framework of singular perturbations may be used to analyze such situations. The lower-order model can be considered as a singularly perturbed version of a higher-order model that would describe the actual physical system more accurately. One allows the perturbation parameter to vary over a continuous range of values, and some limit point corresponds to the singularly perturbed model.

A basic question is of course whether the family of systems so defined depends continuously on the perturbation parameter. In this paper, we shall investigate this question with respect to the so-called graph topology which is defined for linear time-invariant systems with inputs and outputs. This topology was introduced by Vidyasagar [17], and has been shown [21] to be equivalent to the gap topology introduced by Zames and El-Sakkary [20]; it is now widely accepted as appropriate for many purposes including in particular stability robustness [18, 13, 6]. We will recall its definition in the next section. We shall not discuss the problem of determining the range of parameter values for which the singular perturbation approximation is appropriate, although some of the estimates that we prove may be useful for that purpose. The discussion will be limited to linear time-invariant systems, so in particular we shall not deal with nonlinear, time-varying, or stochastic perturbations. Our techniques would allow us to consider perturbations described by linear partial differential equations, but for simplicity we shall restrict ourselves to finite-dimensional systems.

An interesting result concerning continuity of singular perturbations in the graph topology was obtained by Vidyasagar and Cobb [17, 2]. The result pertains to singularly perturbed systems of the form

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \lambda\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= C_1x_1 + C_2x_2 + Du \end{aligned} \tag{1.1}$$

where $\lambda$ is a (small) nonnegative parameter. Natural assumptions under which one could study the continuity of the above family of systems as $\lambda$ tends to zero are that the matrix $A_{22}$ is stable, and that the limit system obtained by setting $\lambda$ equal to zero is stabilizable and detectable. Under these assumptions, Vidyasagar and Cobb showed that continuity at $\lambda = 0$ holds *if and only if* the transfer matrix $C_2(sI - A_{22})^{-1}B_2$ is identically zero. To be precise, the sufficiency was shown by Vidyasagar [16] [17, p. 253] and the necessity by Cobb [2]. Obviously the condition that a certain transfer matrix should be identically zero is very strong; Cobb shows [2] that it is 'generically' not satisfied. This leaves us with a somewhat uncanny situation: approximations that are routinely applied by engineers would seem to be unjustifiable from the point of view of a topology that is generally accepted as the appropriate one for robustness of stability.

In this note, we shall propose a model for singularly perturbed systems that is different from (1.1). We shall show by a simple example in section 3 that this model indeed appears naturally in at least some of the situations in which the approximations that we just referred to are usually made. The model that we suggest is the following:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \lambda\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2(\lambda)u, \quad \lim_{\lambda\downarrow 0} B_2(\lambda) = B_2(0) = 0 \\ y &= C_1x_1 + C_2x_2 + Du. \end{aligned} \tag{1.2}$$

It should be noted that (1.1) and (1.2) are different models; neither of them is a special case of the other. For the model (1.2), we shall again assume that the matrix $A_{22}$ is stable and that the limit system obtained by setting $\lambda$ equal to zero is stabilizable and detectable. The main purpose of this paper is to prove that, under these assumptions, continuity in the sense of the graph topology holds for the model (1.2) *without any further conditions*, in stark contrast to the situation for the model (1.1). This result will be shown in section 3.

First we will collect a number of technical preliminaries in the next section. The proof technique that we shall use for our main result is quite different from the one employed by Vidyasagar and Cobb, and is based on the recent characterization by Qiu and Davison [15] of the graph topology as the topology of uniform convergence of functions of the extended closed right half plane to the Grassmannian manifold of $m$-dimensional subspaces of $(m + p)$-dimensional complex space, where $m$ is the number of inputs and $p$ is the number of outputs. The topology on the Grassmannian is described by the well-known gap function. We shall need a number of lemmas concerning the gap which will be discussed in section 2; some of these are new and may be of interest in their own right. Als already mentioned, the main result is in section 3. Finally, the conclusions are stated briefly in section 4.

## 2 Preliminaries

As is well-known, every rational matrix can be written in coprime factorized form over the ring $RH_\infty$ of proper stable rational functions:

$$P(s) = N(s)D^{-1}(s) \tag{2.3}$$

where $N(s) \in RH_\infty^{p\times m}$, $D(s) \in RH_\infty^{m\times m}$, and $D(s)$ is nonsingular. Vidyasagar [16] (see also [17]) introduced the graph topology as the topology generated by basic neighborhoods of the form

$$\mathbf{N}(N, D; \varepsilon) = \{P_1 \mid P_1 = N_1 D_1^{-1},\;\; N_1,\, D_1 \in RH_\infty,\;\; \| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \| < \varepsilon\}. \tag{2.4}$$

He also proved the fundamental result that the graph topology is the weakest topology in which closed-loop stability is a robust property.

Vidyasagar showed that the graph topology is metrizable by displaying an explicit metric for it. Later it was shown by Zhu [21] that one may also take the gap metric introduced by Zames and El-Sakkary [20] as a metric for the graph topology. An important advantage of the gap metric over the graph metric is that it is much easier to compute [5, 3]. Recently, another interpretation of the graph topology was provided by Qiu and Davison [15]. This interpretation may be explained as follows.

To any coprime factorization (2.3), one can associate a subspace-valued function in the following way (cf. [12]):

$$s \mapsto \text{span} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}. \tag{2.5}$$

Since $N(s)$ and $D(s)$ are proper stable rational matrices, the mapping is defined on at least the closed right half plane including the point at infinity. We shall denote this set by $\mathbb{C}_+$, so

$$\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\} \cup \{\infty\}. \tag{2.6}$$

Note that this is a closed and therefore a compact subset of the Riemann sphere. It is well-known (see for instance [17, Thm. 4.1.43]) that right coprime factorizations over $RH_\infty$ of a given transfer matrix $P(s)$ are unique up to right multiplication of the factors by an $RH_\infty$-unimodular matrix. Since such a modification does not affect the right hand side of (2.5), the subspace-valued function associated to $P(s)$ via (2.3) and (2.5) is unique. Also, it is easy to see that different transfer matrices have different subspace-valued functions associated to them.

The coprimeness assumption is equivalent to saying that the subspace at the right hand side of (2.5) has dimension $m$ for each $s \in \mathbb{C}_+$. Therefore, the mapping (2.5) may be seen as a function from $\mathbb{C}_+$ to the Grassmannian manifold $G^m(\mathbb{C}^{m+p})$ of $m$-dimensional subspaces of $(m+p)$-dimensional complex space. The Grassmannian carries a natural topology, which can be characterized for instance as the quotient topology that is obtained from the Euclidean topology on $\mathbb{C}^{(m+p)\times m}$ by identifying the elements of the Grassmannian with the equivalence classes of full column rank $(m+p) \times m$-matrices modulo right multiplication by nonsingular $m \times m$-matrices. This topology is metrizable, and a natural topology to consider on the space of continuous functions from $\mathbb{C}_+$ to $G^m(\mathbb{C}^{m+p})$ is therefore the topology of uniform convergence. The functions obtained from rational transfer matrices via (2.3) and (2.5) are continuous and so the set of all such functions can also be equipped with the topology of uniform convergence. It was shown by Qiu and Davison [15] (see also [4] for a direct proof) that the topology that is obtained in this way is the same as the graph topology.

A convenient method of describing the topology on $G^m(\mathbb{C}^{m+p})$ is based on the so-called *gap* function (see for instance [9, 7]) which is defined as follows. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^{m+p}$. Then the directed gap between two subspaces $V_1$ and $V_2$ of $\mathbb{C}^{m+p}$ is

$$\delta(V_1, V_2) = \max\{d(x, V_2) \mid x \in V_1,\; \|x\| = 1\} \tag{2.7}$$

where $d(x, V_2) = \min\{\|x - y\| \mid y \in V_2\}$, and the gap between $V_1$ and $V_2$ is

$$\operatorname{gap}(V_1, V_2) = \max\{\delta(V_1, V_2),\, \delta(V_2, V_1)\}. \tag{2.8}$$

Depending on the choice of the norm $\|\cdot\|$, the gap may or may not be a metric, but in any case it describes the topology on $G^m(\mathbb{C}^{m+p})$ in the sense that a sequence $\{V_n\}$ converges to $V$ in $G^m(\mathbb{C}^{m+p})$ if and only if $\operatorname{gap}(V_n, V)$ tends to zero. Actually the directed gap can be used for the same purpose; this is a consequence of the following lemma due to Kato [8, p. 265].

Lemma 2.1 *Let* $V_1$ *and* $V_2$ *be finite-dimensional subspaces of a complex Banach space. If* $\dim V_1 = \dim V_2$*, then*

$$\delta(V_2, V_1) \;\leq\; \frac{1}{1-\delta(V_1,V_2)}\,\delta(V_1,V_2). \tag{2.9}$$

The lemma immediately leads to the following proposition, which will be the basis of our convergence analysis.

Proposition 2.2 *A sequence* $\{V_n\}$ *in* $G^m(\mathbb{C}^{m+p})$ *converges to* $V \in G^m(\mathbb{C}^{m+p})$ *if and only if* $\delta(V, V_n)$ *tends to zero.*

Since we shall work with state space representations, it will be convenient to express the function (2.5) in terms of state space parameters. This is the purpose of the next two lemmas. For the first lemma, recall that a matrix pair $(A, B)$ is said to be *controllable at* $s_0 \in \mathbb{C}$ if the complex matrix $[s_0 I - A \quad B]$ has full row rank, and that a matrix pair $(C, A)$ is said to be *observable at* $s_0 \in \mathbb{C}$ if the complex matrix $\begin{bmatrix} s_0 I - A \\ C \end{bmatrix}$ has full column rank.

Lemma 2.3 *Consider a set of state space parameters* $(A, B, C, D)$*, and let* $m$ *be the number of inputs. Let* $s_0 \in \mathbb{C}$ *be given, and suppose that* $(A, B)$ *is controllable at* $s_0$ *and* $(C, A)$ *is observable at* $s_0$*. Under these conditions, we have*

$$\dim \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \ker\,[s_0 I - A \quad -B] \;=\; m. \tag{2.10}$$

Proof Because $(C, A)$ is observable at $s_0$, we have

$$\ker \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \cap\, \ker\,[s_0 I - A \quad -B] \;=\; \{0\} \tag{2.11}$$

so that

$$\dim \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \ker\,[s_0 I - A \quad -B] \;=\; \dim\ker\,[s_0 I - A \quad -B]. \tag{2.12}$$

Since $(A, B)$ is controllable at $s_0$, the right hand side in this equation equals $m$.

Lemma 2.4 *Consider a set of state space parameters* $(A, B, C, D)$*, and assume that* $(A, B)$ *is stabilizable and that* $(C, A)$ *is detectable. Let* $N(s)D^{-1}(s) = C(sI - A)^{-1}B + D$ *be a right coprime factorization over* $RH_\infty$ *of the corresponding transfer matrix. Under these conditions, one has*

$$\operatorname{span} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \;=\; \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \ker\,[sI - A \quad -B] \tag{2.13}$$

*for all* $s \in \mathbb{C}$ *with* $\operatorname{Re} s \geq 0$*, and*

$$\operatorname{span} \begin{bmatrix} N(\infty) \\ D(\infty) \end{bmatrix} \;=\; \operatorname{span} \begin{bmatrix} D \\ I \end{bmatrix}. \tag{2.14}$$

Proof Take right coprime $RH_\infty$-matrices $P(s)$ and $Q(s)$ such that $(sI - A)^{-1}B = P(s)Q^{-1}(s)$; because of the assumed stabilizability of the pair $(A, B)$, we then have $\ker\,[sI - A \quad -B] = \operatorname{im} \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$. It follows from [10, Lemma 4.1] that (2.13) holds as an equality between rational vector spaces. Therefore, there exist coprime nonsingular polynomial matrices $T(s)$ and $R(s)$ such that

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} T(s) \;=\; \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} R(s). \tag{2.15}$$

This equation is an equation between rational matrices. Since both sides are finite for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$, (2.15) may also be read as an equation between complex matrices for each such $s$. To show that, likewise, (2.13) can be read as an equation between subspaces of $\mathbb{C}^{m+p}$ for all $s$ with $\mathrm{Re}\, s \geq 0$, we have to prove that both $T(s)$ and $R(s)$ are nonsingular for $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$. Suppose that $T(s)x = 0$ for some $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$; then it follows that

$$\begin{bmatrix} CP(s) + DQ(s) \\ Q(s) \end{bmatrix} R(s)x = 0. \tag{2.16}$$

By the previous lemma, the matrix $\begin{bmatrix} CP(s)+DQ(s) \\ Q(s) \end{bmatrix}$ has full column rank, and so the above equation implies that $R(s)x = 0$. Since the matrices $T(s)$ and $R(s)$ are right coprime, it follows that $x = 0$. In the same way, one proves that $R(s)$ is nonsingular for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$, and so the first part of the proof is complete. The formula (2.14) is immediate for instance from the explicit formulas in [17, p. 83].

We note that it is also possible to associate subspace-valued functions as we use them here directly to linear time-invariant *behaviors* in the sense of J.C. Willems [19]. Indeed, if $\mathcal{B}$ is a given behavior with external variable space $W$, then one can define a subspace-valued function $P(s)$ by

$$P(s) = \{w_0 \in W \mid w\colon t \mapsto w_0 e^{st} \text{ belongs to } \mathcal{B}\} \tag{2.17}$$

for $s \in \mathbb{C}$ (if $W$ and $\mathcal{B}$ are real, consider their complexifications). The value of $P(s)$ at infinity can be obtained as a limit when the parameter $s$ tends to infinity. The dimension of the subspace $P(s)$ is generically equal to the number of inputs $m$, and equality holds in fact for every $s \in \mathbb{C}$ if the behavior $\mathcal{B}$ is controllable [19, Thm. V.2]. If equality holds on the right half-plane the behavior might be called 'stabilizable' and the definition above gives a curve in the Grassmannian manifold of $m$-dimensional subspaces of $W$. It is readily verified that, given a minimal representation of the behavior in 'MA' [19, p. 265] or state-space form, the curves defined by these representations via (2.5) or the right-hand side of (2.13–2.14) are the same as the one constructed from (2.17).

In the proof of the main result we shall need a number of lemmas concerning estimates for gaps and condition numbers. Since these lemmas may be useful also in other contexts, we shall prove them in slightly greater generality than would be needed purely for the purposes of the present paper. In particular, we shall work throughout with Banach norms even though some shortcuts would be possible if we would use Hilbert norms instead. We will use the lemmas in finite-dimensional complex spaces, so we shall work mostly in finite dimensions and be sloppy about the distinction between matrices and mappings.

The following two numerical functions of matrices will play an important role in our analysis.

Definition 2.5 For $M \in \mathbb{C}^{k \times \ell}$, we define

$$\alpha(M) = \min\{\|Mx\| \mid \|x\| = 1\} \tag{2.18}$$

and if $M \neq 0$,

$$\gamma(M) = \max\{d(x, \ker M) \mid \|Mx\| = 1\}. \tag{2.19}$$

Another way to phrase these definitions would be to say that $\alpha(M)$ and $\gamma(M)$ are the best possible constants in the estimates

$$\|Mx\| \;\geq\; \alpha(M)\|x\| \tag{2.20}$$

and

$$d(x, \ker M) \;\leq\; \gamma(M)\|Mx\|. \tag{2.21}$$

Note that $\alpha(M)$ is positive if and only if $M$ has full column rank; $\gamma(M)$ is always positive. The quantity $\gamma^{-1}(M)$ has been called the *lower bound of M* by Kato [8, p. 271]. There is a simple relation between $\alpha(\cdot)$ and $\gamma(\cdot)$ which may be stated as follows. The quotient space $\mathbb{C}^\ell/(\ker M)$ with elements $[x] = x + \ker M$ has the natural norm $||[x]|| = d(x, \ker M)$. Defining $\hat{M} : \mathbb{C}^\ell/(\ker M) \to \mathbb{C}^k$ by $\hat{M}[x] = Mx$, we may write $\gamma(M) = \max\{||[x]|| \mid ||\hat{M}[x]|| = 1\}$ so that

$$\gamma(M) = (\alpha(\hat{M}))^{-1}. \tag{2.22}$$

In particular, if $M$ has full column rank then $\gamma(M) = (\alpha(M))^{-1}$. The equality (2.22) also shows that the definition (2.19) is correct in the sense that the maximum is indeed achieved. As will become clear from the development below, both functions can be viewed as 'unnormalized condition numbers'. The term 'unnormalized' refers to the fact that neither $\alpha$ nor $\gamma$ is invariant under scaling; in fact, for $c \in \mathbb{C}$ one has $\alpha(cM) = |c|\alpha(M)$ and $\gamma(cM) = |c|^{-1}\gamma(M)$.

We begin our series of lemmas with a standard result.

LEMMA 2.6 *If $M \in \mathbb{C}^{k\times\ell}$ is of full column rank, then any matrix $\tilde{M}$ satisfying $||M - \tilde{M}|| < \alpha(M)$ also has full column rank.*

PROOF For $x \in \mathbb{C}^\ell$ with $||x|| = 1$, the triangle inequality gives

$$|\,||Mx|| - ||\tilde{M}x||\,| \;\le\; ||Mx - \tilde{M}x|| \;\le\; ||M - \tilde{M}||, \tag{2.23}$$

and so, from the definition (2.18),

$$|\alpha(M) - \alpha(\tilde{M})| \;\le\; ||M - \tilde{M}||. \tag{2.24}$$

If $||M - \tilde{M}|| < \alpha(M)$, this implies that $\alpha(\tilde{M})$ is positive and hence that $\tilde{M}$ has full column rank.

LEMMA 2.7 *The function $M \mapsto \alpha(M)$ defined by (2.18) is continuous on $\mathbb{C}^{k\times\ell}$.*

PROOF This is immediate from the inequality (2.24).

LEMMA 2.8 *For $M_1$, $M_2 \in \mathbb{C}^{k\times\ell}$, we have*

$$\delta(\ker M_1, \ker M_2) \;\le\; \gamma(M_2)||M_1 - M_2||. \tag{2.25}$$

PROOF For $x \in \ker M_1$ with $||x|| \le 1$, we have

$$d(x, \ker M_2) \;\le\; \gamma(M_2)||M_2 x|| \;=\; \gamma(M_2)||(M_1 - M_2)x|| \;\le\; \gamma(M_2)||M_1 - M_2||. \tag{2.26}$$

The following two lemmas are easily formulated for general Banach spaces. The definitions of the distance of a point to a subspace and of the gap between two subspaces are the same as in the finite-dimensional case, except that 'min' is replaced by 'inf' and 'max' by 'sup'.

LEMMA 2.9 *Let $V_1$ and $V_2$ be subspaces of a Banach space $X$. For all $x \in X$, we have*

$$d(x, V_1) \;\le\; (1 + \delta(V_2, V_1))d(x, V_2) + \delta(V_2, V_1)||x||. \tag{2.27}$$

PROOF Take $x \in X$ and $v_2 \in V_2$. For all $v_1 \in V_1$ we have

$$||x - v_1|| \;\le\; ||x - v_2|| + ||v_1 - v_2||. \tag{2.28}$$

Taking infima on both sides over $v_1 \in V_1$, we obtain

$$\begin{aligned} d(x, V_1) &\le ||x - v_2|| + d(v_2, V_1) \\ &\le ||x - v_2|| + ||v_2||\delta(V_2, V_1) \\ &\le ||x - v_2|| + (||x - v_2|| + ||x||)\delta(V_2, V_1) \\ &= (1 + \delta(V_2, V_1))||x - v_2|| + \delta(V_2, V_1)||x||. \end{aligned} \tag{2.29}$$

Taking the infimum over $v_2 \in V_2$ now leads to (2.27).

The next result serves as a substitute for the triangle inequality. See [14, Cor. 2] for a Hilbert space version.

LEMMA 2.10 *Let $V_1$, $V_2$, and $V_3$ be subspaces of a Banach space $X$. We have*

$$\delta(V_3, V_1) \;\le\; (1+\delta(V_2,V_1))\delta(V_3,V_2) + \delta(V_2,V_1). \tag{2.30}$$

PROOF Take suprema in both sides of (2.27) over $x \in V_3$ with $||x|| = 1$.

We defined the number $\alpha(M)$ for linear mappings $M$. In particular, we may apply this to the restriction of a linear mapping to a given subspace:

$$\alpha(M|_V) = \min\{||Mx|| \mid x \in V,\; ||x|| = 1\}. \tag{2.31}$$

We shall need the continuity of this function as a function of $V$.

LEMMA 2.11 *For any mapping $M \in \mathbb{C}^{k\times\ell}$ and $0 \le p \le \ell$, the function $V \mapsto \alpha(M|_V)$ from $G^p(\mathbb{C}^\ell)$ to $[0,\infty)$ is continuous.*

PROOF Let $V_1$ and $V_2$ be subspaces, and take $v_1 \in V_1$ with $||v_1|| = 1$. For all $v_2 \in V_2$, we have

$$\begin{aligned} ||Mv_1|| &\ge ||Mv_2|| - ||M(v_1 - v_2)|| \\ &\ge \alpha(M|_{V_2})||v_2|| - ||M||||v_1 - v_2|| \\ &\ge \alpha(M|_{V_2})(||v_1|| - ||v_1 - v_2||) - ||M||||v_1 - v_2|| \\ &= \alpha(M|_{V_2}) - (\alpha(M|_{V_2}) + ||M||)||v_1 - v_2||. \end{aligned} \tag{2.32}$$

Taking the maximum over $v_2 \in V_2$, we get

$$||Mv_1|| \;\ge\; \alpha(M|_{V_2}) - (\alpha(M|_{V_2}) + ||M||)d(v_1, V_2). \tag{2.33}$$

Now taking the minimum over $v_1 \in V_1$ with $||v_1|| = 1$, we obtain

$$\alpha(M|_{V_1}) \;\ge\; \alpha(M|_{V_2}) - (\alpha(M|_{V_2}) + ||M||)\delta(V_1, V_2) \tag{2.34}$$

or, for $\delta(V_1, V_2) < 1$,

$$\alpha(M|_{V_2}) \;\le\; \frac{1}{1-\delta(V_1,V_2)}\,\alpha(M|_{V_1}) \;+\; \frac{\delta(V_1,V_2)}{1-\delta(V_1,V_2)}\,||M||. \tag{2.35}$$

This inequality, together with the one that follows from it by interchanging the indices 1 and 2, establishes the desired result.

Lemma 2.8 shows that the function $\gamma(\cdot)$ cannot be continuous at $M_1$ if $M_1$ does not have full row rank, since in that case there will be mappings $M_2$ such that $||M_1 - M_2||$ is arbitrarily small but $\dim\ker M_2 < \dim\ker M_1$ so that $\delta(\ker M_1, \ker M_2) = 1$. However, the next lemma shows that we do have continuity on the subset of $\mathbb{C}^{k\times\ell}$ consisting of matrices of full row rank.

LEMMA 2.12 *If $M_0 \in \mathbb{C}^{k\times\ell}$ has full row rank, then the function $\gamma(\cdot)$ as defined in (2.19) is continuous at $M_0$.*

PROOF Let $M_0 \in \mathbb{C}^{k\times\ell}$ be of full row rank. Take $N_0 \in \mathbb{C}^{(\ell-k)\times\ell}$ such that $\begin{bmatrix} N_0 \\ M_0 \end{bmatrix}$ is invertible; then $\ker N_0$ is a complement of $\ker M_0$. Note that we have

$$\gamma(M_0) = \max\{d(x, \ker M_0) \mid x \in \ker N_0,\; ||M_0 x|| = 1\}. \tag{2.36}$$

Now, let $\varepsilon_0 > 0$ be such that the following requirements are satisfied:

(i) for all $M \in \mathbb{C}^{k\times\ell}$ such that $\|M - M_0\| < \varepsilon_0$, $\ker M$ is a complement of $\ker N_0$;

(ii) there is a positive constant $c$ such that $\alpha(M|_{\ker N_0}) \geq \frac{1}{c}$ for all $M$ with $\|M - M_0\| < \varepsilon_0$;

(iii) $\varepsilon_0(1 + c\varepsilon_0)\gamma(M_0) + c\varepsilon_0 < 1$, where $c$ is the constant of (ii).

Item (i) can be satisfied because of Lemma 2.8 and the fact that the set of complements of a given subset is an open subset of the Grassmannian (cf. also [1, Thm. 5.2]). Item (ii) can be satisfied because $\alpha(M_0|_{\ker N_0})$ is positive, and the mapping $M \mapsto \alpha(M|_{\ker N_0})$ is continuous (Lemma 2.7). Note that (i) allows us to write

$$\gamma(M) = \max\{d(x, \ker M) \mid x \in \ker N_0,\ \|Mx\| = 1\} \tag{2.37}$$

when $\|M - M_0\| < \varepsilon_0$. Also note that (ii) implies that $\|x\| \leq c$ when $x \in \ker N_0$, $\|Mx\| \leq 1$, and $\|M - M_0\| < \varepsilon_0$.

Using Lemma 2.9 and Lemma 2.8, we can now write down the following estimates for $\varepsilon \in (0, \varepsilon_0)$, $M \in \mathbb{C}^{k\times\ell}$ such that $\|M - M_0\| \leq \varepsilon$, and $x \in \ker N_0$ with $\|Mx\| = 1$:

$$\begin{aligned} d(x, \ker M) &\leq (1 + \delta(\ker M_0, \ker M))d(x, \ker M_0) + \delta(\ker M_0, \ker M)\|x\| \\ &\leq (1 + \gamma(M)\varepsilon)d(x, \ker M_0) + \gamma(M)\varepsilon\|x\| \\ &\leq (1 + \gamma(M)\varepsilon)\|M_0x\|\gamma(M_0) + \gamma(M)c\varepsilon \\ &\leq (1 + \gamma(M)\varepsilon)(\|Mx\| + \|M - M_0\|\|x\|)\gamma(M_0) + \gamma(M)c\varepsilon \\ &\leq (1 + \gamma(M)\varepsilon)(1 + c\varepsilon)\gamma(M_0) + \gamma(M)c\varepsilon. \end{aligned} \tag{2.38}$$

Re-arranging and maximizing over $x \in \ker N_0$ with $\|Mx\| = 1$, we get

$$\gamma(M) \leq (1 + c\varepsilon)\gamma(M_0) + (\varepsilon(1 + c\varepsilon)\gamma(M_0) + c\varepsilon)\gamma(M). \tag{2.39}$$

Using item (iii), we obtain from this

$$\gamma(M) \leq \frac{1 + c\varepsilon}{1 - c\varepsilon - \varepsilon(1 + c\varepsilon)\gamma(M_0)}\,\gamma(M_0). \tag{2.40}$$

A lower estimate of $\gamma(M)$ can be obtained similarly, as follows. If $x \in \ker N_0$ is such that $\|M_0x\| = 1$, and $\|M - M_0\| \leq \varepsilon$, then

$$\|Mx\| \leq \|M_0x\| + \|M - M_0\|\|x\| = 1 + \varepsilon\|x\|. \tag{2.41}$$

This implies that $\|x\| \leq c(1 + \varepsilon\|x\|)$ so that, if $c\varepsilon < 1$,

$$\|x\| \leq \frac{c}{1 - c\varepsilon} \tag{2.42}$$

and

$$\|Mx\| \leq \frac{1}{1 - c\varepsilon}. \tag{2.43}$$

Using these inequalities with Lemma 2.9 and Lemma 2.8 as above, we get for $\varepsilon \in (0, \varepsilon_0)$, $M \in \mathbb{C}^{k\times\ell}$ such that $\|M - M_0\| \leq \varepsilon$, and $x \in \ker N_0$ with $\|M_0x\| = 1$:

$$\begin{aligned} d(x, \ker M_0) &\leq (1 + \delta(\ker M, \ker M_0))d(x, \ker M) + \delta(\ker M, \ker M_0)\|x\| \\ &\leq (1 + \gamma(M_0)\varepsilon)\frac{1}{1 - c\varepsilon}\gamma(M) + \frac{c\varepsilon}{1 - c\varepsilon}\gamma(M_0). \end{aligned} \tag{2.44}$$

Maximizing over $x \in \ker N_0$ with $\|M_0x\| = 1$ and re-arranging, we obtain

$$\gamma(M) \geq \frac{1 - 2c\varepsilon}{1 + \gamma(M_0)\varepsilon}\,\gamma(M_0). \tag{2.45}$$

Together with (2.40), this proves the continuity of $\gamma(\cdot)$.

A more specialized lemma which will be useful in the proof of our main result is the following.

LEMMA 2.13 *Let $H_1 : \mathbb{C}^k \to \mathbb{C}^\ell$ be a linear mapping, and let $V_1$ be a subspace of $\mathbb{C}^k$. Let $V_0$ be also a subspace of $\mathbb{C}^k$ and let $H_0 : V_0 \to \mathbb{C}^\ell$ be an injective linear mapping defined on $V_0$. Under these conditions, we have*

$$\delta(H_0V_0, H_1V_1) \ \leq \ \gamma(H_0)(||H_0 - H_1|_{V_0}|| + ||H_1||\delta(V_0, V_1)). \tag{2.46}$$

PROOF Take $x \in H_0V_0$ such that $||x|| = 1$, and let $v_0$ be such that $H_0v_0 = x$. Note that

$$||v_0|| \ \leq \ \max\{||v|| \mid v \in V_0, \ ||H_0v_0|| = 1\} \ = \ \gamma(H_0). \tag{2.47}$$

We have, for all $v_1 \in V_1$,

$$\begin{aligned} d(x, H_1V_1) \ &\leq \ ||H_0v_0 - H_1v_1|| \ = \ ||H_0v_0 - H_1v_0 + H_1v_0 - H_1v_1|| \\ &\leq \ ||H_0 - H_1|_{V_0}||||v_0|| + ||H_1||||v_0 - v_1||. \end{aligned} \tag{2.48}$$

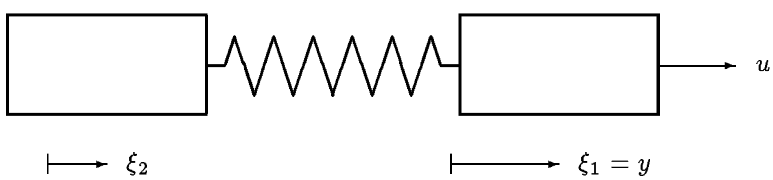
Minimizing over $v_1 \in V_1$, we get

$$\begin{aligned} d(x, H_1V_1) \ &\leq \ ||H_0 - H_1|_{V_0}||||v_0|| + ||H_1||d(v_0, V_1) \\ &\leq \ ||v_0||(||H_0 - H_1|_{V_0}|| + ||H_1||\delta(V_0, V_1)) \\ &\leq \ \gamma(H_0)(||H_0 - H_1|_{V_0}|| + ||H_1||\delta(V_0, V_1)). \end{aligned} \tag{2.49}$$

The right hand side no longer depends on $x$, so by taking the maximum over all $x \in H_0V_0$ with $||x|| = 1$ we obtain (2.46).

## 3 MAIN RESULT

We shall discuss the modeling of singular perturbations in a simple example. Consider two unit masses connected by a spring, as in the figure below. Motion is supposed to be frictionless and to take place only in one direction. The input is the force on one of the two masses; the output is the displacement of that mass (see figure).




Denote the displacements of the two masses by $\xi_1$ and $\xi_2$, let $k > 0$ be the stiffness of the spring, and let $\alpha > 0$ be the damping coefficient of the spring. We can then write down equations of motion as follows:

$$\begin{aligned} \ddot{\xi}_1 \ &= \ u - k(\xi_1 - \xi_2) - \alpha(\dot{\xi}_1 - \dot{\xi}_2) \\ \ddot{\xi}_2 \ &= \ k(\xi_1 - \xi_2) + \alpha(\dot{\xi}_1 - \dot{\xi}_2) \\ y \ &= \ \xi_1. \end{aligned} \tag{3.50}$$

Suppose now that the spring is very stiff. A modeler would then be tempted to consider the connection between the two masses as rigid and to use a second-order model for the relation between input $u$ and output $y$, rather than the fourth-order model (3.50). In order to justify this approximation in a singular perturbation analysis, we introduce a perturbation parameter $\lambda$ and consider the coefficients $k$ and $\alpha$ as functions of $\lambda$. The dependence of $k$ and $\alpha$ on $\lambda$ should be such that small positive values of $\lambda$ correspond to a very stiff spring, whereas the value $\lambda = 0$ should produce the situation in which we have a rigid connection. From the physics of the situation, we expect that we should be able to take $\xi_1 - \xi_2$ as the 'fast' variable (cf. [11] for an analysis of a similar case). From (3.50), one obtains the following equation for $\xi_1 - \xi_2$ in the fast time scale $\tau = t/\lambda$:

$$\frac{d^2}{d\tau^2}(\xi_1 - \xi_2) \;=\; -2\lambda^2 k(\xi_1 - \xi_2) - 2\lambda\alpha\frac{d}{d\tau}(\xi_1 - \xi_2) + \lambda^2 u. \tag{3.51}$$

In view of this, we shall let $k$ and $\alpha$ depend on $\lambda$ according to

$$k = \frac{k_0}{\lambda^2}, \qquad \alpha = \frac{\alpha_0}{\lambda}. \tag{3.52}$$

In order to write the equations (3.50) in a standard first-order form, introduce $x_1 = \xi_1 + \xi_2$, $x_2 = \dot{\xi}_1 + \dot{\xi}_2$, $x_3 = \xi_1 - \xi_2$, and $x_4 = \lambda(\dot{\xi}_1 - \dot{\xi}_2)$ ($x_4$ is the derivative of $x_3$ in the *fast* time scale). One obtains the following equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \lambda\dot{x}_3 \\ \lambda\dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2k_0 & -2\alpha_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \lambda^2 \end{bmatrix} u \tag{3.53}$$

$$y \;=\; [\tfrac{1}{2} \;\; 0 \;\; \tfrac{1}{2} \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{3.54}$$

So we get a system in the form (1.2). Although the example is of course a simple one, we do believe that it is representative of a large class of modeling situations, and that therefore it makes sense to study the continuity of the family (1.2).

So, consider now the family of systems given by (1.2). The systems are practically in state space form when $\lambda > 0$ (the second line ought to be divided by $\lambda$), but when $\lambda = 0$ we get a set of differential-algebraic equations:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ 0 &= A_{21}x_1 + A_{22}x_2 \\ y &= C_1x_1 + C_2x_2 + Du. \end{aligned} \tag{3.55}$$

If we assume that $A_{22}$ is Hurwitz (as we shall do), then in particular $A_{22}$ is invertible and it is easy to re-write (3.55) in state space form. The result is of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.56}$$

with

$$A = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B = B_1, \quad C = C_1 - C_2A_{22}^{-1}A_{21}. \tag{3.57}$$

The family of curves associated with (1.2) is

$$P(s,\lambda) \;=\; \begin{bmatrix} C_1 & C_2 & D \\ 0 & 0 & I \end{bmatrix} \ker \begin{bmatrix} sI - A_{11} & -A_{12} & -B_1 \\ -A_{21} & \lambda sI - A_{22} & -B_2(\lambda) \end{bmatrix}, \tag{3.58}$$

$$P(\infty,\lambda) \;=\; \mathrm{im}\begin{bmatrix} D \\ I \end{bmatrix}. \tag{3.59}$$

Note that this expression is also valid at $\lambda = 0$, that is, the curve obtained from the above formulas by setting $\lambda = 0$ is the same as the one associated to (3.56–3.57) via the formula of Lemma 2.4. As is shown by direct computation, the family of curves (3.58) may also be written in the form

$$P(s,\lambda) = \begin{bmatrix} C(s,\lambda) & D(s,\lambda) \\ 0 & I \end{bmatrix} \ker [sI - A(s,\lambda) \quad -B(s,\lambda)] \tag{3.60}$$

$$P(\infty,\lambda) = \operatorname{im} \begin{bmatrix} D \\ I \end{bmatrix} \tag{3.61}$$

where the matrix functions $A(s,\lambda)$, $B(s,\lambda)$, $C(s,\lambda)$, and $D(s,\lambda)$ are defined by

$$\begin{bmatrix} A(s,\lambda) & B(s,\lambda) \\ C(s,\lambda) & D(s,\lambda) \end{bmatrix} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D_1 \end{bmatrix} + \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} (\lambda sI - A_{22})^{-1} [A_{21} \quad B_2(\lambda)]. \tag{3.62}$$

If the matrix $A_{22}$ is Hurwitz, the matrix functions just defined are uniformly bounded on $\mathbb{C}_+$, and moreover, a comparison of (3.62) and (3.57) shows that

$$A(s,0) = A, \quad B(s,0) = B, \quad C(s,0) = C, \quad D(s,0) = D \tag{3.63}$$

for finite $s$. Furthermore, it is easily seen that the matrix functions $B(s,\lambda)$ and $D(s,\lambda)$ converge *uniformly* to their pointwise limits on $\{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$ as $\lambda$ tends to zero. The same is *not* true, however, for the functions $A(s,\lambda)$ and $C(s,\lambda)$. The key point in the proof of the main theorem is to show that nevertheless, under stabilizability and detectability assumptions, we do have uniform convergence of the subspace-valued functions $P(s,\lambda)$.

Theorem 3.1 *Consider the family of systems (1.2) parametrized by $\lambda \geq 0$. Assume that the matrix $A_{22}$ is Hurwitz, and that (with $A$, $B$, and $C$ defined in (3.57)) the pair $(C, A)$ is detectable and the pair $(A, B)$ is stabilizable. Under these conditions, the family (1.2) is continuous at 0 in the sense of the graph topology.*

Proof In view of Prop. 2.2, we have to show that for each $\varepsilon > 0$ there exists a $\lambda_0 > 0$ such that

$$\dim P(s,\lambda) = \dim P(s,0) \tag{3.64}$$

and

$$\delta(P(s,0), P(s,\lambda)) < \varepsilon \tag{3.65}$$

for all $\lambda \in [0, \lambda_0)$ and for all $s \in \mathbb{C}_+$. The convergence at $s = \infty$ is trivial; it remains to prove the uniform convergence on $\{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$.

Now, fix $\varepsilon > 0$. As already noted above, the assumption that $A_{22}$ is Hurwitz implies that the matrix functions $A(s,\lambda)$, $B(s,\lambda)$, $C(s,\lambda)$, and $D(s,\lambda)$ are uniformly bounded on $\mathbb{C}_+$, so in particular there exists a constant $c_1$ such that

$$||A(s,\lambda)|| \leq c_1 \tag{3.66}$$

for all $s \in \mathbb{C}_+$ and $\lambda \geq 0$. Take $r > c_1$; then the matrix $sI - A(s,\lambda)$ is invertible for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ and $|s| > r$. Denoting this latter set by $\mathbb{C}_+^r$, we have that for all $s \in \mathbb{C}_+^r$ the matrix $[sI - A(s,\lambda) \quad -B(s,\lambda)]$ has full row rank, and

$$\ker \begin{bmatrix} C(s,\lambda) & D(s,\lambda) \\ 0 & I \end{bmatrix} \cap \ker [sI - A(s,\lambda) \quad -B(s,\lambda)] = \{0\}. \tag{3.67}$$

This implies that

$$\dim P(s,\lambda) = m = \dim P(s,0) \tag{3.68}$$

for all $s \in \mathbb{C}_+^r$.

In order to show (3.65) on $\mathbb{C}_+^R$ for a suitable $R > 0$, our strategy will be to apply Lemma 2.13 with

$$\begin{aligned} V_0 &= \ker\,[I \quad 0] && (3.69)\\ V_1 &= \ker\,[sI - A(s,\lambda) \quad -B(s,\lambda)] && (3.70)\\ H_0 &= \begin{bmatrix} D \\ I \end{bmatrix} \qquad (\text{on } V_0) && (3.71)\\ H_1 &= \begin{bmatrix} C(s,\lambda) & D(s,\lambda) \\ 0 & I \end{bmatrix}. && (3.72) \end{aligned}$$

With these definitions, we have $H_0V_0 = \operatorname{im}\begin{bmatrix} D \\ I \end{bmatrix} = P(\infty, 0)$ and $H_1V_1 = P(s,\lambda)$. Note that $H_0$ is injective as required in Lemma 2.13. Introduce a matrix function $\tilde{A}(s,\lambda)$ by

$$\begin{aligned} \tilde{A}(s,\lambda) &= s^{-1}A(s,\lambda) && (\operatorname{Re} s \geq 0,\ |s| > r,\ \lambda \geq 0) \\ \tilde{A}(\infty,\lambda) &= 0 && (\lambda \geq 0) \end{aligned} \tag{3.73}$$

and define $\tilde{B}(s,\lambda)$ likewise. Of course, we have

$$V_1 = \ker\,[I - \tilde{A}(s,\lambda) \quad -\tilde{B}(s,\lambda)]. \tag{3.74}$$

The matrix functions $\tilde{A}(\cdot,\cdot)$ and $\tilde{B}(\cdot,\cdot)$ are continuous; therefore, it follows from Lemma 2.12 that the function $\gamma([I - \tilde{A}(s,\lambda) \quad -\tilde{B}(s,\lambda)])$ is continuous on the compact set $(\mathbb{C}_+^r \cup \{\infty\}) \times [0,1]$, so that it is bounded above on this set. Using this observation together with Lemma 2.8 and the uniform boundedness of the matrix functions introduced in (3.62), we find that there exists a constant $c_2$ such that

$$\delta(V_0, V_1) \leq \frac{c_2}{|s|}. \tag{3.75}$$

Also note that

$$\|H_0 - H_1|_{V_0}\| = \|D - D(s,\lambda)\| = \|C_2(\lambda s I - A_{22})^{-1}B_2(\lambda)\| \leq c_3(\lambda) \tag{3.76}$$

where $\lim_{\lambda\downarrow 0} c_3(\lambda) = 0$. Taking everything together, Lemma 2.13 shows that there exists a function $c_4(\lambda)$ satisfying $\lim_{\lambda\downarrow 0} c_4(\lambda) = 0$ such that

$$\delta(P(\infty,0), P(s,\lambda)) \leq c_4(\lambda) + \frac{c_2}{|s|}. \tag{3.77}$$

for all $s \in \mathbb{C}_+^r$. In particular, we have

$$\delta(P(\infty,0), P(s,0)) \leq \frac{c_2}{|s|}. \tag{3.78}$$

Using the modified triangle inequality (Lemma 2.10) and the modified symmetry property (Lemma 2.1), we see that we can choose $R > 0$ and $\lambda_1 > 0$ such that

$$\delta(P(s,0), P(s,\lambda)) < \varepsilon \tag{3.79}$$

for all $s \in \mathbb{C}_+^R$ and all $\lambda \in [0, \lambda_1]$.

It remains to analyze the situation on $\{s \in \mathbb{C}_+ \mid |s| \leq R\}$. Let us denote this set by $\mathbb{C}_{+R}$. By the stabilizability assumption on the pair $(A, B)$, the matrix $[sI - A \quad -B]$ has full row rank for all $s \in \mathbb{C}_{+R}$. This means that $\alpha([sI - A \quad -B]^T) > 0$ for all $s \in \mathbb{C}_{+R}$, so by the continuity of $\alpha(\cdot)$ we may define

$$\alpha_m = \min\{\alpha([sI - A \quad -B]^T) \mid s \in \mathbb{C}_{+R}\} > 0. \tag{3.80}$$

It is seen from the definition (3.62) that the matrix functions $A(s,\lambda)$ and $B(s,\lambda)$ converge to $A$ and $B$ respectively as $\lambda \downarrow 0$, uniformly on $\mathbb{C}_{+R}$. Therefore, we can find $\lambda_2 > 0$ such that

$$||[A(s,\lambda) - A \quad B(s,\lambda) - B]|| < \alpha_m \tag{3.81}$$

for all $\lambda \in [0,\lambda_2]$ and all $s \in \mathbb{C}_{+R}$. With Lemma 2.6, this implies that $[sI - A(s,\lambda) \quad -B(s,\lambda)]$ has full row rank for all $\lambda \in [0,\lambda_2]$ and all $s \in \mathbb{C}_{+R}$. Similar reasoning shows that there is a $\lambda_3 > 0$ such that $\begin{bmatrix} sI-A(s,\lambda) \\ C(s,\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in [0,\lambda_3]$ and for all $s \in \mathbb{C}_{+R}$. It then follows from Lemma 2.3 that $\dim P(s,\lambda) = m = \dim P(s,0)$ for all $\lambda \in [0,\lambda_4]$, where $\lambda_4 = \min\{\lambda_2,\lambda_3\}$, and for all $s \in \mathbb{C}_{+R}$.

The final estimate that we need is again provided by Lemma 2.13, in which this time we take

$$H_0 = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}\Bigg|_{\ker[sI-A \quad -B]} \tag{3.82}$$

$$V_0 = \ker[sI - A \quad -B] \tag{3.83}$$

$$H_1 = \begin{bmatrix} C(s,\lambda) & D(s,\lambda) \\ 0 & I \end{bmatrix} \tag{3.84}$$

$$V_1 = \ker[sI - A(s,\lambda) \quad -B(s,\lambda)]. \tag{3.85}$$

With these definitions, we have $H_0V_0 = P(s,0)$ and $H_1V_1 = P(s,\lambda)$. Note that $\gamma([sI-A \quad -B])$ is bounded above on $\mathbb{C}_{+R}$ by virtue of the stabilizability assumption and Lemma 2.12. We may therefore use Lemma 2.8 together with Lemma 2.1 and the fact that the matrix functions $A(s,\lambda)$ etc. converge to their respective limits uniformly on $\mathbb{C}_{+R}$, to conclude that there is a $\lambda_5 > 0$ such that

$$\delta(P(s,0), P(s,\lambda)) < \varepsilon \tag{3.86}$$

for all $\lambda \in [0,\lambda_5]$ and for all $s \in \mathbb{C}_{+R}$. This completes the proof.

It is easily verified that the stabilizability assumption of the theorem is equivalent to requiring that the matrix

$$\begin{bmatrix} sI - A_{11} & -A_{12} & -B_1 \\ -A_{21} & -A_{22} & 0 \end{bmatrix}$$

has full row rank for all $s$ with $\mathrm{Re}\, s \geq 0$, and that the detectability assumption comes down to requiring that the matrix

$$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \\ C_1 & C_2 \end{bmatrix}$$

has full column rank for all $s$ with $\mathrm{Re}\, s \geq 0$. One also easily verifies that the conditions of the theorem are satisfied in the example at the beginning of the section, assuming, as we did, that both the spring constant $k_0$ and the damping coefficient $\alpha_0$ are positive.

For purposes of illustration and comparison, we close this section by presenting a proof of the Vidyasagar-Cobb theorem using the techniques of the present paper.

THEOREM 3.2 *Consider the family of systems (1.1) parametrized by $\lambda \geq 0$. Define*

$$A = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{3.87}$$

$$B = B_1 - A_{12}A_{22}^{-1}B_2 \tag{3.88}$$

$$C = C_1 - C_2A_{22}^{-1}A_{21} \tag{3.89}$$

$$D_0 = D - C_2A_{22}^{-1}B_2 \tag{3.90}$$

*so that the quadruple $(A, B, C, D_0)$ is a set of state space parameters for the system obtained from (1.1) by setting $\lambda = 0$. Assume that the matrix $A_{22}$ is stable, that the pair $(A, B)$ is stabilizable, and that the pair $(C, A)$ is detectable. Under these conditions, the family (1.1) is continuous at 0 in the sense of the graph topology if and only if the transfer matrix $C_2(sI - A_{22})^{-1}B_2$ is identically zero.*

PROOF For positive $\lambda$ and finite $s$, the subspace-valued function associated with (1.1) satisfies

$$P(s,\lambda) = \begin{bmatrix} C(s,\lambda) & D(s,\lambda) \\ 0 & I \end{bmatrix} \ker\,[sI - A(s,\lambda) \quad -B(s,\lambda)] \tag{3.91}$$

where the matrix functions $A(s, \lambda)$, $B(s, \lambda)$, $C(s, \lambda)$, and $D(s, \lambda)$ are defined by

$$\begin{bmatrix} A(s,\lambda) & B(s,\lambda) \\ C(s,\lambda) & D(s,\lambda) \end{bmatrix} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix} + \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} (\lambda sI - A_{22})^{-1}[A_{21} \quad B_2]. \tag{3.92}$$

For the 'if' part, assume now that $C_2(sI - A_{22})^{-1}B_2$ is zero so that

$$D(s,\lambda) = D \tag{3.93}$$

for all $\lambda$. By virtue of this property, the proof follows in the same way as for the main theorem, actually with some simplifications (note for instance how the estimate (3.76) becomes trivial).

It remains to prove the 'only if' part. Suppose that $P(\cdot\,,\lambda)$ converges uniformly on $\mathbb{C}_+$ to $P(\cdot\,,0)$ for $\lambda \downarrow 0$. Because the limit function $P(\cdot\,,0)$ is continuous, we should then have that

$$\lim_{\lambda\downarrow 0} P(\frac{s_0}{\lambda},\lambda) \;=\; P(\infty,0) \;=\; \operatorname{span} \begin{bmatrix} -C_2A_{22}^{-1}B_2 + D \\ I \end{bmatrix} \tag{3.94}$$

for each $s_0$ with $\operatorname{Re} s_0 \geq 0$ and $s_0 \neq 0$. One easily computes that in fact

$$\lim_{\lambda\downarrow 0} P(\frac{s_0}{\lambda},\lambda) \;=\; \operatorname{span} \begin{bmatrix} C_2(s_0I - A_{22})^{-1}B_2 + D \\ I \end{bmatrix} \tag{3.95}$$

and combining this with (3.94) we see that the matrix function $C_2(sI - A_{22})^{-1}B_2$ should be constant. Since a proper rational function can only be constant if it is zero, this proves that $C_2(sI - A_{22})^{-1}B_2 = 0$.

## 4 Conclusions

Singular perturbations may be used for the modeling of a large variety of phenomena. It is only natural to expect that the form of the singularly perturbed model may be different in different applications, even within the linear context. Using a simple but representative example, we have suggested that in cases where one wants to analyze the effect of replacing high-frequency modes ('parasitic dynamics') by algebraic equations, the model (1.1) may not be the appropriate one and may have to be replaced by the model (1.2). For the latter model, we have shown that continuity in the graph topology holds under much weaker conditions then were obtained by Vidyasagar and Cobb as being necessary and sufficient for continuity of the first model.

Assuming the validity of the model (1.2) in a given application, the two conditions imposed in the main theorem are that the matrix $A_{22}$ should be Hurwitz, and that we should have stabilizability and detectability for the limit system. The first condition means that the fast dynamics should be stable. The second condition expresses that the algebraic constraint replacing the high-frequency modes should not cause the cancellation of unstable dynamics in the input-output relation. Note that the algebraic relations obtained from (1.2) by setting the perturbation parameter equal to zero do not involve the inputs, in contrast to what happens if one uses the model (1.1). In broad terms, the conclusion of the present paper therefore is that replacing high-frequency modes by algebraic constraints can be justified for sufficiently high and well-damped frequencies under the following conditions: (i) stable fast dynamics, (ii) no cancellation of unstable dynamics by the imposed algebraic constraint, (iii) the algebraic constraint is a pure state constraint.

A natural question which arises is the following. We have been analyzing a specific family of systems, which comes up in a singular perturbation analysis. Much more generally, one might consider a family of systems given by equations of the form

$$\begin{aligned} T(\tfrac{d}{dt},\lambda)\xi &= 0 \\ w &= Q(\tfrac{d}{dt},\lambda)\xi \end{aligned} \tag{4.96}$$

where $T(s,\lambda)$ and $Q(s,\lambda)$ are polynomial matrices in $s$ parametrized by $\lambda$ coming from some topological space $\Lambda$, $\xi$ is a vector of auxiliary variables, and $w$ is a vector of external variables (inputs and outputs). For such families one may ask under what conditions continuity will hold in the sense of the graph topology at a given parameter value $\lambda_0$. This question is currently under investigation.

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