Stochastic grid bundling method for backward stochastic differential equations

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Abstract

In this work, we apply the Stochastic Grid Bundling Method (SGBM) to numerically solve backward stochastic differential equations. The SGBM algorithm is based on conditional expectations approximation by means of bundling of Monte Carlo sample paths and a local regress-later regression within each bundle. The basic algorithm for solving backward stochastic differential equations will be introduced and an upper error bound is established for the local regression. A full error analysis is also conducted for the explicit version of our algorithm and numerical experiments are performed to demonstrate various properties of our algorithm.

1 Introduction

The Stochastic Grid Bundling Method (SGBM) is a Monte Carlo based algorithm designed to solve backward dynamic programming problems, with applications in pricing Bermudan options in [12] and [4]. This algorithm has been further extended computationally by the incorporation of GPU accelation in [13] and generalized to the computation of Credit Valuation Adjustment and Potential Future Exposure in [6]. In this work, we will extend its applicability to the approximation of Backward Stochastic Differential Equations (BSDEs). We shall also study the accompagnying errors in the SGBM algorithm.

The SGBM algorithm is based on the so-called *regress-later technique* and on an adaptive local basis approach. In usual Monte Carlo regression methods for backward-in-time problems, the values of the target function at the end of a time interval are regressed on certain dependent variables that are measured at the beginning of the time interval (which called the regress-now approach). This creates a statistical error. Instead, the dependent variable is projected onto a set of basis functions at the end of the interval in a regress-later method, and a conditional expectation across the interval is then computed for each basis function. This difference removes the statistical error in the regression step. Regress-later schemes have been further discussed in [7]

With an adaptive local basis approach, the regression basis is a set of linear functions defined only on a partition of the domain, and the exact partition depends on the simulated samples themselves. For further application of localization in numerical schemes, the reader may check out [3]. Since each partition is non-overlapping, SGBM is easy to scale up in dimensionality and can

facilitate the application of parallel computing. We would like to test the SGBM algorithm in a new problem setting such that we can take advantage of its nice properties and also get a better understanding of the underlying principles.

The problem that we are interested in is the numerical approximation of BSDEs. These equations form a popular subject of research in quantitative finance ever since their introduction in [15] and related works. The connection between BSDEs and partial differential equations (PDEs) also provides the opportunity of solving PDEs (in high dimensions) with stochastic methods. However, the computational difficulties of solving BSDEs prevents them from being widely used in practice. Therefore, efficient algorithms for the approximation of high-dimensional BSDEs are of great interest. In fact, there are numerous works just focussing on the application of Monte Carlo methods to BSDEs, including [2, 5, 9, 14, 1]. As far as we know, there is no study of a combined approach based on the regress-later scheme and a localization Monte Carlo technique for these equations, which is the goal of this work.

In this article, we consider the application of SGBM to decoupled Forward Backward Stochastic Differential Equations (FBSDEs) of the form

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,\ X_0 = x_0, \\ dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t,\ Y_T = \Phi(X_T), \end{cases} \tag{1.1}$$

defined on $0 \leq t \leq T$. The function $f : [0, T] \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^d$ is called the driver function of the backward process and the process $W_t = (W_{1,t}, \ldots, W_{d,t})^T$ is a d-dimensional Brownian motion. Note that the usual setting of complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ being a filtration applies throughout the article.

The key difference between the two equations in (1.1) is the relation between the known condition and the filtration. For the forward process related to X_t, the initial condition x_0 is adapted to the starting filtration $\mathcal{F}_0$. While for the backward equation, the terminal condition Y_T is given by $\Phi(X_T)$ for a function $\Phi : \mathbb{R}^q \to \mathbb{R}$, which is adapted to the filtration $\mathcal{F}_T$. An adaptive solution to this equation must be constructed (commonly by the expectation condition on the filtration). Precisely, a pair of adapted processes (Y_t, Z_t) is said to be the solution of the FBSDE if Y_t is a continuous real-valued adapted process, Z_t is a real-valued predictable process such that $\int_0^T |Z_t|^2 dt < \infty$ almost surely in $\mathbb{P}$ and the pair satisfies the equation. We wish to find approximate values for (Y_0, Z_0) by the SGBM algorithm.

The functions $\mu : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$ and $\sigma : [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$ refer to the drift and the diffusion coefficients of the forward stochastic process, and x_0 is the initial condition for X. It is assumed that both $\mu(t, x)$ and $\sigma(t, x)$ are measurable functions that are uniformly Lipschitz in x and satisfy a linear growth condition in x. Therefore, a unique strong solution exists for the forward equation. This process also satisfies the Markov property, namely $\mathbb{E}[X_\tau | \mathcal{F}_t] = \mathbb{E}[X_\tau | X_t]$ for $\tau \geq t$, where $\mathbb{E}[\cdot]$ denotes expectation with respect to probability measure $\mathbb{P}$.

The rest of the article is organized as follows. We start in Section 2 with the introduction of the SGBM algorithm, along with the necessary time discretization scheme and assumptions. Section 3 will present an error analysis of a simplified case of SGBM. The proof in this section forms the foundation for the error bound in any full algorithm. Later, in Section 4, we derive the full error bound for a specific choice of discretization scheme as an example. Finally, we finish the article with numerical experiments, and conclusion.

To close off this section, here is some further notation that is used in this article.

- For any vector x, $|x|$ denotes its Euclidean norm and x_r denotes its r-th component.

- Similarly, $X_{r,t}$ denotes the r-th component for any random process X_t.
- The gradient ∇g is defined as $\left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_q}\right)$ for any differentiable function $g : \mathbb{R}^q \to \mathbb{R}$.
- The notations $\mathbb{E}_t[\cdot]$ and $\mathbb{E}_t^x[\cdot]$ are the simplified notations for $\mathbb{E}[\cdot|\mathcal{F}_t]$ and $\mathbb{E}[\cdot|X_t = x]$
- For any set $\mathcal{S}$, the function $\mathbf{1}_S$ is the indicator function which takes value 1 when the input is within set $\mathcal{S}$ and 0 otherwise.
- For any function space H containing functions $\phi : \mathbb{R}^q \to \mathbb{R}$, H is defined as the set $\{\{(x, y) \in \mathbb{R}^q \times \mathbb{R} : \phi(x) \geq y\} : \phi \in H\}$.
- For any function ϕ and compact set $\mathcal{A}$, the constant $C_{\phi,\mathcal{A}}$ is an extended real number defined as $\sup_{x\in\mathcal{A}} |\phi(x)|$.

2 Assumption and Algorithm

In this section, we shall introduce the SGBM algorithm and its application to the approximation of BSDEs. To begin, we need to discretize the BSDEs.

2.1 Discretization Scheme

We denote a time grid $\pi = \{0 = t_0 < \ldots < t_N = T\}$ on the interval $[0, T]$ and let $\Delta_k := t_{k+1} - t_k$, $\Delta W_{l,k} := W_{l,t_{k+1}} - W_{l,t_k}$, and $\Delta W_k := (\Delta W_{1,k}, \ldots, \Delta W_{d,k})^T$ be the time-step, the Brownian motion increment along the l-th dimension and the Brownian motion increment, respectively, for $k \in \{0, \ldots, N-1\}$.

For the forward process X_t, we shall apply a Markovian approximation $X_{t_k}^\pi, t_k \in \pi$. The most common choice is the Itô-Euler scheme, which will be explained in Section 5. However, our algorithm can work with any simulation method where the conditional expectations over one time step are known for some specific functions.

The backward time discretizations (Y^π, Z^π) are based on the theta-scheme from [18] and [17]. This scheme is formulated as follows:

$$
\begin{aligned}
Y_{t_N}^\pi =& \Phi(X_{t_N}^\pi), \quad Z_{t_N}^\pi = \nabla\Phi(X_{t_N}^\pi)\sigma(t_N, X_{t_N}^\pi),\\
Z_{t_k}^\pi =& -\theta_2^{-1}(1-\theta_2)\mathbb{E}_{t_k}\left[Z_{t_{k+1}}^\pi\right] + \frac{1}{\Delta_k}\theta_2^{-1}\mathbb{E}_{t_k}\left[Y_{t_{k+1}}^\pi \Delta W_k^T\right]\\
&+ \theta_2^{-1}(1-\theta_2)\mathbb{E}_{t_k}\left[f_{k+1}(Y_{t_{k+1}}^\pi, Z_{t_{k+1}}^\pi)\Delta W_k^T\right],\ k = N-1, \ldots, 0,\\
Y_{t_k}^\pi =& \mathbb{E}_{t_k}\left[Y_{t_{k+1}}^\pi\right] + \Delta_k\theta_1 f_k(Y_{t_k}^\pi, Z_{t_k}^\pi)\\
&+ \Delta_k(1-\theta_1)\mathbb{E}_{t_k}\left[f_{k+1}(Y_{t_{k+1}}^\pi, Z_{t_{k+1}}^\pi)\right],\ k = N-1, \ldots, 0,
\end{aligned}
$$

where $f_k(y, z) := f(t_k, X_{t_k}^\pi, y, z)$, $0 \leq \theta_1 \leq 1$ and $0 < \theta_2 \leq 1$.

By picking various parameters (θ_1, θ_2), we can construct different types of one-step dynamic programming schemes. For example, the choice $(\theta_1, \theta_2) = (0, 1)$ would result in an explicit scheme, while the choice $(\theta_1, \theta_2) = (0.5, 0.5)$ would give the Crank-Nicolson scheme. Using a general construction means that our algorithm can be applied to various types of schemes and we may adjust our algorithms towards the specific problem.

2.2 Standing Assumptions

To ensure the existence and uniqueness of the solution of the continuous BSDEs, some basic assumptions are required. Moreover, these assumptions will affect the algorithm designed in changing the admissible choice of π and (θ_1, θ_2) and the error bound of the scheme. In this work, we assume the global Lipschitz condition as stated in Assumption 2.1. The reader should note that this assumption will affect the derivation and the result of the error bound for the complete algorithm, one example can be seen is Section 4. Assumption 2.1 is in force here as it is the most common assumption in the BSDE literature. Alternative assumptions can be found, for instance, in [10].

Assumption 2.1 (Globally Lipschitz driver)**.**

($\mathbf{A}_\xi$) i.) Φ is a measurable function and when $\theta_2 \neq 1$, Φ is differentiable.

ii.) the constant $C_{\Phi,A} < \infty$ for any given compact set A, and when $\theta_2 \neq 1$, $C_{\Phi',A} < \infty$ for any given compact set A.

($\mathbf{A}_F$) i.) $(t, x, y, z) \mapsto f(t, x, y, z)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.

ii.) For every $k \leq N$, $f_k(y, z)$ as defined in the last subsection is $\mathcal{F}_{t_k} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and there exists an $L_f \in [0, +\infty)$ such that

$$|f_k(y, z) - f_k(y', z')| \leq L_f(|y - y'| + |z - z'|), \qquad \forall k \in \{0, \dots, N\},$$

for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

iii.) There exists a $C_f \in [0, \infty)$ such that

$$|f_k(0, 0)| \leq C_f, \qquad \forall k \in \{0, \dots, N\}.$$

iv.) The time discretization is such that

$$\limsup_{N\to\infty} R_\pi < +\infty, \quad \text{where } R_\pi = \sup_{0\leq k\leq N-2} \frac{\Delta_k}{\Delta_{k+1}}.$$

Again, the assumption here is for the consistency of our derivation and does not imply that our algorithm can only be applied when these assumptions are satisfied.

2.3 Stochastic Grid Bundling Method

We now introduce SGBM. Due to the Markovian setting of $(X^\pi_{t_k}, \mathcal{F}_{t_k})_{t_k\in\pi}$, there exist functions $y_k^{(\theta_1,\theta_2)}(x)$ and $z_k^{(\theta_1,\theta_2)}(x)$ such that

$$Y^\pi_{t_k} = y_k^{(\theta_1,\theta_2)}(X^\pi_{t_k}),\ Z^\pi_{t_k} = z_k^{(\theta_1,\theta_2)}(X^\pi_{t_k}).$$

Our method is based on estimating these functions $(y_k^{(\theta_1,\theta_2)}(x), z_k^{(\theta_1,\theta_2)}(x))$ recursively backward in time by a local least-squares regression technique onto a finite function space with basis functions $(p_l)_{0\leq l\leq Q}$.

As a Monte Carlo based algorithm, our program starts with the simulation of M independent samples of $(X^\pi_{t_k})_{0\le k\le N}$, denoted by $(X^{\pi,m}_{t_k})_{1\le m\le M, 0\le k\le N}$. Note that in this basic algorithm, the simulation is only performed once. This scheme is therefore a non-nested Monte Carlo scheme.

The next step is the backward recursion. At initialization, we assign the terminal values to each path for our approximations, i.e.,

$$y_N^{(\theta_1,\theta_2),R,I}(X^{\pi,m}_{t_N}) = \Phi(X^{\pi,m}_{t_N}), \quad z_N^{(\theta_1,\theta_2),R}(X^{\pi,m}_{t_N}) = \nabla\Phi(X^{\pi,m}_{t_N})\sigma(t_N, X^{\pi,m}_{t_N}), \quad m = 1,\dots,M.$$

The following steps are performed recursively, backward in time, at t_k, $k = N-1,\dots,0$. First, we bundle all paths into $\mathcal{B}_{t_k}(1),\dots,\mathcal{B}_{t_k}(B)$ non-overlapping partitions based on the result of $(X^{\pi,m}_{t_k})$. We do not specify the exact bundling technique here as our algorithm and the error analysis are compatible with various bundling techniques. We shall specify the scheme later when we design a full scheme and perform numerical tests. This is the partition step.

Next, we perform the regress-later approximation separately within each bundle. The regress-later technique we are using combines the least-squares regression with the (analytical) expectations of the basis functions to calculate the necessary expectations.

Generally speaking, for M Monte Carlo paths, a standard regress-now algorithm for a dynamic programming problem finds a function ι within the space spanned by the regression basis such that it minimizes the value $\frac{1}{M}\sum_{i=1}^M (g(X^i_{t+\delta}) - \iota(X^i_t))$ and approximates the expectation $\mathbb{E}_t[g(X_{t+\delta})]$ by $\mathbb{E}_t[\iota(X_t)] = \iota(X_t)$. As a projection from a function of $X_{t+\delta}$ to a function of X_t is performed then, it would introduce a statistical bias to the approximation.

Instead, the regress-later technique we employ picks out a function κ such that it minimizes $\frac{1}{M}\sum_{i=1}^M (g(X^i_{t+\delta}) - \kappa(X^i_{t+\delta}))$ and approximates the expectation $\mathbb{E}_t[g(X_{t+\delta})]$ by $\mathbb{E}_t[\kappa(X_{t+\delta})]$. By using functions on the same variable in the regression basis, we can remove the statistical bias in the regression. However, the expectation of all basis functions must preferably be known in order to apply the regress-later technique efficiently.

In the context of our algorithm, we define the bundle-wise regression parameters $\alpha_{k+1}(b)$, $\beta_{k+1}(b)$, $\gamma_{k+1}(b)$ as

$$\alpha_{k+1}(b) = \arg\min_{\alpha\in\mathbb{R}^Q} \frac{\sum_{m=1}^M (p(X^{\pi,m}_{t_{k+1}})\alpha - y_{k+1}^{(\theta_1,\theta_2),R,I}(X^{\pi,m}_{t_{k+1}}))^2 \mathbf{1}_{\mathcal{B}_{t_k}(b)}(X^{\pi,m}_{t_k})}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}_{t_k(b)}}(X^{\pi.m}_{t_k})},$$

$$\beta_{i,k+1}(b) = \arg\min_{\beta\in\mathbb{R}^Q} \frac{\sum_{m=1}^M (p(X^{\pi,m}_{t_{k+1}})\beta - z_{i,k+1}^{(\theta_1,\theta_2),R}(X^{\pi,m}_{t_{k+1}}))^2 \mathbf{1}_{\mathcal{B}_{t_k}(b)}(X^{\pi,m}_{t_k})}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}_{t_k(b)}}(X^{\pi.m}_{t_k})},$$

$$\gamma_{k+1}(b) = \arg\min_{\gamma\in\mathbb{R}^Q} \frac{\sum_{m=1}^M (p(X^{\pi,m}_{t_{k+1}})\gamma - f_{k+1}(y_{k+1}^{(\theta_1,\theta_2),R,I}(X^{\pi,m}_{t_{k+1}}), z_{k+1}^{(\theta_1,\theta_2),R}(X^{\pi,m}_{t_{k+1}})))^2 \mathbf{1}_{\mathcal{B}_{t_k}(b)}(X^{\pi,m}_{t_k})}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}_{t_k(b)}}(X^{\pi.m}_{t_k})}.$$

The approximate functions within the bundle at time k are defined by the above parameters and the expectations $\mathbb{E}^x_{t_k}[p(X^\pi_{t_{k+1}})]$ and $\mathbb{E}^x_{t_k}\left[p(X^\pi_{t_{k+1}})\frac{\Delta W_{r,k}}{\Delta_k}\right]$:

$$\begin{aligned} z_{r,k}^{(\theta_1,\theta_2),R}(b,x) &= -\theta_2^{-1}(1-\theta_2)\mathbb{E}^x_{t_k}\left[p(X^\pi_{t_{k+1}})\right]\beta_{k+1}(b) \\ &\quad + \theta_2^{-1}\mathbb{E}^x_{t_k}\left[\frac{\Delta W_{r,k}}{\Delta_k}p(X^\pi_{t_{k+1}})\right](\alpha_{k+1}(b) + (1-\theta_2)\Delta_k\gamma_{k+1}(b)), \quad r = 1,\dots,d; \\ y_k^{(\theta_1,\theta_2),R,0}(b,x) &= \mathbb{E}^x_{t_k}\left[p(X^\pi_{t_{k+1}})\right]\alpha_{k+1}(b), \end{aligned}$$

$$y_k^{(\theta_1,\theta_2),R,i}(b,x) = \Delta_k\theta_1 f_k(y_k^{\pi,R,i-1}(x), z_k^{\pi,R}(x)) + h_k(x),$$
$$h_k(b,x) = \mathbb{E}_{t_k}^x\left[p(X_{t_{k+1}}^\pi)\right](\alpha_{k+1}(b) + \Delta_k(1-\theta_1)\gamma_{k+1}(b)), \quad i = 1,\dots,I.$$

Note that a Picard iteration is performed at each time step for each bundle if the choice of (θ_1,θ_2) results in an implicit scheme. For further details on the application of the Picard iteration, readers may refer to [8] or [17] and the references therein.

There are two details we would like to mention for the regress-later technique. First, as the expectations related to the basis functions are the foundation of the scheme, we assume all the necessary expectations to be known, either analytically or empirically. Furthermore, we assume that the following assumptions are satisfied.

Assumption 2.2. The regression basis $\{p_1,\dots,p_Q\}$ is assumed to satisfy the following assumptions.

($\mathbf{A}_p$) i.) $\mathbb{E}_{t_k}^x[p_l(X_{t_{k+1}}^\pi)]$ and $\mathbb{E}_{t_k}^x\left[p_l(X_{t_{k+1}}^\pi)\frac{\Delta W_{r,k}}{\Delta_k}\right]$ are known functions for all $k = 0,\dots N-1$, $l = 1,\dots,Q$ and $r = 1,\dots,d$.

ii.) For any given compact set A in $\mathbb{R}^q$, the constant $C_{p,A} := \max_{l=1,\dots,Q} C_{p_l,A}$. Moreover, there exists a constant $C_{M,p}$ such that

$$\sum_{l=1}^{Q}\mathbb{E}_{t_k}^x[p_l(X_{t_{k+1}}^\pi)] \le C_{M,A}, \qquad \forall x\in A, \text{ and } k = 0,\dots,N-1;$$

and

$$\sum_{l=1}^{Q}\mathbb{E}_{t_k}^x[p_l(X_{t_{k+1}}^\pi)\frac{\Delta W_{r,k}}{\Delta_k}] \le C_{M,A}, \qquad \forall x\in A, \text{ and } k = 0,\dots,N-1.$$

Next, to ensure the stability of our algorithm, $|\alpha_k(b)|$, $|\beta_{r,k}(b)|$ and $|\gamma_k(b)|$ must be bounded above for all k, b, r. In practice, this means that an error notion should be given by the program when the Euclidean norm of any regression coefficient vector is greater than a predetermined constant L. Further details on this requirement will be described in Section 3.

Finally, to simplify notation, we define the notations below for the regression result across the bundles.

$$\tilde{y}_{k+1}^{(\theta_1,\theta_2),R,I}(x_1,x_2) := \sum_{b=1}^{B}\mathbf{1}_{\mathcal{B}_{t_k}(b)}(x_1)p(x_2)\alpha_{k+1}(b),$$
$$\tilde{z}_{r,k+1}^{(\theta_1,\theta_2),R}(x_1,x_2) := \sum_{b=1}^{B}\mathbf{1}_{\mathcal{B}_{t_k}(b)}(x_1)p(x_2)\beta_{r,k+1}(b),$$
$$\tilde{f}_{k+1}^{(\theta_1,\theta_2),R}(x_1,x_2) := \sum_{b=1}^{B}\mathbf{1}_{\mathcal{B}_{t_k}(b)}(x_1)p(x_2)\gamma_{k+1}(b).$$

3 Refined Regression

In this section, we derive a proof of an error bound for our regress-later strategy. In order to ensure the stability of our algorithm, we have introduced a sample selection step into the algorithm and

modified the classical proof for nonparametric regression from [11], which was used in [10], for the derivation of the error bound to SGBM.

In order to simplify expressions, different notations are used in this section. We consider a random vector (X, Y), where X and Y are both $\mathbb{R}^q$, following the probability measure ν. A cloud of simulation paths can be generated by independently simulating M copies, $\{(X^m, Y^m) : m = 1, \dots, M\}$, defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$.

Let Π_B be a family of partitions of $\mathbb{R}^q$ into B non-overlapping sets. Denote by $\mathbb{B}$ a specific partition with $\mathbb{B} := \{\mathcal{B}_1, \dots, \mathcal{B}_B\}$ and $\bigcup_{b=1}^{B} \mathcal{B}_b = \mathbb{R}^d$. The partition which is used in the regression estimates is based on the simulation data X^m in our setting and to which bundle a sample belongs solely depends on X^m.

For a given partition and samples, we may define the estimate $\tilde{v}$ as

$$\begin{aligned}\tilde{v}(x, y) :=& \sum_{b=1}^{B} \mathbf{1}_{\mathcal{B}_b}(x) \arg\min_{\phi \in H} \left\{ \frac{\sum_{m=1}^{M} \mathbf{1}_{\mathcal{B}_b}(X^m) |v(Y^m) - \phi(Y^m)|^2}{\sum_{m=1}^{M} \mathbf{1}_{\mathcal{B}_b}(X^m)} \right\} \\ =& \sum_{b=1}^{B} \mathbf{1}_{\mathcal{B}_b}(x) \sum_{k=1}^{K} \alpha_k(b) p_k(y) =: \sum_{b=1}^{B} \mathbf{1}_{\mathcal{B}_b}(x) \tilde{v}_b(y), \end{aligned} \tag{3.1}$$

where H is a linear vector space spanned by continuous functions $\{p_1, \dots, p_Q\}$, with $p_l : \mathbb{R}^q \to \mathbb{R}$, $\forall l = 1, \dots, Q$. In fact, if we denote the total number of samples in a given bundle by $\#\mathcal{B}(b)$ and let $\{(X^{b,1}, Y^{b,1}), \dots, (X^{b,\#\mathcal{B}(b)}, Y^{b,\#\mathcal{B}(b)})\}$ be the samples in this bundle, the coefficients $\alpha(b)$ satisfy

$$\mathcal{I}^T \mathcal{I} \alpha(b) = \mathcal{I}^T v(Y^b),$$

with

$$\mathcal{I} = (p_j(Y^{b,i}))_{1 \le i \le \#\mathcal{B}(b), 1 \le j \le K} \text{ and } v(Y^b) = (v(Y^{b,1}), \dots, v(Y^{b,\#\mathcal{B}(b)}))^T.$$

Since the choice of partition solely depends on the simulation of X and all the regression coefficients are determined by the value of Y inside a bundle, the coefficients α in each bundle can be seen as random variables with respect to $(X^m, Y^m)_{m=1,\dots,M}$. Reversely, we may select the simulation cloud based on the regression coefficients. Let the set S be the set containing all possible collections of $(x_m, y_m)_{1 \le m} \in (\mathbb{R}^q \times \mathbb{R}^q)^M$ such that $|\alpha(b)| \le L$ for all b given that $(X^m, Y^m) = (x_m, y_m)_{1 \le m}$. We modify the probability of the simulation cloud by only accepting those results that are in S. We denote the modified expectation by $\hat{\mathbb{E}}_S$ and it is related to the original expectation by $\hat{\mathbb{E}}_S[\mathbf{1}_A] = \frac{\hat{\mathbb{E}}[\mathbf{1}_A \mathbf{1}_S]}{\hat{\mathbb{E}}[\mathbf{1}_S]}$. [1]

Remark 3.1. In a regress-now scheme, especially in a recursion scheme, the resulting approximation is truncated such that its value is within a bounded interval $[M_1, M_2]$. The truncation guarantees the convergence and the stability of the scheme. However, truncation is not feasible in our regress-later scheme as we have to keep the full function for further operation. Therefore, we must instead control the output by limiting the admissible samples.

The following random norms (depending on the simulation cloud (X^m, Y^m)) are used to quantify the error of approximation.

[1] The situation of $\frac{0}{0}$ should be understood as 0 and $\frac{K}{0}$ as ∞ in the rest of this article.

Definition 3.2. Let $\varphi : \hat{\Omega} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ or $\mathbb{R}^d$ be measurable. For any $\mathcal{B} \subset \mathbb{B}$, we define the following random norms

$$||\varphi||^2_{\mathcal{B},\infty} := \frac{\int_{\mathcal{B}} \int |\varphi(x,y)|^2 \nu(dx,dy)}{\int_{\mathcal{B}} \int \nu(dx,dy)}; \qquad ||\varphi||^2_{\mathcal{B},\#} := \frac{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)|\varphi(X^m,Y^m)|^2}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)}.$$

We derive the following theorem for the estimation of the error.

Theorem 3.3. *Assume a given compact set $A \subset \mathbb{R}^q$, $C_{v,A} \leq \infty$ and $\int v^2(x)\nu(dx) \leq \infty$, then, for the real function v, we have*

$$\begin{aligned}
&\hat{\mathbb{E}}_S \left[\iint |v(y) - \tilde{v}(x,y)|^2 \nu(dx,dy) \right] \\
\leq &\frac{\vartheta(L')}{\hat{\mathbb{E}}[\mathbf{1}_S]} \hat{\mathbb{E}} \left[\sum_{B \in \mathbb{B}} \int_B \int \nu(dx,dy) \frac{(\log(\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)) + 1)Q}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)} \right] \\
&+ \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]} \hat{\mathbb{E}} \left[\sum_{B \in \mathbb{B}} \int_B \int \nu(dx,dy) (\inf_{\phi \in H} \sup_{x \in B} \mathbb{E}\left[|v(Y) - \phi(Y)|^2 | X = x \right] \wedge L') \right] \\
&+ \hat{\mathbb{E}}_S \left[\iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy) \right],
\end{aligned}$$

for $L' := LQC^2_{p,A} + C_{v,A}$, and $\vartheta(L')$ a function depending on L'. Note that the set A is introduced here to avoid the restrictive assumption of v being bounded. It plays no role in the actual algorithm.

Proof. For any given partition $\mathbb{B}$, we start with the decomposition

$$\begin{aligned}
&\iint |v(y) - \tilde{v}(x,y)|^2 \nu(dx,dy) \\
\leq &\sum_{\mathcal{B} \in \mathbb{B}} \int_{\mathcal{B}} \int |v(y) - \tilde{v}(x,y)|^2 \mathbf{1}_{\mathcal{A}}(y) \nu(dx,dy) + \iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy) \\
= &\sum_{\mathcal{B} \in \mathbb{B}} \int_{\mathcal{B}} \int \nu(dx,dy) \Big(||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\infty} - 2||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#} + 2||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#} \Big)^2 \\
&+ \iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy) \\
\leq &\sum_{\mathcal{B} \in \mathbb{B}} \int_{\mathcal{B}} \int \nu(dx,dy) \Big(\max\{||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\infty} - 2||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#}, 0\} + 2||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#} \Big)^2 \\
&+ \iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy) \\
\leq &\sum_{\mathcal{B} \in \mathbb{B}} \int_{\mathcal{B}} \int \nu(dx,dy) 2 \max\{||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\infty} - 2||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#}, 0\}^2 \\
&+ \sum_{\mathcal{B} \in \mathbb{B}} 8 \int_{\mathcal{B}} \int \nu(dx,dy) ||(v - \tilde{v})\mathbf{1}_{\mathcal{A}}||^2_{\mathcal{B},\#} + \iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy) \\
=: &\sum_{\mathcal{B} \in \mathbb{B}} \int_{\mathcal{B}} \int \nu(dx,dy) T_{1,\mathcal{B}} + \sum_{\mathcal{B} \in \mathbb{B}} T_{2,\mathcal{B}} + \iint |v(y) - \tilde{v}(x,y)|^2 (1 - \mathbf{1}_{\mathcal{A}}(y)) \nu(dx,dy). \qquad (3.2)
\end{aligned}$$

In this step, we separate the last term, which is the integration with respect to the probability measure outside of a compact set $\mathcal{A}$ for Y, from the rest. The last term should be bounded based on our assumption that the original integration is finite and our choice of basis functions. We only have to deal with the first two terms.

First of all, under our choice of acceptable simulations, we have a bound for the function $|(v-\tilde{v})\mathbf{1}_{\mathcal{A}}|^2$. Given that $|\alpha(b)|^2 \leq L$, we have

$$\forall y, b,\ |\tilde{v}_{\mathcal{B}_b}(y)\mathbf{1}_{\mathcal{A}}(y)|^2 \leq \left(\sum_{l=1}^{Q}|\alpha_l(b)|^2\right)\left(\sum_{l=1}^{Q}|p_l(y)\mathbf{1}_{\mathcal{A}}(y)|^2\right) \leq LQ \max_{l=1\ldots,Q}\max_{y\in\mathcal{A}}|p_l(y)|^2.$$

As the result, $|(v-\tilde{v})\mathbf{1}_{\mathcal{A}}|^2$ is bounded from above by L', and $\forall\mathcal{B}$, we have $||(v-\tilde{v})\mathbf{1}_{\mathcal{A}}||^2_{\mathcal{B},\#} \leq L'$.

On the other hand, it is obvious that

$$||\mathbf{1}_{\mathcal{A}}(v-\tilde{v})||_{\mathcal{B},\#} \leq ||v-\tilde{v}||_{\mathcal{B},\#} = \min_{\phi\in H}||v-\phi||^2_{\mathcal{B},\#},$$

for any $\mathcal{B}$. If we define the function

$$\phi_{\mathcal{B}} := \arg\inf_{\phi\in H}\sup_{x\in B}\mathbb{E}\left[|v(Y)-\phi(Y)|^2|X=x\right],$$

and apply a limiting argument, we can show that, for $T_{2,\mathcal{B}}$ in (3.2),

$$\begin{aligned}
\hat{\mathbb{E}}_S\left[\sum_{\mathcal{B}\in\mathbb{B}}T_{2,\mathcal{B}}\right] &= 8\hat{\mathbb{E}}_S\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)||(v-\tilde{v})\mathbf{1}_{\mathcal{A}}||^2_{\mathcal{B},\#}\right] \\
&\leq 8\hat{\mathbb{E}}_S\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)||v-\phi_{\mathcal{B}}||^2_{\mathcal{B},\#}\right] \\
&\leq \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\frac{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)|v(Y^m)-\phi_{\mathcal{B}}(Y^m)|^2}{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)}\right] \\
&\leq \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\frac{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)\hat{\mathbb{E}}[|v(Y^m)-\phi_{\mathcal{B}}(Y^m)|^2|X^1,\ldots,X^M]}{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)}\right] \\
&= \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\frac{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)\mathbb{E}[|v(Y)-\phi_{\mathcal{B}}(Y)|^2|X=X^m]}{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)}\right] \\
&\leq \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\frac{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)\sup_{x\in\mathcal{B}}\mathbb{E}\left[|v(Y)-\phi_{\mathcal{B}}(Y)|^2|X=x\right]}{\sum_{m=1}^{M}\mathbf{1}_{\mathcal{B}}(X^m)}\right] \\
&\leq \frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{B\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\inf_{\phi\in H}\sup_{x\in\mathcal{B}}\mathbb{E}\left[|v(Y)-\phi(Y)|^2|X=x\right]\right].
\end{aligned} \tag{3.3}$$

In this equation, we first use the fact that the regression function is the best projection function under the empirical norm in the first two lines. Then we expand the denominator of our adjusted probability by also including the rejected cases. Finally, we finish this part by taking a continuous distribution.

Moreover, it is possible that $\inf_{\phi\in H}\sup_{x\in\mathcal{B}}\mathbb{E}\left[|v(Y)-\phi(Y)|^2|X=x\right]\geq L'$. For these bundles, we shall use the constant bound L' instead in Equation (3.3). We apply the same argument as before for the other bundles and have

$$\hat{\mathbb{E}}_S\left[\sum_{B\in\mathbb{B}}T_{2,\mathcal{B}}\right]\leq\frac{8}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{B\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)(\inf_{\phi\in H}\sup_{x\in\mathcal{B}}\mathbb{E}\left[|v(Y)-\phi(Y)|^2|X=x\right]\wedge L')\right].$$

Next, we should consider the term $T_{1,\mathcal{B}}$ in (3.2). Remember again that S denotes the modified probability, based on the regression coefficients, while $\mathcal{A}$ is a compact set defined with respect to Y only. We have,

$$\begin{aligned}
&\hat{\mathbb{E}}_S\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_B\int\nu(dx,dy)T_{1,\mathcal{B}}\right]\\
=&\frac{1}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\hat{\mathbb{E}}[T_{1,\mathcal{B}}\mathbf{1}_S|\mathbb{B}]\right]\\
=&\frac{1}{\hat{\mathbb{E}}[\mathbf{1}_S]}\hat{\mathbb{E}}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)\hat{\mathbb{E}}_{\mathcal{B}}[\mathbf{1}_S2\max\{||\mathbf{1}_{\mathcal{A}}(v-\tilde{v})||_{\mathcal{B},\infty}-2||(v-\tilde{v})\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#},0\}^2]\right].
\end{aligned}$$

Given a (random) partition $\mathbb{B}$, we may consider the two norms $||\cdot||_{\mathcal{B},\infty}$ and $||\cdot||_{\mathcal{B},\#}$ as the theoretical and empirical L^2 norms of a random process satisfying the probability distribution $\mathbb{P}_{\mathcal{B}}:=\frac{\int_{\mathcal{B}}\nu(dx,\cdot)}{\int_{\mathcal{B}}\int\nu(dx,dy)}$ and extend our notation on expectations to this measure.

Assume that $\sum_{m=1}^M\mathbf{1}_{\mathcal{B}}(X^m)=N$ and let $u>576L'/N$ be arbitrary, by Theorem 11.2 in [11], we find

$$\begin{aligned}
&\hat{\mathbb{P}}_{\mathcal{B}}\{2\max\{||\mathbf{1}_{\mathcal{A}}(v-\tilde{v})||_{B,\infty}-2||(v-\tilde{v})\mathbf{1}_{\mathcal{A}}||_{B,\#},0\}^2>u\text{ and the event }S\text{ is true}\}\\
\leq&\hat{\mathbb{P}}_{\mathcal{B}}\{\exists\phi\in H_L:||\mathbf{1}_{\mathcal{A}}(v-\phi)||_{B,\infty}-2||(v-\phi)\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#}>\sqrt{u/2}\text{ and the event }S\text{ is true}\}\\
\leq&\hat{\mathbb{P}}_B\{\exists\phi\in H_L:||\mathbf{1}_{\mathcal{A}}(v-\phi)||_{B,\infty}-2||(v-\phi)\mathbf{1}_{\mathcal{A}}||_{\mathcal{B},\#}>\sqrt{u/2}\}\\
\leq&3\hat{\mathbb{E}}_B[\mathcal{N}_2(\sqrt{u}/24,H_{L,\mathcal{A}},Y^{2N})]\exp\left(-\frac{Nu}{576L'}\right)\\
\leq&3\hat{\mathbb{E}}_B[\mathcal{N}_2(\sqrt{L'}/\sqrt{N},H_{L,\mathcal{A}},Y^{2N})]\exp\left(-\frac{Nu}{576L'}\right),
\end{aligned}$$

where H_L is the set of all functions in H whose coordinates with respect to the basis $(p_l)_{1\leq l\leq Q}$, has a Euclidean norm no greater than L and $H_{L,\mathcal{A}}$ the set containing all functions of the form $\mathbf{1}_{\mathcal{A}}(\phi-v)$, where ϕ belongs to H_L. $\mathcal{N}_2$ is called the covering number and it is bounded by Lemma 9.2 and Theorem 9.4 of [11]:

$$\mathcal{N}_2(\sqrt{L'}/\sqrt{N},H_{L,\mathcal{A}},Y^{2N})\leq3\left(\frac{2eL'}{L'/N}\log(\frac{3eL'}{L'/N})\right)^{V_{H^+_{L,\mathcal{A}}}}\leq3\left(3eN\right)^{2V_{H^+_{L,\mathcal{A}}}},$$

where V denotes the Vapnik-Chervonenkis dimension, which represents the number of elements in the largest set that can be shattered by a class of subset in $\mathbb{R}^q$. The readers are referred to section 9.4 of [11] for further information on $\mathcal{N}_2$ and V. Next, we notice that $V_{H^+_{L,\mathcal{A}}}\leq V_{H^+_L}$, which can be

shown by the following argument. Let $(y,z) \in \mathbb{R}^q \times \mathbb{R}$, if $y \notin \mathcal{A}$ and $z \geq 0$, then (y,z) is contained in none of the sets in $H^+_{L,\mathcal{A}}$ and if $y \notin \mathcal{A}$ and $z \leq 0$, then (y,z) is contained in each set of $H^+_{L,\mathcal{A}}$. Hence, if $H^+_{L,\mathcal{A}}$ shatters a set of points, then the x-coordinates of these points must lie in $\mathcal{A}$ and H^+_L also shatters this set of points.

In addition, we have the fact that $H_L \subset H$ and observe that

$$H^+ \subseteq \{\{(x,t) : \phi(x) + a_0 t \geq 0\} : \phi \in H, a_0 \in \mathbb{R}\},$$

which is a linear vector space of dimension less than or equal to $Q+1$, thus Theorem 9.5 of [11] implies

$$V_{H^+_L} \leq Q+1.$$

It follows that, for any $u > 576L'/N$, the probability under consideration is bounded by $9(3eN)^{2(Q+1)} \exp\left(-\frac{Nu}{576L'}\right)$, and for any $w > 576L'/N$,

$$\begin{aligned}\hat{\mathbb{E}}[T_{1,\mathcal{B}}\mathbf{1}_S|\mathbb{B}, \sum_{m=1}^{M} \mathbf{1}_{\mathcal{B}}(X^m) = N] &\leq w + 9(3eN)^{2(Q+1)} \int_w^\infty \exp\left(-\frac{Nt}{576L'}\right) dt \\ &= w + 9(3eN)^{2(Q+1)} \frac{576L'}{N} \exp\left(-\frac{Nw}{576L'}\right).\end{aligned}$$

Setting

$$w = \frac{576L'}{N} \log\left(9(3eN)^{2(Q+1)}\right),$$

this implies

$$\hat{\mathbb{E}}_S\left[\sum_{\mathcal{B}\in\mathbb{B}} \int_B \int \nu(dx,dy) T_{1,\mathcal{B}}\right] \leq \frac{\vartheta(L')}{\hat{\mathbb{E}}[\mathbf{1}_S]} \hat{\mathbb{E}}\left[\sum_{B\in\mathbb{B}} \int_B \int \nu(dx,dy) \frac{(\log(\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)) + 1)Q}{\sum_{m=1}^M \mathbf{1}_{\mathcal{B}}(X^m)}\right],$$

which in turn implies the assertion by an appropriate choice of ϑ. Note that ϑ is independent of the number of samples in a bundle. □

4 Explicit Scheme

To gain a better understanding of the error propagation over the recurring steps, one particular scheme, the explicit scheme, will be studied in detail in this section and a complete error description of the algorithm with respect to the application of SGBM towards BSDEs will be derived.

4.1 Algorithm

We select $(\theta_1, \theta_2) = (0,1)$, and have the following explicit discretization:

$$\begin{aligned}
&y_N^{(0,1)}(x) = \Phi(x), \quad z_N^{(0,1)}(x) = \nabla\Phi(x)\sigma(t_N, x), \\
&z_k^{(0,1)}(x) = \frac{1}{\Delta_k} \mathbb{E}^x_{t_k}\left[y_{k+1}^{(0,1)}(X^\pi_{t_{k+1}}) \Delta W_k^T\right],\ k = N-1, \ldots, 0, \\
&y_k^{(0,1)}(x) = \mathbb{E}^x_{t_k}\left[y_{k+1}^{(0,1)}(X^\pi_{t_{k+1}})\right] + \Delta_k \mathbb{E}^x_{t_k}\left[f_{k+1}(y_{k+1}^{(0,1)}(X^\pi_{t_{k+1}}), z_{k+1}^{(0,1)}(X^\pi_{t_{k+1}}))\right],\ k = N-1, \ldots, 0,
\end{aligned}$$

with the corresponding explicit algorithm for any given regression basis, sample set (and corresponding partition):

$$y_N^{(0,1),R}(x) = \Phi(x), \quad z_N^{(0,1),R}(x) = \nabla\Phi(x)\sigma(t_N, x);$$
$$z_k^{(0,1),R}(x) = \frac{1}{\Delta_k}\mathbb{E}_{t_k}^x\left[\tilde{y}_{k+1}^{(0,1),R}(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\Delta W_k^T\right],\ k = N-1,\dots,0,$$
$$y_k^{(0,1),R}(x) = \mathbb{E}_{t_k}^x\left[\tilde{y}_{k+1}^{(0,1),R}(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right] + \Delta_k\mathbb{E}_{t_k}^x\left[\tilde{f}_{k+1}^{(0,1),R}(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right],\ k = N-1,\dots,0.$$

We will suppress the $(0,1)$ notation for the rest of the section as there is no confusion of the scheme used.

4.2 Error analysis

In order to derive an explicit error bound, we also have to specify the bundling method. Our choice of bundling technique is the equal-partitioning technique as stated in [13]. To perform an equal partitioning, we consider a value function $v : \mathbb{R}^q \to \mathbb{R}$ and sort our sample paths according to the value $v(X_{t_n}^m)$. If we assume that M is divisible by B, then we perform the bundling by putting the first M/B samples from the sorted list into the first bundle, the following M/B paths into the second bundle, so on and so forth, until we have distributed all of the samples. If M is not divisible by B, we can just distribute the remainder samples such that we put 1 extra sample in some bundles. In order to simplify notation, we assume that M is always divisible by B in this section.

Moreover, we wish to apply the theorem from the last section to establish an error bound for the expectation of our approximation with respect to the selected simulation cloud. We need to check that after rejecting all the simulations that generate regression coefficients that are "too large", our approximation functions are bounded in the recursion. We notice that for any $k \le N$,

$$|y_k^R(x)| \le \max\{C_{M,A}L\sqrt{2(1+C_\pi^2)}, C_{\Phi,A}\} =: C_{Y,A}$$

and

$$|z_k^R(x)| \le \max\{C_{M,A}L\sqrt{2(1+C_\pi^2)}, C_{\sigma,A}C_{\Phi',A}\} =: C_{Z,A}$$

for all x in a compact set A. The constant C_π is defined as $\max_{k=0,\dots,N-1}\Delta_k$. These bounds can be proven by Assumption 2.1 and some simple inequalities. Furthermore, we have $\forall x \in A$,

$$f_k^R(x) := f_k(y_k^R(x), z_k^R(x)) \le C_f + L_f(C_{Y,A} + C_{Z,A}) =: C_{f,A},$$

which follows from the Lipschitz assumptions of f. Therefore, Theorem 3.3 applies.

We denote by S the set of all simulation cloud values $(X_{t_k}^{\pi,m})_{\substack{1\le m\le M\\ 0\le k\le N}}$ such that the Euclidean norm of the regression coefficients at each time step in each bundle is bounded by L, and the expectation is adjusted accordingly. With the application of Theorem 3.3, we know that for any given compact set $\mathcal{A}$,

$$\begin{aligned}
&\hat{\mathbb{E}}_{t_k,S}^x\left[\mathbb{E}_{t_k}^x\left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2\right]\right]\\
\le&\frac{\vartheta(L_y')}{\hat{\mathbb{E}}_{t_k}^x[\mathbf{1}_S]}\frac{(\log(M/B)+1)Q}{M/B} + \hat{\mathbb{E}}_{t_k,S}^x\left[\mathbb{E}_{t_k}^x\left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2(1-\mathbf{1}_{\mathcal{A}}(X_{t_{i+1}}^\pi))\right]\right]\\
&+\frac{8}{\hat{\mathbb{E}}_{t_k}^x[\mathbf{1}_S]}\hat{\mathbb{E}}_{t_k}^x\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)(\inf_{\phi\in H}\sup_{\theta\in\mathcal{B}}\mathbb{E}_{t_i}^\theta\left[|y_{t_{i+1}}^R(X_{t_{i+1}}^\pi) - \phi(X_{t_{i+1}}^\pi)|^2\right]\wedge L_y')\right] =: \Xi_{t_k}^x(i,y).
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\mathbb{E}}^x_{t_k,S}[\mathbb{E}^x_{t_k}\left[|f^R_{i+1}(X^\pi_{t_{i+1}})-\tilde{f}^R_{i+1}(X^\pi_{t_i},X^\pi_{t_{i+1}})|^2\right]]\\
\leq&\frac{\vartheta(L'_f)}{\hat{\mathbb{E}}^x_{t_k}[\mathbf{1}_S]}\frac{(\log(M/B)+1)Q}{M/B}+\hat{\mathbb{E}}^x_{t_k,S}\left[\mathbb{E}^x_{t_k}\left[|f^R_{i+1}(X^\pi_{t_{i+1}})-\tilde{f}^R_{i+1}(X^\pi_{t_i},X^\pi_{t_{i+1}})|^2(1-\mathbf{1}_{\mathcal{A}}(X^\pi_{t_{i+1}}))\right]\right]\\
&+\frac{8}{\hat{\mathbb{E}}^x_{t_k}[\mathbf{1}_S]}\hat{\mathbb{E}}^x_{t_k}\left[\sum_{\mathcal{B}\in\mathbb{B}}\int_{\mathcal{B}}\int\nu(dx,dy)(\inf_{\phi\in H}\sup_{\theta\in\mathcal{B}}\mathbb{E}^\theta_{t_i}\left[|f^R_{t_{i+1}}(X^\pi_{t_{i+1}})-\phi(X^\pi_{t_{i+1}})|^2\right]\wedge L'_f)\right]=:\Xi^x_{t_k}(i,f),
\end{aligned}$$

where $L'_y=LQC^2_{p,A}+C_{Y,A}$ and $L'_f=LQC^2_{p,A}+C_{f,A}$.

The following proposition summarizes the error bound for our scheme:

$$\Delta z_k(x):=z_k(x)-z^R_k(x);\quad \Delta y_k(x):=y_k(x)-y^R_k(x).$$

Proposition 4.1. *Given Assumption 2.1, and the time-grid π and an N-dimensional vector $\gamma\in(0,+\infty)^N$ satisfying $12q(L^2_fR_\pi\vee 1)(\Delta_k+\frac{1}{\gamma_k})\leq 1$ for all $k\leq N-1$, we have, for $0\leq k\leq N$,*

$$\begin{aligned}
&\hat{\mathbb{E}}^x_{t_k,S}[|\Delta y_k(x)|^2]\\
\leq&6qe^{T/4}\sum_{i=k}^{N-2}(\Delta_i+\gamma_i^{-1})\Gamma_iL^2_f\Xi^x_{t_k}(i+1,y)+3e^{T/4}\sum_{i=k}^{N-1}(\Delta_i+\gamma_i^{-1})\Gamma_i\frac{1}{\Delta_i}\Xi^x_{t_k}(i,y)\\
&+3e^{T/4}\sum_{i=k}^{N-1}(\Delta_i+\gamma_i^{-1})\Gamma_i\Delta_i\Xi^x_{t_k}(i,f),
\end{aligned}$$

where $\Gamma_i:=\prod_{i=0}^{k-1}(1+\gamma_i\Delta_i)$, and

$$\begin{aligned}
&\hat{\mathbb{E}}^x_{t_k,S}\left[\sum_{i=k}^{N-1}\Delta_i\mathbb{E}^x_{t_k}\left[|\Delta z_i(X^\pi_{t_i})|^2\right]\Gamma_i\right]\\
\leq&(12q+3Te^{T/4})\sum_{i=k+1}^{N-1}\left(\Delta_i+\gamma_i^{-1}\right)\frac{1}{\Delta_i}\Xi^x_{t_k}(i,y)\Gamma_i+6qTe^{T/4}\sum_{i=k}^{N-2}(\Delta_i+\gamma_i^{-1})\Gamma_iL^2_f\Xi^x_{t_k}(i+1,y)\\
&+(12q+3Te^{T/4})\sum_{i=k+1}^{N-1}\left(\Delta_i+\gamma_i^{-1}\right)\Delta_i\Xi^x_{t_k}(i,f)\Gamma_i+4\sum_{i=k}^{N-1}q\Xi^x_{t_k}(i,y)\Gamma_i.
\end{aligned}$$

Proof. The proof is fairly simular to the one used in [10] with the necessary modification for our present algorithm.

We shall derive an a-priori estimate of the error propagation in the recursion steps and we start with an estimate of $\Delta z_k(x)$. Note that we add an extra term in the formula which is equal to zero due to the expectation of the Brownian motion being equal to zero. This term is added here to facilitate future steps of the proof. We have

$$\begin{aligned}
|\Delta_k\Delta z_k(x)|^2=&\left(\mathbb{E}^x_{t_k}\left[\left(\Delta y_{k+1}(X^\pi_{t_{k+1}})-\mathbb{E}^x_{t_k}\left[\Delta y_{k+1}(X^\pi_{t_{k+1}})\right]\right)\Delta W^T_k\right]\right.\\
&\left.+\mathbb{E}^x_{t_k}\left[\left(y^R_{k+1}(X^\pi_{t_{k+1}})-\tilde{y}^R_{k+1}(X^\pi_{t_k},X^\pi_{t_{k+1}})\right)\Delta W^T_k\right]\right)^2\\
\leq&2\left(\mathbb{E}^x_{t_k}\left[\left(\Delta y_{k+1}(X^\pi_{t_{k+1}})-\mathbb{E}^x_{t_k}\left[\Delta y_{k+1}(X^\pi_{t_{k+1}})\right]\right)\Delta W^T_k\right]\right)^2\\
&+2\left(\mathbb{E}^x_{t_k}\left[\left(y^R_{k+1}(X^\pi_{k+1})-\tilde{y}^R_{k+1}(X^\pi_{t_k},X^\pi_{t_{k+1}})\right)\Delta W^T_k\right]\right)^2.
\end{aligned}$$

The inequality follows from the inequality $(\sum_{n=1}^{N} a_n)^2 \leq \sum_{n=1}^{N} N a_n^2$, which will be frequently used in the proof and will not be specified again. By applying the Cauchy-Schwarz inequality, we can derive bounds for the two terms separately, where

$$\begin{aligned}&\left|\mathbb{E}_{t_k}^x\left[\left(\Delta y_{k+1}(X_{t_{k+1}}^\pi) - \mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right]\right)\Delta W_k^T\right]\right|^2\\ \leq & q\Delta_k\left(\mathbb{E}_{t_k}^x\left[(\Delta y_{k+1}(X_{t_{k+1}}^\pi))^2\right] - \left(\mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right]\right)^2\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\mathbb{E}_{t_k}^x\left[\left(y_{k+1}^R(X_{k+1}^\pi) - \tilde{y}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right)\Delta W_k^T\right]\right|^2\\ \leq & q\Delta_k\mathbb{E}_{t_k}^x\left[\left|y_{k+1}^R(X_{k+1}^\pi) - \tilde{y}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right|^2\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_k|\Delta z_k(x)|^2 \leq & 2q\left(\mathbb{E}_{t_k}^x\left[(\Delta y_{k+1}(X_{t_{k+1}}^\pi))^2\right] - \left(\mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right]\right)^2\right)\\ &+ 2q\mathbb{E}_{t_k}^x\left[\left|y_{k+1}^R(X_{k+1}^\pi) - \tilde{y}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right|^2\right].\end{aligned} \tag{4.1}$$

Combining the fact that $(a+b)^2 \leq (1+\gamma_k\Delta_k)a^2 + (1+\gamma_k^{-1}\Delta_k^{-1})b^2$ for $(a,b) \in \mathbb{R}^2$, $\gamma_k > 0$, and the Lipschitz property of f, one deduces with Equation (4.1) that, for $0 \leq k \leq N-2$:

$$\begin{aligned}|\Delta y_k(x)|^2 \leq & \left(\mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right] + \mathbb{E}_{t_k}^x\left[y_{k+1}^R(X_{t_{k+1}}^\pi) - \tilde{y}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right]\right.\\ &+ \mathbb{E}_{t_k}^x\left[f_{k+1}(y_{k+1}(X_{t_{k+1}}^\pi), z_{k+1}(X_{t_{k+1}}^\pi)) - f_{k+1}^R(X_{t_{k+1}}^\pi)\right]\Delta_k\\ &\left.+\mathbb{E}_{t_k}^x\left[f_{k+1}^R(X_{t_{k+1}}^\pi) - \tilde{f}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)\right]\Delta_k\right)^2\\ \leq & (1+\gamma_k\Delta_k)\left(\mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right]\right)^2\\ &+ 3\left(\Delta_k + \gamma_k^{-1}\right)\Delta_k\left[L_f^2\mathbb{E}_{t_k}^x\left[(\Delta y_{k+1}(X_{t_{k+1}}^\pi))^2\right] + L_f^2\mathbb{E}_{t_k}^x\left[(\Delta z_{k+1}(X_{t_{k+1}}^\pi))^2\right]\right.\\ &\qquad+\frac{1}{\Delta_k^2}\mathbb{E}_{t_k}^x\left[|y_{k+1}(X_{t_{k+1}}^\pi) - \tilde{y}_{k+1}(X_{t_k}^\pi, X_{t_{k+1}}^\pi)|^2\right]\\ &\qquad\left.+\mathbb{E}_{t_k}^x\left[|f_{k+1}^R(X_{t_{k+1}}^\pi) - \tilde{f}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)|^2\right]\right]\\ \leq & (1+\gamma_k\Delta_k)\left(\mathbb{E}_{t_k}^x\left[\Delta y_{k+1}(X_{t_{k+1}}^\pi)\right]\right)^2\\ &+ 3(\Delta_k + \gamma_k^{-1})\Delta_k L_f^2\mathbb{E}_{t_k}^x\left[(\Delta y_{k+1}(X_{t_{k+1}}^\pi))^2\right]\\ &+ 6q(\Delta_k + \gamma_k^{-1})L_f^2 R_\pi\left(\mathbb{E}_{t_k}^x\left[(\Delta y_{k+2}(X_{t_{k+2}}^\pi))^2\right] - \mathbb{E}_{t_k}^x\left[\left(\mathbb{E}_{t_{k+1}}\left[\Delta y_{k+2}(X_{t_{k+2}}^\pi)\right]\right)^2\right]\right)\\ &+ 6q(\Delta_k + \gamma_k^{-1})L_f^2\mathbb{E}_{t_k}^x\left[\left|y_{k+2}^R(X_{t_{k+2}}^\pi) - \tilde{y}_{k+2}^R(X_{t_{k+1}}^\pi, X_{t_{k+2}}^\pi)\right|^2\right]\\ &+ 3(\Delta_k + \gamma_k^{-1})\Delta_k\frac{1}{\Delta_k^2}\mathbb{E}_{t_k}^x\left[|y_{k+1}^R(X_{t_{k+1}}^\pi) - \tilde{y}_{k+1}^R(X_{t_k}^\pi, X_{t_{k+1}}^\pi)|^2\right]\end{aligned}$$

$$+ 3(\Delta_k + \gamma_k^{-1})\Delta_k \mathbb{E}^x_{t_k}\left[|f^R_{k+1}(X^\pi_{t_{k+1}}) - \tilde{f}^R_{k+1}(X^\pi_{t_k}, X^\pi_{t_{k+1}})|^2\right], \tag{4.2}$$

while

$$\begin{aligned}|\Delta y_{N-1}(x)|^2 \leq & 3\left(\Delta_k + \gamma_k^{-1}\right)\Delta_k \left[\frac{1}{\Delta_k^2}\mathbb{E}^x_{t_k}\left[|y^R_{k+1}(X^\pi_{t_{k+1}}) - \tilde{y}^R_{k+1}(X^\pi_{t_k}, X^\pi_{t_{k+1}})|^2\right]\right. \\ & \left. + \mathbb{E}^x_{t_k}\left[|f^R_{k+1}(X^\pi_{t_{k+1}}) - \tilde{f}^R_{k+1}(X^\pi_{t_k}, X^\pi_{t_{k+1}})|^2\right]\right]. \end{aligned} \tag{4.3}$$

Next, we define the following sequence

$$\lambda_k := \left[1 + \left(\gamma_{k-1} + \frac{1}{4}\right)\Delta_{k-1}\right]\lambda_{k-1}, \text{ where } \lambda_0 := 1,$$

consider the sum of $|\Delta y_i^{(0,1)}(X^\pi_{t_i})|\lambda_i$, from $i = 1$ to $N-1$, and take conditional expectations with respect to $\mathcal{F}_k$. Applying Equation (4.3) for the case $k = N-1$ and Equation (4.2) otherwise, we have:

$$\begin{aligned}\sum_{i=k}^{N-1}\mathbb{E}^x_{t_k}\left[|\Delta y_i(X^\pi_{t_i})|^2\lambda_i\right] \leq & \sum_{i=k}^{N-2}\lambda_{i+1}\mathbb{E}^x_{t_k}\left[\left(\Delta y_{i+1}(X^\pi_{t_{i+1}})\right)^2\right] \\ & + \sum_{i=k}^{N-2} 6q(\Delta_i + \gamma_i^{-1})L_f^2\lambda_i\mathbb{E}^x_{t_k}\left[\left|y^R_{i+2}(X^\pi_{i+2}) - \tilde{y}^R_{i+2}(X^\pi_{i+1}, X^\pi_{i+2})\right|^2\right] \\ & + \sum_{i=k}^{N-1} 3(\Delta_i + \gamma_i^{-1})\Delta_i\frac{1}{\Delta_i^2}\lambda_i\mathbb{E}^x_{t_k}\left[|y^R_{i+1}(X^\pi_{t_{i+1}}) - \tilde{y}^R_{i+1}(X^\pi_{t_i}, X^\pi_{t_{i+1}})|^2\right] \\ & + \sum_{i=k}^{N-1} 3(\Delta_i + \gamma_i^{-1})\Delta_i\lambda_i\mathbb{E}^x_{t_k}\left[|f^R_{i+1}(X^\pi_{t_{i+1}}) - \tilde{f}^R_{i+1}(X^\pi_{t_i}, X^\pi_{t_{i+1}})|^2\right].\end{aligned}$$

By rearranging the terms, we have:

$$\begin{aligned}|\Delta y_k(x)|^2\lambda_k \leq & \sum_{i=k}^{N-2} 6q(\Delta_i + \gamma_i^{-1})L_f^2\lambda_i\mathbb{E}^x_{t_k}\left[\left|y^R_{i+2}(X^\pi_{t_{i+2}}) - \tilde{y}^R_{i+2}(X^\pi_{t_{i+1}}, X^\pi_{t_{i+2}})\right|^2\right] \\ & + \sum_{i=k}^{N-1} 3(\Delta_i + \gamma_i^{-1})\Delta_i\frac{1}{\Delta_i^2}\lambda_i\mathbb{E}^x_{t_k}\left[|y^R_{i+1}(X^\pi_{t_{i+1}}) - \tilde{y}^R_{i+1}(X^\pi_{t_i}, X^\pi_{t_{i+1}})|^2\right] \\ & + \sum_{i=k}^{N-1} 3(\Delta_i + \gamma_i^{-1})\Delta_i\lambda_i\mathbb{E}^x_{t_k}\left[|f^R_{i+1}(X^\pi_{t_{i+1}}) - \tilde{f}^R_{i+1}(X^\pi_{t_i}, X^\pi_{t_{i+1}})|^2\right].\end{aligned}$$

It follows from the simple inequality $\Gamma_k \leq \lambda_k = \exp(\sum_{i=0}^k \log(1 + (\gamma_i + 0.25)\Delta_i) \leq e^{T/4}\Gamma_k$ that, for

all $k \in \{0, \dots, N\}$,

$$
\begin{aligned}
|\Delta y_k(x)|^2 \leq & 6qe^{T/4} \sum_{i=k}^{N-2} (\Delta_i + \gamma_i^{-1}) \Gamma_i L_f^2 \mathbb{E}_{t_k}^x \left[\left| y_{i+2}^R(X_{t_{i+2}}^\pi) - \tilde{y}_{i+2}^R(X_{t_{i+1}}^\pi, X_{t_{i+2}}^\pi) \right|^2 \right] \\
& + 3e^{T/4} \sum_{i=k}^{N-1} (\Delta_i + \gamma_i^{-1}) \Gamma_i \frac{1}{\Delta_i} \mathbb{E}_{t_k}^x \left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \\
& + 3e^{T/4} \sum_{i=k}^{N-1} (\Delta_i + \gamma_i^{-1}) \Gamma_i \Delta_i \mathbb{E}_{t_k}^x \left[|f_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{f}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right]. \qquad (4.4)
\end{aligned}
$$

We can take expectations with respect to the simulation cloud and apply Theorem 3.3, which finishes the calculation for Δy.

Regarding the error term Δz, $\sum_{i=k}^{N-1} \Delta_i \mathbb{E}_{t_k}^x \left[|\Delta z_i(X_{t_i}^\pi)|^2 \right] \Gamma_i$ is bounded from above by

$$
\begin{aligned}
& \sum_{i=k}^{N-1} \Delta_i \mathbb{E}_{t_k}^x \left[|\Delta z_i(X_{t_i}^\pi)|^2 \right] \Gamma_i \\
\leq & \sum_{i=k}^{N-1} 2q \left(\mathbb{E}_{t_k}^x \left[(\Delta y_{i+1}(X_{t_{i+1}}^\pi))^2 \right] - \mathbb{E}_{t_k}^x \left[\left(\mathbb{E}_{t_i} \left[\Delta y_{i+1}(X_{t_{i+1}}^\pi) \right] \right)^2 \right] \right) \Gamma_{i+1} \\
& + \sum_{i=k}^{N-1} 2q \mathbb{E}_{t_k}^x \left[\left| y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi) \right|^2 \right] \Gamma_i \\
\leq & 2q \Gamma_N \mathbb{E}_{t_k}^x \left[(\Delta y_N(X_{t_N}^\pi))^2 \right] \\
& + \sum_{i=k+1}^{N-1} 2q \Gamma_i \left(\mathbb{E}_{t_k}^x \left[(\Delta y_i(X_{t_i}^\pi))^2 \right] - (1 + \gamma_i \Delta_i) \mathbb{E}_{t_k}^x \left[\left(\mathbb{E}_{t_i} \left[\Delta y_{i+1}(X_{t_{i+1}}^\pi) \right] \right)^2 \right] \right) \\
& + \sum_{i=k}^{N-1} 2q \mathbb{E}_{t_k}^x \left[\left| y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi) \right|^2 \right] \Gamma_i,
\end{aligned}
$$

because of Equation (4.1), and from (4.2), we have

$$
\begin{aligned}
\sum_{i=k}^{N-1} \Delta_i \mathbb{E}_{t_k}^x \left[|\Delta z_i(X_{t_i}^\pi)|^2 \right] \Gamma_i \leq & 6 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \Delta_i L_f^2 \mathbb{E}_{t_k}^x \left[(\Delta y_{i+1}(X_{t_{i+1}}^\pi))^2 \right] \Gamma_i \\
& + 6 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \Delta_i L_f^2 \mathbb{E}_{t_k}^x \left[(\Delta z_{i+1}(X_{t_{i+1}}^\pi))^2 \right] \Gamma_i \\
& + 6 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \frac{1}{\Delta_i} \mathbb{E}_{t_k}^x \left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + 6 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \Delta_i \mathbb{E}_{t_k}^x \left[|f_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{f}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + \sum_{i=k}^{N-1} 2q \mathbb{E}_{t_k}^x \left[\left| y_{i+1}^R(X_{i+1}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi) \right|^2 \right] \Gamma_i.
\end{aligned}
$$

Using the assumptions of the proposition statement, it follows that

$$
\begin{aligned}
&\sum_{i=k}^{N-1} \Delta_i \mathbb{E}_{t_k}^x \left[|\Delta z_i(X_{t_i}^\pi)|^2 \right] \Gamma_i \\
\leq & 12 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \frac{1}{\Delta_i} \mathbb{E}_{t_k}^x \left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + 12 \sum_{i=k+1}^{N-1} q \left(\Delta_i + \gamma_i^{-1} \right) \Delta_i \mathbb{E}_{t_k}^x \left[|f_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{f}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + 4 \sum_{i=k}^{N-1} q \mathbb{E}_{t_k}^x \left[\left| y_{i+1}^R(X_{i+1}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{i+1}^\pi) \right|^2 \right] \Gamma_i + \sum_{i=k+1}^{N-1} \Delta_i \mathbb{E}_{t_k}^x \left[(\Delta y_{i+1}(X_{t_{i+1}}^\pi))^2 \right] \Gamma_{i+1} \\
\leq & (12q + 3Te^{T/4}) \sum_{i=k+1}^{N-1} \left(\Delta_i + \gamma_i^{-1} \right) \frac{1}{\Delta_i} \mathbb{E}_{t_k}^x \left[|y_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + 6qTe^{T/4} \sum_{i=k}^{N-2} (\Delta_i + \gamma_i^{-1}) \Gamma_i L_f^2 \mathbb{E}_{t_k}^x \left[\left| y_{i+2}^R(X_{t_{i+2}}^\pi) - \tilde{y}_{i+2}^R(X_{t_{i+1}}^\pi, X_{i+2}^\pi) \right|^2 \right] \\
& + (12q + 3Te^{T/4}) \sum_{i=k+1}^{N-1} \left(\Delta_i + \gamma_i^{-1} \right) \Delta_i \mathbb{E}_{t_k}^x \left[|f_{i+1}^R(X_{t_{i+1}}^\pi) - \tilde{f}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi)|^2 \right] \Gamma_i \\
& + 4 \sum_{i=k}^{N-1} q \mathbb{E}_{t_k}^x \left[\left| y_{i+1}^R(X_{i+1}^\pi) - \tilde{y}_{i+1}^R(X_{t_i}^\pi, X_{t_{i+1}}^\pi) \right|^2 \right] \Gamma_i,
\end{aligned}
$$

in which we used the estimate from Equation (4.4). Again, taking expectations with respect to the simulation cloud finishes the proof. □

5 Numerical Experiments

In this section, we conduct our numerical experiments with the Euler discretization scheme, which is a common standard in the literature.

Definition 5.1 (Euler scheme)**.** The Euler scheme is defined by

$$X_{t_{k+1}}^\pi = X_{t_k}^\pi + b(t_k, X_{t_k}^\Delta)\Delta_k + \sigma(t_k, X_{t_k}^\pi)\Delta W_k =: d(X_{t_k}^\pi, \Delta W_k).$$

Note that the conditional expectation $\mathbb{E}_{t_k}^x \left[\frac{\Delta W_{l,k}}{\Delta_k} p(X_{t_{k+1}}^\pi) \right]$ can be calculated by:

$$
\begin{aligned}
\mathbb{E}_{t_k}^x \left[\frac{\Delta W_{l,k}}{\Delta_k} p(X_{t_{k+1}}^\pi) \right] =& \frac{1}{\sqrt{(2\pi)^q \Delta_k^q}} \int_{\mathbb{R}^q} p(d(x,y)) \frac{\partial}{\partial y_l} \left(-\exp\left(-\frac{1}{2} \sum_{r=1}^q \frac{y_r^2}{\Delta_k} \right) \right) dy \\
=& \frac{1}{\sqrt{(2\pi)^q \Delta_k^q}} \int_{\mathbb{R}^q} \exp\left(-\frac{1}{2} \sum_{r=1}^q \frac{y_r^2}{\Delta_p} \right) \nabla p(d(x,y)) \frac{\partial d(x,y)}{\partial y_l} dy \\
=& \mathbb{E}_{t_k}^x \left[\nabla p(X_{t_{k+1}}^\pi) \right] \sigma_l(t_k, x),
\end{aligned}
$$

where σ_l is the l-th column of the matrix σ.

For example, for the one-dimensional monomial x^r, $r \in \mathbb{N}$ and a forward process discretized by the Euler scheme, we have

$$\mathbb{E}^x_{t_k}\left[\frac{\Delta W_k}{\Delta_k}(X^\pi_{t_{k+1}})^r\right] = \mathbb{E}^x_{t_k}\left[r(X^\pi_{t_{k+1}})^{r-1}\right]\sigma(t_k, x).$$

Our numerical experiments will be conducted for the following examples.

5.1 Example 1

This example is originally from [18]. The considered FBSDE is given by

$$\begin{cases} dX_t = d\omega_t, \\ dY_t = -(Y_t Z_t - Z_t + 2.5Y_t - \sin(t + X_t)\cos(t + X_t) - 2\sin(t + X_t))dt + Z_t d\omega_t. \end{cases}$$

We take the initial and terminal conditions $x_0 = 0$ and $Y_T = \sin(X_T + T)$.

The exact solution is given by

$$(Y_t, Z_t) = (\sin(X_t + t), \cos(X_t + t)).$$

The terminal time is set to be $T = 1$ and $(Y_0, Z_0) = (0, 1)$. We use the set $\{1, x, x^2\}$ as the regression base for this example. We apply equal partitioning bundling for all our tests with the sample paths sorted by the value function x.

Table 1 shows the tests that we have run. Basically, our test cases can be placed into two groups. Test cases 1a, 1b, 1c are tests for the explicit version of our algorithm, while test cases 1d, 1e, 1f are for the Cracn-Nicolson version. Within each group, the three tests are run for identical test settings, except for the constant L, i.e., the pre-set limit for the Euclidean norm so that we may check the influence of the factor L. Within each test, the factors M, N and B are linked to a common factor J such that when J tends infinity, N, B and M/B tend to infinity as well. The error of our algorithm should converge with respect to J according to our theoretical derivations.

Test Case	θ_1	θ_2	I	M	N	B	L
1a	0	1	-	2^{2J}	2^J	2^J	100
1b	0	1	-	2^{2J}	2^J	2^J	10000
1c	0	1	-	2^{2J}	2^J	2^J	–
1d	0.5	0.5	4	2^{2J}	2^J	2^J	100
1e	0.5	0.5	4	2^{2J}	2^J	2^J	10000
1f	0.5	0.5	4	2^{2J}	2^J	2^J	–

Table 1: Test cases for Example 1

5.2 Example 2: Black-Scholes European option

The second example under consideration is the calculation of the price $v(t, S_t)$ of a European option under the q-dimensional Black-Scholes model by solving a FBSDE. We consider a market where the assets satisfy:

$$dS_{i,t} = \mu_i S_{i,t} dt + \sigma_i S_{i,t} dB_{i,t},\ 1 \le i \le q,$$

where B_t is a correlated q-dimensional Wiener process, with

$$dB_{i,t}dB_{j,t} = \rho_i j dt.$$

The parameters ρ_{ij} form a symmetric matrix ρ,

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1q} \\ \rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2q} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{q1} & \rho_{q2} & \rho_{q3} & \cdots & 1 \end{pmatrix},$$

and we assume it is invertible. By performing a Cholesky decomposition on ρ such that $LL^T = \rho$, where L is a lower triangular matrix with real and positive diagonal entries, we may relate the correlated and standard Brownian motions, as follows,

$$B_t = LW_t.$$

Along the line of reasoning in [17], we assume the financial market is complete, there is no trading restriction and a derivative can be perfectly hedged. To derive the corresponding pricing BSDE for a European option with terminal payoff $g(S_t)$, we construct a replicating portfolio Y_t, containing $\omega_{i,t}$ of asset $S_{i,t}$ and bonds with risk-free return rate r. Applying the self-financing assumption, the portfolio follows the SDE:

$$dY_t = -(-rY_t - \sum_{i=1}^{q} \omega_{i,t}(\mu_i - r)S_{i,t})dt + \sum_{i=1}^{q} \omega_{i.t}\sigma_i S_i dB_{i,t}.$$

If we set $Z_t = (\omega_{1,t}\sigma_1 S_{1,t}, \ldots, \omega_{q,t}\sigma_q, S_{q,t})L$, then (Y, Z) solves the BSDE,

$$\begin{cases} dY_t = -\left(-rY_t - Z_t L^{-1}\left(\frac{\mu - r}{\sigma}\right)\right)dt + Z_t dW_t; \\ Y_T = g(S_T), \end{cases}$$

where $\left(\frac{\mu - r}{\sigma}\right) = \left(\frac{\mu_1 - r}{\sigma_1}, \cdots, \frac{\mu_q - r}{\sigma_q}\right)^T$.

We test our algorithm for the next two cases.

5.2.1 Arithmetic Basket Put Option

In this numerical test, we use the 5-dimensional example from [16], which is designed as a tractable representation for the German stock index DAX at that time. All μ_i are assumed to be r here. The volatilities are given by

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (0.518, 0.648, 0.623, 0.570, 0.530),$$

while the correlations ρ are given by

$$\rho = \begin{pmatrix} 1.00 & 0.79 & 0.82 & 0.91 & 0.84 \\ 0.79 & 1.00 & 0.73 & 0.80 & 0.76 \\ 0.82 & 0.73 & 1.00 & 0.77 & 0.72 \\ 0.91 & 0.80 & 0.77 & 1.00 & 0.90 \\ 0.84 & 0.76 & 0.72 & 0.90 & 1.00 \end{pmatrix}.$$

We would consider a European weighted basket put option for $T = 1$ year, with the payoff function g given by

$$g(S) = \left(1 - \sum_{i=1}^{5} w_i S_i\right)^+,$$

where $(w_1, w_2, w_3, w_4, w_5) = (38.1, 6.5, 5.7, 27.0, 22.7)$. The risk free interest rate is $r = 0.05$ and all the stocks have starting value 0.01. The reference price is given as 0.175866 in [16].

We perform the equal-partitioning bundling technique and sort the paths in different bundles according to the ordering of the value $\sum_{i=1}^{5} w_i S_{i,t_p}^m$. The regression basis is chosen to be $p_k(x) = \left(\sum_{i=1}^{5} w_i x_i\right)^{k-1}$ for $k = 1, \ldots, K$.

Table 2 shows the tests that we have run. In these tests, we keep most of the parameters fixed but vary the number of bundles. We test our algorithm for explicit scheme with second-order regression basis and the Crank-Nicolson scheme with third-order regression basis. The change of basis is made to test the impact of the regression basis to our algorithm. We just keep these two sets of tests to demonstrate the impact of the number of bundles.

Test Case	θ_1	θ_2	I	M	N	B	L	K
2.1a	0.5	0.5	4	2^{12}	10	2^{2J}	-	3
2.1b	0	1	-	2^{12}	10	2^{2J}	-	2

Table 2: Test cases for Example 2.1

5.2.2 Example 2.2: Geometric Basket Put Option

Here we also consider the problem of pricing q-dimensional geometric basket options with initial state $S_0 = (40, \ldots, 40) \in \mathbb{R}^q$; strike $K = 40$; risk-free interest rate $r = 0.06$; volatility $\sigma_i = 0.2, i = 1, \ldots, q$; correlation $\rho_{ij} = 0.25, i, j = 1, \ldots, q, i \neq j$; and maturity $T = 1.0$. The final payoff function is given by

$$g(S) = \left(K - \left(\prod_{i=1}^{q} S_i\right)^{\frac{1}{q}}\right)^+.$$

This is the same setting as in [13] but for European options instead of Bermudan options.

We again use the equal-partitioning technique and sort the paths in different bundles according to the ordering of the values $\left(\prod_{i=1}^{q} S_{i,t_p}^m\right)^{\frac{1}{q}}$. The regression basis is chosen to be $p_k(x) = \left(\prod_{i=1}^{q} x_i\right)^{\frac{k-1}{q}}$ for $k = 1, \ldots, 3$.

The exact solution is given in this case by applying Itô's lemma to $\left(\prod_{i=1}^{q} S_{i,t_p}^m\right)^{\frac{1}{q}}$ and using the martingale pricing property.

Table 3 shows the tests that we have run. In these sets of tests, we fixed all the parameters but change the number of stocks in our test. This example is used to test the *scalability* of our methodology. Tests are performed for both explicit and Crank-Nicolson schemes.

Test Case	θ_1	θ_2	I	M	N	B	L	q
2.2a	0	1	-	2^{12}	20	16	-	J
2.2b	0.5	0.5	4	2^{12}	20	16	-	J

Table 3: Test cases for Example 2.2

5.3 Results

The results are given as the average values of 10 separated runs of the algorithm.

We first consider the results of the explicit version of our algorithm applied to Example 1, namely test cases 1a, 1b and 1c, in Table 4. This test can be seen as a proof of concept. As suggested in our proof of error, our algorithm converges when the number of steps N, the number of bundles B and the ratio M/B tend to infinity. Moreover, the total variation of the absolute errors among each successful run converges with respect to J too, as the reader can read from the second part of Table 4. It is defined as the sum of the individual differences between the Monte Carlo result of each run (which is not rejected) and the analytic solution, divided by the total number of successful runs.

$|Y_0 - y_0^{(\theta_1,\theta_2),R}(x_0)|$

J	2	3	4	5
1a	0.023535	0.20392	0.046947	0.057056
1b	0.18360	0.17807	0.098821	0.030159
1c	0.41648	0.14362	0.10368	0.04658
J	6	7	8	
1a	0.026622	0.018172	0.016179	
1b	0.028840	0.019621	0.0057568	
1c	0.018068	0.019175	0.0098448	

Total Variation/Successful Run

J	2	3	4	5
1a	0.28203	0.20392	0.081031	0.057056
1b	0.31030	0.17807	0.098884	0.044555
1c	0.60090	0.15673	0.10368	0.054715
J	6	7	8	
1a	0.027255	0.018172	0.016179	
1b	0.028840	0.020392	0.0079454	
1c	0.019420	0.019175	0.011833	

Table 4: Test result for Example 1 with explicit scheme.

While we have not shown the proof of convergence for the Crank–Nicolson scheme, where $\theta_1 = \theta_2 = 0.5$, our numerical tests for test cases 1d, 1e, 1f, in Table 5, suggest that it works well in our framework.

A specific point of interest is the impact of factor L introduced in Section 3 for the samples selection. It can be seen in Table 5 that when the number of paths or the bundles are few, a smaller value of L preserves the stability of our algorithm. In test case $1d$, where the factor L is relatively small, our algorithm rejected all tests for $J = 8$. One of the explanations is that the regression

$|Y_0 - y_0^{(\theta_1,\theta_2),R}(x_0)|$

J	2	3	4	5
1a	0.0053401	0.032606	0.18142	0.025799
1b	3.6788	0.24551	0.34892	0.069220
1c	4.6822×10^8	3.5241×10^{137}	1.0773×10^{44}	0.051122
J	6	7	8	
1a	0.0060404	0.020565	NA	
1b	0.012861	0.0013653	0.0024095	
1c	0.0050518	0.011735	0.0030526	

Total Variation/Successful Run

J	2	3	4	5
1a	0.23450	0.032606	0.18142	0.025799
1b	4.5732	0.37590	0.34892	0.075550
1c	4.6822×10^8	3.5241×10^{137}	1.0773×10^{44}	0.058288
J	6	7	8	
1a	0.012630	0.020565	NA	
1b	0.014571	0.012470	0.010903	
1c	0.020924	0.014260	0.0078873	

Table 5: Test result for Example 1 with Crank–Nicolson scheme

coefficients converge to the analytic projection coefficients on the basis space but the norm of these analytic coefficient is greater than L. The effect of the factor L actually can be seen in Table 4 too. Some runs for test case 1a were rejected when $J = 8$ and the result for $J = 8$ is worse than either 1b or 1c. On the contrary, if we remove the restriction on L altogether, the results are non-satisfactory when the value of J is low but converge when the number of time steps and samples are high enough. Heuristically, the regression coefficients should converge to the actual projection coefficients on the basis space, which results in a function that is bounded in a compact set. This in turns satisfies the conditions of the proof of convergence with respect to the regression.

Next, we shall move on to the result for the more practical and higher-dimensional Example 2. The results for Example 2.1 in Table 7 show that our method can be easily applied to a practical problem.

$|Y_0 - y_0^{(\theta_1,\theta_2),R}(x_0)|$

J	0	1	2
2.1a	2.0321×10^{-3}	2.2567×10^{-3}	1.9883×10^{-3}
2.1b	2.9314×10^{-3}	1.8934×10^{-3}	2.2151×10^{-4}

Table 6: Test result for Example 2.1

With respect to the problem of dimensionality, we can check the results in Table 7. Since the analytical solution is known to this problem, we compare our result to the actual value. It can be seen that under our choice of bundling and regression basis, the accuracy of our method is similar across all choices of problem dimensions. This suggested that with appropriate setting, our algorithm can easily scale up to tackle high-dimensional problems.

	$\lvert Y_0 - y_0^{(\theta_1,\theta_2),R}(x_0)\rvert$				
J	1	2	3	4	5
2.2a	6.5482×10^{-3}	7.3015×10^{-3}	6.6827×10^{-3}	8.0384×10^{-3}	7.1308×10^{-3}
2.2b	5.1918×10^{-3}	6.9460×10^{-3}	6.4038×10^{-3}	6.9507×10^{-3}	7.4937×10^{-3}
J	6	7	8	9	10
2.2a	6.9885×10^{-3}	7.5067×10^{-3}	6.9271×10^{-3}	6.9993×10^{-3}	7.5682×10^{-3}
2.2b	7.2034×10^{-3}	7.1633×10^{-3}	7.0850×10^{-3}	7.2023×10^{-3}	6.7595×10^{-3}
J	11	12	13	14	15
2.2a	6.9549×10^{-3}	7.4005×10^{-3}	7.5329×10^{-3}	7.1437×10^{-3}	7.1364×10^{-3}
2.2b	8.4614×10^{-3}	7.1430×10^{-3}	7.6267×10^{-3}	7.8998×10^{-3}	7.2455×10^{-3}

Table 7: Test result for Example 2.2

More generally, all the results from Example 2 suggest that linking the bundling criterion and the regression basis to the terminal condition can deliver an accurate algorithm. Adapting our algorithm to a specific problem to improve the performance could be a promising direction of further research. In fact, the choice of basis itself deserves further study. Even in our localised setting, regression with respect to the linear basis scheme fails to converge for Example 1. A more sophisticated way to pick the regression basis may be important to put our algorithm into actual applications.

To sum up, we have developed a new algorithm for approximating BSDEs based on SGBM and our numerical tests showed that this new algorithm can deliver accurate estimation results.

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